Dirichlet problem on unbounded domains and at infinity

Aleksi Vähäkangas

Abstract

We consider one formulation of the Dirichlet problem for $\mathcal{A}$-harmonic functions on an unbounded domain of a Riemannian manifold. More specifically if M is a connected Riemannian manifold, $\Omega \subset M$ is an unbounded domain, and $\theta : M \to \mathbb{R}$ is a bounded Lipschitz function, then we provide a sufficient condition so that there exists an $\mathcal{A}$-harmonic function $u : \Omega \to \mathbb{R}$ such that $\lim_{x \to x_0} u(x) = \theta(x_0)$ for every $x_0 \in \partial\Omega$ and $|u(x) - \theta(x)| \to 0$ as $d(x, o) \to \infty$, where $o \in M$ is a fixed basepoint. This condition involves geometric inequalities for M and an integral condition for $|\nabla\theta|$. We then apply this results in the context of the Dirichlet problem at infinity on a Cartan-Hadamard manifold and prove new solvability results.

The existence of globally defined bounded nonconstant harmonic functions on a given Riemannian manifold $M = M^n$ depends heavily on the manifold. Yau [13] proved that if M has nonnegative Ricci curvature, then there are no positive (or bounded) harmonic functions other than the constants. This fact and the work of Greene and Wu [5] has motivated people to study the existence of bounded harmonic functions on Cartan-Hadamard manifolds, in other words, complete simply connected Riemannian manifold with nonpositive sectional curvature. Greene and Wu conjectured in [5] that a Cartan-Hadamard manifold M admits a bounded nonconstant harmonic function if

$$K_M \leq -C/\rho^2$$

outside a compact set, where $C > 0$ is a constant and ρ is the distance to a fixed base-point. This conjecture is still open in dimensions $n \geq 3$.

Date: May 13, 2009.

2000 Mathematics Subject Classification. Primary 58J32; Secondary 31C45.

Keywords: Dirichlet problem, Cartan-Hadamard manifold.

The author was supported by the Graduate School in Mathematical Analysis and Its Applications and by the Academy of Finland, project 123633.

A Cartan-Hadamard manifold has a natural geometric boundary, the sphere at infinity $M(\infty)$, such that $\bar{M} = M \cup M(\infty)$, equipped with so called cone topology, is homeomorphic to the closed ball $\bar{B}(0,1) \subset \mathbb{R}^n$. Dirichlet problem at infinity is to find for a given continuous function $\theta : M(\infty) \to \mathbb{R}$ a harmonic function $u : M \to \mathbb{R}$ such that $\lim_{x \to x_0} u(x) = \theta(x_0)$ for every $x_0 \in M(\infty)$ and we say that it is solvable if such u exists for every continuous $\theta : M(\infty) \to \mathbb{R}$. Anderson [1] and Sullivan [11] proved that if M is a Cartan-Hadamard manifold and

$$-b^2 \le K_M \le -a^2, \tag{0.1}$$

where $a, b > 0$ are constants, then the Dirichlet problem at infinity is solvable. There are numerous generalizations of this result in the literature. In 2003 Hsu [10] proved that the Dirichlet problem at infinity is solvable on a Cartan-Hadamard manifold satisfying either

$$-\rho^{2\phi-4-\varepsilon} \le K_M \le -\phi(\phi-1)/\rho^2 \tag{0.2}$$

outside a compact set where $\varepsilon > 0$ and $\phi > 2$ are constants, or

$$-h(\rho)^2 e^{2k\rho} \le K_M \le -k^2, \tag{0.3}$$

where $k > 0$ is a constant and h is a positive and nonincreasing function with $\int_0^\infty r h(r)\, dr < \infty$.

The Dirichlet problem at infinity has been recently studied in the context of p-harmonic and $\mathcal{A}$-harmonic functions. A continuous function u is $\mathcal{A}$-harmonic if it is a weak solution to the equation

$$-\operatorname{div} \mathcal{A}(\nabla u) = 0, \tag{0.4}$$

where $\mathcal{A}$ is an operator satisfying $\langle \mathcal{A}(v), v \rangle \approx |v|^p$ $(1 < p < \infty)$ and other conditions. Equation (0.4) is modelled after the p-Laplace equation where $\mathcal{A}(v) = |v|^{p-2} v$ and in this case the continuous weak solutions to (0.4) are called p-harmonic functions. Note that a function is 2-harmonic if and only if it is harmonic. Holopainen [7] proved that the Dirichlet problem at infinity for p-harmonic functions is solvable under the pinching condition (0.1). This problem is defined analogously by replacing the requirement that u is harmonic with the requirement that it is p-harmonic. In [9] Holopainen and the author showed that the Dirichlet problem at infinity for p-harmonic functions is solvable under curvature condition $-\rho^{-2-\varepsilon} e^{2k\rho} \le K_M \le -k^2$ and, on the other hand, under (0.2) in conjunction with the condition $p < 1 + (n-1)\phi$. By generalizing a proof method by Cheng [2] the author proved in [12] that the Dirichlet problem at infinity for $\mathcal{A}$-harmonic functions is solvable under curvature upper bound $K_M \le -\phi(\phi-1)/\rho^2$ and pointwise pinching condition $|K_M(P)| \le C|K_M(P')|$, where $P, P' \subset T_xM$ are 2-planes containing the radial vector $\nabla\rho(x)$.

In this paper we generalize the results proven in [9] to include $\mathcal{A}$-harmonic functions. Our new results concerning the Dirichlet problem at infinity are Theorem 3.6 and corollaries 3.7 and 3.8. More specifically, we prove that if $x_0 \in M(\infty)$ and U is a neighborhood of x_0 in the cone topology, then x_0 is $\mathcal{A}$-regular point at infinity if $k>0$, $\varepsilon>0$, and

$$-\rho(x)^{-4-\varepsilon}\exp\bigl(2k\rho(x)\bigr) \le K_M(P) \le -k^2$$

for every $x \in U\cap M$ and radial 2-plane $P \subset T_xM$. On the other hand we prove that x_0 is $\mathcal{A}$-regular if $\phi>1$, $1<p<\bigl(1+\phi(n-1)\bigr)\alpha/\beta$, $\varepsilon>0$, and

$$-\rho(x)^{2\phi-4}\bigl(\log\rho(x)\bigr)^{-2-\varepsilon} \le K_M(P) \le -\phi(\phi-1)/\rho(x)^2$$

for every $x \in U\cap M$ and every radial 2-plane $P\subset T_xM$. Here α, β are the structure constants of $\mathcal{A}$ and the Dirichlet problem at infinity is solvable if and only if every point at infinity is $\mathcal{A}$-regular.

We prove these results concerning the Dirichlet problem at infinity by first considering the following formulation of a Dirichlet problem on (possibly) unbounded domains. Suppose that M is a connected Riemannian manifold (not necessarily Cartan-Hadamard) and that $\Omega\subset M$ is an open set and $\theta : M\to\mathbb{R}$ is a continuous function. Is there an $\mathcal{A}$-harmonic function $u:\Omega\to\mathbb{R}$ such that $\lim_{x\to x_0} u(x)=\theta(x_0)$ for every $x_0\in\partial\Omega$ and $|u(x)-\theta(x)|\to 0$ as $\rho(x)\to\infty$? We provide a sufficient condition for this to be the case in our main result, Theorem 2.4. In this result a crucial role is played by an integral condition

$$\int_\Omega F\bigl(|\nabla\theta|w\bigr)<\infty,$$

where $F(t)\le t^{p+\varepsilon}\exp(-t^{-1+\varepsilon})$ and w is a weight function related to the geometry of Ω. To illustrate this result we note that in the case of the Laplacian on a Cartan-Hadamard manifold $M=\Omega$ of dimension $n\ge 3$ it implies the following.

0.5 Theorem. *Fix $\varepsilon\in(0,1)$. Suppose that M is a Cartan-Hadamard n-manifold with $n\ge 3$. Let $\theta : M\to\mathbb{R}$ be bounded and Lipschitz satisfying*

$$\int_M F\bigl(|\nabla\theta|\rho\bigr)<\infty, \tag{0.6}$$

where $F:[0,\infty)\to[0,\infty)$ is given by the formula $F(t)=t^{2+\varepsilon}\exp\bigl(-t^{-1+\varepsilon}\bigr)$. Then there exists a unique harmonic function $u:M\to\mathbb{R}$ such that $|u(x)-$

$\theta(x)| \to 0$ as $\rho(x) \to \infty$. Recall that ρ is distance from a fixed point on the manifold.

The integral condition (0.6) is sharp in the sense that if one allows the case $\varepsilon = 0$, then the result no longer holds by Example 2.9.

Considering this kind of Dirichlet problem on unbounded domains, as an explicit intermediate step towards solving the Dirichlet problem at infinity, is a new approach as far as we are aware. The main techniques that we use to prove Theorem 2.4 and Theorem 3.6 are based on ideas used by Cheng [2].

The paper is divided into four sections. Section 1 contains the preliminaries for this work. In Section 2 we consider the Dirichlet problem on unbounded domains and prove our main result 2.4. In Section 3 we apply this result to the Dirichlet problem at infinity for $\mathcal{A}$-harmonic functions. The last section is Appendix, where we deal with the construction of certain auxiliary functions needed in the proof of Theorem 2.4.

Throughout the paper c denotes an arbitrary positive constant that may vary even within a line. All manifolds are without boundary.

1 Preliminaries

In this section we recall definitions for the basic concepts that we use: $\mathcal{A}$-harmonic functions, Cartan-Hadamard manifolds and the Dirichlet problem at infinity, and finally $\mathcal{A}$-regular points at infinity on Cartan-Hadamard manifolds.

1.1 $\mathcal{A}$-harmonic functions

Let M be a Riemannian manifold and $1 < p < \infty$. Suppose that $\mathcal{A} : TM \to TM$ is an operator that satisfies the following assumptions for some $0 < \alpha \leq \beta < \infty$: the mapping $\mathcal{A}_x = \mathcal{A}|T_xM : T_xM \to T_xM$ is continuous for almost every $x \in M$ and the mapping $x \mapsto \mathcal{A}_x(V_x)$ is measurable for all measurable vectorfields V on M; for almost every $x \in M$ and every $v \in T_xM$:

$$\langle \mathcal{A}_x(v), v \rangle \geq \alpha |v|^p,$$

$$|\mathcal{A}_x(v)| \leq \beta |v|^{p-1},$$

$$\big\langle \mathcal{A}_x(v) - \mathcal{A}_x(w), v - w \big\rangle > 0,$$

whenever $w \in T_xM \setminus \{v\}$, and

$$\mathcal{A}_x(\lambda v) = \lambda |\lambda|^{p-2} \mathcal{A}_x(v)$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$. We denote the set of all such operators by $\mathcal{A}^p(M)$. The constants α and β are called *structure constants* of $\mathcal{A}$.

Suppose that $U \subset M$ is an open set and $\mathcal{A} \in \mathcal{A}^p(M)$. A function $u \in C(U) \cap W^{1,p}_{\text{loc}}(U)$ is *$\mathcal{A}$-harmonic* in U if it is a weak solution of the equation

$$-\operatorname{div} \mathcal{A}(\nabla u) = 0, \tag{1.2}$$

in other words, if

$$\int_U \langle \mathcal{A}(\nabla u), \nabla \varphi \rangle = 0 \tag{1.3}$$

for every test function $\varphi \in C_0^\infty(U)$. If $|\nabla u| \in L^p(U)$, then it is equivalent to require (1.3) for all $\varphi \in W_0^{1,p}(U)$ by approximation.

A lower semicontinuous function $u : U \to (-\infty, \infty]$ is *$\mathcal{A}$-superharmonic* if $u \not\equiv \infty$ in each component of U, and for each open $D \subset\subset U$ and each $h \in C(\bar{D})$, $\mathcal{A}$-harmonic in D, $h \le u$ on ∂D implies $h \le u$ in D.

In the case of the *p-Laplacian* $\mathcal{A}(v) = |v|^{p-2}v$, the continuous weak solutions of (1.2) are called *p-harmonic functions*. In this case $\alpha = \beta = 1$. A function $u \in C(U) \cap W^{1,2}_{\text{loc}}(U)$ is 2-harmonic if and only if it belongs to $C^\infty(U)$ and $\Delta u \equiv 0$ in U, i.e. u is harmonic in the usual sense.

The $\mathcal{A}$-harmonic functions have many features in common with harmonic functions. See [6] for properties and theory of $\mathcal{A}$-harmonic and $\mathcal{A}$-superharmonic functions in $\mathbb{R}^n$.

1.4 Cartan-Hadamard manifolds and Dirichlet problem at infinity

A *Cartan-Hadamard manifold* M is a complete simply connected Riemannian n-manifold, $n \ge 2$, with nonpositive sectional curvature. Cartan-Hadamard theorem then implies that the exponential map $\exp_x : T_xM \to M$ is a diffeomorphism for every $x \in M$. In particular, M is diffeomorphic to $\mathbb{R}^n$.

Let us recall the definition of cone topology. For details and proofs, see [4]. We say that two unit speed geodesics $\gamma, \sigma : \mathbb{R} \to M$ are *asymptotic* if $\sup_{t \ge 0} d\big(\gamma(t), \sigma(t)\big) < \infty$. This defines an equivalence relation. Denote the equivalence class of γ by $\gamma(\infty)$ and the set of all equivalence classes by $M(\infty)$. We call elements of $M(\infty)$ *points at infinity* and denote $\bar{M} = M \cup M(\infty)$. For every $x \in M$ and $y \in \bar{M} \setminus \{x\}$ there exists a unique unit speed geodesic $\gamma^{x,y}$ such that $\gamma^{x,y}(0) = x$ and $y \in \gamma^{x,y}(0, \infty]$. Given $x \in M$, $v \in T_xM \setminus \{0\}$, $\delta > 0$, and $r > 0$, we define a *cone*

$$C(v, \delta) = \{y \in \bar{M} \setminus \{x\} : \sphericalangle(v, \dot{\gamma}_0^{x,y}) < \delta\}$$

and a *truncated cone* $T(v,\delta,r) = C(v,\delta) \setminus \bar{B}(x,r)$ with vertex x. The collection

$$\{\text{open balls}\} \cup \{\text{truncated cones with vertex } o\}$$

is a basis for *cone topology* on $\bar{M}$. This topology is independent of o and, equipped with this topology, $\bar{M}$ is homeomorphic to the closed unit ball $\bar{B}(0,1) \subset \mathbb{R}^n$ and $M(\infty)$ to $\mathbb{S}^{n-1}$. We always equip $\bar{M}$ with this topology.

Suppose that $p \in (1,\infty)$ and $\mathcal{A} \in \mathcal{A}^p(M)$. The *Dirichlet problem at infinity* (for $\mathcal{A}$-harmonic functions) is to find for a given $\theta \in C(M(\infty))$ a function $u \in C(\bar{M})$ such that $u|M$ is $\mathcal{A}$-harmonic and $u|M(\infty) = \theta$. We say that the Dirichlet problem at infinity is *solvable* if such u exists for every $\theta \in C(M(\infty))$.

1.5 Perron's method and regular points at infinity

Suppose now that M is a Cartan-Hadamard manifold. We approach the Dirichlet problem at infinity using Perron's method. The definitions of the upper and lower Perron solutions follow [6], where such concepts are defined for $\mathcal{A}$-harmonic functions in the Euclidean setting. Fix $p \in (1,\infty)$ and $\mathcal{A} \in \mathcal{A}^p(M)$.

1.6 Definition. A function $u : M \to (-\infty,\infty]$ belongs to the *upper class* $\mathcal{U}_f$ of $f : M(\infty) \to [-\infty,\infty]$ if

(i) u is $\mathcal{A}$-superharmonic in M,

(ii) u is bounded below, and

(iii) $\liminf_{x\to x_0} u(x) \geq f(x_0)$ for all $x_0 \in M(\infty)$.

The function

$$\overline{H}_f = \inf\{u : u \in \mathcal{U}_f\}$$

is called the *upper Perron solution.*

1.7 Theorem. *One of the following is true:*

(i) $\overline{H}_f$ *is* $\mathcal{A}$*-harmonic in* M*,*

(ii) $\overline{H}_f \equiv \infty$ *in* M*,*

(iii) $\overline{H}_f \equiv -\infty$ *in* M*.*

Proof. As in [6, Theorem 9.2]. □

Note that if f is bounded, then $\overline{H}_f$ is also bounded and it is then $\mathcal{A}$-harmonic in M by Theorem 1.7. Hence the upper Perron solution is a good candidate to be the solution of the Dirichlet problem at infinity with boundary data f.

1.8 Definition. A point $x_0 \in M(\infty)$ is $\mathcal{A}$-*regular*, if

$$\lim_{x \to x_0} \overline{H}_f(x) = f(x_0)$$

for each continuous $f : M(\infty) \to \mathbb{R}$.

Define the *lower class* $\mathcal{L}_f = -\mathcal{U}_{-f}$ and the *lower Perron solution* $\underline{H}_f = -\overline{H}_{-f}$. Then $\overline{H}_f \geq \underline{H}_f$.

The concept of regularity is related to the Dirichlet problem at infinity in the way that the Dirichlet problem at infinity is solvable for $\mathcal{A}$-harmonic functions if and only if every point at infinity is $\mathcal{A}$-regular.

2 Dirichlet problem on unbounded domains

In this section we formulate and prove our main result Theorem 2.4 that concerns the existence of a bounded $\mathcal{A}$-harmonic function on a domain of a Riemannian manifold with prescribed boundary values and behavior at infinity. Throughout the section M is a complete connected Riemannian n-manifold, $p \in (1, \infty)$, and $\mathcal{A} \in \mathcal{A}^p(M)$ is an operator as defined in Section 1.1. Note that we do not assume M to be Cartan-Hadamard. Let $o \in M$ be a fixed basepoint and denote $\rho = d(o, \cdot)$.

Let us first define two geometric inequalities that we need in formulation of our main result. We say that an open subset U of M satisfies a *weighted* $(1,1)$-*Sobolev inequality with weight* $w : U \to [0, \infty)$ if

$$\int_U |\eta| \leq \int_U |\nabla \eta| w \tag{2.1}$$

for every test function $\eta \in C_0^\infty(U)$. We say that M satisfies a *local Sobolev inequality* if there exist constant $r_S > 0$ and $C_S < \infty$ such that

$$\Big(\int_B |\eta|^{n/(n-1)} \Big)^{(n-1)/n} \leq C_S \int_B |\nabla \eta| \tag{2.2}$$

holds for every ball $B = B(x, r_S) \subset M$ of radius r_S and every $\eta \in C_0^\infty(B)$.

2.3 Remark. (i) Suppose that M is a Cartan-Hadamard n-manifold. Then (2.1) holds with $U = M \setminus \{o\}$ and $w(x) = \rho(x)/n$. In particular this applies if $M = \mathbb{R}^n$. If $K_M \le -a^2$ for some constant $a > 0$, then (2.1) holds on M with constant weight $w = 1/(n-1)a$. If $K_M \le -\phi(\phi-1)/\rho^2$ holds outside a compact set with some constant $\phi > 1$ and if $\lambda < 1 + (n-1)\phi$, then (2.1) holds with weight function $w(x) = \rho(x)/\lambda$ on $M \setminus K$, where K is some compact set. We prove the last claim in Lemma 3.1 and the others can be proved similarly.

(ii) The local Sobolev inequality (2.2) holds on a complete connected Riemannian manifold M if $\operatorname{inj} M > 0$. To see this one can use [3, Theorem 11] to prove a local isoperimetric inequality and from this the local Sobolev inequality. In particular (2.2) holds on any Cartan-Hadamard manifold.

The following is our main theorem.

2.4 Theorem. *Suppose that M is a complete connected Riemannian n-manifold with $n \ge 2$, $p \in (1,\infty)$, and $\mathcal{A} \in \mathcal{A}^p(M)$.*

Let $\Omega \subset M$ be a nonempty open set and $w : \Omega \to [0,\infty)$ a nonnegative Lipschitz function with constant satisfying

$$p \operatorname{Lip} w < \alpha/\beta, \tag{2.5}$$

where α and β are the structure constants of $\mathcal{A}$ as in 1.1. Suppose that $K \subset \Omega$ is a compact set and $\Omega \setminus K$ satisfies the weighted $(1,1)$-Sobolev inequality (2.1) with weight w. Suppose that M satisfies the local Sobolev inequality (2.2).

Let $\theta : M \to \mathbb{R}$ be a continuous function in $W^{1,\infty}(M)$ (i.e. $\|\theta\|_\infty, \|\nabla\theta\|_\infty < \infty$) satisfying

$$\int_\Omega F\big(|\nabla\theta| w\big) < \infty, \tag{2.6}$$

where $F : [0,\infty) \to [0,\infty)$ is given by

$$F(t) = t^{p+\varepsilon} \exp\Big(-\frac{1}{t}\Big(\log\big(e + \frac{1}{t}\big)\Big)^{-1-\varepsilon}\Big)$$

for some constant $\varepsilon > 0$.

Then there exists a bounded $\mathcal{A}$-harmonic function $u : \Omega \to \mathbb{R}$ such that $u(x) \to \theta(x_0)$ whenever $x \to x_0$ and $x_0 \in \partial\Omega$ is an $\mathcal{A}$-regular boundary point. Also, $|u(x_k) - \theta(x_k)| \to 0$ for every sequence (x_k) in Ω with $\rho(x_k) \to \infty$ as $k \to \infty$.

2.7 Remark. (i) The special case $\Omega = M$ can be interesting in itself. For example, Theorem 0.5 is an immediate corollary of Theorem 2.4 with $\Omega = M$.

(ii) One cannot in general allow equality in (2.5), see Example 2.8 below.

(iii) The behavior of F near 0 is close to being sharp in the sense that the result does not hold in general if one allows the possibility $\varepsilon = 0$. This is shown by Example 2.9. We could, however, replace F in Theorem 2.4 with any function in the class $\mathcal{F}_p$ as defined in the Appendix. This is clear from the proof that we present.

(iv) The function u in Theorem 2.4 is unique if every boundary point $x_0 \in \partial\Omega$ is $\mathcal{A}$-regular and either M is noncompact or $\partial\Omega \neq \emptyset$. This follows easily from the comparison principle.

We illustrate Theorem 2.4 with the following easy example.

2.8 Example. Glue two copies of $\mathbb{R}^n$, $n \geq 2$, together with a compact pipe to form a complete connected Riemannian n-manifold M. Denote the pipe by P and the two components of $M \setminus P$ by U_1 and U_2 so that $M = U_1 \cup P \cup U_2$ is a union of disjoint sets and $\bar{U}_1 \cap \bar{U}_2 = \emptyset$.

Consider the p-Laplacian on $\Omega = M$ with $p \in (1, \infty)$. By Remark 2.3(i) the weighted $(1,1)$-Sobolev inequality holds on $M \setminus K$ for some compact set K and weight function w with Lipschitz constant $1/n$. The local Sobolev inequality holds on M by Remark 2.3(ii). Since $p \operatorname{Lip} w = p/n$, (2.5) holds if and only if $p < n$.

Choose a Lipschitz function $\theta : M \to \mathbb{R}$ such that $\theta | U_1 \equiv 0$ and $\theta | U_2 \equiv 1$. Then the integral condition (2.6) holds trivially since the integrand vanishes outside P.

Theorem 2.4 then implies that if $p \in (1, n)$, then there exists a bounded p-harmonic function u on M with $u(x) \to 0$ as $\rho(x) \to \infty$, $x \in U_1$, and $u(x) \to 1$ as $\rho(x) \to \infty$, $x \in U_2$.

On the other hand, if $p \geq n$, then M is p-parabolic by [8, Theorem 1.4(i)] and in particular there are no nonconstant bounded p-harmonic functions on M. This shows that one cannot in general allow equality in (2.5).

2.9 Example. In this example we show that if $F_0 : [0, \infty) \to [0, \infty)$ is any function satisfying

$$F_0(t) \leq \exp\Big(-\frac{1}{t}\Big(\log\frac{1}{t}\Big)^{-1}\Big) \tag{2.10}$$

for all sufficiently small t, then Theorem 2.4 can fail if instead of (2.6) one assumes

$$\int_\Omega F_0\big(|\nabla\theta| w\big) < \infty. \tag{2.11}$$

To show this note that one can assume without loss of generality that equality holds in (2.10) for all small t. Consider the Laplacian ($p = 2$) on

$\Omega = M = \mathbb{R}^3$. Then (2.2) holds and (2.1) holds with $w(x) = |x|/3$ and $K = \{0\}$ by Remark 2.3(i). Also, $p \operatorname{Lip} w = 2/3$ so that (2.5) holds.

Suppose that $\Phi : [0, \infty) \to [0, \infty)$ is a smooth function with $\Phi(t) = 2^{-1} \log\log\log t$ for all large enough t and let $\theta : \mathbb{R}^3 \to \mathbb{R}$, $\theta(x) = \sin\bigl(\Phi(\rho(x))\bigr)$. Then $\theta \in W^{1,\infty}(\mathbb{R}^3)$ is a continuous function.

Now (2.10) implies that

$$F_0\bigl(\Phi'(t)t/3\bigr) = F_0\bigl(6^{-1}(\log t)^{-1}(\log\log t)^{-1}\bigr) \le \exp\Bigl(-\frac{6(\log t)(\log\log t)}{\log\bigl(6(\log t)(\log\log t)\bigr)}\Bigr)$$
$$\le \exp\Bigl(-\frac{6(\log t)(\log\log t)}{(3/2)\log\log t}\Bigr) = t^{-4}$$

for all large enough t so that

$$\int_{\mathbb{R}^3} F_0\bigl(|\nabla\theta| w\bigr) \le c \int_0^\infty F_0\bigl(\Phi'(t)t/3\bigr)t^2\,dt \le c + \int_c^\infty t^{-4+2}\,dt < \infty.$$

Therefore (2.11) holds.

Since there exist no nonconstant bounded harmonic functions on $\mathbb{R}^3$ by Liouville's theorem, the conclusion in Theorem 2.4 fails. This shows that one cannot in general replace F in Theorem 2.4 with any function F_0 satisfying (2.10) for all small enough t.

Our goal for the rest of this section is to prove Theorem 2.4. Therefore until the end of this section we use the assumptions and notation from Theorem 2.4 in order to prove it.

In the proof we need several auxiliary functions defined on the positive real axis. We need functions $F_0, G_0, \varphi : [0, \infty) \to [0, \infty)$ that are all homeomorphisms and smooth on $(0, \infty)$. They need to satisfy the following conditions: $F_0 \le F$ simply so that we can replace F with F_0 that suits us better,

$$G_0 \circ \varphi' = \varphi \tag{2.12}$$

that is a condition used in the proof of Lemma 2.17 below and that ties G_0 and φ together, and

$$\lim_{t\to 0+} \frac{\varphi''(t)\varphi(t)}{\varphi'(t)^2} = 1 \tag{2.13}$$

for technical reasons. In addition we need to use Young's inequality in the proof of Lemma 2.17 and for this reason we require that $F_0(\cdot^{1/p})$ and $G_0(\cdot^{1/p})^p$ are complementary Young functions. The construction of these functions is done in the Appendix and we refer the reader interested in the details there.

We define one more auxiliary function $\psi := (\varphi')^{p-1}\varphi$. Then $\psi : [0,\infty) \to [0,\infty)$ is also a homeomorphism that is smooth on $(0,\infty)$. It follows from (2.13) that

$$\lim_{t\to 0+} \frac{\psi'(t)}{\varphi'(t)^p} = p. \tag{2.14}$$

The following Caccioppoli type inequality plays a central role in what follows. It is a technical tool that yields information on $\mathcal{A}$-harmonic functions.

2.15 Lemma. *Let $U \subset M$ be open and relatively compact. Suppose that $\eta \geq 0$ is a Lipschitz function on U. Suppose that $\tilde{\theta}, u \in L^\infty(U) \cap W^{1,p}(U)$ are continuous functions and that u is $\mathcal{A}$-harmonic in U. Denote $h = |u - \tilde{\theta}|$ and suppose that*

$$\eta^p \psi(h) \in W_0^{1,p}(U).$$

Then

(2.16)

$$\Big(\int_U \eta^p \psi'(h)|\nabla u|^p\Big)^{1/p} \leq \frac{\beta}{\alpha}\Big(\int_U \eta^p \psi'(h)|\nabla \tilde{\theta}|^p\Big)^{1/p} + \frac{p\beta}{\alpha}\Big(\int_U \frac{\psi^p}{(\psi')^{p-1}}(h)|\nabla \eta|^p\Big)^{1/p}$$

Proof. Denote $f = \eta^p\psi((u-\tilde{\theta})^+) - \eta^p\psi((u-\tilde{\theta})^-)$. Then $f \in W^{1,p}(U)$ and its gradient is

$$\nabla f = \eta^p\psi'(h)(\nabla u - \nabla\tilde{\theta}) + p\eta^{p-1}\operatorname{sgn}(u-\tilde{\theta})\psi(h)\nabla\eta.$$

Since $|f| = \eta^p\psi(h) \in W_0^{1,p}(U)$ by assumption, we have $f \in W_0^{1,p}(U)$, cf. [6, Lemma 1.25(iii)]. Testing $\mathcal{A}$-harmonicity of u with the test function f and using of Hölder's inequality we get

$$\begin{aligned}
\int_U \eta^p\psi'(h)|\nabla u|^p &\leq \frac{1}{\alpha}\int_U \big\langle \mathcal{A}(\nabla u), \eta^p\psi'(h)\nabla u\big\rangle \\
&= \frac{1}{\alpha}\int_U \big\langle \mathcal{A}(\nabla u), \eta^p\psi'(h)\nabla\tilde{\theta}\big\rangle - \frac{p}{\alpha}\int_U \big\langle \mathcal{A}(\nabla u), \eta^{p-1}\operatorname{sgn}(u-\tilde{\theta})\psi(h)\nabla\eta\big\rangle \\
&\leq \frac{\beta}{\alpha}\int_U \eta^p\psi'(h)|\nabla u|^{p-1}|\nabla\tilde{\theta}| + \frac{p\beta}{\alpha}\int_U \eta^{p-1}\psi(h)|\nabla u|^{p-1}|\nabla\eta| \\
&\leq \frac{\beta}{\alpha}\Big(\int_U \eta^p\psi'(h)|\nabla u|^p\Big)^{(p-1)/p}\Big(\int_U \eta^p\psi'(h)|\nabla\tilde{\theta}|^p\Big)^{1/p} \\
&\qquad + \frac{p\beta}{\alpha}\Big(\int_U \eta^p\psi'(h)|\nabla u|^p\Big)^{(p-1)/p}\Big(\int_U \frac{\psi^p}{(\psi')^{p-1}}(h)|\nabla\eta|^p\Big)^{1/p}.
\end{aligned}$$

We simplify this to finish the proof. □

The following lemma establishes an integral estimate for $|u_r - \theta|$, where u_r is the unique $\mathcal{A}$-harmonic function in $\Omega \cap B(o,r)$ with boundary data θ. It is important that this estimate is uniform with respect to r as this makes a limiting argument possible later on.

2.17 Lemma. *Suppose that $\tilde\theta \in W^{1,\infty}(M)$ is a continuous function with $\|\tilde\theta\|_\infty \le 1$. Let $r > 0$ be so large that $U := \Omega \cap B(o,r) \neq \emptyset$. Let u be the unique $\mathcal{A}$-harmonic function in U that satisfies $u - \tilde\theta \in W_0^{1,p}(U)$. Then*

$$\int_U \varphi\big(|u-\tilde\theta|/c_0\big)^p \le c_0 + c_0 \int_U F\big(c_0|\nabla\tilde\theta|w\big),$$

where $c_0 > 1$ is a constant that is independent of r and $\tilde\theta$.

Proof. By assumption (2.5) there exists a constant $\delta > 0$ so that $(1+\delta)^2 p \operatorname{Lip} w < \alpha/\beta$.

By using the assumption $\|\tilde\theta\|_\infty \le 1$ and by replacing $\tilde\theta$ with $c\tilde\theta$ and u with cu if necessary, we can assume without loss of generality for the rest of this proof that $\|\tilde\theta\|_\infty$ is smaller than a given constant that does not depend on r or $\tilde\theta$. For this reason and (2.14) we can assume without loss of generality that $\|\tilde\theta\|_\infty$ is so small that

$$\psi'(t)/2p \le \varphi'(t)^p \le (1+\delta)^p p^{-1}\psi'(t) \tag{2.18}$$

and

$$\frac{\psi(t)^p}{\psi'(t)^{p-1}} \le (1+\delta)^p p^{1-p}\varphi(t)^p \tag{2.19}$$

for every $t \in (0, 2\|\tilde\theta\|_\infty]$.

We denote $h = |u - \tilde\theta| : U \to [0,\infty)$. Fix a function $\eta \in C_0^\infty(M)$ such that $0 \le \eta \le 1$ and $\eta|K \equiv 1$. Then $(1-\eta)\varphi(h)^p \in W_0^{1,1}(U \setminus K)$ so that (2.1) implies

$$\begin{aligned}\int_U \varphi(h)^p &= \int_U \eta\varphi(h)^p + \int_U (1-\eta)\varphi(h)^p\\ &\le c + \int_U \big|\nabla\big((1-\eta)\varphi(h)^p\big)\big| w \le c + p\int_U \varphi(h)^{p-1}\varphi'(h)|\nabla h| w,\end{aligned}$$

where the constants depend on the manifold, p, w, φ, and η. Hölder's inequality then implies

$$\Big(\int_U \varphi(h)^p\Big)^{1/p} \le c + p\Big(\int_U \varphi'(h)^p|\nabla h|^p w^p\Big)^{1/p}.$$

Using (2.18), Lemma 2.15, and (2.19) we get

$$\begin{aligned}
\Big(\int_U \varphi(h)^p\Big)^{1/p} &\le c + p\Big(\int_U \varphi'(h)^p|\nabla h|^p w^p\Big)^{1/p} \\
&\le c + (1+\delta)p^{1-1/p}\Big(\int_U \psi'(h)|\nabla h|^p w^p\Big)^{1/p} \\
&\le c + c\Big(\int_U \psi'(h)|\nabla\tilde\theta|^p w^p\Big)^{1/p} + (1+\delta)p^{2-1/p}(\operatorname{Lip} w)\frac{\beta}{\alpha}\Big(\int_U \frac{\psi^p}{(\psi')^{p-1}}(h)\Big)^{1/p} \\
&\le c + c\Big(\int_U \varphi'(h)^p|\nabla\tilde\theta|^p w^p\Big)^{1/p} + (1+\delta)^2 p(\operatorname{Lip} w)\frac{\beta}{\alpha}\Big(\int_U \varphi(h)^p\Big)^{1/p}.
\end{aligned}$$

Since $(1+\delta)^2 p(\operatorname{Lip} w)\beta/\alpha < 1$, we can combine the left side term and the last term to get

$$\int_U \varphi(h)^p \le c + c\int_U \varphi'(h)^p|\nabla\tilde\theta|^p w^p.$$

The auxiliary functions F_0 and G_0 are chosen such that $F_0(\cdot^{1/p})$ and $G_0(\cdot^{1/p})^p$ are complementary Young functions. Therefore we can use Young's inequality

$$xy \le kG_0(x^{1/p})^p + kF_0(k^{-1/p}y^{1/p}), \qquad \text{for all } x, y \ge 0 \text{ and } k > 0,$$

and identity (2.12) to obtain

$$\begin{aligned}
\int_U \varphi(h)^p &\le c + c\int_U \varphi'(h)^p|\nabla\tilde\theta|^p w^p \\
&\le c + ck\int_U G_0(\varphi'(h))^p + ck\int_U F_0\big(k^{-1/p}|\nabla\tilde\theta|w\big) \\
&\le c + \frac{1}{2}\int_U \varphi(h)^p + c\int_U F_0\big(c|\nabla\tilde\theta|w\big)
\end{aligned}$$

if we choose k appropriately small. The claim follows. □

In order to pass from the integral estimate in Lemma 2.17 to a pointwise estimate we need the following lemma. Its proof is based on the idea of Moser iteration.

2.20 Lemma. *Suppose that $\|\theta\|_\infty \le 1$. Suppose that $r \in (0, r_S)$ is a constant and $x \in M$. Denote $B = B(x,r)$. Suppose that $u \in W^{1,p}_{\mathrm{loc}}(M)$ is a function that is $\mathcal{A}$-harmonic in the open set $\Omega \cap B$, satisfies $u - \theta \in W^{1,p}_0(\Omega)$, and $u = \theta$ a.e. in $M \setminus \Omega$. Then*

$$\operatorname*{ess\,sup}_{B/2} \varphi\big(|u-\theta|\big)^{p(n+1)} \le c\int_B \varphi\big(|u-\theta|\big)^p,$$

where the constant is independent of x.

Proof. We denote $h = |u-\theta|$ and $\Phi = \varphi^p$. Without loss of generality we can assume that $\Omega \cap B \neq \emptyset$.

Suppose that $\eta \geq 0$ is a Lipschitz function with $\operatorname{supp} \eta \subset B$. Since $u - \theta \in W_0^{1,p}(\Omega)$ and $\operatorname{supp} \eta \subset B$, we have $\eta^p \Phi(h)^m \in W_0^{1,p}(\Omega \cap B)$ for all $m \geq 1$. Hence we can use Lemma 2.15 (with Φ^m replacing ψ) to obtain

$$
\begin{aligned}
\int_B \eta^p (\Phi^m)'(h) |\nabla h|^p &= \int_{\Omega\cap B} \eta^p (\Phi^m)'(h) |\nabla h|^p \\
&\leq c \int_{\Omega\cap B} \eta^p (\Phi^m)'(h) |\nabla \theta|^p + c \int_{\Omega \cap B} \frac{\Phi^{mp}}{(m\Phi'\Phi^{m-1})^{p-1}}(h) |\nabla \eta|^p \\
&\leq c \int_B \eta^p (\Phi^m)'(h) |\nabla \theta|^p + c m^{1-p} \int_B \Phi(h)^m |\nabla \eta|^p
\end{aligned}
\tag{2.21}
$$

for all $m \geq 1$. Here we used the fact that $\Phi'(t) \geq c\Phi(t)$ for all $t \in (0,2]$ that holds since $\varphi'(t) \geq c\varphi(t)$ for all $r \in (0,2]$ by (2.12).

Denote $\kappa = n/(n-1)$. We denote $r_j = r(1+\kappa^{-j})/2$ and $B_j = B(x, r_j)$. Note that $r_j \to r/2$ as $j \to \infty$. Let η_j be a Lipschitz function with $0 \leq \eta_j \leq 1$, $\eta_j | B_{j+1} \equiv 1$, and $\eta_j | M \setminus B_j \equiv 0$. We choose it to be $(r_j - r_{j+1})^{-1}$-Lipschitz so that $|\nabla \eta_j| \leq c\kappa^j$.

If $m \geq 1$, then

$$
\begin{aligned}
\left|\nabla\big(\eta_j^p \Phi(h)^m\big)\right| &\leq p\eta_j^{p-1}\Phi(h)^m |\nabla \eta_j| + m\eta_j^p \Phi'(h)\Phi(h)^{m-1}|\nabla h| \\
&\leq c\kappa^j \eta_j^{p-1}\Phi(h)^m + m\eta_j^p\Phi'(h)\Phi(h)^{m-1}\big(1 + |\nabla h|^p\big) \\
&\leq c(\kappa^j + m)\eta_j^{p-1}\Phi(h)^{m-1} + \eta_j^p (\Phi^m)'(h)|\nabla h|^p
\end{aligned}
\tag{2.22}
$$

so that the local Sobolev inequality (2.2) and (2.21) imply for $m \geq 1$ that

$$
\begin{aligned}
&\Big(\int_{B_{j+1}} \Phi(h)^{\kappa m}\Big)^{1/\kappa} \leq \Big(\int_{B_j} \big(\eta_j^p \Phi(h)^m\big)^\kappa\Big)^{1/\kappa} \leq C_S \int_{B_j} \left|\nabla\big(\eta_j^p\Phi(h)^m\big)\right| \\
&\quad \leq c(\kappa^j + m)\int_{B_j} \eta_j^{p-1}\Phi(h)^{m-1} + c\int_{B_j} \eta_j^p (\Phi^m)'(h)|\nabla h|^p \\
&\quad \leq c(\kappa^j + m)\int_{B_j} \Phi(h)^{m-1} + c\int_{B_j} \eta_j^p(\Phi^m)'(h)|\nabla\theta|^p + cm^{1-p}\int_{B_j}\Phi(h)^m|\nabla\eta_j|^p \\
&\quad \leq c(\kappa^j + m + m^{1-p}\kappa^{jp})\int_{B_j}\Phi(h)^{m-1}.
\end{aligned}
$$

We apply this with $m = m_j + 1$, where $m_j := (n+1)\kappa^j - n$. Note that $m_{j+1} = \kappa(m_j+1)$ so that we get

$$
\Big(\int_{B_{j+1}} \Phi(h)^{m_{j+1}}\Big)^{1/\kappa} \leq c\kappa^j \int_{B_j} \Phi(h)^{m_j}.
\tag{2.23}
$$

By denoting $I_j = \big(\int_{B_j}\Phi(h)^{m_j}\big)^{1/\kappa^j}$ we can write (2.23) as a recursion formula $I_{j+1} \le c^{1/\kappa^j}\kappa^{j/\kappa^j} I_j$. Since

$$\limsup_{j\to\infty} I_j \ge \lim_{j\to\infty}\Big(\int_{B/2}\Phi(h)^{m_j}\Big)^{(n+1)/m_j} = \|\Phi(h)\|^{n+1}_{L^\infty(B/2)},$$

we get

$$\operatorname*{ess\,sup}_{B/2}\Phi(h)^{n+1} \le \limsup_{j\to\infty} I_j \le c^{\sum_{k=0}^\infty 1/\kappa^k}\kappa^{\sum_{k=0}^\infty k/\kappa^k} I_0 \le c\int_B \Phi(h)$$

as claimed. □

Proof of Theorem 2.4. Note first that by scaling θ (and u) if necessary, we can assume without loss of generality that $\|\theta\|_\infty \le 1/c_0$, where c_0 is as in Lemma 2.17. By scaling we can also assume without loss of generality instead of (2.6) that $\int_\Omega F(c_0^2|\nabla\theta|w) < \infty$.

Let $j_0 \in \mathbb{N}$ be so large that $\Omega \cap B(o, j_0) \neq \emptyset$. If $j \ge j_0$, we denote $\Omega_j = \Omega \cap B(o, j)$. Let u_j be the unique $\mathcal{A}$-harmonic function in Ω_j that satisfies $u_j - \theta \in W_0^{1,p}(\Omega_j)$. Now $(u_j)_{j\ge j_0}$ is a bounded sequence of $\mathcal{A}$-harmonic functions and it follows that it is equicontinuous. Ascoli's theorem implies that this sequence has a locally uniformly converging subsequence. We denote this subsequence by (u_{i_j}). The limit function $u : \Omega \to \mathbb{R}$ is an $\mathcal{A}$-harmonic function in Ω. Suppose that $x_0 \in \partial\Omega$ is an $\mathcal{A}$-regular boundary point of Ω and that (x_k) is a sequence of points in Ω so that $x_k \to x_0$ as $k \to \infty$. Using the fact that x_0 is $\mathcal{A}$-regular and standard potential theoretic arguments one can prove that $u(x_k) \to \theta(x_0)$ as $k \to \infty$. We omit this proof as it is very similar to an argument in the proof of [12, Theorem 4.1]. This takes care of the first claim in Theorem 2.4.

Fix a sequence (x_k) of points in Ω so that $\rho(x_k) \to \infty$ as $k \to \infty$. We have to prove that $|u(x_k) - \theta(x_k)| \to 0$ as $k \to \infty$. Denote $\tilde\theta = c_0\theta$ and $\tilde u = c_0 u$. Fatou's lemma together with Lemma 2.17 applied with $U = \Omega_{k_j}$ imply

$$\begin{aligned}\int_\Omega \varphi\big(|u-\theta|\big)^p &= \int_\Omega \varphi\big(|\tilde u - \tilde\theta|/c_0\big)^p \le \liminf_{j\to\infty}\int_{\Omega_{i_j}}\varphi\big(|\tilde u_{i_j} - \tilde\theta|/c_0\big)^p \\ &\le c_0 + c_0\int_\Omega F\big(c_0|\nabla\tilde\theta|w\big) = c_0 + c_0\int_\Omega F\big(c_0^2|\nabla\theta|w\big) < \infty.\end{aligned} \tag{2.24}$$

Let $x \in \Omega$. Since $u_{i_j} - \theta \in W_0^{1,p}(\Omega_{i_j})$, we can extend u_{i_j} to a function in $W^{1,p}_{\mathrm{loc}}(M)$ by setting $u_{i_j}(y) = \theta(y)$ if $y \in M\setminus\Omega_{i_j}$. If j is large enough then the extension u_{i_j} satisfies the assumptions of Lemma 2.20 for fixed $r \in (0, r_S)$ and hence

$$\operatorname*{ess\,sup}_{B(x,r/2)} \varphi\big(|u_{i_j} - \theta|\big)^{p(n+1)} \le c \int_{B(x,r)} \varphi\big(|u_{i_j} - \theta|\big)^p.$$

Here ess sup is needed instead of sup since $|u_{i_j} - \theta|$ need not be continuous on the boundary of Ω. This and the dominated convergence theorem imply

$$\begin{aligned}\sup_{\Omega\cap B(x,r/2)} \varphi\big(|u - \theta|\big)^{p(n+1)} &= \sup_{\Omega\cap B(x,r/2)} \lim_{j\to\infty} \varphi\big(|u_{i_j} - \theta|\big)^{p(n+1)} \\ &\le \limsup_{j\to\infty} \operatorname*{ess\,sup}_{B(x,r/2)} \varphi\big(|u_{i_j} - \theta|\big)^{p(n+1)} \\ &\le c \limsup_{j\to\infty} \int_{B(x,r)} \varphi\big(|u_{i_j} - \theta|\big)^p = c \int_{\Omega\cap B(x,r)} \varphi\big(|u - \theta|\big)^p.\end{aligned}$$

We apply this with $x = x_k$ and note that (2.24) implies that

$$\lim_{k\to\infty} \int_{\Omega\cap B(x_k,r)} \varphi\big(|u - \theta|\big)^p = 0$$

to see that $|u(x_k) - \theta(x_k)| \to 0$ as $k \to \infty$. □

3 Solving the Dirichlet problem at infinity using Theorem 2.4

In this section we apply Theorem 2.4 to obtain new solvability results for the Dirichlet problem at infinity on Cartan-Hadamard manifolds or more specifically the $\mathcal{A}$-regularity of a point at infinity $x_0 \in M(\infty)$ under some curvature conditions. The results we obtain are Theorem 3.6 and corollaries 3.7 and 3.8. Within this section M is a Cartan-Hadamard n-manifold, $o \in M$ is a fixed basepoint, $\rho = d(o, \cdot)$, and $\mathcal{A} \in \mathcal{A}^p(M)$ with $p \in (1, \infty)$.

If $a : [0, \infty) \to [0, \infty)$ is a smooth function that is constant in a neighborhood of 0, we denote by f_a the solution to the initial value problem $f_a(0) = 0$, $f_a'(0) = 1$, and $f_a'' = a^2 f_a$. These functions are valuable to us because they can be used to bound growth of normalized Jacobi fields on a unit speed geodesic ray γ with $\gamma(0) = o$ provided that $K_M(\gamma(t))$ is bounded from above (or below) by $a(t)$ for every $t \ge 0$, see [9, Proposition 2.5(a)]. Some basic properties that we will use in this section are proved for these functions in [9, Section 2.2].

The following result tells us that the weighted $(1, 1)$-Sobolev inequality in a truncated cone can be obtained from a suitable curvature upper bound.

3.1 Lemma. *Let $\phi > 1$. Let $T(v, \delta, r_0)$ be a truncated cone with $v \in S_o M$, $\delta > 0$, and $r_0 > 0$. Suppose that*

$$K_M(P) \le -\phi(\phi - 1)/\rho(x)^2$$

for every $x \in T(v, \delta, r_0) \cap M$ and every 2-dimensional subspace $P \subset T_x M$ that contains the radial vector $\nabla\rho(x)$. Suppose that $0 < \lambda < 1 + (n-1)\phi$. Then there exists $r_1 > r_0$ such that

$$\lambda \int_M |\eta| \le \int_M |\nabla\eta|\rho$$

for every $\eta \in C_0^\infty\big(T(v, \delta, r_1) \cap M\big)$.

Proof. Let $a : [0, \infty) \to [0, \infty)$ be a smooth function such that

$$a(t) \begin{cases} = 0 & \text{if } t \in [0, r_0], \\ \le \sqrt{\phi(\phi-1)}/t & \text{if } t \in [r_0, r_0 + 1], \\ = \sqrt{\phi(\phi-1)}/t & \text{if } t \ge r_0 + 1. \end{cases}$$

Then the radial curvatures in $C(v, \delta) \cap M$ are bounded from above by $-a(\rho)^2$. It follows that $\Delta\rho \ge (n-1) f_a'(\rho)/f_a(\rho)$ in $C(v, \delta) \cap M$ by [9, Proposition 2.5(b)]. Since $a(t) = \sqrt{\phi(\phi-1)}/t$ for all $t \ge r_0 + 1$, there exist constants $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that $f_a(t) = c_1 t^\phi + c_2 t^{1-\phi}$ for all $t \ge r_0 + 1$. From this and the assumption $\lambda < 1 + \phi(n-1)$ we can conclude that there exists a constant $r_1 \ge r_0 + 1$ such that $\Delta\rho \ge (\lambda - 1)/\rho$ in $T(v, \delta, r_1) \cap M$.

Now let $\eta \in C_0^\infty\big(T(v, \delta, r_1) \cap M\big)$. Without loss of generality we can assume that $\eta \ge 0$. Then

$$(\lambda - 1) \int_M \eta \le \int_M \eta\rho\Delta\rho = -\int_M \big\langle \nabla(\eta\rho), \nabla\rho \big\rangle = -\int_M \eta - \int_M \rho\langle \nabla\eta, \nabla\rho \rangle$$

so that $\lambda \int_M \eta \le \int_M |\nabla\eta|\rho$. $\square$

In the following lemma we apply Theorem 2.4 to show that a point at infinity is $\mathcal{A}$-regular assuming a curvature upper bound and control on the growth of various Jacobi fields.

For $x \in M\setminus\{o\}$ we denote by $J(x)$ the supremum and by $j(x)$ the infimum of $|V(\rho(x))|$ over all Jacobi fields V along the unit speed geodesic $\gamma^{o,x}$ from o to x that satisfy $V(0) = 0$, $|V'(0)| = 1$, and $V'(0) \perp \dot\gamma_0^{o,x}$.

3.2 Lemma. *Let $x_0 \in M(\infty)$ be a point at infinity and $\phi > 1$. Suppose that x_0 has a neighborhood U in the cone topology such that*

$$K_M(P) \le -\phi(\phi - 1)/\rho(x)^2$$

for every $x \in U \cap M$ and every 2-dimensional subspace $P \subset T_xM$ that contains the radial vector $\nabla\rho(x)$.

Let $h : [0,\infty) \to [0,\infty)$ be a function that satisfies $\int_0^\infty h(t)\,dt < \infty$. Suppose that $F \in \mathcal{F}_p$ is as in Definition 4.1 in Appendix and

$$F\Big(\frac{\rho(x)}{j(x)}\Big) J(x)^{n-1} \le h\big(\rho(x)\big)$$

for every $x \in U \cap M$. Suppose that

$$1 < p < \frac{\alpha}{\beta}\big(1 + (n-1)\phi\big),$$

where α and β are the structure constants of $\mathcal{A}$. Then x_0 is an $\mathcal{A}$-regular point at infinity.

Proof. Let $f : M(\infty) \to \mathbb{R}$ be a continuous function. We have to prove that

$$\lim_{x\to x_0} \overline{H}_f(x) = f(x_0).$$

Fix $\lambda \in \big(p\beta/\alpha, 1 + (n-1)\phi\big)$ and $\varepsilon > 0$. Denote $v = \dot\gamma_0^{o,x_0}$ and let $\delta \in (0,\pi)$ be so small and $r_0 > 0$ so large that $T(v,\delta,r_0) \subset U$ and $|f(x_1) - f(x_0)| < \varepsilon$ whenever $x_1 \in C(v,\delta) \cap M(\infty)$. Let $r_1 > r_0$ be as in Lemma 3.1. We denote $\Omega = T(v,\delta,r_1) \cap M$. Then (2.1) holds for every $\eta \in C_0^\infty(\Omega)$ with weight $w = \rho/\lambda$.

We define $\theta \in C(\bar M)$ by the formula

$$\theta(x) = \min\Big(1, \max\big(r_1 + 1 - \rho(x), \delta^{-1}\sphericalangle_o(x_0,x)\big)\Big).$$

By [12, Lemma 2] there exists a constant $c_1 > 0$ such that $|\nabla\theta(x)| \le c_1/j(x)$ for all $x \in \Omega$. Denote $\tilde F = F(c_1^{-1}\lambda\cdot)$. Then $\tilde F \in \mathcal{F}_p$ by Remark 4.2 and

$$\begin{aligned}
\int_\Omega \tilde F\big(|\nabla\theta| w\big) &\le \int_\Omega F\big(c_1^{-1}|\nabla\theta|\rho\big)\\
&= \int_{r_1}^\infty \int_{S_oM\cap C(v,\delta)} F\big(c_1^{-1}|\nabla\theta(r,\xi)|r\big)\lambda_M(r,\xi)\,d\xi\,dr\\
&\le \int_{r_1}^\infty \int_{S_oM\cap C(v,\delta)} F\Big(\frac{r}{j(r,\xi)}\Big) J(r,\xi)^{n-1}\,d\xi\,dr \le c\int_{r_1}^\infty h(r)\,dr < \infty.
\end{aligned}$$

Now the assumptions of Theorem 2.4 are satisfied with $\tilde F \in \mathcal{F}_p$ replacing F (see Remark 2.7(iii)). Thus there exists an $\mathcal{A}$-harmonic function $u : \Omega \to \mathbb{R}$ so that

$$\lim_{x\to x_0} u(x) = \theta(x_0) = 0 \tag{3.3}$$

and

$$\lim_{x\to y} u(x) = \theta(y) = 1 \tag{3.4}$$

for every $y \in M \cap \partial\Omega$.

Now we define $\tilde{u} : M \to \mathbb{R}$,

$$\tilde{u}(x) = \begin{cases} \min(1, 2u)(x), & \text{if } x \in \Omega, \\ 1, & \text{if } x \in M \setminus \Omega. \end{cases}$$

Since the minimum of two $\mathcal{A}$-superharmonic functions is $\mathcal{A}$-superharmonic and (3.4) holds for every $y \in M \cap \partial\Omega$, we see that $\tilde{u}$ is continuous and $\mathcal{A}$-superharmonic in some neighborhood of each point in M. Since $\mathcal{A}$-superharmonicity is a local property, $\tilde{u}$ is $\mathcal{A}$-superharmonic. Now

$$\overline{H}_f \le f(x_0) + \varepsilon + 2(\sup|f|)\tilde{u}$$

by the definition of $\overline{H}_f$. By this and (3.3) we get $\limsup_{x\to x_0} \overline{H}_f(x) \le f(x_0) + \varepsilon$. Similarly one proves that $\liminf_{x\to x_0} \underline{H}_f(x) \ge f(x_0) - \varepsilon$. Taking into account $\overline{H}_f \ge \underline{H}_f$ and that $\varepsilon > 0$ is arbitrary, we get $\lim_{x\to x_0} \overline{H}_f(x) = f(x_0)$. □

We would like to get rid of the technical functions j and J involving Jacobi fields in the assumptions in Lemma 3.2 and replace the assumption $F\big(\rho(x)/j(x)\big)J(x)^{n-1} \le h\big(\rho(x)\big)$ with some curvature condition. The following gives us one way to do this assuming a curvature bound of the type $-(b\circ\rho)^2 \le K_M \le -(a\circ\rho)^2$.

3.5 Lemma. *Suppose that $a, b : [0,\infty) \to [0,\infty)$ are smooth functions that are constant in some neighborhood of 0. Let $U = T(v,\delta,r_0)$ be a truncated cone at o with $v \in S_oM$, $\delta > 0$, and $r_0 > 0$. Suppose that*

$$-b\big(\rho(x)\big)^2 \le K_M(P) \le -a\big(\rho(x)\big)^2$$

for all 2-dimensional subspaces $P \subset T_xM$, $x \in U \cap M$, containing the radial vector. Suppose that

$$a(t) \ge \sqrt{\phi(\phi-1)}/t$$

for some constant $\phi > 1$ and all sufficiently large t.

Suppose that $\lim_{t\to\infty} b'(t)/b(t)^2 = 0$ and that there exists a constant $\varepsilon > 0$ such that

$$b(t) \le \frac{f_a'(t)}{t\big(\log f_a(t)\big)^{1+\varepsilon}}$$

for all sufficiently large t. Then there exists $F \in \mathcal{F}_p$ such that

$$F\Big(\frac{\rho(x)}{j(x)}\Big)J(x)^{n-1} \le \rho(x)^{-2}$$

for every $x \in U \cap M$ outside a compact set.

Proof. We have $j(x) \ge c_1 f_a\big(\rho(x)\big)$, where $c_1 > 0$ is a constant, and $J(x) \le c f_b\big(\rho(x)\big)$ for every $x \in U \cap M$. This follows from [9, Proposition 2.5(a)] if the curvature bound holds in $C(v,\delta)\cap M$ and in the general case by modifying a and b in a bounded set so that the curvature bound holds in $C(v,\delta)\cap M$ and by applying [9, Lemma 2.4]. Fix $\varepsilon_0 \in (0,\varepsilon)$. By Proposition 4.3 there exists $F \in \mathcal{F}_p$ such that $F \le \tilde{F}$, where

$$\tilde{F}(t) = \exp\Big(-\frac{1}{t}\Big(\log\frac{1}{t}\Big)^{-1-\varepsilon_0}\Big)$$

for all small t. We denote

$$\begin{aligned}\Phi(t) &= \Big(t^{-2}/\tilde{F}\Big(\frac{t}{c_1 f_a(t)}\Big)\Big)^{1/(n-1)}\\ &= t^{-2/(n-1)}\exp\Big(\frac{1}{n-1}\frac{c_1 f_a(t)}{t}\Big(\log\frac{c_1 f_a(t)}{t}\Big)^{-1-\varepsilon_0}\Big)\end{aligned}$$

for all large t. One can use [9, Lemma 2.2] to obtain $\liminf_{s\to\infty} s f_a'(s)/f_a(s) \ge \phi > 1$ and if t is sufficiently large, then this and $f_a(t) \ge ct^{\phi}$ imply that

$$\begin{aligned}\frac{\Phi'(t)}{\Phi(t)} &= \frac{-2t + c_1\Big(1-(1+\varepsilon_0)\Big(\log\frac{c_1 f_a(t)}{t}\Big)^{-1}\Big)\big(tf_a'(t)-f_a(t)\big)\Big(\log\frac{c_1 f_a(t)}{t}\Big)^{-1-\varepsilon_0}}{(n-1)t^2}\\ &\ge \frac{cf_a'(t)\Big(\log\frac{cf_a(t)}{t}\Big)^{-1-\varepsilon_0}}{t} \ge \frac{cf_a'(t)}{t\big(\log f_a(t)\big)^{1+\varepsilon_0}} \ge 2\frac{f_a'(t)}{t\big(\log f_a(t)\big)^{1+\varepsilon}} \ge 2b(t).\end{aligned}$$

Since $b'(t)/b(t)^2 \to 0$ as $t \to \infty$, we have

$$\lim_{t\to\infty}\frac{f_b'(t)/f_b(t)}{b(t)} = 1$$

by [9, Lemma 2.3]. Therefore $\Phi'(t)/\Phi(t) \ge 2b(t) \ge f_b'(t)/f_b(t)$ for all large t. From this we see that $\Phi(t) \ge cf_b(t)$ for all sufficiently large t. Therefore

$$\begin{aligned}F\Big(\frac{\rho(x)}{j(x)}\Big)J(x)^{n-1} &\le \tilde{F}\Big(\frac{\rho(x)}{j(x)}\Big)J(x)^{n-1} \le c\tilde{F}\Big(\frac{\rho(x)}{c_1 f_a\big(\rho(x)\big)}\Big)f_b\big(\rho(x)\big)^{n-1}\\ &\le c\tilde{F}\Big(\frac{\rho(x)}{c_1 f_a\big(\rho(x)\big)}\Big)\Phi\big(\rho(x)\big)^{n-1} = \rho(x)^{-2}\end{aligned}$$

for all $x \in U \cap M$ outside a compact set. □

By combining lemmas 3.2 and 3.5 we now obtain the following result that gives sufficient curvature conditions so that a point at infinity is $\mathcal{A}$-regular.

3.6 Theorem. *Let $a, b : [0,\infty) \to [0,\infty)$ be smooth functions that are constant in some neighborhood of 0. Let $\phi > 1$ and $\varepsilon > 0$ be constants such that $\lim_{t\to\infty} b'(t)/b(t)^2 = 0$,*

$$a(t) \geq \sqrt{\phi(\phi-1)}/t,$$

$$b(t) \leq \frac{f_a'(t)}{t\big(\log f_a(t)\big)^{1+\varepsilon}}$$

for all sufficiently large t.

Suppose that M is a Cartan-Hadamard manifold, $o \in M$, $\rho = d(\cdot, o)$, and $\mathcal{A} \in \mathcal{A}^p(M)$, where

$$1 < p < \frac{\alpha}{\beta}\big(1 + (n-1)\phi\big)$$

and α, β are the structure constants of $\mathcal{A}$. Let $x_0 \in M(\infty)$ and let U be a neighborhood of x_0 in the cone topology such that

$$-b\big(\rho(x)\big)^2 \leq K_M(P) \leq -a\big(\rho(x)\big)^2$$

for all $x \in U \cap M$ and all 2-dimensional subspaces $P \subset T_xM$ that contain the radial vector. Then x_0 is an $\mathcal{A}$-regular point at infinity. □

Now if we are given a curvature upper bound $K_M \leq -(a \circ \rho)^2$, we can try to find a corresponding function b for the lower bound so that the assumptions of Theorem 3.6 are satisfied. The following corollaries cover the two most natural special cases of this result in this way.

3.7 Corollary. *Suppose that M is a Cartan-Hadamard manifold, $o \in M$, $\rho = d(\cdot, o)$, and $\mathcal{A} \in \mathcal{A}^p(M)$, where*

$$1 < p < \frac{\alpha}{\beta}\big(1 + (n-1)\phi\big).$$

Fix $\phi > 1$ and $\varepsilon > 0$. Let $x_0 \in M(\infty)$ and let U be a neighborhood of x_0 in the cone topology such that

$$-\rho(x)^{2\phi-4}\big(\log\rho(x)\big)^{-2-\varepsilon} \leq K_M(P) \leq -\phi(\phi-1)/\rho(x)^2$$

for all $x \in U \cap M$ and all 2-dimensional subspaces $P \subset T_xM$ that contain the radial vector. Then x_0 is an $\mathcal{A}$-regular point at infinity.

Proof. This follows from Theorem 3.6 by choosing $a(t) = \sqrt{\phi(\phi-1)}/t$ and $b(t) = t^{\phi-2}(\log t)^{-1-\varepsilon/2}$ for all large t. □

3.8 Corollary. *Suppose that M is a Cartan-Hadamard manifold, $o \in M$, $\rho = d(\cdot, o)$, and $\mathcal{A} \in \mathcal{A}^p(M)$, where $p \in (1, \infty)$ is an exponent. Fix $k > 0$ and $\varepsilon > 0$. Let $x_0 \in M(\infty)$ and let U be a neighborhood of x_0 in the cone topology such that*

$$-\rho(x)^{-4-\varepsilon} \exp\big(2k\rho(x)\big) \le K_M(P) \le -k^2$$

for all $x \in U \cap M$ and all 2-dimensional subspaces $P \subset T_xM$ that contain the radial vector. Then x_0 is an $\mathcal{A}$-regular point at infinity.

Proof. This follows from Theorem 3.6 by choosing $a(t) = k$ and $b(t) = t^{-2-\varepsilon/2} \exp(kt)$ for all large t. □

3.9 Remark. (i) One can replace the curvature bound in Corollary 3.8 with $-\rho(x)^{-2-\varepsilon} \exp\big(2k\rho(x)\big) \le K_M(P) \le -k^2$. This is done by rewriting the earlier results in this section with curvature bound $K_M \le -k^2$ instead of $K_M \le -\phi(\phi - 1)/\rho(x)^2$ and making appropriate changes.

(ii) These corollaries (together with (i)) generalize those given in [9, Corollary 3.22] and [9, Corollary 3.23] in two important ways. First, we handle all $\mathcal{A} \in \mathcal{A}^p(M)$ instead of just the p-Laplacian. Second, the curvature bounds now involve only radial curvatures. Furthermore, the proof method used here is more natural in the nonlinear potential theoretic setting than the pointwise argument used in [9].

4 Appendix: Auxiliary functions

In this section we define auxiliary functions F, G, and φ on the positive real axis and prove their properties that we need in the proof of our main result. Fix an exponent $p \in (1, \infty)$.

4.1 Notation. Suppose that G is a Young function that is a homeomorphism $[0, \infty) \to [0, \infty)$ and is a diffeomorphism $(0, \infty) \to (0, \infty)$ and satisfies

(A1)
$$\int_0^1 \frac{dt}{G^{-1}(t)} < \infty$$

and

(A2)
$$\lim_{t \to 0} \frac{tG'(t)}{G(t)} = 1.$$

Then $G(\cdot^{1/p})^p$ is also a Young function and we can define a function $F : [0, \infty) \to [0, \infty)$ so that $F(\cdot^{1/p})$ and $G(\cdot^{1/p})^p$ form a complementary Young pair. In this case write $F \in \mathcal{F}_p$.

4.2 Remark. If G is as in Notation 4.1, then λG and $G(\lambda\cdot)$ satisfy (A1) and (A2) as well for every $\lambda > 0$. It follows that if $F \in \mathcal{F}_p$, then $\lambda F \in \mathcal{F}_p$ and $F(\lambda\cdot) \in \mathcal{F}_p$ for every $\lambda > 0$.

The existence of functions in $\mathcal{F}_p$ is guaranteed by the following.

4.3 Proposition. *Fix $\varepsilon \in (0,1)$. Then there exists $F \in \mathcal{F}_p$ such that*

$$F(t) \le t^{p+\varepsilon} \exp\Big(-\frac{1}{t}\Big(\log\big(e+\frac{1}{t}\big)\Big)^{-1-\varepsilon}\Big)$$

for all $t \in [0,\infty)$.

Proof. We choose $\lambda \in (1, 1+\varepsilon)$ and a homeomorphism $H : [0,\infty) \to [0,\infty)$ that is a diffeomorphism $(0,\infty) \to (0,\infty)$ and satisfies

$$H(t) = \begin{cases} \big(\log \frac{1}{t}\big)^{-1}\big(\log\log\frac{1}{t}\big)^{-\lambda} & \text{if } t \text{ is small enough,} \\ t^{p/\varepsilon} & \text{if } t \text{ is large enough.} \end{cases}$$

We define $G(t) = \int_0^t H(s)\,ds$ and denote $\tilde{G}(t) = G(t^{1/p})^p$. Then $\tilde{G}$ is a Young function and we denote its Young conjugate by $\tilde{F}$ and also $F(t) = \tilde{F}(t^p)$.

Now $tH'(t)/H(t) \to 0$ as $t \to 0$ so that $tH(t)/G(t) \to 1$ as $t \to 0$ by l'Hospital's rule and we see that G satisfies (A2). Denote $R(t) = t/H(t)$. Then $R(k\cdot) \approx R$ for every constant $k > 0$ so that

$$R\big(G(t)\big) \approx R\big(tH(t)\big) = t\frac{H(t)}{H\big(tH(t)\big)} \approx t$$

for all small enough t. Hence $G^{-1}(t) \approx R(t)$ for all small enough t. Since $\lambda > 1$, it follows that G satisfies (A1) and therefore $F \in \mathcal{F}_p$.

We still have to estimate F from above. Since $G(t) \approx tH(t)$ for all t, we get

$$\tilde{G}'(t) = t^{1/p-1}G(t^{1/p})^{p-1}H(t^{1/p}) \approx H(t^{1/p})^p \tag{4.4}$$

for all t. Since $H(t^{1/p})^p \ge cH(t)^p$ for all sufficiently small t, $\tilde{F}'(t) = (\tilde{G}')^{-1}(t) \le H^{-1}(ct^{1/p})$ for all small enough t. Thus

$$F(t) = p\int_0^t s^{p-1}\tilde{F}'(s^p)\,ds \le p\int_0^t s^{p-1}H^{-1}(cs)\,ds \le cH^{-1}(ct)$$

for all small enough t. Now

$$H^{-1}(t) = \exp\Big(-\exp\Big(\lambda W\big(\lambda^{-1}t^{-1/\lambda}\big)\Big)\Big),$$

for all sufficiently small t, where W is the Lambert W function defined by the identity $W(s)e^{W(s)} = s$. It is easy to verify that W satisfies $W(s) \geq \log s - \log\log s$ for all $s \geq e$ so that we can estimate

$$\begin{aligned} F(t) &\leq cH^{-1}(ct) \leq c\exp\Big(-\exp\Big(\lambda W\big(ct^{-1/\lambda}\big)\Big)\Big) \\ &\leq c\exp\Big(-\exp\Big(\lambda\log(ct^{-1/\lambda}) - \lambda\log\log(ct^{-1/\lambda})\Big)\Big) \\ &= c\exp\Big(-ct^{-1}\Big(\log(ct^{-1/\lambda})\Big)^{-\lambda}\Big) \leq c\exp\Big(-c\frac{1}{t}\Big(\log\frac{1}{t}\Big)^{-\lambda}\Big) \end{aligned}$$

for all small enough t. On the other hand, if t is large enough, then (4.4) implies that $\tilde{G}'(t) \approx H(t^{1/p})^p = ct^{p/\varepsilon}$ and hence $\tilde{F}'(t) = (\tilde{G}')^{-1}(t) \leq ct^{\varepsilon/p}$ for all large enough t. From this we see that

$$F(t) = p\int_0^t s^{p-1}\tilde{F}'(s^p)\,ds \leq ct^{p+\varepsilon}$$

if t is large enough. Putting all the above together with $\lambda < 1+\varepsilon$ we obtain

$$F(t) \leq ct^{p+\varepsilon}\exp\Big(-\frac{1}{t}\Big(\log\big(e+\frac{1}{t}\big)\Big)^{-1-\varepsilon}\Big)$$

for all t. The claim follows since $kF \in \mathcal{F}_p$ for all $k > 0$ by Remark 4.2. □

We define an additional auxiliary function in the following way

$$\varphi(t) := \Big(\int_0^{\cdot}\frac{1}{G^{-1}}\Big)^{-1}(t).$$

This function is well defined by (A1).

4.5 Lemma. *The function* φ *is a homeomorphism* $[0,\infty) \to [0,\infty)$ *that is smooth on* $(0,\infty)$ *and satisfies*

$$G \circ \varphi' = \varphi$$

and

$$\lim_{t\to 0}\frac{\varphi''(t)\varphi(t)}{\varphi(t)^2} = 1.$$

Proof. Since G is convex, $G(t) \geq ct$ for all $t \geq 1$ and hence we have $G^{-1}(t) \leq ct$ for all large enough t. It follows that $\int_0^\infty 1/G^{-1} = \infty$ and hence $\int_0^{\cdot} 1/G^{-1}$ is a homeomorphism $[0,\infty) \to [0,\infty)$ that is a diffemorphism $(0,\infty) \to (0,\infty)$. Hence the same holds for its inverse φ.

In order to see that $G \circ \varphi' = \varphi$, notice that $G^{-1} = 1/(\varphi^{-1})' = \varphi' \circ \varphi^{-1}$ and compose this from left with G and from right with φ.

Finally, since $G \circ \varphi' = \varphi$, we can differentiate both sides to obtain $(G' \circ \varphi')\varphi'' = \varphi'$ and using this we can write

$$\frac{\varphi''\varphi}{(\varphi')^2} = \frac{\frac{\varphi'}{G'\circ\varphi'} \cdot (G \circ \varphi')}{(\varphi')^2} = \frac{G}{\mathrm{id} \cdot G'} \circ \varphi'$$

Since $\lim_{t\to 0} \varphi'(t) = \lim_{t\to 0} G^{-1}\big(\varphi(t)\big) = 0$, (A2) implies that $\varphi''(t)\varphi(t)/\varphi'(t)^2 \to 1$ as $t \to 0$. □

Acknowledgement. The author would like to thank Ilkka Holopainen for his help with preparing this manuscript.

References

[1] M. T. Anderson. The Dirichlet problem at infinity for manifolds of negative curvature. *J. Differ. Geom.*, 18(4):701–721, 1983.

[2] S. Y. Cheng. The Dirichlet problem at infinity for non-positively curved manifolds. *Comm. Anal. Geom.*, 1(1):101–112, 1993.

[3] C. B. Croke. Some isoperimetric inequalities and eigenvalue estimates. *Ann. Sci. École. Norm. Sup. (4)*, 13(4):419–435, 1980.

[4] P. Eberlein and B. O'Neill. Visibility manifolds. *Pacific J. Math.*, 46:45–109, 1973.

[5] R. E. Greene and H. Wu. *Function theory on manifolds which possess a pole*, volume 699 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[6] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.

[7] I. Holopainen. Nonlinear potential theory and quasiconformal mappings on Riemannian manifolds. *Ann. Acad. Sci. Fenn. Ser. A I. Math. Diss.*, 74:1–45, 1990.

[8] I. Holopainen. Solutions of elliptic equations on manifolds with roughly Euclidean ends. *Math. Z.*, 217:459–477, 1994.

[9] I. Holopainen and A. Vähäkangas. Asymptotic Dirichlet problem on negatively curved spaces. *J. Analysis*, 15:63–110, 2007. Proceedings of the International Conference on Geometric Function Theory, Special Functions and Applications (ICGFT) (R. W. Barnard and S. Ponnusamy, eds.).

[10] E. P. Hsu. Brownian motion and Dirichlet problems at infinity. *Ann. Probab.*, 31(3):1305–1319, 2003.

[11] D. Sullivan. The Dirichlet problem at infinity for a negatively curved manifold. *J. Differ. Geom.*, 18(4):723–732, 1984.

[12] A. Vähäkangas. Dirichlet problem at infinity for $\mathcal{A}$-harmonic functions. *Potential Anal.*, 27:27–44, 2007.

[13] S. T. Yau. Harmonic functions on complete Riemannian manifolds. *Comm. Pure Appl. Math.*, 28:201–228, 1975.

Address: Aleksi Vähäkangas, Department of Mathematics and Statistics, P.O. Box 68 (Gustaf Hällströmin katu 2b) FI-00014 University of Helsinki, Finland.

E-mail: `Aleksi.Vahakangas@Helsinki.FI`