LIPSCHITZ EQUIVALENCE OF SUBSETS OF SELF-CONFORMAL SETS

MARTA LLORENTE AND PERTTI MATTILA

Abstract. We give sufficient conditions to guarantee that if two self-conformal sets E and F have Lipschitz equivalent subsets of positive measure, then there is a bilipschitz map of E into, or onto, F.

1. Introduction

In this note we shall consider the following question: suppose that E and F are self-conformal (see below for terminology) subsets of $\mathbb{R}^n$ of the same Hausdorff dimension s. If there are measurable subsets $E' \subset E$ and $F' \subset F$ of positive s-dimensional Hausdorff measure which are Lipschitz equivalent, are then also E and F Lipschitz equivalent? By Lipschitz equivalence we mean that there is a bilipschitz map of E onto F. We shall prove that this is true for some Cantor type sets, more precisely, when E and F satisfy the strong separation condition and one of these sets is generated by two maps. If E and F satisfy the open set condition we shall show that there is a bilipschitz map of E into F, but not necessarily onto.

Lipschitz equivalence of self-similar and self-conformal sets has been considered in [FM], [RRX] and [RRY], and a general study of sets having many Lipschitz equivalent subsets can be found in [DS]. In particular, Falconer and Marsh gave necessary algebraic conditions on the similarity ratios of the generating maps in order that two sets satisfying strong separation condition could be Lipschitz equivalent. These imply, for example, that many self similar subsets of Hausdorff dimension $\log 2/\log 3$ are not Lipschitz equivalent with the classical $1/3$-Cantor set. Hence by our result in such a case neither are any of their subsets of positive measure. In [MS] it was shown for the much larger

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class of Ahlfors-David regular sets $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^n$ of dimensions $s < t < 1$, respectively, that E is Lipschitz equivalent to some subset of F. Combining the results of Falconer and Marsh and the results of this paper we see that this cannot be extended to the case $s = t$.

2. Preliminaries

We shall denote the closed ball with center x and radius r by $B(x,r)$. The diameter of a set A is denoted by $d(A)$. A map $h : A \to B, A \subset \mathbb{R}^n, B \subset \mathbb{R}^p$, is said to be bilipschitz, or L-bilipschitz, if there is $L < \infty$ such that

$$|x-y|/L \leq |h(x)-h(y)| \leq L|x-y| \text{ for all } x,y \in A.$$

The smallest such a constant L is denoted by $\operatorname{bilip}(h)$. Note that we don't require h to be onto.

We shall make use of the following simple lemma:

2.1. **Lemma.** *Let $A_k \subset A \subset \mathbb{R}^n, B_k \subset B \subset \mathbb{R}^p$, be compact and $h_k : A_k \to B_k, k = 1,2,\dots,$ be such that for some $L, 0 < L < \infty$,*

$$|x-y|/L \leq |h_k(x)-h_k(y)| \leq L|x-y| \textit{ for all } x,y \in A_k, k = 1,2,\dots.$$

If for every $x \in A$ there are $x_k \in A_k$ with $x_k \to x$, then there is an L-bilipschitz map $h : A \to B$. If also $h_k(A_k) = B_k$ and for every $y \in B$ there are $y_k \in B_k$ with $y_k \to y$, then $h(A) = B$.

Proof. We can extend the maps h_k to L-Lipschitz maps $\mathbb{R}^n \to \mathbb{R}^p$; we shall denote by h_k also the extended maps. By the Arzela-Ascoli theorem the sequence (h_k) has a subsequence which converges uniformly on compact subsets of $\mathbb{R}^n$ to an L-Lipschitz map h. It is easy to check that $h|A : A \to B$ is L-bilipschitz, and also the last claim is simple. □

We denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure. We shall use the fact (see, e.g. Theorem 6.2 in [M]) that for any $\mathcal{H}^s$ measurable sets $A \subset E \subset \mathbb{R}^n$ with $\mathcal{H}^s(E) < \infty$, $\mathcal{H}^s$ almost all points $x \in A$ are density points of A with respect to E in the sense that

$$\lim_{r\to 0} r^{-s}\mathcal{H}^s(B(x,r) \cap E \setminus A) = 0. \tag{2.2}$$

We shall consider a conformal iterated function system $\{f_1,\dots,f_N\}$ in $\mathbb{R}^n$ following the scheme of [MU]. By this we mean that $N \geq 2$ and there is an open connected set $V \subset \mathbb{R}^n$ such that each $f_i : V \to V$ is an injective conformal contraction;

$$|f_i(x) - f_i(y)| \leq L_0 < 1 \text{ for all } x,y \in V, i = 1,\dots,N, \tag{2.3}$$

of class $C^{1+\gamma}$ for some fixed $\gamma > 0$, that is, the partial derivatives are Hölder continuous with exponent γ. Of course, the last condition is only needed when $n = 1$, since the conformal maps in higher dimensions are C^∞. We shall also assume that there are positive constants c_1 and C_1 such that $0 < c_1 < C_1 < 1$ and

$$c_1 \le ||Df_i(x)|| \le C_1 \text{ for all } x \in V, i = 1, \dots, N.$$

Here $||\cdot||$ is the operator norm of a linear map. Then there is a unique compact invariant set E such that (see [H])

$$E = \bigcup_{i=1}^{N} f_i(E).$$

We shall use the following notation. Let $\mathcal{N} = \{1, 2, ..., n\}$ and

$$\mathcal{N}^k = \{\mathbf{i} = (i_1, ..., i_k) : i_j \in \mathcal{N} \ \forall \ j = 1, ...k\}.$$

For $\mathbf{i} = (i_1, ..., i_k) \in \mathcal{N}^k$ let

$$f_{\mathbf{i}} = f_{i_1} \circ f_{i_2} \circ ... \circ f_{i_k},$$
$$E_{\mathbf{i}} = f_{\mathbf{i}}(E),$$
$$d_{\mathbf{i}} = d(E_{\mathbf{i}}).$$

Then for every $k = 1, 2, \dots,$

$$E = \bigcup_{\mathbf{i} \in \mathcal{N}^k} f_{\mathbf{i}}(E).$$

We shall assume that the system $\{f_i\}$ satisfies the open set condition, that is, there is a non-empty bounded open set $O \subset V$ such that the closure of O is contained in V and the sets $f_i(O)$ are disjoint subsets of O. Then the following bounded distortion property holds, see [MU], Remark 2.3: there is constant K such that for $\mathbf{i} \in \mathcal{N}^k$,

$$||Df_{\mathbf{i}}(x)|| \le K ||Df_{\mathbf{i}}(y)|| \text{ for all } x, y \in V.$$

Let s be the Hausdorff dimension of E. The open set condition implies (and it is in fact equivalent to) that $0 < \mathcal{H}^s(E) < \infty$, see [PRSS]. From the bounded distortion property one can conclude that there exist positive constants $c < 1, C$ and R such that for all $\mathbf{i} = (i_1, \dots, i_k) \in \mathcal{N}^k$ and $\mathbf{i}i = (i_1, \dots, i_k, i) \in \mathcal{N}^{k+1}$,

$$cd_{\mathbf{i}} \le ||Df_{\mathbf{i}}(x)|| \le Cd_{\mathbf{i}} \text{ for } x \in V, \tag{2.4}$$

$$B(f_{\mathbf{i}}(x), cd_{\mathbf{i}}r) \subset f_{\mathbf{i}}(B(x, r)) \text{ for } x \in E, 0 < r < R, \tag{2.5}$$

$$cd_{\mathbf{i}}|x - y| \le |f_{\mathbf{i}}(x) - f_{\mathbf{i}}(y)| \le Cd_{\mathbf{i}}|x - y| \text{ for } x, y \in E, \tag{2.6}$$

$$d_{\mathbf{i}} \le Cd_{\mathbf{i}i}, \tag{2.7}$$

$$d_{\mathbf{i}} \le L_0^k d(E), d_{\mathbf{i}} \to 0 \text{ when } \mathbf{i} \in \mathcal{N}^k, k \to \infty, \tag{2.8}$$

(2.9) $$cr^s \leq \mathcal{H}^s(B(x,r)\cap E) \leq Cr^s \text{ for } x \in E, 0 < r < R.$$

Here (2.4) and (2.5) are proven in Section 2 of [MU], (2.6) follows easily from (2.4), (2.7) follows from (2.6), (2.8) follows from (2.3), and (2.9) is proven in Lemma 3.14 of [MU] (where a measure m is used instead of $\mathcal{H}^s$, but this is equivalent).

We shall assume all the time that $f_{\mathbf{i}}, E, E_{\mathbf{i}}, d_{\mathbf{i}}, s, c$ and C are as above. We shall also consider another conformal iterated function system $\{g_1, \dots, g_P\}$ in $\mathbb{R}^p$ and use the corresponding notation $g_{\mathbf{i}}, F, F_{\mathbf{i}}, e_{\mathbf{i}}, s, c$ and C; in particular we assume that E and F have the same Hausdorff dimension s and we choose the constants c and C so that they match both systems.

3. Open set condition

We shall use the result of Peres, Rams, Simon and Solomyak from [PRSS] (proven first by Schief in [S] for self-similar sets) according to which the open set condition is equivalent to the strong open set condition: for some open set O as in the open set condition $E \cap O \neq \emptyset$. Both are also equivalent with $0 < \mathcal{H}^s(E) < \infty$.

It is easy to see that if O is as in the open set condition, then $E \subset \bar{O}$, and so for all $\mathbf{i} \in \mathcal{N}^k, E_{\mathbf{i}} \subset f_{\mathbf{i}}(\bar{O})$. Using the strong open set condition, choose $x_0 \in E \cap O$ and $r_0, 0 < r_0 < 1$, such that $B(x_0, r_0) \subset O$. Then for all $\mathbf{i} \in \mathcal{N}^k$ by (2.5), $B(f_{\mathbf{i}}(x_0), cd_{\mathbf{i}}r_0) \subset f_{\mathbf{i}}(B(x_0,r_0)) \subset f_{\mathbf{i}}(O)$, from which it follows that

(3.1) $$(E \setminus E_{\mathbf{i}}) \cap B(f_{\mathbf{i}}(x_0), cd_{\mathbf{i}}r_0) = \emptyset.$$

3.2. **Lemma.** *Suppose that the system $\{f_1, \dots, f_N\}$ satisfies the open set condition. Let $b = cr_0/2$ where r_0 is as above. Then for $\mathcal{H}^s$ almost all $x \in E$ there are $\mathbf{i}_k, k = 1, 2, \dots,$ such that $x \in E_{\mathbf{i}_k}, d_{\mathbf{i}_k} \to 0$ and $(E \setminus E_{\mathbf{i}_k}) \cap B(x, bd_{\mathbf{i}_k}) = \emptyset$ for all $k = 1, 2, \dots$.*

Proof. Let $A_m, m = 1, 2, \dots,$ be the set of $x \in E$ such that $(E \setminus E_{\mathbf{i}}) \cap B(x, bd_{\mathbf{i}}) \neq \emptyset$ whenever $x \in E_{\mathbf{i}}$ and $d_{\mathbf{i}} < 1/m$. We shall show that $\mathcal{H}^s(A_m) = 0$ which implies the lemma.

Let $x \in A_m$. If $x \in E_{\mathbf{i}}$ and $d_{\mathbf{i}} < 1/m$, then by (3.1) and the choice of b,

$$(E \setminus E_{\mathbf{i}}) \cap B(f_{\mathbf{i}}(x_0), 2bd_{\mathbf{i}}) = \emptyset,$$

and so

$$(E \setminus E_{\mathbf{i}}) \cap B(y, bd_{\mathbf{i}}) = \emptyset \text{ for all } y \in B(f_{\mathbf{i}}(x_0), bd_{\mathbf{i}}),$$

whence

$$B(f_{\mathbf{i}}(x_0), bd_{\mathbf{i}}) \subset B(x, (1+b)d_{\mathbf{i}}) \setminus A_m.$$

Therefore by (2.9),

$$\mathcal{H}^s(B(x,(1+b)d_{\mathbf{i}})\cap E\setminus A_m)\geq c(bd_{\mathbf{i}})^s.$$

It follows that x cannot be a density point of A_m and proves that $\mathcal{H}^s(A_m)=0$. □

3.3. **Theorem.** *Suppose that the systems* $\{f_1,\dots,f_N\}$ *and* $\{g_1,\dots,g_P\}$ *satisfy the open set condition. If there are an* $\mathcal{H}^s$ *measurable subset* E' *of* E *with* $\mathcal{H}^s(E')>0$ *and a bilipschitz map* $h:E'\to F$*, then there exists a bilipschitz map* $\tilde{h}:E\to F$*.*

Proof. We may assume that E' is compact. Let $x\in E'$ be a density point of E' such that also $y=h(x)$ is a density point of $F'=h(E')$ and that, using Lemma 3.2, there are $b>0$ and $\mathbf{j}_k$ such that $y\in F_{\mathbf{j}_k}, e_{\mathbf{j}_k}\to 0$ and $(F\setminus F_{\mathbf{j}_k})\cap B(y,be_{\mathbf{j}_k})=\emptyset$ for all $k=1,2,\dots$. Let L be the bilipschitz constant of h. For $k=1,2,\dots$, let $\mathbf{i}_k$ be a multi-index of shortest length such that $x\in E_{\mathbf{i}_k}$ and $Ld_{\mathbf{i}_k}\leq be_{\mathbf{j}_k}$. Then

$$h(E'\cap E_{\mathbf{i}_k})\subset B(y,Ld_{\mathbf{i}_k})\cap F'\subset B(y,be_{\mathbf{j}_k})\cap F'\cap F_{\mathbf{j}_k},$$

and, by the minimality of $\mathbf{i}_k$ and (2.7), if $\mathbf{i}_k=\mathbf{i}i$, then

$$d_{\mathbf{i}_k}\geq C^{-1}d_{\mathbf{i}}\geq (CL)^{-1}be_{\mathbf{j}_k}.$$

Denote

$$h_k=g_{\mathbf{j}_k}^{-1}\circ h\circ f_{\mathbf{i}_k}:A_k:=f_{\mathbf{i}_k}^{-1}(E'\cap E_{\mathbf{i}_k})\to F.$$

Then by (2.6) h_k is bilipschitz with $\operatorname{bilip}(h_k)\leq L'$ with L' independent of k. To complete the proof we shall check that the condition of Lemma 2.1 holds for A_k and $A=E$.

Suppose it doesn't. Then there are $a\in E$ and $r, 0<r<R$, such that for some subsequence of $(\mathbf{i}_k)$, which we assume to be the full sequence, we have $B(a,r)\cap f_{\mathbf{i}_k}^{-1}(E'\cap E_{\mathbf{i}_k})=\emptyset$. Then by (2.5)

$$B(f_{\mathbf{i}_k}(a),cd_{\mathbf{i}_k}r)\cap E_{\mathbf{i}_k}\subset f_{\mathbf{i}_k}(B(a,r))\cap E_{\mathbf{i}_k}\subset B(x,d_{\mathbf{i}_k})\cap(E\setminus E').$$

This gives by (2.9) that

$$\mathcal{H}^s(B(x,d_{\mathbf{i}_k})\cap(E\setminus E'))\geq c^{s+1}(d_{\mathbf{i}_k}r)^s.$$

This contradicts the fact that x is a density point and proves the theorem. □

3.4. *Remark.* We would like to thank Tamas Keleti for the following observation: in Theorem 3.3 we cannot always get the map $\tilde{h}$ to be onto. To see this, take in Theorem 3.3 $E'=E\subset\mathbb{R}$ to be a self-similar set satisfying the open set condition with positive Lebesgue measure and E not being an interval, F a compact interval containing E and

$h = id$, the identity map. That such a set E exists can be seen for example from [B].

Also note that, as a straightforward consequence of Theorem 3.3, we find that a self-similar set in $\mathbb{R}^n$ with open set condition and positive Lebesgue measure must have a non-empty interior. A simple direct proof for this fact can be found in [S].

4. Strong separation condition

We say that the strong separation condition holds if the sets $f_i(E), i = 1, \dots, N$, are disjoint. Then we can choose the constant c so that, in addition to the previous properties,

$$\operatorname{dist}(E_{\mathbf{i}i}, E_{\mathbf{i}j}) \geq c d_{\mathbf{i}} \text{ for } i \neq j. \tag{4.1}$$

4.2. **Theorem.** *Suppose that $p = 2$ and the systems $\{f_1, \dots, f_N\}$ and $\{g_1, g_2\}$ satisfy the strong separation condition. If there are an $\mathcal{H}^s$ measurable subset E' of E with $\mathcal{H}^s(E') > 0$ and a bilipschitz map $h : E' \to F$, then there exists a bilipschitz map $\tilde{h} : E \to F$ with $\tilde{h}(E) = F$.*

Proof. By Theorem 3.3, we may assume that $E' = E$. Let $x \in E$ be such that $y = h(x)$ is a density point of $F' = h(E)$. For every $k \in \mathbb{N}$, choose $E_{\mathbf{i}_k}$ such that $x \in E_{\mathbf{i}_k}$ and $d_{\mathbf{i}_k} \to 0$. Then there are sets $F_{k,l} = F_{\mathbf{j}_{k(l)}}$ and the corresponding maps $g_{k,l} = g_{\mathbf{j}_{k(l)}}, l = 1, \dots m_k$, such that $d(F_{k,l}) \geq c_1 d_{\mathbf{i}_k}$ and $m_k \leq m$, where $c_1 > 0$ and m are independent of k, $F' \cap F_{k,l} \neq \emptyset$ and

$$h(E_{\mathbf{i}_k}) = \bigcup_{l=1}^{m_k} F' \cap F_{k,l}.$$

This is essentially Lemma 3.2 in [FM] but we give a quick proof. By (4.1) $\operatorname{dist}(E_{\mathbf{i}_k}, E_\lambda) \geq c d_{\mathbf{i}_k}$ whenever $E_{\mathbf{i}_k} \cap E_\lambda = \emptyset$. If for such λ and for some $\mathbf{j}$, $h(E_{\mathbf{i}_k}) \cap F_{\mathbf{j}} \neq \emptyset$ and $h(E_\lambda) \cap F_{\mathbf{j}} \neq \emptyset$, then $d(F_{\mathbf{j}}) \geq \operatorname{dist}(h(E_{\mathbf{i}_k}), h(E_\lambda)) \geq (c/L) d_{\mathbf{i}_k}$ where L is the bilipschitz constant of h. Therefore we can take as $F_{k,l}$ all the maximal sets $F_{\mathbf{j}}$ such that $h(E_{\mathbf{i}_k}) \cap F_{\mathbf{j}} \neq \emptyset$ and $d(F_{\mathbf{j}}) < (c/L) d_{\mathbf{i}_k}$. Denoting $d_{k,l} = d(F_{k,l})$ we have then by (2.7) (as $c/L \leq 1$),

$$c(LC)^{-1} d_{\mathbf{i}_k} \leq d_{k,l} \leq d_{\mathbf{i}_k}. \tag{4.3}$$

Since $p = 2$, we can choose disjoint sets $\tilde{F}_{k,i} = F_{\mathbf{j}_{l(i)}}, i = 1, \dots, m_k$, such that $d(\tilde{F}_{k,i}) \geq c_2$, with $c_2 > 0$ independent of k, and $F = \cup_{i=1}^{m_k} \tilde{F}_{k,i}$. Let $\tilde{g}_{k,i}$ be the corresponding maps. Note that the maps $\tilde{g}_{k,i}$ are selected from a fixed finite family, so their bilipschitz constants have an upper bound independent of k. Define

$$h_k : E \to F$$

by setting

$$h_k(x) = \tilde{g}_{k,i}(g_{k,i}^{-1}(h(f_{\mathbf{i}_k}(x)))) \text{ if } x \in f_{\mathbf{i}_k}^{-1}(h^{-1}(F' \cap F_{k,i})).$$

Then $\mathrm{bilip}(h_k) \leq L'$ where L' is independent of k. Namely, the case when $x,\, y \in f_{\mathbf{i}_k}^{-1}(h^{-1}(F' \cap F_{k,i}))$, follows by composition. Furthermore, the strong separation condition provides us with constants $M_1, M_2 > 0$ such that

$$\mathrm{dist}(\tilde{F}_{k,i}, \tilde{F}_{k,j}) \geq M_1$$

and

$$\mathrm{dist}(F_{k,i}, F_{k,j}) \geq M_2 d_{\mathbf{i}_k}$$

for all $i \neq j, i, j = 1, ...m_k$. This takes care of the case $x \in f_{\mathbf{i}_k}^{-1}(h^{-1}(F' \cap F_{k,i}))$, $y \in f_{\mathbf{i}_k}^{-1}(h^{-1}(F' \cap F_{k,j}))$ with $i \neq j$.

We still need to check that the sets

$$h_k(E) = \bigcup_{i=1}^{m_k} \tilde{g}_{k,i}(g_{k,i}^{-1}(F' \cap F_{k,i}))$$

and F satisfy the condition for B_k and B in Lemma 2.1. Suppose this is not so. Then there are $a \in F$ and $r, 0 < r < R$, such that $B(a, r) \cap h_k(E) = \emptyset$ for some arbitrarily large k. For such a k, a belongs to some $\tilde{F}_{k,i_0}$. For some $c_3 > 0$ independent of k, $B(\tilde{g}_{k,i_0}^{-1}(a), c_3 r) \subset \tilde{g}_{k,i_0}^{-1}(B(a, r))$ and so, with $b = g_{k,i_0}(\tilde{g}_{k,i_0}^{-1}(a))$ by (2.5),

$$B(b, c_3 c r d_{k,i_0}) \subset g_{k,i_0}(\tilde{g}_{k,i_0}^{-1}(B(a, r))) \subset \mathbb{R}^n \setminus F'.$$

Since $y \in h(E_{\mathbf{i}_k}), b \in F_{k,i_0}, F_{k,i_0} \cap h(E_{\mathbf{i}_k}) \neq \emptyset$, and $d_{k,i_0} \leq d_{\mathbf{i}_k}$, by (4.3), we have $d(y, b) \leq (L + 1)d_{\mathbf{i}_k}$ and so by (4.3),

$$B(b, c_4 d_{\mathbf{i}_k}) \subset B(b, c_3 c r d_{k,i_0}) \subset B(y, (L + 1)d_{\mathbf{i}_k} + c_3 r d_{k,i_0}) \subset B(y, c_5 d_{\mathbf{i}_k})$$

with $c_4 = c_3 c^2 (LC)^{-1} r, c_5 = L + 1 + c_3 r$. Hence by (2.9),

$$\mathcal{H}^s(B(y, c_5 d_{\mathbf{i}_k}) \cap (F \setminus F')) \geq \mathcal{H}^s(B(b, c_4 d_{\mathbf{i}_k}) \cap F) \geq c(c_4 d_{\mathbf{i}_k})^s$$

contradicting the fact that y is a density point of F'.

□

4.4. *Remark.* We don't know if the condition $p = 2$ is needed in Theorem 4.2. Clearly the proof gives for general p that there is a bilipschitz map of E onto F_0 where F_0 is a finite union of sets $F_{\mathbf{j}}$. This raises a question: under what conditions is such a union Lipschitz equivalent with F?

Falconer and Marsh proved in [FM] that for self-similar sets E and F satisfying the strong separation condition the Lipschitz equivalence of E and F implies certain algebraic conditions on the similarity ratios of the generating maps. Possibly their method could be modified to prove the same conditions already if E and F have Lipschitz equivalent measurable subsets of positive measure. However, we could not deduce from this the Lipschitz equivalence of E and F in general since it is not clear when the necessary conditions of Falconer and Marsh are also sufficient.

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Marta Llorente: Departamento de Análisis Económico: Economía Cuantitativa, Universidad Autònoma de Madrid, Campus de Cantoblanco 28049 Madrid, Spain,

E-mail address: `m.llorente@uam.es`

Pertti Mattila: Department of Mathematics and Statistics, FI-00014 University of Helsinki, Finland,

E-mail address: `pertti.mattila@helsinki.fi`