MAPPINGS OF BOUNDED MEAN DISTORTION AND COHOMOLOGY

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Abstract. We obtain a quantitative cohomological boundedness theorem for closed manifolds receiving entire mappings of bounded mean distortion and finite lower order. We prove also an equidistribution theorem for mappings of finite distortion.

1. Introduction

By the classical *Uniformization Theorem*, the sphere $\mathbb{S}^2$ and the torus $\mathbb{T}^2$ are the only closed Riemann surfaces admitting nonconstant conformal mappings from the complex plane. The same rigidity is present in higher dimensions; closed manifolds admitting conformal mappings from $\mathbb{R}^n$ are quotients of $\mathbb{S}^n$ and $\mathbb{T}^n$, see e.g. [2, Prop. 1.4]. However, if the distortion of the conformal geometry is allowed, simple examples show that spaces $\mathbb{S}^{k_1} \times \mathbb{S}^{k_2} \times \cdots \times \mathbb{S}^{k_\ell}$ ($k_1 + \cdots + k_\ell = n$) receive nonconstant mappings of bounded distortion from $\mathbb{R}^n$. A mapping $f\colon M \to N$ between oriented Riemannian n-manifolds is said to be a *mapping of bounded distortion*, or *quasiregular*, if f is a Sobolev mapping in $W^{1,n}_{\mathrm{loc}}(M;N)$ and there exists a constant $K \ge 1$ so that

$$|Df|^n \le K J_f \quad \text{a.e. in } M,$$

where $|Df|$ is the operator norm of the differential Df and J_f is the Jacobian determinant of f. By Reshetnyak's theorem [18, p. 163], quasiregular mappings are discrete and open, and therefore examples of generalized branched covers.

A connected and oriented Riemannian n-manifold receiving a nonconstant (K-)quasiregular mapping from $\mathbb{R}^n$ is called (K-)*quasiregularly elliptic*. By the Uniformization Theorem and the measurable Riemann Mapping Theorem, the only closed quasiregularly elliptic 2-manifolds are $\mathbb{S}^2$ and $\mathbb{T}^2$. For $n = 3$, closed quasiregularly elliptic manifolds are by Jormakka's theorem [11] quotients of $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$, and $\mathbb{T}^3$. In higher dimensions such characterizations are not known. In dimension $n = 4$, a construction of Rickman [20]

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gives a positive answer to a question of Gromov [3, 2.41] on quasiregular ellipticity of $\mathbb{S}^2 \times \mathbb{S}^2 \# \mathbb{S}^2 \times \mathbb{S}^2$.

The fundamental group and the de Rham cohomology ring yield obstructions for quasiregular ellipticity of a closed manifold. More precisely, by Varopoulos' theorem [24, Theorem X.11] the order of growth of the fundamental group of a closed quasiregularly elliptic manifold cannot exceed the dimension of the manifold. Similarly, by a theorem of Bonk and Heinonen [2, Theorem 1.1]: *Given $n \geq 2$ and $K \geq 1$ there exists a constant $C = C(n,K) > 0$ so that the dimension of the de Rham cohomology ring of a closed K-quasiregularly elliptic n-manifold is at most C.*

Local versions of these theorems show that analogous results hold for mappings that are quasiregular in a neighborhood of the infinity; see [16]. The quasiregularity assumption can, however, be further relaxed. With Onninen we showed in [15] that Varopoulos' theorem holds for a larger class of mappings, a subclass of *mappings of finite distortion*. In this vein, we show that a cohomological boundedness phenomenon of Bonk-Heinonen type holds for a subclass of *mappings of bounded mean distortion*. To state our main results, we give some definitions.

We say that a nonconstant continuous mapping $f\colon \mathbb{R}^n \to N$ is a *mapping of finite distortion* if f belongs to the Sobolev space $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n; N)$ and there exists a measurable function $K\colon \mathbb{R}^n \to [1,\infty)$ so that

$$\|Df\|^n \leq K J_f \quad \text{a.e. in } \mathbb{R}^n.$$

We set the *outer distortion function* K_f of f to be the function $K_f(x) = |Df(x)|^n / J_f(x)$ whenever $J_f(x) > 0$ and $K_f(x) = 1$ otherwise.

We say that a mapping of finite distortion f has *$\mathcal{K}$-bounded p-mean distortion*, $p \geq 1$, if there exist constants $\mathcal{K} \geq 1$ and $r_0 > 0$ so that

$$\left(\fint_{B^n(r)} K_f^p \right)^{1/p} \leq \mathcal{K}$$

for every $r \geq r_0$. Here $B^n(r)$ is the open ball of radius r about the origin in $\mathbb{R}^n$. We also say that f has *finite lower order* λ if

$$\lambda = \liminf_{r\to\infty} \frac{\log A_f(r)}{\log r} < \infty.$$

Here and in what follows A_f is the *averaged counting function*

$$A_f(r) = \int_{B^n(r)} J_f.$$

Theorem 1. *For every $n \geq 2$ there exists $p = p(n) > n-1$ with the following property. Let N be a closed, connected, and oriented Riemannian n-manifold and let $f\colon \mathbb{R}^n \to N$ be a mapping of $\mathcal{K}$-bounded p-mean distortion having finite lower order λ. Then there exists $C = C(n,\lambda,\mathcal{K}) > 0$ so that $\dim H^*(N) \leq C$.*

Let us recall that by the rescaling principle, see e.g. [2, Section 2], a quasiregularly elliptic manifold always admits a quasiregular mapping having finite lower order at most n. Thus we recover the Bonk-Heinonen theorem from Theorem 1. As in the case of quasiregular mappings, the constant C in Theorem 1 is known only for $n = 2$ and $n = 3$. By Varopoulos' theorem, the first de Rham cohomology has dimension at most n, and hence we obtain the bound $C = 2^n$ for $n = 2, 3$.

The proof of the main theorem relies on two ingredients of possible independent interest. In Section 2, we give a very simple proof for an extension of a special case of the Mattila-Rickman equidistribution theorem [14, Theorem 5.1]. For mappings of bounded mean distortion, our result reads as follows.

Theorem 2. *Let N be a closed, connected, and oriented Riemannian n-manifold, $n \geq 2$, $u \in L^q(N)$, $q > n$, and suppose that $f\colon \mathbb{R}^n \to N$ is a mapping of bounded $(n-1)$-mean distortion. Then there exists a set $E \subset [1,\infty)$ of finite logarithmic measure so that*

$$\frac{1}{A_f(r)} \int_{B^n(r)} (u \circ f) J_f \to \fint_N u \tag{1.1}$$

as $r \to \infty$, $r \notin E$.

We find it interesting that the proof of this equidistribution theorem does not rely on discreteness and openness, as the sharper result of Mattila and Rickman for quasiregular mappings does, but uses only the change of variables methods. In fact, it is not known to us whether the mappings in question are discrete and open. A result of Manfredi and Villamor [12] states that mappings of finite distortion having distortion in L^p_{loc} for $p > n-1$ are discrete and open and hence branched covers. For $n = 2$, Iwaniec and Šverák [10] proved that distortion in L^p_{loc} for $p \geq n-1$ implies discreteness and openness. They also conjecture the same result in all dimensions $n > 2$. For recent results in this direction, see [6].

In Section 3 we consider Caccioppoli type potential estimates for pull-back forms under mappings of finite distortion. Instead of focusing on the solutions of degenerate $\mathcal{A}$-harmonic equations arising in the pull-back, we consider pairs of closed forms (ξ, ζ) satisfying a nonnegativity condition $\star(\xi \wedge \zeta) \geq 0$. Such pairs arise naturally in the Hodge theory and in the nonlinear Hodge theory; pairs $(\xi, \star\xi)$ and $(\xi, \star|\xi|^{p-2}\xi)$ are nonnegative if ξ is a harmonic or a p-harmonic form, respectively. These pairs are a special case of *Cartan forms* of Hajłasz, Iwaniec, Malý, and Onninen [4].

Having the equidistribution result and a Caccioppoli type estimate at our disposal, we finish the proof of Theorem 1 in Section 4. The argument follows closely the proof of Bonk and Heinonen. The main difference is in the replacement of conformal exponents by exponents within a range determined by a Sobolev-embedding theorem for differential forms.

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2. A Mattila-Rickman type value distribution theorem

In this section we give a weak Mattila-Rickman type equidistribution theorem for mappings of finite distortion. Since the natural class of mappings in this theorem is larger than mappings of bounded mean distortion, we introduce first some notations. Let $f\colon \mathbb{R}^n \to N$ be a mapping of finite distortion.

The outer distortion function of f gives a raise to a logarithmic type measure m_f on $(0,\infty)$ defined by

$$m_f(E) = \int_E \frac{\mathrm{d}r}{r k_f(r)},$$

where $k_f\colon (0,\infty) \to [1,\infty]$ is the *spherical mean distortion function*

$$k_f(r) = \left(\fint_{S^{n-1}(r)} K_f^{n-1} \right)^{\frac{1}{n-1}}.$$

For quasiregular mappings and mappings of $\mathcal{K}$-bounded $(n-1)$-mean distortion, the measure m_f is comparable, with a constant depending only on $\mathcal{K}$, to the *logarithmic measure* $m_{\log}$,

$$m_{\log}(E) = \int_E \frac{\mathrm{d}r}{r}.$$

For a more detailed discussion on the logarithmic measures in the value distribution theory of quasiregular mappings, see e.g. [19, V.9.16].

The main theorem of this section reads as follows.

Theorem 3. *Let N be a closed, connected, and oriented Riemannian n-manifold and suppose that $f\colon \mathbb{R}^n \to N$ is such a mapping of finite distortion that $m_f([1,\infty)) = \infty$. Then for every n-form ω in $L^q(\bigwedge^n N)$, $q > n$, there exists a set $E \subset [1,\infty)$ of finite m_f-measure so that*

$$\frac{1}{A_f(r)} \int_{B^n(r)} f^*\omega \to \fint_N \omega \tag{2.2}$$

as $r \to \infty$, $r \notin E$.

Theorem 2 can now be obtained as a special case of Theorem 3. Indeed, as the measure m_f is comparable to the logarithmic measure for mappings of bounded $(n-1)$-mean distortion, Theorem 2 is obtained by considering n-forms $\omega = u\mathrm{vol}_N$. Here vol_N is the Riemannian volume form on N.

Theorem 3 is analogous to the Euclidean version of the Mattila-Rickman equidistribution theorem [14, Theorem 5.11]. For our applications, it suffices to have the following version of this result.

Theorem 4. *Let N be a closed, connected, and oriented Riemannian n-manifold and suppose that $f\colon \mathbb{R}^n \to N$ is a mapping of finite distortion. Then for every $\varepsilon > 0$ and every n-form ω in $L^q(\bigwedge^n N)$, $q > n$, there exists a set $E \subset [1,\infty)$ of finite m_f-measure so that*

$$\left(\fint_N \omega - \varepsilon\right)\int_{B^n(r)} J_f < \int_{B^n(r)} f^*\omega < \left(\fint_N \omega + \varepsilon\right)\int_{B^n(r)} J_f \tag{2.3}$$

for $r \in [1,\infty) \setminus E$.

The proofs of Theorems 3 and 4 can be reduced to the following lemma corresponding to the case of exact n-forms.

Lemma 5. *Let N and f be as in Theorem 4. Then for every $\delta > (n-1)/n$, $\varepsilon > 0$, and every bounded $(n-1)$-form τ in $W^{1,q}(\bigwedge^{n-1} N)$, $q > n$, there exists a set $E \subset [1,\infty)$ of finite m_f-measure so that*

$$\left|\int_{S^{n-1}(r)} f^*\tau\right| < \varepsilon\left(\int_{B^n(r)} J_f\right)^\delta \tag{2.4}$$

for $r \in [1,\infty) \setminus E$.

The mappings we consider have the Lusin property (N) and hence support the change of variables formula, see e.g. [5] and [13] or [8]. We use these properties frequently in what follows.

Proof of Theorem 4 assuming Lemma 5. Let

$$\tilde\omega = \omega - \left(\fint_N \omega\right)\mathrm{vol}_N,$$

where vol_N is the volume form on N. Since

$$\int_N \tilde\omega = 0,$$

$\tilde\omega$ is weakly exact; see e.g. [17, Section 3]. Thus, by the Poincaré inequality [9, Theorem 6.4], there exists an $(n-1)$-form $\tau \in W^{1,q}(\bigwedge^{n-1} N)$ so that $d\tau = \tilde\omega$. Since $q > n$, τ is Hölder continuous, and hence bounded, by the Sobolev embedding theorem.

Let $\varepsilon > 0$. Since

$$\int_{B^n(r)} f^*\tilde\omega = \int_{S^{n-1}(r)} f^*\tau$$

for almost every $r > 0$, we may apply Lemma 5 with $\delta = 1$ and we obtain a set $E \subset [1,\infty)$ of finite m_f-measure so that

$$\left|\int_{B^n(r)} f^*\tilde\omega\right| < \varepsilon \int_{B^n(r)} J_f$$

for $r \in [1,\infty) \setminus E$. Since

$$\int_{B^n(r)} f^*\tilde\omega = \int_{B^n(r)} f^*\omega - \left(\fint_N \omega\right)\int_{B^n(r)} J_f,$$

the claim follows. □

Proof of Theorem 3 assuming Lemma 5. Suppose first that the avaraged counting function A_f is bounded. Let k be an integer so that the set $X = \{y \in N \colon \mathrm{card}\, f^{-1}(y) = k\}$ has positive measure. Thus

$$\int_{B^n(r)} (\chi_X \circ f) J_f = \int_X n(y, B^n(r); f) dy \to k|X|$$

as $r \to \infty$, where $n(\cdot, \cdot; f)$ is the counting function of f and $|X|$ the Lebesgue measure of X.

By an application of Theorem 4 to $\chi_X \mathrm{vol}_N$ and by the change of variables, there exists for every $\varepsilon > 0$ a set $E \subset [1, \infty)$ of finite m_f measure so that

$$(k - \varepsilon)\mathrm{vol}(N) < A_f(r) < (k + \varepsilon)\mathrm{vol}(N)$$

for $r \in [1, \infty) \setminus E$. Since $m_f([1, \infty)) = \infty$, we have that $A_f(r) \to k\mathrm{vol}(N)$ as $r \to \infty$. Hence $N \setminus X$ is a zero set and f is a k-to-1 map. Thus

$$\int_{\mathbb{R}^n} f^*\omega = k \int_N \omega = \left(\fint_N \omega \right) k\mathrm{vol}(N)$$

for all $\omega \in L^q(N)$, $q > n$, by the change of variables.

Suppose now that the avaraged counting function A_f is unbounded. Let ω and $\tilde{\omega}$ be n-forms as in the proof of Theorem 4. Then an application of Lemma 5 with $(n-1)/n < \delta < 1$ yields a set $E \subset [1, \infty)$ of finite m_f-measure so that

$$\left| \int_{B^n(r)} f^*\tilde{\omega} \right| < \left(\int_{B^n(r)} J_f \right)^\delta$$

for $r \in [1, \infty) \setminus E$. Thus

$$\left| \frac{1}{A_f(r)} \int_{B^n(r)} f^*\tilde{\omega} \right| < A_f(r)^{\delta - 1} \to 0,$$

as $r \to \infty$, $r \notin E$. The claim follows. □

Proof of Lemma 5. Let $E \subset [1, \infty)$ be the set of such radii $r \geq 1$ that (2.4) does not hold. Then, for almost every $r \in E$, Hölder's inequality yields

$$\begin{aligned} \varepsilon \left(\int_{B^n(r)} J_f \right)^\delta &\leq \int_{S^{n-1}(r)} |Df|^{n-1} (|\tau| \circ f) \\ &\leq \|\tau\|_\infty \left(\int_{S^{n-1}(r)} K_f^{n-1} \right)^{\frac{1}{n}} \left(\int_{S^{n-1}(r)} J_f \right)^{\frac{n-1}{n}}. \end{aligned}$$

Thus the averaged counting function A_f satisfies the differential inequality

$$\varepsilon^{\frac{n}{n-1}} A_f(r)^{\delta \frac{n}{n-1}} \leq \|\tau\|_\infty^{\frac{n}{n-1}} r k_f(r) A_f'(r)$$

for almost every $r \in E$. Since

$$\left(\frac{\|\tau\|_\infty}{\varepsilon}\right)^{\frac{n}{n-1}} \int_1^\infty \frac{A_f'(r)}{A_f(r)^{\delta\frac{n}{n-1}}}\,\mathrm{d}r \quad \geq \quad \int_E \frac{\mathrm{d}r}{r k_f(r)} = m_f(E)$$

by the differential inequality and

$$\int_1^\infty \frac{A_f'(r)}{A_f(r)^{\delta\frac{n}{n-1}}}\,\mathrm{d}r \quad \leq \quad \frac{1}{\delta\frac{n}{n-1}-1} A_f(1)^{1-\delta\frac{n}{n-1}},$$

the claim follows. □

3. A Caccioppoli type estimate

In this section we deduce a counterpart for the Caccioppoli type estimate used in the proof of Bonk and Heinonen. In what follows we use the notation q' to denote the Hölder conjugate $q/(q-1)$ of $q > 1$.

We begin with a potential theoretic lemma of Caccioppoli type for non-negative pairs of forms.

Lemma 6. *Let $0 < r < R$, $1 \leq \ell \leq n-1$, and $q > 1$. Let ω and ω' be closed forms in $L^q(\bigwedge^\ell B^n(R))$ and $L^{q'}(\bigwedge^{n-\ell} B^n(R))$, respectively, so that $\star(\omega \wedge \omega') \geq 0$. Then there exists $C = C(n) > 0$ so that*

$$\int_{B^n(r)} \omega \wedge \omega' \leq \frac{C}{R-r}\left(\int_{B^n(R)} |\tau|^q\right)^{1/q} \left(\int_{B^n(R)} |\omega'|^{q'}\right)^{1/q'} \tag{3.5}$$

for every $\tau \in W^{d,q}(\bigwedge^{\ell-1} B^n(R))$ satisfying $d\tau = \omega$.

Proof. Since $\star(\omega \wedge \omega') \geq 0$, we have, by Stokes' theorem and Hadamard's and Hölder's inequalities,

$$\begin{aligned}
\int_{B^n(r)} \omega \wedge \omega' \quad &\leq \quad \frac{1}{R-r}\int_r^R \int_{B^n(t)} \omega \wedge \omega' = \frac{1}{R-r}\int_r^R \int_{S^{n-1}(t)} \tau \wedge \omega' \\
&\leq \quad \frac{1}{R-r}\int_r^R \int_{S^{n-1}(t)} |\tau \wedge \omega'| \leq \frac{C}{R-r}\int_{B^n(R)} |\tau||\omega'| \\
&\leq \quad \frac{C}{R-r}\left(\int_{B^n(R)} |\tau|^q\right)^{1/q} \left(\int_{B^n(R)} |\omega'|^{q'}\right)^{1/q'},
\end{aligned}$$

where $C = C(n) > 0$. □

The main result of this section combines the Caccioppoli type estimate with value distribution results for mappings of finite distortion having finite lower order. The proof uses the following observation, typical in the value distribution theory; see e.g. [2, Lemma 4.14]. For the reader's convenience we give a simple proof.

Lemma 7. *Let $\lambda > 0$ and $\varphi\colon (0,\infty) \to (0,\infty)$ be a nondecreasing function so that*

$$\lambda = \liminf_{r\to\infty} \frac{\log\varphi(r)}{\log r} < \infty. \tag{3.6}$$

Then for every $r_0 > 0$ there exists $r_1 \ge r_0$ so that

$$\varphi(r) \le 5^\lambda \varphi(r/2) \tag{3.7}$$

for all $r_1/2 \le r \le r_1$. In particular, (3.7) *holds in a set of infinite logarithmic measure.*

Proof. Let $r_0 > 0$. We show first that there exists $r_1 \ge r_0$ so that $\varphi(r_1) \le 5^\lambda\varphi(r_1/4)$. Should this not be the case, $\varphi(4^k r_0) \ge 5^{k\lambda}\varphi(r_0)$ for every $k \ge 0$. To show that this is a contradiction, let $k_0 > 0$ to be fixed later. Let $k \ge k_0$ and $4^k r_0 \le r \le 4^{k+1} r_0$. Then

$$\frac{\log\varphi(r)}{\log(r)} \ge \frac{\log(\varphi(4^k r_0))}{\log(4^{k+1}r_0)} \ge \frac{\log(5^{k\lambda}\varphi(r_0))}{\log(4^k 4r_0)} = \frac{k\lambda\log 5 + \log\varphi(r_0)}{k\log 4 + \log(4r_0)} > C\lambda,$$

where $C = C(k_0) > 1$ for k_0 large. This contradicts (3.6). Thus there exists $r_1 \ge r_0$ so that $\varphi(r_1) \le 5^\lambda\varphi(r_1/4)$. Then

$$\varphi(r) \le \varphi(r_1) \le 5^\lambda\varphi(r_1/4) \le 5^\lambda\varphi(r/2).$$

for every $r_1/2 \le r \le r_1$. □

The following proposition combines the Caccioppoli type estimate with the value distribution and finite lower order.

Proposition 8. *Let N be a closed, connected, and oriented Riemannian n-manifold so that $\mathrm{vol}(N) = 1$, and let $f\colon \mathbb{R}^n \to N$ be a mapping of $\mathcal{K}$-bounded p-mean distortion for $p > n-1$ having finite lower order λ.*

Let $1 \le \ell \le n-1$ and suppose that $q > n/\ell$ satisfies

$$q' = \frac{p}{p+1}\left(\frac{n}{\ell}\right)'.$$

Let also ξ and ζ be closed forms in $L^\infty(\bigwedge^\ell N)$ and in $L^\infty(\bigwedge^{n-\ell} N)$, respectively, so that $\star(\xi\wedge\zeta) \ge 0$. Then there exists a constant $C = C(n,\ell,\lambda,\|\xi\wedge\zeta\|_1, \|\zeta\|_{\frac{n}{n-\ell}}) > 0$ and a set $F \subset [1,\infty)$ of infinite logarithmic measure so that for every $\alpha \in W^{d,q}_{\mathrm{loc}}\left(\bigwedge^{\ell-1} B^n(r)\right)$ satisfying $d\alpha = f^*\xi$ we have

$$\left(\int_{B^n(r)} J_f\right)^{\ell/n} \le C\mathcal{K}^{\frac{n-\ell}{n}} r^{\ell-1-\frac{n}{q}} \left(\int_{B^n(r)} |\alpha|^q\right)^{1/q}$$

for $r \in F$.

Proof. We show first that

$$\int_{B^n(r)} f^*\xi \wedge f^*\zeta \leq \frac{C}{R-r}\left(\int_{B^n(R)} K_f^{\frac{1}{s-1}}\right)^{\frac{s-1}{sq'}} \left(\int_{B^n(R)} |\alpha|^q\right)^{1/q} \left(\int_{B^n(r)} |\zeta|^{\frac{n}{n-\ell}} \circ f J_f\right)^{\frac{n-\ell}{n}} \tag{3.8}$$

for $R > r \geq 1$, where $s = \frac{n}{n-\ell}\frac{1}{q'}$.

Since $\star(f^*\xi \wedge f^*\zeta) = \star(\xi\wedge\zeta)\circ f J_f \geq 0$, it suffices, by Lemma 6, to show that

$$\left(\int_{B^n(R)} |f^*\zeta|^{q'}\right)^{1/q'} \leq \left(\int_{B^n(R)} K_f^{\frac{1}{s-1}}\right)^{\frac{s-1}{sq'}} \left(\int_{B^n(r)} |\zeta|^{\frac{n}{n-\ell}} \circ f J_f\right)^{\frac{n-\ell}{n}}.$$

By Hölder's inequality, we obtain

$$\begin{aligned}\int_{B^n(R)} |f^*\zeta|^{q'} &\leq \int_{B^n(R)} (K_f J_f)^{\frac{n-\ell}{n}q'} (|\zeta|\circ f)^{q'} \\ &= \int_{B^n(R)} K_f^{\frac{1}{s}} (|\zeta|\circ f)^{q'} J_f^{\frac{1}{s}} \\ &\leq \left(\int_{B^n(R)} K_f^{\frac{1}{s-1}}\right)^{\frac{s-1}{s}} \left(\int_{B^n(R)} (|\zeta|\circ f)^{\frac{n}{n-\ell}} J_f\right)^{\frac{1}{s}}.\end{aligned}$$

The inequality (3.8) now follows.

Since f has finite lower order λ, we have, by Lemma 7, that there exists $C = C(\lambda) > 0$ so that

$$A_f(r) \leq C A_f(r/2) \tag{3.9}$$

for $r \in F'$, where $F' \subset [1,\infty)$ is a set of infinite logarithmic measure.

By Theorem 4, we can also fix a set $E \subset [1,\infty)$ of finite logarithmic measure so that

$$\frac{C}{2}\int_{B^n(r/2)} J_f \leq \int_{B^n(r/2)} f^*\xi \wedge f^*\zeta \tag{3.10}$$

and

$$\int_{B^n(r)} (|\zeta|\circ f)^{\frac{n}{n-\ell}} J_f \leq 2C \int_{B^n(r)} J_f \tag{3.11}$$

for $r \in [1,\infty)\setminus E$, where $C = C(n,\ell,\|\xi\wedge\zeta\|_1, \|\zeta\|_{\frac{n}{n-\ell}}) > 0$. We set $F = F'\setminus E$.

Thus (3.8) and (3.9) together with (3.10) yield

$$\begin{aligned}\int_{B^n(r)} J_f &\le C\int_{B^n(r/2)} J_f \le C\int_{B^n(r/2)} f^*\xi\wedge f^*\zeta \\ &\le \frac{C}{r}\left(\int_{B^n(r)} K_f^{\frac{1}{s-1}}\right)^{\frac{s-1}{sq'}}\left(\int_{B^n(r)}|\alpha|^q\right)^{1/q}\left(\int_{B^n(r)}|\zeta|^{\frac{n}{n-\ell}}\circ f J_f\right)^{\frac{n-\ell}{n}} \\ &\le \frac{C}{r}\left(\int_{B^n(r)} K_f^{\frac{1}{s-1}}\right)^{\frac{s-1}{sq'}}\left(\int_{B^n(r)}|\alpha|^q\right)^{1/q}\left(\int_{B^n(r)} J_f\right)^{\frac{n-\ell}{n}},\end{aligned}$$

where $C = C(n,\ell,\lambda,\|\xi\wedge\zeta\|_1,\|\zeta\|_{\frac{n}{n-\ell}}) > 0$.

Since q satisfies $q' = \frac{p}{p+1}\left(\frac{n}{\ell}\right)'$, we have

$$\frac{1}{s-1} = \frac{1}{\frac{n}{n-\ell}\frac{1}{q'}-1} = \frac{q'}{\left(\frac{n}{\ell}\right)'-q'} = p$$

and

$$\left(\int_{B^n(r)} K^{\frac{1}{s-1}}\right)^{\frac{s-1}{sq'}} = \left(\int_{B^n(r)} K^p\right)^{\frac{1}{sq'p}} \le \mathcal{K}_p^{\frac{n-\ell}{n}} r^{\frac{n}{sq'p}} = \mathcal{K}_p^{\frac{n-\ell}{n}} r^{\frac{n-\ell}{p}}.$$

This concludes the proof. □

4. Proof of Theorem 1

By the Poincaré duality, it suffices to consider cohomology groups $H^\ell(N)$ for $1\le\ell\le n/2$. We may also assume that $\mathrm{vol}(N)=1$.

Suppose $d = \dim H^\ell(N) > 0$. By the non-linear Hodge theory [21], we may fix (n/ℓ)-harmonic ℓ-forms $\xi_1,\ldots,\xi_d$ on N so that the cohomology classes of the forms span $H^\ell(N)$ and that the forms satisfy

$$\|\xi_i\|_{n/\ell} = 1 \quad\text{and}\quad \|\xi_i-\xi_j\|_{n/\ell}\ge 1$$

for all i and $j\ne i$. For every i, we set ζ_i to be the (n/ℓ)-harmonic conjugate of ξ_i, i.e., $\zeta_i = \star|\xi_i|^{\frac{n}{\ell}-2}\xi_i$. Then $\|\zeta_i\|_{\frac{n}{n-\ell}} = 1$. Forms ξ_i and ζ_i are Hölder continuous by results of Uhlenbeck [22] and Ural'tseva [23]. Especially, they are bounded.

Since $n/\ell\ge 2$, we have, by a pointwise monotonicity estimate (see e.g. [1, p. 288]), that

$$\int_N (\xi_i-\xi_j)\wedge(\zeta_i-\zeta_j) \ge C\int_N |\xi_i-\xi_j|^{\frac{n}{\ell}} \ge C,$$

where $C = C(n,\ell) > 0$. By Hölder's inequality, we also obtain

$$\int_N (\xi_i-\xi_j)\wedge(\zeta_i-\zeta_j) \le C\|\xi_i-\xi_j\|_{n/\ell}\left(\|\zeta_i\|_{\frac{n}{n-\ell}}+\|\zeta_j\|_{\frac{n}{n-\ell}}\right)\le C,$$

where $C = C(n) > 0$. For brevity, we set $\xi_{ij} = \xi_i-\xi_j$ and $\zeta_{ij} = \zeta_i-\zeta_j$ for every $i\ne j$.

To obtain an estimate for the number of forms ξ_i, we show that there exists exponents s and q and a radius $r > 0$ so that the Poincaré homotopy operator $T\colon L^s(\bigwedge^\ell B^n) \to L^q(\bigwedge^{\ell-1} B^n)$ of Iwaniec and Lutoborski [7] is compact and that we have the estimates

$$\|T\lambda_r^* f^* \xi_i\|_q \le C \left(\int_{B^n(r)} J_f \right)^{\ell/n}$$

and

$$\|T\lambda_r^* f^* \xi_i - T\lambda_r^* f^* \xi_j\|_q \ge \frac{1}{C} \left(\int_{B^n(r)} J_f \right)^{\ell/n},$$

where $C = C(n,\ell,\lambda,\mathcal{K}) > 0$ and $\lambda_r\colon \mathbb{R}^n \to \mathbb{R}^n$ is the similarity mapping $x \mapsto rx$. Then, by the compactness of T, the number of forms $\lambda_r^* f^* \xi_i$, and hence also ξ_i, is bounded by a constant depending only on n, λ, and $\mathcal{K}$.

We fix

$$s = \frac{1}{2}\left(\frac{n}{\ell+1} + \frac{n}{\ell} \right) \quad \text{and} \quad q = \frac{1}{2}\left(\frac{n}{\ell} + s^* \right),$$

where $s^* = ns/(n-s)$ is the Sobolev conjugate of s. Since $s > n/(\ell+1)$, we have $s^* > n/\ell$ and $n/\ell < q < s^*$. Thus $q' < (n/\ell)'$ and there exists $\tilde{p} > 1$ so that

$$q' = \frac{\tilde{p}}{\tilde{p}+1} \left(\frac{n}{\ell} \right)'.$$

Since $1 \le \ell \le n/2$, we may fix $p = p(n) > n-1$ so that

$$p \ge \max\left\{ \tilde{p}, \frac{s\ell}{n - s\ell} \right\}.$$

Suppose now that $f\colon \mathbb{R}^n \to N$ is a mapping of $\mathcal{K}$-bounded p-mean distortion. We fix a set $E \subset [1,\infty)$ of finite logarithmic measure so that

$$\int_{B^n(r)} (|\xi_i|^{n/\ell} \circ f) J_f \le C \int_{B^n(r)} J_f$$

for every i and every $r \in [1,\infty) \setminus E$, where $C = C(n,\ell) > 0$.

Using Hölder's inequality, we obtain

(4.12)

$$\begin{aligned}
\left(\int_{B^n(r)} |f^*\xi_i|^s \right)^{1/s} &\le \left(\int_{B^n(r)} K_f^{\frac{\ell s}{n-\ell s}} \right)^{\frac{n-\ell s}{ns}} \left(\int_{B^n(r)} (|\xi_i|^{\frac{n}{\ell}} \circ f) J_f \right)^{\ell/n} \\
&\le C \left(\fint_{B^n(r)} K_f^p \right)^{\frac{\ell}{np}} r^{\frac{n-\ell s}{s}} \left(\int_{B^n(r)} J_f \right)^{\ell/n} \\
&\le C \mathcal{K}^{\frac{\ell}{n}} r^{\frac{n-\ell s}{s}} \left(\int_{B^n(r)} J_f \right)^{\ell/n}
\end{aligned}$$

for every $r \in [1,\infty) \setminus E$, where $C = C(n) > 0$. Thus

$$\left(\int_{B^n} |\lambda_r^* f^* \xi_i|^s \right)^{1/s} \le C \mathcal{K}^{\frac{\ell}{n}} \left(\int_{B^n(r)} J_f \right)^{\ell/n}$$

for every $r \in [1,\infty) \setminus E$.

Since

$$dT\lambda_r^* f^* \xi_{ij} = \lambda_r^* f^* \xi_{ij} - T d\lambda_r^* f^* \xi_{ij} = \lambda_r^* f^* \xi_{ij},$$

we may set $\alpha_{ij} = \lambda_{1/r}^* T \lambda_r^* f^* \xi_{ij}$ and we have

$$d\alpha_{ij} = f^* \xi_{ij}.$$

for every $i \neq j$. By Proposition 8, there exists a set $F \subset [1,\infty)$ of infinite logarithmic measure and $C = C(n,\ell,\lambda) > 0$ so that

$$\left(\int_{B^n(r)} J_f \right)^{\ell/n} \le C \mathcal{K}^{\frac{n-\ell}{n}} r^{\ell - 1 - \frac{n}{q}} \left(\int_{B^n(r)} |\alpha_{ij}|^q \right)^{1/q} \tag{4.13}$$

for $r \in F$.

Since E has finite logarithmic measure, we may fix $r \in F \setminus E$ so that (4.12) and (4.13) hold for every $i \neq j$.

Since

$$\left(\int_{B^n} |T\lambda_r^* f^* \xi_{ij}|^q \right)^{1/q} = r^{\ell - 1 - \frac{n}{q}} \left(\int_{B^n(r)} |\alpha_{ij}|^q \right)^{1/q},$$

we obtain the last required estimate

$$\|T\lambda_r^* f^* \xi_{ij}\|_q \ge C \left(\int_{B^n(r)} J_f \right)^{\ell/n}.$$

This concludes the proof of Theorem 1.

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