# Capacity estimates for quasiminimizers

O. Martio
University of Helsinki

**Abstract.** Let $E = (C, \Omega)$ be a condenser in $\mathbf{R}^n$ where $\Omega$ is an open set and $C$ a compact subset of $\Omega$. The $p$–Dirichlet integral of the potential function $u$ of $C$ in $\Omega$ associated with the $p$–harmonic equation gives the $p$–capacity $cap_p E$ of the condenser $E$. Moreover, $u$ satisfies the basic $p$–capacity equation $t^{p-1} cap_p(C_t, \Omega) = cap_p E$ where $C_t = \{x \in \Omega : u(x) \geq t\}$. A counterpart of this equation for quasiminimizers is considered. The estimates make use of one dimensional quasiminimizers.

**Key words:** Quasiminimizers; capacity estimates.

**2000 Mathematics Subject Classification:** 31B15, 31C15.

## 1 Introduction

Let $\Omega$ be an open and bounded subset of $\mathbf{R}^n$, $n \geq 1$ and $p > 1$. If $C$ is a compact subset of $\Omega$, then the pair $E = (C, \Omega)$ is called a *condenser*. The *p–capacity* of E is defined as

$$cap_p E = \inf \int_\Omega |\nabla \varphi|^p \, dx \tag{1}$$

where the infimum is taken over all functions $\varphi \in C_o^\infty(\Omega)$ such that $\varphi = 1$ on $C$. Let $\varphi$ be as above. Now there is a unique function $u \in W^{1,p}(\Omega)$ such that $u - \varphi \in W_0^{1,p}(\Omega \setminus C)$ and

$$cap_p E = \int_\Omega |\nabla u|^p \, dx.$$

Here $W^{1,p}(\Omega)$ is the Sobolev space of functions in $L^p(\Omega)$ with distributional partial derivatives in $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$ stands for the space of functions in $W^{1,p}(\Omega)$ with zero boundary values in the Sobolev sense. Moreover, the function $u$ is $p$–harmonic in $\Omega \setminus C$, i.e. $u$ is a solution of the $p$–harmonic equation

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 \tag{2}$$

in $\Omega \setminus C$. The function $u$ is called the *$p$–potential* of $C$ in $\Omega$, see [HKM, Section 6.10].

Let $u$ be the $p$–potential of $C$ in $\Omega$ and $t \in (0,1)$. Set $C_t = \{x \in \Omega : u(x) \geq t\}$. Although $C_t$ need not be a compact set in $\Omega$ we can define the $p$–capacity of the pair, also called a condenser, $(C_t, \Omega)$ as

$$cap_p(C_t, \Omega) = \inf \int_\Omega |\nabla v|^p \, dx \tag{3}$$

where the infimum is now taken over all functions $v$ such that $v - u/t \in W_0^{1,p}(\Omega \setminus C_t)$ and $v = 1$ on $C$. Here we use the refined version of the the space $W_0^{1,p}(\Omega \setminus C_t)$ consisting of all functions $w \in W^{1,p}(\mathbf{R}^n)$ such that $w = 0$ and $w = 1$ $p$–quasieverywhere in the complement of $\Omega$ and in $C_t$, respectively. For this theory see [HKM, Chapter 4].

The basic equation between the $p$–capacities of the condensers $(C_t, \Omega)$ and $E$ is

$$t^{p-1} cap_p(C_t, \Omega) = cap_p E. \tag{4}$$

Equation (4) becomes a double inequality

$$(\frac{\alpha}{\beta})^{p+1} t^{p-1} cap_p(C_t, \Omega) \leq cap_p E \leq (\frac{\beta}{\alpha})^{p+1} t^{p-1} cap_p(C_t, \Omega) \tag{5}$$

if, instead of a $p$–potential, an $A$–potential $u$ of $C$ in $\Omega$ is used. Here $A$ refers to the degenerate second order partial differential equation

$$\nabla \cdot A(x, \nabla u) = 0 \tag{6}$$

where the operator $A$ satisfies

$$\alpha |h|^p \leq A(x,h) \cdot h \leq \beta |h|^p, \; 0 < \alpha \leq \beta < \infty, \tag{7}$$

see [HKM, Lemma 6.19]. Equation (4) and inequality (5) are important tools in the study of boundary behavior of $p$– and $A$–harmonic functions as well as in the study of polar sets. The purpose of this paper is to find the corresponding estimates for quasiminimizers.

We recall the definition. Let $\Omega$ be an open subset of $\mathbf{R^n}$, $n \geq 1$, $p > 1$ and $K \geq 1$. A function $u$ in the local Sobolev space $W_{loc}^{1,p}(\Omega)$ is called a $(p,K)$–*quasiminimizer* in $\Omega$ if for all open sets $\Omega' \subset\subset \Omega$

$$\int_{\Omega'} |\nabla u|^p \, dx \leq K \int_{\Omega'} |\nabla v|^p \, dx \tag{8}$$

for all functions $v$ such that $v - u \in W_0^{1,p}(\Omega')$. In general we keep the number $p$ fixed and use the abbreviation $K$–quasiminimizer. For $K = 1$ the function $u$ is minimizer and hence a $p$–harmonic function. Note that an $A$–potential $u$ of $C$ in $\Omega$ is a $(\beta/\alpha)^p$– quasiminimizer in $\Omega \setminus C$. For the theory of quasiminimizers see [GG] and [KiM].

Since quasimininimizers neither form a sheaf nor obey the comparison principle, which is fundamental in Potential Theory, the capacity estimates of the type (5) for quasiminimizers should be based on other methods than the proof for (5). In this paper we develop a method which uses one dimensional quasiminimizers. These are considered in Section 3 and Section 4 is devoted to the main result. The sharpness of the capacity estimates for quasiminimizers is considered at the end of Section 4.

# 2 Preliminaries and a characterization for quasiminimizers

We first introduce an alternative definition for a quasiminimizer. In particular this is useful for quasiminimizers on the real line. The definition does not involve the $p$–Dirichlet integral of the quasiminimizer $u$ itself but only that of minimizers, i.e. solutions of the $p$–harmonic equation with the same boundary values as $u$.

Let $u \in W^{1,p}_{loc}(\Omega), p > 1$. For each open set $\Omega' \subset\subset \Omega$ we let $u_{\Omega'}$ denote the minimizer of the $p$–Dirichlet integral in $\Omega'$ with boundary values $u$, i.e. $u_{\Omega'} - u \in W_0^{1,p}(\Omega')$ and $u_{\Omega'}$ is a solution of the $p$–harmonic equation (2) in $\Omega'$. Condition (8) can now be rewritten as

$$\int_{\Omega'} |\nabla u|^p \, dx \le K \int_{\Omega'} |\nabla u_{\Omega'}|^p \, dx. \tag{9}$$

**Theorem 2.1** *Suppose that $u$ belongs to $W^{1,p}_{loc}(\Omega)$. Then $u$ is a $K$–quasiminimizer in $\Omega$ if and only if for each open set $\Omega' \subset\subset \Omega$ and all disjoint open sets $\Omega_1, ..., \Omega_k \subset \Omega'$ it holds*

$$\sum_i \int_{\Omega_i} |\nabla u_{\Omega_i}|^p \, dx \le K \int_{\Omega'} |\nabla u_{\Omega'}|^p \, dx. \tag{10}$$

*Proof.* The necessity of condition (10) is immediate.

For the converse we have to show (9) in every open set $\Omega' \subset\subset \Omega$. Fix an open set $\Omega'$ and note that (10) holds for a countable collection of open subsets $\Omega_1, \Omega_2, ...$ of $\Omega'$ as well. Form a Whitney decomposition $\{Q_i\}$ of $\Omega'$ where the open cubes $Q_i'$s are disjoint and $\cup \overline{Q_i} = \Omega'$. For each $j = 1, 2, ...$ subdivide every cube $Q_i$, if necessary, to disjoint cubes to obtain a new sequence $\{Q_i^j\}$ of disjoint cubes so that every cube $Q_i^j$ satisfies $diam(Q_i^j) \leq 1/j$.

Next define for each $j$ the function $v_j$ as

$$\begin{aligned} v_j(x) &= u_{Q_i^j}(x), x \in Q_i^j, \, i = 1, 2, ... \\ &= u(x), x \in \overline{\Omega'} \setminus \cup_i Q_i^j. \end{aligned}$$

Now it easily follows that $v_j - u \in W_0^{1,p}(\Omega')$ and by (10)

$$\int_{\Omega'} |\nabla v_j|^p \, dx \leq \int_{\Omega'} |\nabla u|^p \, dx$$

for each $j$. Since the sequence $\nabla v_j$ is bounded in $L^p(\Omega')$ and $v_j - u \in W_0^{1,p}(Q_i^j)$ for each $i$ and $j$, the Sobolev inequality yields

$$\int_{Q_i^j} |v_j - u|^p \, dx \leq C diam(Q_i^j)^p \int_{Q_i^j} |\nabla (v_j - u)|^p \, dx$$

$$\leq C j^{-p} \int_{Q_i^j} |\nabla (v_j - u)|^p \, dx$$

where $C$ depends only on $p$ and $n$. Summing over $i$ we obtain

$$\int_{\Omega'} |v_j - u|^p \, dx \leq \sum_i \int_{Q_i^j} |v_j - u|^p \, dx \leq 2^{p+1} C j^{-p} \int_{\Omega'} |\nabla u|^p \, dx$$

because

$$\int_{\Omega'} |\nabla v_j|^p \, dx \leq \int_{\Omega'} |\nabla u|^p \, dx$$

by he minimizing property of the function $v_j$ in each $Q_i^j$. Thus $v_j \to u$ in $L^p(\Omega')$.

Since the sequence $\nabla v_j$ is bounded in $L^p(\Omega')$, passing to a subsequence if necessary, we may assume that $\nabla v_j \to \nabla u$ weakly in $L^p(\Omega')$. By the lower semicontinuity of the $L^p$–norm in the weak convergence we see that

$$\int_{\Omega'} |\nabla u|^p \, dx \leq \liminf_{j \to \infty} \int_{\Omega'} |\nabla v_j|^p \, dx \leq K \int_{\Omega'} |\nabla u_{\Omega'}|^p \, dx$$

where (11) is used in the last step. This yields (9) and the proof is complete.

There is a version of Theorem 2.1 where the assumption $u \in W_{loc}^{1,p}(\Omega)$ is not needed. To formulate the result we introduce some notation. Let $w$ be a continuous real valued function defined on the boundary $\partial\Omega$ of a bounded open set $\Omega$ of $\mathbf{R}^n$. We let $H_w^\Omega$ denote the Perron–Wiener–Brelot solution associated with the $p$–harmonic equation (2) and with the boundary values $w$, see [HKM, Chapter 9]. Since $\Omega$ is bounded and $w$ is continuous a unique Perron–Wiener–Brelot solution $H_w^\Omega$ with boundary values $w$ exists, see [HKM, Theorem 9.26].

If $u$ is a quasiminimizer, then $u$ is locally Hölder continuous, see [CC] and [KiM]. Hence the continuity assumption in Theorem 2.2 is not an essential restriction.

**Theorem 2.2** *Suppose that $u$ is continuous in $\Omega$. Then $u$ is a $K$–quasi–minimizer in $\Omega$ if and only if for each open set $\Omega' \subset\subset \Omega$ and all disjoint open sets $\Omega_1, ..., \Omega_k \subset \Omega'$ it holds*

$$\sum_i \int_{\Omega_i} |\nabla H_u^{\Omega_i}|^p \, dx \leq K \int_{\Omega'} |\nabla H_u^{\Omega'}|^p \, dx < \infty. \tag{11}$$

*Proof.* The proof is similar to that of Theorem 2.1. For the sufficiency replace in the definition of the sequence $H_u^{Q_i^j}$ the functions $u_{Q_i^j}$ by the functions $H_u^{Q_i^j}$. Note that a cube is regular domain for the $p$–Dirichlet problem and hence the function $v_j$ is continuous in $\Omega'$. Now it is easy to see that the sequence $v_j, j = 1, 2, ...$, converges locally uniformly to $u$ in $\Omega'$ and hence no Poincaré inequality is needed.

# 3 Quasiminimizers in R

In the one dimensional case Theorem 2.1 takes a simple form.

**Theorem 3.1** *Suppose that $p > 1, K \geq 1$, $\Delta$ is an open interval in $\mathbf{R}$ and $u : \Delta \to \mathbf{R}$ is a function. Then $u$ is a $K$–quasiminimizer if and only if for all intervals $[a, b] \subset \Delta$ it holds*

$$\sum_{i=1}^{k} \frac{|u(x_{i+1}) - u(x_i)|^p}{(x_{i+1} - x_i)^{p-1}} \leq K \frac{|u(b) - u(a)|^p}{(b-a)^{p-1}} \tag{12}$$

*whenever $a = x_1 < x_2 < ... < x_{k+1} = b$ is a partition of $[a, b]$.*

*Proof.* Since affine functions are minimizers in the 1–dimensional case for all $p$, see [GG], and

$$\int_c^d |f'(t)|^p \, dt = \frac{|f(d) - f(c)|^p}{(d-c)^{p-1}} \tag{13}$$

for an affine function $f$, (12) follows from Theorem 2.1 for a $K$–quasiminimizer $u$.

To prove the sufficiency of (12) we first show that $u$ is absolutely continuous in any closed interval $[a,b] \subset \Delta$. Let $(a_i,b_i), i=1,...,k$ be a collection of disjoint intervals in $[a,b]$. By the Hölder inequality and by (12) we obtain

$$(\sum_i |u(b_i)-u(a_i)|)^p \leq (\sum_i \frac{|u(b_i)-u(a_i)|^p}{|b_i-a_i|^{p-1}})(\sum_i |b_i-a_i|)^{p-1}$$

$$\leq K\frac{|u(b)-u(a)|^p}{(b-a)^{p-1}}(\sum_i |b_i-a_i|)^{p-1}$$

and this clearly implies absolute continuity of $u$ on $[a,b]$.

Condition (12) also implies that $u' \in L^p_{loc}(\Delta)$. Indeed, let $[a,b] \subset \Delta$ and subdivide $[a,b]$ into intervals of equal length $< 1/i$. Approximate $u$ on $[a,b]$ by a piecewise linear function $v_i$ which equals $u$ at the endpoints of subintervals. Then $v_i$ converges uniformly to $u$ in $[a,b]$ and it follows from (12), as in the proof of Theorem 2.1, that $v_i' \to u'$ weakly in $L^p([a,b])$, at least for a subsequence. Hence $u' \in L^p([a,b])$ and the inequality (8) follows from the lower semicontinuity of the norm with respect to the weak convergence, see the proof for Theorem 2.1. The proof follows.

**Remark 3.2** The condition (12) should be compared to the condition

$$\sum_{i=1}^{j} \frac{|u(x_{i+1})-u(x_i)|^p}{(x_{i+1}-x_i)^{p-1}} < \infty \tag{14}$$

for a function $u : [a,b] \to \mathbf{R}$. Here $x_1, x_2, ...x_{j+1}$ is any partition of $[a,b]$. Now (14) is equivalent to the fact that $u$ is absolutely continuous on $[a,b]$ and $u' \in L^p([a,b])$, $p > 1$. The sufficiency of (14) follows as in the proof for Theorem 3.1 and the necessity is due to the Hölder inequality and the estimate

$$|u(x_{i+1})-u(x_i)| \leq \int_{x_{i+1}}^{x_i} |u'|\,dt.$$

Inequality (12) has a reverse nature since it gives the bound for the sum in (14) in terms of the values of $u$ at the endpoints of each interval $[a,b]$. Note that (12) is much stronger than (14). In particular (12) implies that $u'$ is locally integrable to some exponent $q > p$ and that u is either constant or strictly monotone. Some of these properties are considered in the following.

A natural domain of definition for a quasiminimizer in $\mathbf{R}$ is a closed interval $[a,b]$. Indeed, if $u$ is a $K$–quasiminimizer in an open interval $(a,b)$, then $u$ has a continuous extension to $[a,b]$ and

$$\int_c^d |u'(t)|^p\,dt \leq K\frac{|u(d)-u(c)|^p}{(d-c)^{p-1}} \tag{15}$$

holds for all intervals $[c,d] \subset [a,b]$. For this and other properties of one dimensional quasiminimizers see [MS].

We say that a $K$–quasiminimizer $u : [0,1] \to [0,1]$ is a *normalized* $K$–quasiminimizer if $u(0) = 0$ and $u(1) = 1$. The following lemma is immediate.

**Lemma 3.3** *Suppose that $u$ is a normalized $(p,K)$–quasiminimizer. Then*

$$u(t) \leq K^{1/p} t^{(p-1)/p}$$

*for each $t \in [0,1]$.*

*Proof.* By the Hölder inequality

$$u(t) = \int_0^t u'(s)\,ds \leq t^{(p-1)/p} (\int_0^1 u'(s)^p\,ds)^{1/p}$$

$$\leq t^{(p-1)/p} [K \frac{(u(1)-u(0))^p}{(1-0)^{p-1}}]^{1/p} = K^{1/p} t^{(p-1)/p}$$

as required.

Next we review some results from [MS] which will be needed in the sequel. We use the same notation as in [MS] where the higher regularity properties of one dimensional quasiminimizers were considered in detail. From [MS, Theorem 4] it follows that there is a function $p_1 : (1,\infty) \times [1,\infty) \to (1,\infty]$ and for each triple $(p,K,s) \in (1,\infty) \times [1,\infty) \times (1, p_1(p, K^{1/p}))$ a number $K_1 = K_1(p,K,s)$ such that if $u$ is a normalized $(p,K)$–quasiminimizer, then $u$ is also a $(s,K_1)$–quasiminimizer. The function $p_1$ satisfies for each $p \in (1,\infty)$and $K \geq 1$:

$$\begin{aligned} p_1(p,K) &> p && (16) \\ \lim_{K\to 1} p_1(p,K) &= \infty = p_1(p,1) && (17) \\ \lim_{K\to\infty} p_1(p,K) &= p && (18) \end{aligned}$$

and the number $K_1$ has the property $K_1(p,1,s) = 1$ for each $p, s > 1$.

From the above property we obtain an improved version of Lemma 3.3:

**Corollary 3.4** *Suppose that $u$ is a normalized $(p,K)$–quasiminimizer. Then for each $s \in (1, p_1(p, K^{1/p}))$*

$$u(t) \leq K_1(p,K,s)^{1/s} t^{(s-1)/s} \tag{19}$$

*for every $t \in [0,1]$.*

**Remark 3.5** The only normalized 1–quasiminimizer is $u(t) = t$. Note that (19) reduces to $u(t) \le t$ for $K = 1$.

Suppose that $u$ is a normalized $(p, K)$–quasiminimizer. Then $u$ is strictly increasing and continuous. By [MS, Theorem 14] there is a function $p_2 : (1, \infty) \times [1, \infty) \to (1, \infty]$ and for each $q \in p_2(p/(p-1), K^{1/(p-1)})$ a number $K_2 = K_2(p, K, q)$ such that the inverse function $v$ of $u$ is for each $q \in (1, p_2(p/(p-1), K^{1/(p-1)}))$ also a $(q, K_2)$–quasiminimizer. The function $p_2$ and the number $K_2 = K_2(p, K, q)$ satisfy

$$\lim_{K\to 1} p_2(p/(p-1), K) = \infty = p_2(p/(p-1), 1) \tag{20}$$

$$\lim_{K\to \infty} p_2(p/(p-1), K) = 1 \tag{21}$$

$$K_2(p, 1, q) = 1. \tag{22}$$

**Remark 3.6** For $p = 2$ the functions $p_1(2, K)$ and $p_2(2, K)$ have explicit expressions, see [MS],

$$p_1(2, \sqrt{K}) = 1 + \sqrt{K/(K-1)}$$

$$p_2(2, K) = \sqrt{K/(K-1)}$$

for $K > 1$. The aforementioned functions $p_1$, $p_2$ and the numbers $K_1$, $K_2$ can be numerically computed for all argument values, see [D'AS] and [MS]. Moreover, all these results are sharp. Note the open ended property for the exponents $s$ and $q$. Hence there is no best Hölder exponent in (19).

# 4 Capacity estimates

We state the counterpart to (3) as two separate theorems although their proofs follow from the same principle. We let the functions $p_1, p_2$ and the numbers $K_1, K_2$ be as in the previous section.

Suppose that $E = (C, \Omega)$ be a condenser in $\mathbf{R}^n, n \ge 1$, where $\Omega$ is a bounded open set. Let $u$ be a $(p, K)$– quasiminimizer in $\Omega \setminus C$ with boundary values 0 on $\partial\Omega$, 1 on $C$, i.e. $u - \varphi \in W_0^{1,p}(\Omega \setminus C)$ where $\varphi \in C_0^\infty(\Omega)$ and $\varphi = 1$ on $C$. Write $C_t = \{x \in \Omega : u(x) \ge t\}$.

**Theorem 4.1** *For each $s \in (1, p_1(p/(p-1), K^{1/p(p-1)})$ there is a number $\kappa_1 = \kappa_1(p, K, s) < \infty$ such that*

$$cap_p E \le \kappa_1 t^{p-s/(s-1)}\, cap_p(C_t, \Omega). \tag{23}$$

*The number $\kappa_1$ has the following property for each $p > 1$ and $s \in (1, p_1(p/(p-1), K^{1/p(p-1)})$:*

$$\lim_{K \to 1} \kappa_1(p, K, s) = 1 = \kappa_1(p, 1, s)$$

**Remark 4.2** For $K = 1$ inequality (23) and the properties of $p_1$ give $cap_p E \leq t^{p-1}\, cap_p(C_t, \Omega)$. Inequality (23) does not reduce to the right hand side of (5) if the $A$–potential $u$ of $C$ in $\Omega$ is considered as a quasiminimizer because now an $A$–potential of $C$ in $\Omega$ is a $(\beta/\alpha)^p$–quasiminimizer in $\Omega \setminus C$. Since $p_1(p/(p-1), K^{1/p(p-1)}) > p/(p-1)$, the exponent $s$ can be chosen $> p/(p-1)$.

*Proof for Theorem 4.1* Set $u = 1$ on $C$. Then $u \in W^{1,p}(\Omega)$ and write

$$\Omega_t = \{x \in \Omega : u(x) > t\},\ 0 \leq t < 1.$$

Although $\Omega_t$ need not be an open subset of $\Omega$, for each $0 \leq t < t' \leq 1$ we can define a condenser $(C_{t'}, \Omega_t)$ and its $p$–capacity $cap_p(C_{t'}, \Omega_t)$ as before using the refined Sobolev functions. Note that $\Omega_0 = \Omega$ provided that $C$ is a set of positive $p$–capacity and $\Omega$ is a domain. These we can assume without loss of generality. For each $0 \leq t < t' \leq 1$ the quasiminimizing property of $u$ yields

$$cap_p(C_{t'}, \Omega_t) \leq (t' - t)^{-p} \int_{\Omega_t \setminus C_{t'}} |\nabla u|^p\, dx \leq K cap_p(C_{t'}, \Omega_t). \tag{24}$$

Fix an interval $[a, b] \subset [0, 1]$ and let $a = t_0 < t_1 < ... < t_k = b$ be a partition of $[a, b]$. Next we employ the well known separation inequality for capacities of condensers, see [HKM, Theorem 2.6]: For the condensers $(C_{t_i}, \Omega_{t_{i-1}})$, $i = 1, 2, ..., k$, this gives

$$\sum_{i=1}^{k} cap_p(C_{t_i}, \Omega_{t_{i-1}})^{-1/(p-1)} \leq cap_p(C_{t_k}, \Omega_{t_0})^{-1/(p-1)} \tag{25}$$

because

$$C_{t_k} \subset \Omega_{t_{k-1}} \subset C_{t_{k-1}} \subset \Omega_{t_{k-2}} \subset\ ... \subset \Omega_{t_0}.$$

Set

$$\varphi(t) = \int_{\{u(x) < t\}} |\nabla u|^p\, dx.$$

Then $\varphi : [0, 1] \to [0, \beta]$,

$$\beta = \int_{\Omega \setminus C} |\nabla u|^p\, dx,$$

is a continuous strictly increasing function with $\varphi(0) = 0$ and $\varphi(1) = \beta$. Now (24) and (25) yield

$$\begin{aligned} &\sum_{i=1}^{k} (t_i - t_{i-1})^{p/(p-1)} (\varphi(t_i) - \varphi(t_{i-1}))^{1/(1-p)} \\ &\leq K^{1/(p-1)} (b - a)^{p/(p-1)} (\varphi(b) - \varphi(a))^{1/(1-p)}. \end{aligned} \tag{26}$$

Note that $\nabla u = 0$ almost everywhere on the set where $u$ =const.

Let $\psi : [0, \beta] \to [0, 1]$ denote the inverse function of $\varphi$. Writing (26) for the inverse function $\psi$ we obtain

$$\sum_{i=1}^{k} (\psi(t'_i) - \psi(t'_{i-1}))^{p/(p-1)} (t'_i - t'_{i-1})^{1/(1-p)}$$
$$\leq K^{1/(p-1)} (\psi(b') - \psi(a'))^{p/(p-1)} (b' - a')^{1/(1-p)} \tag{27}$$

where $\varphi(a) = a'$, $\varphi(b) = b'$ and $\varphi(t_i) = t'_i, i = 0, 1, \dots, k$. Thus (27) holds for an arbitrary partition

$$a' = t'_0 < t'_1 < \dots < t'_k = b'$$

of the interval $[a', b'] \subset [0, \beta]$. By Theorem 3.1 the function $\psi$ is a $(p/(p-1), K^{1/(p-1)})$–quasiminimizer. Since $\psi(\beta t)$ is a normalized quasiminimizer, Corollary 3.4 yields for each $s \in (1, p_1(p/(p-1), K^{1/p(p-1)}))$

$$\psi(\beta t) \leq c_1^{1/s} t^{(s-1)/s} \tag{28}$$

where $c_1 = c_1(p, K, s)$. For the inverse function $\varphi$ of $\psi$ this means that

$$\varphi(t) \geq \beta c_1^{1/(1-s)} t^{s/(s-1)},\ t \in [0, 1]. \tag{29}$$

By the quasiminimizing property of $u$

$$cap_p(C_t, \Omega) \geq K^{-1} \int_{\{u<t\}} |\nabla (\frac{u}{t})|^p \, dx = K^{-1} t^{-p} \varphi(t)$$

and hence we obtain from (29)

$$cap_p(C, \Omega) \leq \beta \leq c_1^{1/(s-1)} t^{-s/(s-1)} \varphi(t)$$
$$\leq K c_1^{1/(s-1)} t^{p-s/(s-1)} cap_p(C_t, \Omega).$$

It is easy to check that the number

$$\kappa_1 = \kappa_1(p, K, s) = K c_1^{1/(1-s)}$$

has the required property. The proof follows.

The next theorem gives the counterpart of the left hand side of (5) for quasiminimizers.

**Theorem 4.3** *For each $q \in (1, p_2(p, K))$ there is $\kappa_2 = \kappa_2(p, K, q)$ such that*

$$\frac{t^{p-(q-1)/q}}{\kappa_2}\, cap_p(C_t, \Omega) \leq cap_p E. \tag{30}$$

*The number $\kappa_2$ satisfies for each $p > 1$ and $q \in (1, p_2(p, K))$:*

$$\lim_{K \to 1} \kappa_2(p, K, q) = 1 = \kappa_2(p, 1, q).$$

*Proof.* We proceed as in the proof of Theorem 4.1. Since the function $\varphi$ is the inverse function of $\psi$ and $\psi$ is a $(p/(p-1), K^{1/(p-1)})$–quasiminimizer, for each $q \in (1, p_2(p, K))$, see Section 3, there is a number $K_2 = K_2(p/(p-1), K^{1/(p-1)}, q)$ such that the function $\varphi$ is a $(q, K_2)$–quasiminimizer in $[0, 1]$. Since $\varphi/\beta$ is a normalized quasiminimizer, Corollary 3.4 yields

$$\varphi(t) \leq K_2^{1/q} t^{(q-1)/q} \beta,\ t \in [0, 1]$$

and since

$$cap_p(C_t, \Omega) \leq t^{-p} \varphi(t)$$

we obtain from the quasiminimizing property of $u$ that

$$cap_p(C, \Omega) \geq \frac{\beta}{K} \geq \frac{t^{p-(q-1)/q}}{K K_2^{1/q}} cap_p(C_t, \Omega)$$

$$= \frac{t^{p-(q-1)/q}}{\kappa_2} cap_p(C_t, \Omega)$$

where the number $\kappa_2 = K K_2^{1/q}$ has the required property. The proof follows.

**Remark 4.4** Remark 4.2 also applies to Theorem 4.3: For $K = 1$ inequality (30) reduces to $cap_p E \geq t^{p-1}\, cap_p(C_t, \Omega)$. However, the exponents and the constants on the right and left hand side of (23) and (30) are different.

The inequalities (23) and (30) are sharp for $n = 1$. It remains an open question if they are sharp for $n \geq 2$. For $p = 2$ in the one dimensional case the sharpness can be easily checked by considering the 2–capacity of the condenser $E = ([1, 2], (0, 3))$. The function $u(x) = x^\alpha$, $x \in [0, 1]$ and $u(x) = 1 - (x-2)^\alpha$, $x \in [2, 3]$ is a $(2, K)$–quasiminimizer in $[0, 1] \cup [2, 3]$ where $\alpha \in (1/2, \infty)$ and $K = \alpha^2/(2\alpha - 1)$. For these computations see [M] and [MS]. Note that $u$ is not $(2, K')$–quasiminimizer for any $K' < \infty$ if $\alpha \leq 1/2$ and that the number $K$ is the smallest possible.

Set $u = 1$ in $[1, 2]$ and let $E_t$, $t \in (0, 1)$, be the condenser $(\{u(x) \geq t\}, (0, 3))$. Now $cap_2 E = 2$ and $cap_2 E_t = 2/t^{1/\alpha}$ and hence

$$cap_2 E = t^{1/\alpha} cap_2 E_t. \tag{31}$$

Since

$$p_1(2, \sqrt{K}) = 1 + \sqrt{K/(K-1)},$$

a direct computation shows that

$$\begin{aligned} p_1(2, \sqrt{K}) &= \frac{2\alpha - 1}{\alpha - 1},\ 1 < \alpha, & (32) \\ &= \frac{1}{1-\alpha},\ 1/2 < \alpha < 1. & (33) \end{aligned}$$

By Theorem 4.1 the exponent $s$ in (23) can now be chosen so that

$$\begin{aligned} s &\in (1, \frac{2\alpha - 1}{\alpha - 1}),\ 1 < \alpha, \\ s &= \infty,\ \alpha = 1, \\ s &\in (1, \frac{1}{1-\alpha}),\ 1/2 < \alpha < 1. \end{aligned}$$

It is easy to see that the endpoints $\frac{2\alpha-1}{\alpha-1}$ and $\frac{1}{1-\alpha}$ of these intervals cannot be used for $s$ in (23) and that these values correspond to the values of $K$ in Theorem 4.1. This is no accident because the function $u$ is a normalized $(2, K)$–quasiminimizer in $[0, 1]$ and $u$ is an optimal function with respect to the higher integrability property of one dimensional quasiminimizers, see [MS, Section 5].

Similar computations show that the estimate in Theorem 4.3 is optimal for $p = 2$ as well.

# References


[D'AS] D'Apuzzo, L. and C. Sbordone, Reverse Hölder inequalities. A sharp result. Rend. Matem. Ser. VII, 10 (1990), 357–366.

[GG] Giaquinta, M. and E. Giusti, Quasiminima, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), 79–104.

[HKM] Kilpeläinen, T., J. Heinonen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations,* Dover 2006.

[KiM] Kinnunen, J. and O. Martio, Potential theory of quasiminimizers, Ann. Acad. Sci. Fenn. Math. 28 (2003), 459–490.

[M] Martio, O., Reflection principle for quasiminimizers, Preprint 459, Reports in Mathematics, Department of Mathematics and Statistics, 2007, University of Helsinki, 1-10.

[MS] Martio, O. and C. Sbordone, Quasiminimizers in one dimension: integrability of the derivative, inverse function and obstacle problems, Annali di Matematica 186 (2007), 579-590. DOI 10.1007/s10231-006-0020-3,electronic version: http://dx.doi.org/10.1007/s10231-006-0020-3