THERMODYNAMIC FORMALISM OF STOCHASTIC EQUILIBRIUM ECONOMICS

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Abstract

Due to the difficulty of constructing "perfect models" (in the sense of physical sciences) in economics in general, it follows that the aposteriori observed value for an economic equilibrium may deviate significantly from its model-based apriori expected value. Mathematically this means that the aposteriori observed equilibrium may represent a *large deviation*, namely, fall outside the region of validity of the *central limit theory.*

With this simple fact as the motivating starting point for the present study, we propose a *new approach* to the theory of stochastic economic equilibrium. Drawing on recent developments in probability theory, we advocate the relevance of the *theory of large deviations* for stochastic equilibrium economics.

It follows from the proposed approach that the formalism of stochastic equilibrium economics becomes analogous with the formalism of classical *statistical mechanics.* This analogy is due to the fact that the theory of large deviations forms also the mathematical basis of statistical mechanics. Due to the thermodynamic analogy, new "thermodynamic" concepts such as, e.g., *entropy, partition function, canonical probability,* can be introduced to stochastic equilibrium economics. These concepts play a cental role in the economic analogs of classical thermodynamic principles.

We focus here on the economic analogs of two fundamental principles, namely, of the *second law of thermodynamics* and of *Gibbs conditioning principle.*

The (integral form of the) classical second law expresses thermodynamic entropy with the aid of temperature, internal energy and partition function. The *economic analog* of the concept of *entropy* is defined as the *information content* in the observation of the realized equilibrium. The *"second law of stochastic equilibrium economics"* expresses the economic entropy in terms of an "economic partition function".

In view of its definition as information content, economic entropy is a "measure of rareness" for the observed value of the equilibrium. It follows that the economic second law can be interpreted as providing an *information theoretic measure of goodness* for a random equilibrium model.

The formulation of the economic second law gives rise to the definition of a new type of economic equilibrium. Namely, the second law is valid at any "apriori possible equilibrium". At an *apriori possible equilibrium* zero is a "possible value" (in the sense of large deviation theory) for the random total excess demand.

We discuss also the economic analog of the classical thermodynamic Gibbs conditioning principle characterizing the canonical thermodynamic probability law as the governing probability law for a thermodynamic system at a measured temperature. Its economic analog characterizes a "canonical" probability law as the *aposteriori probability law* governing a random economy, conditionally on the observation of the realized equilibrium.

In order to illustrate the new concepts and principles in a mathematically simple context, we study in detail the economic analog of the classical *ideal gas*, namely, the special case where the participating economic agents are supposed to be statistically identical and independent ("ideal" random economy). As an example we will deal with "ideal" random Cobb-Douglas economies.

We suggest a systematic, broad study of the analogy of the formalisms of statistical mechanics and stochastic equilibrium economics in general. At the end of the introductory chapter we will outline such a program by suggesting some concrete topics for later research.

1 Introduction

1.1 The set-up

The subject matter of modern economic theory concerns the building of models, written in the language of mathematics, which purport to describe "real" economic systems. The ideal is to achieve such an exact correspondence between the model and reality which is paradigmatically characteristic for the physical sciences.

Classical *equilibrium theory* is concerned with the following fundamental problem:

0. Does there exist an equilibrium, i.e., a price vector at which the total excess demand in the economy vanishes?

In the classical theory the total excess demand is regarded as a *deterministic* function (see Debreu [13]). The definition of the equilibrium prices as zeros of the total excess demand function can be regarded as the *fundamental axiom* of equilibrium economics. Namely, while in reality the total excess demand were not known as an explicit function, it is still implicitely thought that the equilibrium prices are zeros of a "hypothetical" total excess demand function.

After the pioneering works by Arrow, Debreu and McKenzie in the 1950's ([2], [3], [13], [22]) the theory of general (deterministic) equilibrium became a major subject of research developing in the course of the following couple of decades to its present established mature state.

As human behaviour in general, economic behaviour is always more or less unpredictable and often also irrational. It follows that - in contrast with the case of physical sciences - in economics it is never possible to achieve a perfect correspondence between the model and reality. So an economic model, however sophistically built, will at its best present only an approximative image of the real world.

An economic model can be tested by making "objective" aposteriori observations on the real economic system which the model aims to describe.

Now, due to the imperfectness in the modelling, the observations on the real economy may well contradict with the predictions by the apriori model. Thus, even while the model was initially built as deterministic, these contradictions indicate the presence of apriori uncertainty in the model.

Taking into account uncertainty means that, instead of one single "configuration" of the economy, there is an "ensemble" of configurations, each configuration having a "weight" (apriori probability) indicating the "apriori degree of belief" to its truth. These beliefs can be of "subjective type" ("rational guesses") as well as of "objective type" (based, e.g., on an apriori statistical study of the microeconomic behaviour of a suitably chosen representative sample from the set of economic agents). Mathematically these beliefs can be described by specifying an *apriori probability law.*

In view of the discussion above it is natural to consider equilibrium models which are *stochastic* rather than deterministic. This means that we will assume that the *parameters* in the equilibrium model are *random variables.* In particular, it then follows that the *equilibria* of the model are *random*, too.

We also assume that the economy is *large*, viz. the number of participating agents is "big".

The problem of the existence of an equilibrium (with probability 1) is relevant for random economies, too. The seminal paper is due to Hildenbrand [15].

The first special problem one can pose for a large random economy is as follows:

1. What is the apriori relation of a random equilibrium with its model-based apriori expected value? More precisely, does there exist a *law of large numbers (LLN)* and a *central limit theorem (CLT)*?

The seminal work dealing with the solution to Problem 1 is due to Bhattacharya and Majumdar [6].

In contrast with the case of deterministic equilibrium, after the pioneering papers by Hildenbrand, Bhattacharya and Majumdar in the early 1970's, the theory of random economic equilibrium has remained in the marginal of research in economic science.

Drawing on recent developments in general probability theory the purpose of the present paper is to suggest a new method of approach to the theory of random equilibrium. Namely, we advocate the relevance of the *theory of large deviations* for the equilibrium theory of random economies.

We formulate two basic problems concerning the *Bayesian* type of interplay between an apriori random equilibrium model with the aposteriori observations on the realized equilibrium.

By an *aposteriori (macroeconomic) observation* we mean the observation of some realized macroeconomic variable or quantity like, e.g.: of the equilibrium prices; of the consumption or production of some commodities in the economy (or in some economic sector); of the number of economic agents being of a certain "type". In this paper we will focus on the case where the observations concern the *realized equilibrium prices* in the system.

The first new problem to be addressed here is concerned with the apriori probability of an aposteriori observation of the equilibrium:

2. Suppose that the realized equilibrium is aposteriori observed. What is the *apriori probability* of this observation?

Note that, if the apriori model is "good", then (due to the LLN and CLT), this probability ought to be "big". Moreover, in this case its value can be approximated with the aid of the CLT, see Section 2.5.

However, since economic models usually are not perfect (in the sense of physical sciences), the aposteriori observed equilibrium may deviate significantly from its model-based apriori expected value. It follows that then the apriori probability of the observed value is likely to be "small". In fact, it can be argued that in this case the observed equilibrium may represent a *large deviation (LD)*, namely, fall *outside the region of the validity of the CLT*. This is so, because the CLT is valid only on a region which has the asymptotically small order of the standard deviation, cf. the discussion in Section 2.5.

We address also the problem of the inference from the aposteriori observation of the equilibrium (possibly representing a large deviation):

3. What is the *aposteriori probability law* governing a random economy, conditionally on an observation of the realized equilibrium?

This inference problem arises also in the classical non-stochastic equilibrium theory. Namely, due to the (inavoidable) contradictions of the aposteriori observations on the real system with their model-based predictions remodelling becomes necessary. However, in the case of a deterministic apriori model, the remodelling can be based only on some kind of ad hoc arguments.

1.2 Description of the results

We argue that the set-up of Problems 2 and 3 has an *analogy* in the formalism of classical *statistical mechanics.* This is due to the common ground of stochastic equilibrium economics and statistical mechanics in the *theory of large deviations.*

The set-up of Problem 2 turns out to be an analog of the *second law of thermodynamics.* To this end, recall that the (integral form of the) classical second law relates thermodynamic entropy with temperature, internal energy and thermodynamic partition function, see (2.1.7).

We argue that there is an analogy in stochastic equilibrium economics which suggests the definition of the *information content* in (defined as the logarithm of the inverse probability of) the observation of an equilibrium as the economic analog of thermodynamic entropy (Definition 1 in Section 2.3). The *"second law of stochastic equilibrium economics"* relates economic entropy with the "economic partition function" (see Section 2.3).

The economic second law is obtained at the ideal limit of an "infinitely large economy" (the analog of the *thermodynamic limit*, cf. Section 2.1) from a *theorem of large deviations (TLD)* concerning the random equilibrium prices (Theorem 2 in Section 4.3).

The TLD gives a formula with the aid of the *Laplace transform* of the random total excess demand for the apriori probability of the observation of an apriori non-expexted equilibrium. The TLD is valid at any "apriori possible" equilibrium price, and therefore its use is *mathematically legitimate* even *for an imperfect apriori model.* (For the precise definition of the concept of an *apriori possible equilibrium price* see Definition 2 in Section 3.2.)

The economic second law can be interpreted as providing an *information theoretic measure of goodness* for the apriori equilibrium model, see Remark 2.3(ii).

The set-up of Problem 3 is an analog of the thermodynamic *Gibbs conditioning principle (GCP)* characterizing the *thermodynamic canonical probability law* as the governing probability law of a thermodynamic system at a measured temperature. The canonical law is a member of the *exponential probability distribution family* generated by the total energy (see Section 3.1).

According to the economic analog of Gibbs conditioning principle, conditionally on the observation of an equilibrium, the aposteriori probability law governing a random economy is a "canonical member" in the exponential family generated by the random total excess demand (Section 3.2).

As an exact mathematical theorem GCP is a *conditional law of large numbers* concerning *macroeconomic random variables* (Theorem 3 in Section 5.2).

We organize the paper in two parts. In the first part we develop the "thermodynamic formalism" of large random economies focusing on the economic analogs of the second law of thermodynamics and of Gibbs conditioning principle. The second part comprises "exact results" in that we formulate the economic second law and Gibbs conditioning principle as exact mathematical theorems within the framework of the theory of large deviations.

We will investigate in detail the special case of an *"ideal" random economy* where the economic agents are supposed to be statistically identical and independent. (This is the analog of the classical *thermodynamic ideal gas.*) As a special illustrating example we study *"ideal" random Cobb-Douglas economies.*

There seems to be no easily accessible treatment on the LD theoretic foundations of statistical mechanics. Therefore, for the convenience of a possibly unacquainted reader and in order to point out the proposed analogy of stochastic equilibrium economics with statistical mechanics, we will begin each chapter with a short review on the relevant basics of statistical mechanics and its LD theoretic foundations.

The presentation attempts to be as self-contained as possible. (This goal implies that there will be some overlapping with [22-25]. We also try to avoid reference to the general LD theory. For a reader who is interested in the general LD theory we recommend the monographs by Bucklew [10], and by Dembo and Zeitouni [14].

We plan to study *stochastic finance markets* and the so-called *survival model* as applications of the proposed formalism in the forthcoming papers [28], [29].

1.3 General background on LD theory and statistical mechanics

We advocate in this study the relevance of the theory of large deviations for the equilibrium theory of large random economies comprising a big number of participating agents.

In probability theory in general, by a *large deviation* is meant the occurrence of a value for a random variable, which falls *outside* the region of validity of the *central limit theorem.* Large deviations are bound to occur in a large random system, if the apriori model is imperfect (as is the case usually, e.g., in economics, cf. Remark 2.3(ii) and Section 2.5).

The standard LD theory is concerned with the probabilities of large deviations of "extensive" random variables, viz. of random variables which result from the accumulation of a big number of "micro" random variables. The basic example is the sum of independent and identically distributed random variables. The extensivity can also mean temporal extensivity, i.e., time is regarded as the size parameter.

Theorems of large deviations type express the probabilities of large deviations in an exponential form where the exponent is proportional to the size parameter of the system. The negative of the coefficient of proportionality is referred to as the *rate function.*

Due to the thermodynamic analogy, the rate function is also referred to as the *entropy function.* The estimate yielded by the TLD is *valid* also *outside the region of validity of the CLT.*

Gibbs conditioning principle is concerned with the aposteriori inference from the observation of a large deviation, viz. how to take into account such an observation in a mathematically legitimate way. According to GCP the aposteriori probability law belongs to the exponential family generated by the random variable under consideration. As a mathematical theorem GCP is a *conditional law of large numbers.*

Since the seminal classical work by H. Cramér [12]), LD theory has become a major subject in probability theory, and subsequently also an important tool in stochastic modelling, like e.g., in: statistics, information theory, engineering problems, see [10], [14]; modelling of large communication networks [33]; risk theory [21]; dynamical macroeconomic phenomena [1]; calibration of asset prices [4].

The theory of large deviations can be regarded as the mathematical abstraction of the inherent mathematical structure of *statistical mechanics* in that large stochastic systems are understood as "thermodynamic" systems. Due to this relation, the proposed formalism for stochastic economic equilibrium theory is conceptually and structurally similar to the formalism of statistical mechanics.

Classical thermodynamic systems are physical systems which comprise a big number of particles (see e.g. [20]) or [30]). The goal is to describe the macroscopic behaviour of the system in terms of a few thermodynamic variables. Standard thermodynamic variables are, e.g., *volume, pressure, internal energy, temperature* and *entropy.* The thermodynamic variables can be classified either as *extensive* (i.e., proportional to the volume) or *intensive* (i.e., independent of the volume). Examples of extensive variables are volume, internal energy and entropy, while pressure and temperature are intensive.

The relations between the thermodynamic variables are described by *thermodynamic equations of state.* Thus, e.g., (the integral form of) the *second law of thermodynamics* relates thermodynamic entropy with temperature, internal energy and partition function, see formula (2.1.7).

According to the paradigm, the thermodynamic laws are assumed to hold true *universally*, i.e., to govern the behaviour of any thermodynamic system (even if the behaviour of the system is not mathematically fully understood).

Like a thermodynamic system, a large economic system comprising a big number of economic agents has many "degrees of freedom", only a few of which are "observable". As in statistical mechanics one can distinguish extensive (proportional to the size of the economy) and intensive (independent of the size) variables. Examples of extensive variables are, e.g., the total demand and supply on some commodities, while *prices* are typical *intensive* variables.

Despite of their intensive character, however, as zeros of the (extensive) random total excess demand function, the *random equilibrium prices still obey the principles of large deviation theory*, see [22-25]. It follows that stochastic equilibrium theory becomes a potential object of application of the LD method.

We argue that, in analogy with statistical mechanics, the "thermodynamic formalism" of stochastic equilibrium theory reflects certain underlying "universal principles" which hold true for a larger class of random economies than are those for which the exact mathematical conditions can be validated.

1.4 Suggestions for future research

As the present author sees it, the purpose of mathematical economics is not only to study mathematical structures which already are merited as being relevant to economics, but - as importantly - also theories which only seem to have such potential. The aim of the present work is to advocate certain recent developments in probability theory which the author believes to be useful to the stochastic equilibrium theory, but of which mathematical economists may not be well aware.

As the main thesis of this study we argue that there is a structural analogy between the formalisms of statistical mechanics and stochastic equilibrium economics which is worthwhile to be investigated.

In this paper we focus on the economic analogs of two basic thermodynamic principles, namely, of the second law of thermodynamics and of Gibbs conditioning principle. The goal is also to provide economic content to these analogs.

We suggest a systematic investigation and economic interpretation of the analogy of statistical mechanics and stochastic equilibrium economics in general. Below we make an outline for such a program by suggesting some concrete topics of research. We add comments on the possible solutions and interpretations.

(i) Consider an economic system which comprises *economic sectors*. This could be regarded as the economic analog of a thermodynamic system which comprises subsystems. As a natural question one may now ask whether there is a meaningful analog of the *first law of thermodynamics*, namely of thermodynamic *temperature equilibrium* (cf. [18]; for a risk-theoretic analog, see [21].)

(ii) The *principle of minimum entropy* (viz. the analog of the thermodynamic principle of *maximum entropy*) will be shortly discussed in Remarks 2.3(iii) and 3.2(iv). We suggest its systematic study and comparison with the thermodynamic analog as a subject of further study.

(iii) Due to the fact that economic models are bound to be non-perfect, it follows that approximations play an important role. Therefore it is desirable to investigate the (second order) approximations in a systematic way. In analogy with thermodynamics, the second order approximation ought to concern the *stability and fluctuations* of the economic equilibrium (cf. [30]: Section 1.5).

(iv) The dichotomy of *reversibility* and *irreversibility* plays a central role in thermodynamics. A thermodynamic process is reversible, if it is *quasistatic*, viz. "infinitely slow", and such that its path can be reversed ([30]: Section 1.2.1).

This suggests to conjecture that a "reversibile economic process" ought to be defined as a (random) dynamic economic process which is "quasistatic", i.e., such that the parameters change so slowly that the agents' economic behaviour is based solely on the present state of the economy and not on the agents' expectations on the future states of the economy, and the path of which can be "reversed". (This might be related to the economic theory of *rational expectations.*)

(v) The ultimate test for the relevance of any abstract theory is its applicability and explanatory power in the analysis of the real world. Thus, e.g., classical statistical mechanics explains the thermodynamic phenomena in the physical world, and the classical theory of general equilibrium forms the basis for the equilibrium econometrics.

The proposed "thermodynamic formalism" of economic equilibrium is also subject to this test. Suggested by the results of this paper, we propose the use of the second law in the evaluation of the quality of an econometric model which is subject to aposteriori "objective" macroeconomic observations, cf. Remark 2.3(iii). Similarly, in view of its economic interpretation, Gibbs conditioning principle could possibly work as an updating tool in the stochastic modelling of economic equilibrium.

PART I: THERMODYNAMIC FORMALISM

2 The Second Law

2.1 The second law of thermodynamics

The proposed formalism for stochastic equilibrium theory is analogous with the formalism of statistical mechanics. Therefore, in order to point out this analogy, we review in Sections 2.1 and 3.1 the classical second law of thermodynamics and Gibbs conditioning principle.

Consider a physical system which comprises n *particles* $i = 1, 2, ..., n$. Their positions $\rho_i \in R^3$ and momenta $\theta_i \in R^3$ form a *particle configuration* ω in the *thermodynamic ensemble* $\Omega \doteq R^{6n}$. Associated with each particle configuration ω there is the *energy* $U(\omega)$ ($\doteq$ the sum of the kinetic energies of the particles and of the potential energy associated with the particle configuration ω).

According to *Liouville's theorem*, *Lebesgue's measure* $d\omega$ (= the Euclidean *volume*) in R^{6n} is *invariant under the Hamiltonian dynamics* (see [20]: Chapter 1). This means that Lebesgue's measure can be regarded as the "apriori probability law". So in statistical mechanics the "apriori model" is "completely imperfect" in that, apriori, all configurations ω are "equiprobable".

The *observation (measurement)* of the *temperature* T restricts the thermodynamic system to a compact *energy shell*:

$$\{|U - E| < \Delta\} \doteq \{\omega \in \Omega : \ |U(\omega) - E| < \Delta\}. \tag{2.1.1}$$

Here E denotes the *internal energy* at the temperature T, Δ denotes the thickness of the "infinitesimally thin" energy shell. (The symbol "dot" indicates equality by definition.) The *thermodynamic entropy* S is defined as the logarithm of the volume of this energy shell:

$$S \doteq \log \mathrm{Vol}\{|U - E| < \Delta\}, \tag{2.1.2}$$

([30]: p.32).

The *thermodynamic partition function* is defined as the *Laplace transform* of the energy:

$$\Lambda(\beta) \doteq \int e^{-\beta U(\omega)} d\omega, \ \beta > 0. \tag{2.1.3}$$

The *conjugate variable* β has the meaning of *inverse temperature*: $T \doteq \frac{1}{\beta}$. The *internal energy* $E(\beta)$ associated with the inverse temperature β is defined as the derivative

$$E(\beta) \doteq -\frac{d}{d\beta}\log\Lambda(\beta). \tag{2.1.4}$$

The *second law of thermodynamics* introduces entropy as an extensive thermodynamic variable. According to it an infinitesimal reversible addition dQ of heat leads to a proportional increase in entropy with the inverse temperature as the coefficient of proportionality:

$$dS = \beta dQ. \tag{2.1.5}$$

For a system in constant volume (doing no work) the heat adds solely the internal energy of the system. Therefore, for a system in constant volume we have

$$dS = \beta dE. \tag{2.1.6}$$

Integrating by parts in (2.1.6) and taking into account the relation (2.1.4) leads to the equivalent *integral form* of the second law:

$$\begin{aligned} S(\beta) &= \beta E(\beta) - \int E(\beta)d\beta \\ &= \beta E(\beta) + \log\Lambda(\beta). \end{aligned} \tag{2.1.7}$$

The second law is obtained at the ideal limit $n = \infty$ (the socalled *thermodynamic limit*) from a *theorem of large deviations* concerning the total energy U, see Section 4.1.

Example: The classical ideal gas

In the classical *ideal gas* there is no interaction between the particles so that the energy comprises solely the kinetic energies of the individual particles, see [20]: Section 2.1. Therefore it is sufficient to include in the ensemble the momenta of the particles only:

$$\Omega \doteq \{\omega = (\theta_1, ..., \theta_n) : \ \theta_i \in R^3\} = R^{3n}.$$

The kinetic energy u_i of particle i is given by the "structure function"

$$u_i = u(\theta_i) = \frac{|\theta_i|^2}{2m}$$

from its momentum θ_i ("thermodynamic characteristics") and mass m. (The non-standard terms "structure function" and "thermodynamic characteristics" refer to their economic analogs, cf. Section 2.4.) Thus the total energy becomes simply the sum

$$U(\omega) = \sum_{i=1}^{n} u(\theta_i) = \sum_{i=1}^{n} \frac{|\theta_i|^2}{2m}. \tag{2.1.8}$$

Consequently, the partition function of the ideal gas is the n'th power of the partition function associated with a single particle:

$$\Lambda(\beta) \doteq \int \cdots \int e^{-\sum_{i=1}^{n} \frac{|\theta_i|^2}{2m}} d\theta_1 \cdots d\theta_n = \lambda(\beta)^n, \tag{2.1.9}$$

where

$$\lambda(\beta) \doteq \int\limits_{R^3} e^{-\beta u(\theta)} d\theta = \int\limits_{R^3} e^{-\frac{\beta|\theta|^2}{2m}} d\theta = (\int\limits_{R} e^{-\frac{\beta x^2}{2m}} dx)^3 = (2\pi m)^{\frac{3}{2}} \beta^{-\frac{3}{2}}. \quad (2.1.10)$$

The internal energy of the ideal gas is given by

$$E(\beta) = ne(\beta), \quad (2.1.11)$$

where

$$e(\beta) \doteq -\frac{d}{d\beta} \log \lambda(\beta) = \frac{3}{2\beta} \quad (2.1.12)$$

denotes the internal energy of a single particle.

It follows that the integral form (2.1.7) of the second law for the ideal gas obtains the form

$$S(\beta) = n(\log \lambda(\beta) + \beta e(\beta)) = \frac{3n}{2}(-\log \beta + \log 2\pi e m). \quad (2.1.13)$$

2.2 Random economies and their equilibria

We consider an *economic system* (shortly, *economy*), where certain *commodities* $j = 1, ..., l+1$ are traded by a set of *economic agents.*

We assume that there is a parameter n, to be called the *size parameter.* (Typically n is simply equal to the number of economic agents.) We assume that we are dealing with a "large economy"; namely, in the exact theorems we let $n \to \infty$.

Let $Z^j(p)$ denote the *total excess demand* on the commodity $j \in \{1, ..., l+1\}$ at the *price* $p \in R_+^{l+1}$. (Superscripts will refer to commodities.) We assume that, for each j and p, the total excess demand is a *random variable* defined on an underlying *probability space* $(\Omega, \mathcal{F}, P)$, i.e.,

$$Z^j(p) = \{Z^j(\omega; p);\ \omega \in \Omega\},$$

where $Z^j(\omega; p)$ is a measurable map of the variable ω, see e.g. [8]: p.182.

We will refer to Ω as the *macroeconomic ensemble* and to its elements ω as the *macroeconomic configurations.* (This somewhat peculiar terminology is due to the analogy with statistical mechanics, cf. Section 2.1.) The underlying probability measure P will be called the *apriori macroeconomic probability law.*

We make the following two standard assumptions:

(i) $Z^j(ap) \equiv Z^j(p)$ for every constant $a > 0$ (*homogeneity of degree 0*); and

(ii) $\sum_{j=1}^{l+1} p^j Z^j(p) \equiv 0$ (*Walras' law*).

(As is common the symbol ω will be usually omitted.)

Due to the homogeneity of degree 0, the prices can without loss of generality be normed to belong to the *price simplex*

$$S^l \doteq \{p \in R_+^{l+1} :\ \sum_{j=1}^{l+1} p^j = 1\}.$$

(The symbol "dot" indicates equality by definition.)

Walras' law implies that, for any price $p \in S^l$ such that $p^{l+1} > 0$, the total excess demand on the $l+1$'st commodity is determined by the total excess demands on the other commodities:

$$Z^{l+1}(p) = -(p^{l+1})^{-1} \sum_{j=1}^{l} p^j Z^j(p).$$

Thus we will omit the $l+1$'st component and call the vector

$$Z(p) \doteq (Z^1(p), ..., Z^l(p)) \in R^l$$

comprising the total excess demands on the commodities $j = 1, ..., l$ simply the *(random) total excess demand*.

The *random equilibrium prices (r.e.p.'s)* are defined as those price vectors $p^* = \{p^*(\omega);\ \omega \in \Omega\}$ at which the random total excess demand vanishes:

$$Z(\omega; p^*(\omega)) = 0, \tag{2.2.1}$$

or, shortly:

$$Z(p^*) = 0.$$

Let

$$EZ(p) \doteq \int Z(\omega; p) P(d\omega)$$

denote the (deterministic) *expected total excess demand function.* Its zeros p_e^* will be called the *(apriori) expected equilibrium prices*:

$$EZ(p_e^*) = 0.$$

Remarks:

(i) Formally, the random equilibrium prices comprise a *random set*:

$$\pi^* = \{\pi^*(\omega); \omega \in \Omega\},$$

where

$$\pi^*(\omega) = \{p \in S^l :\ Z(\omega; p) = 0\}$$

denotes the set of equilibrium prices at the realized macroeconomic configuration ω. Thus $p^*(\omega)$ denotes an arbitrary element of $\pi^*(\omega)$.

(ii) There is a more general concept of random equilibrium:

A random variable $X(p) = \{X(\omega; p);\ \omega \in \Omega\} \in R^d$ (for some $d \geq 1$) which depends on the macroeconomic configuration ω and on the price p will be called a *macroeconomic random variable*. Examples are, e.g., the total demand, supply, production or share of these by the whole economy or by some macroeconomic sector. Also, if the economic agents can be classified into a finite set of different *types,* then the numbers of agents belonging to these classes can be regarded as macroeconomic random variables, cf. Example (iii) below.

For any r.e.p. p^*, let

$$x^* \doteq n^{-1} X(p^*)$$

denote the *mean* of the macroeconomic random variable at this equilibrium price. The pair (p^*, x^*) is called a *random composite equilibrium (r.c.e.)*, see [27]: Section 4.3.

Examples:

(i) In the standard *Cobb-Douglas exchange economy* (shortly, *CD economy*) comprising n economic agents $i = 1, ..., n$, the *individual excess demand* by agent i is given by the vector

$$\zeta_i(p) \doteq \frac{a_i^j}{p^j} p \cdot e_i - e_i^j;\ j = 1, ..., l), \tag{2.2.2}$$

where $a_i \doteq (a_i^1, ..., a_i^{l+1}) \in S^l$ denotes the *share parameter*, $e_i \doteq (e_i^1, ..., e_i^{l+1}) \in R^{l+1}$ denotes the *initial endowment*, and $p \cdot e_i \doteq \sum_{k=1}^{l+1} p^k e_i^k$ denotes the *initial wealth* of the agent i.

In a *random CD economy* the share parameters a_i as well as the initial endowments e_i will be random variables.

The random total excess demand equals the sum of the random individual excess demands $\zeta_i(p)$:

$$Z(p) \doteq \sum_{i=1}^n \zeta_i(p) = ((p^j)^{-1} \sum_{i=1}^n \sum_{k=1}^{l+1} a_i^j e_i^k p^k - \sum_{i=1}^n e_i^j;\ j = 1, ..., l).$$

It follows that the random equilibrium prices are obtained from the formula

$$p^* = ((\sum_{i=1}^n e_i^j)^{-1} (W^*)^j;\ j = 1, ..., l+1), \tag{2.2.3}$$

where $W^* = ((W^*)^1, ..., (W^*)^{l+1})$ is a left eigenvector (associated with the eigenvalue 1) of the *(random) stochastic matrix*

$$A \ = (a^{jk};\ j, k = 1, ..., l+1) \doteq ((\sum_{i=1}^n e_i^j)^{-1} \sum_{i=1}^n a_i^k e_i^j;\ j, k = 1, ..., l+1), \tag{2.2.4}$$

normalized so that $p^* \in S^l$. (In particular, it follows that, if the matrix A is *irreducible* (see e.g. [32]), then there will be only one unique random equilibrium price p^*.)

The expected total excess demand on the commodity j is given by

$$EZ(p) = \sum_{i=1}^n E\zeta_i(p) = ((p^j)^{-1} \sum_{k=1}^{l+1} M_{a;e}^{jk} p^k - M_e^j;\ j = 1, ..., l),$$

where

$$M_{a;e}^{jk} \doteq \sum_{i=1}^n E(a_i^j e_i^k),$$

$$M_e^k \doteq \sum_{i=1}^n Ee_i^k = \sum_{j=1}^{l+1} \mu_{a;e}^{jk}.$$

It follows that the expected equilibrium prices are given by

$$p_e^* = ((M_e^j)^{-1}(w_e^*)^j;\ j = 1, ..., l), \tag{2.2.5}$$

where w_e^* is a left eigenvector of the stochastic matrix

$$A_e = (a_e^{jk};\ j, k = 1, ..., l+1) \doteq ((M_e^j)^{-1} M_{a;e}^{kj};\ j, k = 1, ..., l+1),$$

subject to the normalization $p_e^* \in S^l$. If A_e is irreducible, then there will be only one unique expected equilibrium price p_e^*.

(ii) In a *random (one-period) random financial market* ([27]: Example 2, [28]) the parameters a_i ($\doteq$ the risk aversion of agent i), μ_i ($\doteq$ the vector of subjective expectations by i of the values of the assets at the end of the period), Σ_i ($\doteq$ the matrix of subjective expectations by i of the correlations of the values of the assets at the end of the period) and e_i ($\doteq$ the initial endowment of i) are random variables. (There is "double stochasticity" in that the agents' subjective expectations μ_i and Σ_i are regarded as random variables.)

(iii) The concept of a random composite equilibrium can be illustrated with the aid of the following *survival model* (see e.g. [7]).

Consider a random Cobb-Douglas economy as described above. Suppose that at each price p there is a *survival level* $\underline{w}(p)$ such that an agent i having initial endowment $e_i \in R^{l+1}$ can survive only if his initial wealth exceeds this level: $p \cdot e_i \geq \underline{w}(p)$. Let $\chi_{\{p \cdot e_i < \underline{w}(p)\}}$ denote the *indicator of non-survival,* i.e., it is 1 or 0 according as the inequality $p \cdot e_i < \underline{w}(p)$ characterizing the non-survival is satisfied or not. Thus we can write the *total number* $N(p)$ *of non-surviving agents* as the sum

$$N(p) = \sum_{i=1}^{n} \chi_{\{p \cdot e_i < \underline{w}(p)\}}.$$

The pair (p^*, n^*) comprising the equilibrium price p^* and the mean number $n^* \doteq n^{-1} N(p^*)$ of non-surviving agents (at this equilibrium) is now an example of a random composite equilibrium.

2.3 The second law of stochastic equilibrium economics

Let p be any price belonging to the price simplex S^l.

By the *observation* of an equilibrium price *in the* $\delta-$*neighborhood* of the price p we mean the observation of a random equilibrium price which is at a distance less than δ from p, i.e., the occurrence of the event

$$\{\omega \in \Omega :\ \text{there exists a r.e.p. } p^*(\omega) \text{ such that } |p^*(\omega) - p| < \delta\}. \tag{2.3.1}$$

We think (2.3.1) as the observation of the *realized* ("true") equilibrium price in the $\delta-$neighborhood of the price p. The "tolerance" δ is supposed to be "negligible", and therefore we may speak about the observation of the equilibrium price *at* the price p. In the sequel we will write the observation (2.3.1) shortly as

$$\exists p^* : |p^* - p| < \delta. \tag{2.3.2}$$

(In the ensuing exact theorems we let $\delta \to 0$.)

Let A be any event having probability $P(A)$. The *information content* $\mathcal{I}(A)$ in the observation of the event A is defined as the logarithm of the inverse of its probability:

$$\mathcal{I}(A) \doteq \log \frac{1}{P(A)} = -\log P(A)$$

(see e.g. [11]). Thus the observation of a "common" event having high probability has low information content whereas the observation of a "rare" event has high information content.

Definition 1. The *economic entropy* $I(p)$ is defined as the *information content* in the observation of a random equilibrium price at the price p:

$$I(p) \doteq \mathcal{I}(\exists p^* : |p^* - p| < \delta) \doteq -\log P(\exists p^* : |p^* - p| < \delta). \qquad (2.3.3)$$

The *Laplace transform (L.t.)* of the total excess demand $Z(p)$ is defined as the function

$$\Lambda(\alpha; p) \doteq Ee^{\alpha \cdot Z(p)} = \int e^{\alpha \cdot Z(\omega;p)} P(d\omega), \;\; \alpha \in R^l. \qquad (2.3.4)$$

The *(macroeconomic) partition function* $\Lambda(p)$ is defined as its minimum:

$$\Lambda(p) \doteq \min_{\alpha \in R^l} \Lambda(\alpha; p).$$

The logarithm of the L.t. of a random variable (the socalled *cumulate generating function*) is known to be a convex function, see e.g. [8] p.148. It follows, in particular, that if there exists a parameter $\alpha = \alpha(p) \in R^l$ such that

$$\frac{\partial \log \Lambda}{\partial \alpha}(\alpha(p); p) = 0, \qquad (2.3.5)$$

then necessarily,

$$\Lambda(p) = \Lambda(\alpha(p); p). \qquad (2.3.6)$$

Due to the thermodynamic analogy with the concept of inverse temperature, the variable $\alpha(p)$ will be called the *conjugate variable* (associated with the price p). (Later we will show that the existence of a conjugate variable is equivalent to p being a socalled *possible equilibrium price*, see Theorem 1 in Section 4.2.)

According to the *second law of stochastic equilibrium economics* the economic entropy equals the negative of the logarithm of the partition function:

The Second Law of Stochastic Equilibrium Economics:

$$I(p) = -\log \Lambda(p). \qquad (2.3.7)$$

Remarks: (i) The observation (2.3.1) is the economic analog of the observation (2.1.1) of temperature in thermodynamics.

(ii) The second law is obtained at the limit $n = \infty$, $\delta = 0$, from a *theorem of large deviations* concerning the random equilibrium prices. According to the TLD

$$\mathcal{I}(\exists p^* : |p^* - p| < \delta) + \log \Lambda(p) = \varepsilon(n, \delta)n,$$

where $\varepsilon(n, \delta) \to 0$ as $n \to \infty$ and $\delta \to 0$, see Theorem 2 in Section 4.3.

(iii) Entropy is a "measure of rareness" for the possible values p for the random equilibrium prices. Namely, apriori non-expected "rare" values for the r.e.p. have big entropy whereas apriori expected "common" values have small entropy.

Thus the second law of can be interpreted as providing an *information theoretic measure of goodness* for the apriori equilibrium model:

Namely, if the apriori equilibrium model is "good", then due to the LLN, the observed value p for the random equilibrium price p^* is near to its apriori expected value p^*_e, and, therefore, has high apriori probability. Thus its entropy $I(p)$, which by the definition equals its information content, is small. On the other hand, the realization of an apriori rare value for the random equilibrium price has big entropy indicating the "badness" of the apriori model.

(iv) A *partial observation* of the r.e.p. means the observation of the r.e.p. p^* in some subset of the price simplex:

$$\exists p^* \in B, \quad \text{where } B \subset S^l.$$

This is the case, e.g., if the prices of some subset of the commodities are observed, see [27]: Ex.3.1, [28]. Also, if in a dynamical finance market the prices of the assets are observed only up to some time, then this represents a partial observation (of the whole price process), see [27]: Ex. 3.2, [28].

According to the *principle of minimum entropy*, the entropy of a partial observation of the equilibrium price is equal to the entropy of the *entropy minimizing price* in the observation set:

$$I(B) \doteq \mathcal{I}(\exists p^* \in B) = I(p^*_B),$$

where

$$p^*_B \doteq \text{argmin}\{I(p) : \ p \in B\}.$$

As a mathematical theorem this is a *large deviation theorem concerning partial observations*, see [27]: Theorem 3.4.

(v) By an observation of a *random composite equilibrium* at the price-variable pair $(p, x) \in S^l \times R^d$ we mean the occurrence of the event

$$\{\omega \in \Omega : \text{ there exists a r.e.p. } p^*(\omega) \text{ such that } |p^*(\omega) - p| < \delta \text{ and } |x^*(\omega) - x| < \delta\}, \tag{2.3.8}$$

where $x^*(\omega) \doteq n^{-1} X(\omega; p^*(\omega))$ denotes the mean of the macroeconomic random variable $X(\omega; p)$ at the equilibrium. The observation (2.3.8) will be written shortly as

$$\exists p^* : \ |p^* - p| < \delta, |x^* - x| < \delta. \tag{2.3.9}$$

(Also here $\delta > 0$ is an "infinitesimally small" constant.)

The *(bivariate) entropy* $I(p,x)$ associated with the observation of a r.c.e. at (p,x) is defined as the information content in this observation:

$$\begin{aligned} I(p,x) &\doteq \mathcal{I}(\exists p^* : |p^*-p| < \delta, |x^*-x| < \delta) \\ &\doteq -\log P(\exists p^* : |p^*-p| < \delta, |x^*-x| < \delta). \end{aligned}$$

The *bivariate partition function* $\Lambda(p,x)$ is defined by the formula

$$\Lambda(p,x) \doteq \Lambda(\alpha(p,x), \beta(p,x); p),$$

where

$$\Lambda(\alpha,\beta;p) \doteq E e^{\alpha\cdot Z(p)+\beta\cdot X(p)} = \int e^{\alpha\cdot Z(\omega;p)+\beta\cdot X(\omega;p)} P(d\omega),\ \ \alpha \in R^l, \beta \in R^d,$$

denotes the *bivariate Laplace transform*, and the *bivariate conjugate variables* $\alpha(p,x)$ and $\beta(p,x)$ are the solutions of the equations

$$\begin{aligned} \frac{\partial}{\partial\alpha}\log\Lambda(\alpha(p,x),\beta(p,x);p) &= 0, \\ \frac{\partial}{\partial\beta}\log\Lambda(\alpha(p,x),\beta(p,x);p) &= nx, \end{aligned} \tag{2.3.10}$$

cf. [27]: Section 4.3.

According to a *generalized second law* the entropy of a bivariate composite equilibrium can be expressed in terms of the bivariate partition function $\Lambda(p,x)$ and the conjugate variable $\beta(p,x)$:

$$I(p,x) = -\log\Lambda(p,x) + n\beta(p,x)\cdot x, \tag{2.3.11}$$

cf. [27]: Section 4.3.

In the case of Example 2.2(ii), the observation of a random composite equilibrium (p^*, n^*) at (p,x) has the meaning of a simultaneous observation of the r.e.p. p^* at p and of the proportion $n^* \doteq n^{-1}N(p^*)$ of non-surviving agents at x. The bivariate entropy $I(p,x)$ is the information content of this observation.

2.4 Ideal random economies

We will illustrate the general results with the aid of a special class of simple random economies which, due to their analogy with the classical *ideal gas* of statistical mechanics, will be called *ideal random economies*:

We assume that the *individual excess demand* $\zeta_i(p)$ by agent i is obtained with the aid of a deterministic *structure function* $z(\theta_i;p)$ from a random parameter θ_i (the *economic characteristics* of the agent i) and from the price p:

$$\zeta_i(p) = z(\theta_i;p).$$

The economic characteristics θ_i are supposed to form a sequence of R^m-valued (for some $m \geq 1$) *independent and identically distributed (i.i.d.)* random variables.

Note that, since statistical independence is preserved under deterministic transformations, it follows that, for each price p, the individual excess demands are i.i.d., too. Thus in an ideal economy the total excess demand is the sum of i.i.d. random variables:

$$Z(\omega;p) = \sum_{i=1}^{n} z(\theta_i;p). \tag{2.4.1}$$

(The macroeconomic configuration ω is defined now as the vector of the individual characteristics: $\omega \doteq (\theta_1, ..., \theta_n) \in \Omega \doteq R^{mn}$.)

Let $f(\theta)$ denote the common *probability distribution function (p.d.f.)* of the economic characteristics, i.e.,

$$P(\theta_i \in A) = \int_A f(\theta)d\theta \quad \text{for } i = 1, 2, ..., \; A \subset R^m. \tag{2.4.2}$$

We call $f(\theta)$ the *apriori microeconomic p.d.f.*. It follows that the apriori macroeconomic probability law $P(d\omega)$ is the product probability law, under which the economic characteristics θ_i are i.i.d. $f(\theta)$-distributed random variables, viz.

$$P(d\omega) = f(\theta_1)\cdots f(\theta_n)d\theta_1\cdots d\theta_n. \tag{2.4.3}$$

Let

$$\mu(p) \doteq \; E\zeta_i(p) = \int z(\theta;p)f(\theta)d\theta$$

denote the *expected individual excess demand*. Since

$$EZ(p) = n\mu(p),$$

the expected equilibrium prices are also zeros of the expected individual excess demand:

$$\mu(p_e^*) = 0. \tag{2.4.4}$$

Due to the independence of the individual total excess demands, the Laplace transform of the total excess demand in an ideal random economy is equal to the n'th power of the L.t. of the individual excess demand:

$$\begin{aligned}
\Lambda(\alpha;p) &\doteq Ee^{\alpha\cdot Z(p)} \\
&= Ee^{\alpha\cdot\sum_{i=1}^{n}\zeta_i(p)} \\
&= Ee^{\alpha\cdot\zeta_1(p)}\cdots Ee^{\alpha\cdot\zeta_n(p)} \\
&= \lambda(\alpha;p)^n,
\end{aligned}$$

where

$$\lambda(\alpha;p) \doteq Ee^{\alpha\cdot\zeta_i(p)} = \int e^{\alpha\cdot z(\theta;p)}f(\theta)d\theta.$$

It follows that the partition function is the n'th power of the *individual partition function* $\lambda(p)$:

$$\Lambda(p) = \lambda(p)^n, \tag{2.4.5}$$

where

$$\lambda(p) \doteq \min_\alpha \lambda(\alpha;p) = \lambda(\alpha(p);p), \tag{2.4.6}$$

and the conjugate variable $\alpha(p)$ satisfies the equation

$$\frac{\partial}{\partial \alpha} \log \lambda(\alpha(p); p) = 0, \tag{2.4.7}$$

cf. (2.3.5).

In view of (2.3.7) the *second law* for an ideal economy obtains the form

$$I(p) = -n \log \lambda(p). \tag{2.4.8}$$

Examples:

(i) In an *ideal random Cobb-Douglas economy* the economic characteristics

$$\theta_i \doteq (a_i, e_i) \in S^l \times R^{l+1}$$

form an i.i.d. sequence of random variables. In view of the formula (2.2.2), the *structure function* is

$$z(\theta; p) = z(a, e; p) \doteq (\frac{a^j}{p^j} p \cdot e - e^j;\ j = 1, ..., l),\ \theta \doteq (a, e) \in S^l \times R^{l+1}. \tag{2.4.9}$$

It follows that the expected individual excess demand on the commodity j is given by

$$\mu(p) = ((p^j)^{-1} \sum_{k=1}^{l+1} \mu_{a;e}^{jk} p^k - \mu_e^j;\ j = 1, ..., l), \tag{2.4.10}$$

where

$$\mu_{a;e}^{jk} \doteq E(a_i^j e_i^k) = \int \int a^j e^k f(a, e) dade,$$

$$\mu_e^k \doteq E e_i^k = \sum_{j=1}^{l+1} \mu_{a;e}^{jk},$$

and $f(\theta) = f(a, e)$ denotes the microeconomic p.d.f..

In view of equations (2.4.4) and (2.4.10) the expected equilibrium price is given by

$$p_e^* = (\frac{(w_e^*)^j}{\mu_e^j};\ j = 1, ..., l),$$

where w_e^* is a left eigenvector of the stochastic matrix

$$A_e \doteq (\frac{\mu_{a;e}^{kj}}{\mu_e^j};\ j, k = 1, ..., l+1),$$

subject to the normalization $p_e^* \in S^l$, cf. (2.2.3) and (2.2.4). If A_e is irreducible, then there will be only one unique expected equilibrium price p_e^*.

The Laplace transform of the individual excess demand in an ideal random CD economy is given by

$$\begin{aligned} \lambda(\alpha; p) &= \int_{S^l} \int_{R^{l+1}} e^{\alpha \cdot z(a,e;p)} f(a, e) dade \\ &= \int_{S^l} \int_{R^{l+1}} e^{\sum_{j=1}^{l} \alpha^j ((p^j)^{-1} a^j p \cdot e - e^j)} f(a, e) dade. \end{aligned}$$

The conjugate parameter $\alpha(p) \in R^l$ is the solution of the equation (2.4.7) , viz., presently, of the system

$$\int\limits_{S^l}\int\limits_{R^{l+1}} ((p^j)^{-1}a^j p \cdot e - e^j) e^{\Sigma_{j=1}^l \alpha^j(p)((p^j)^{-1}a^j p\cdot e - e^j)} f(a,e) dade = 0, \; j = 1, ..., l.$$

For the individual partition function we obtain the formula

$$\begin{aligned} \lambda(p) &= \lambda(\alpha(p); p) \\ &= \int\limits_{S^l}\int\limits_{R^{l+1}} e^{\sum\limits_{j=1}^{l} \alpha^j(p)((p^j)^{-1}a^j p\cdot e - e^j)} f(a,e) dade, \end{aligned}$$

cf. (2.4.6). In view of (2.4.8) the second law for an ideal random CD economy obtains the form

$$I(p) = -n \log \int\limits_{S^l}\int\limits_{R^{l+1}} e^{\sum\limits_{j=1}^{l} \alpha^j(p)((p^j)^{-1}a^j p\cdot e - e^j)} f(a,e) dade.$$

(ii) An *ideal random financial market* is formed by n statistically independent and identical financial agents. The natural choice for the economic characteristics θ_i of agent i will be the m-dimensional ($m = l^2 + 2l + 2$) vector comprising his risk parameter $a_i \in R$, the vector of his subjective expectations $\mu_{\psi;i} \in R^l$ and covariances $\Sigma_{\psi;i} \in R^{l\times l}$, and his initial endowment $e_i \in R^{l+1}$:

$$\theta_i \doteq (a_i, \mu_{\psi;i}, \Sigma_{\psi;i}, e_i), \; i = 1, ..., n.$$

The structure function of the individual excess demand is (cf. [9])

$$\begin{aligned} z(\theta; p) &= z(a, \mu_\psi, \Sigma_\psi, e) \\ &\doteq (a\Sigma_\psi)^{-1}\mu - \frac{p^T(a\Sigma_\psi)^{-1}\mu - p^T e}{p^T(a\Sigma_\psi)^{-1}p}(a\Sigma_\psi)^{-1}p. \end{aligned}$$

We will not pursue further with this example but plan to investigate it in a later study [28].

(iii) Consider the *survival model* as described in Section 2.2 (Example (iii) therein).

Due to the LLN, under appropriate regularity conditions, the proportion $n^* = n^{-1}N(p^*)$ of non-surviving agents at the equilibrium price p^* is apriori near to the *probability of non-survival* of a randomly chosen agent at the expected equilibrium price:

$$n^* \approx n_e^* \doteq P(p_e^* \cdot e < \underline{w}(p_e^*)).$$

However, again, due to the imperfectness of the apriori model, the actually realized proportion n^* may well represent a large deviation within the apriori model. The information content of the simultaneous observation of the r.e.p. at the price p and the proportion of non-surviving agents at x is given by the generalized second law (2.3.11) . Due to the postulated statistical independence of the agents, the generalized second law obtains now the form (cf. (2.4.8))

$$I(p,x) = -n \log \lambda(p,x) + n\beta(p,x) \cdot x,$$

where $\lambda(p,x)$ and $\beta(p,x)$ denote the *individual partition function* and *conjugate parameter* defined as the solutions of the equations 2.3.10.

We will neither pursue further with this example but plan to investigate it in a later study [29].

2.5 The second law and the central limit theorem

According to the *law of large numbers*, a r.e.p. is apriori "near to" its expected value:

$$p^* \to p_e^* \ \text{ as } n \to \infty,$$

([6], [24]).

The *central limit theorem* for the r.e.p.'s ([6]) characterizes the "small deviations" of the r.e.p. p^* from its expected value p_e^* as asymptotically normally distributed. Namely, under appropriate regularity conditions

$$\sqrt{n}(p^* - p_e^*) \to \mathcal{N}(0, \Sigma) \ \text{ as } n \to \infty,$$

where $\mathcal{N}(0,\Sigma)$ denotes a multinormal random vector having mean zero and covariance Σ. Thus

$$p^* \approx \mathcal{N}(p_e^*, n^{-1}\Sigma) \ \text{ for big } n \tag{2.5.1}$$

so that the standard deviation of the distribution of the r.e.p. p^* itself is of the asymptotically small order $\frac{1}{\sqrt{n}}$. This means that the CLT describes the random fluctuations at the *"mesoeconomic intermediate scale"* $\frac{1}{\sqrt{n}}$ between the *"micro-"* and *"macroeconomic scales"* $\frac{1}{n}$ and 1.

It follows that if the observed value p for the r.e.p. p^* happens to fall within a distance of the order $\frac{1}{\sqrt{n}}$ from its expected value p_e^*, then, due to (2.5.1), the probability of this observation can be approximated with the aid of the CLT:

$$\begin{aligned} P(\exists p^* : |p^* - p| < \delta) &= P(\exists p^* : |\sqrt{n}p^* - \sqrt{n}p| < \sqrt{n}\delta) \\ &\approx Cn^{\frac{l}{2}}\delta^l e^{-\frac{n}{2}(p-p_e^*)^T\Sigma^{-1}(p-p_e^*)}, \end{aligned}$$

where C is a constant and, again, the "tolerance" $\delta > 0$ is supposed to be small. Furthermore, since $\frac{\log(Cn^{\frac{l}{2}}\delta^l)}{n} \to 0$ as $n \to \infty$, this probability has the exponential order

$$e^{-\frac{n}{2}(p-p_e^*)^T\Sigma^{-1}(p-p_e^*)}.$$

Therefore, at the distance of the order $\frac{1}{\sqrt{n}}$ from p_e^* the entropy is approximately

$$I(p) \doteq -\log P(\exists p^* : |p^* - p| < \delta) \approx \frac{n}{2}(p - p_e^*)^T\Sigma^{-1}(p - p_e^*).$$

This CLT-based approximation is consistent with the second law (as it ought to be). Namely,

$$-\log \Lambda(p) \approx \frac{n}{2}(p - p_e^*)^T\Sigma^{-1}(p - p_e^*),$$

when p is close to p_e^* (cf. [25]: formula (3.2)] and [6]: Theorem 4.1(iii)]).

Outside its (narrow) region of validity this CLT-based approximation of the second law is no more valid and its use is therefore mathematically unjustified.

Remark. The *mesoscopic scale* in *statistical mechanics* refers to the small *Gaussian random fluctuations* of the thermodynamic equilibrium.

3 Gibbs Conditioning Principle

3.1 Gibbs conditioning principle in thermodynamics

Let $\beta > 0$ be an arbitrary fixed inverse temperature.

In view of the definition (2.1.3) of the thermodynamic partition function $\Lambda(\beta)$,

$$P(d\omega|\beta) \doteq \Lambda(\beta)^{-1} e^{-\beta U(\omega)} d\omega \tag{3.1.1}$$

is a probability law on the thermodynamic ensemble Ω. It is called the *canonical probability law* (at β). The corresponding probability distribution function (p.d.f.)

$$p(\omega|\beta) \doteq \Lambda(\beta)^{-1} e^{-\beta U(\omega)} \tag{3.1.2}$$

is called the *canonical probability distribution function*.

The energy $U = (U(\omega); \omega \in \Omega)$ can be regarded as a *random variable* under the canonical probability law $P(d\omega|\beta)$, and the internal energy $E(\beta)$ can be interpreted as its *expectation*. Namely, in view of (2.1.3) and (2.1.4) we have

$$\begin{aligned} E(U|\beta) &\doteq \int U(\omega) P(d\omega|\beta) \\ &= \Lambda(\beta)^{-1} \int U(\omega) e^{-\beta U(\omega)} d\omega \\ &= -\frac{d}{d\beta} \log \int e^{-\beta U(\omega)} d\omega \\ &= E(\beta). \end{aligned} \tag{3.1.3}$$

According to *Gibbs conditioning principle (GCP)* a thermodynamic system is governed by the canonical probability law at the measured temperature. (Recall that the measurement of the temperature $T = \frac{1}{\beta}$ means that the energy $U(\omega)$ is in the neighborhood of the associated internal energy $E(\beta)$, see (2.1.1).)

Example: The ideal gas

In view of (2.1.9) and (2.1.10) the canonical probability law for the ideal gas has the following product form:

$$P(d\omega|\beta) = \Lambda(\beta)^{-1} e^{-\beta \sum\limits_{i=1}^{n} u(\theta_i)} d\omega = f(\theta_1|\beta) \cdots f(\theta_n|\beta) d\theta_1 \cdots d\theta_n,$$

where

$$f(\theta|\beta) = \lambda(\beta)^{-1} e^{-\beta u(\theta)} = (2\pi m)^{-\frac{3}{2}} \beta^{\frac{3}{2}} e^{-\frac{\beta|\theta|^2}{2m}}, \ \theta \in R^3, \tag{3.1.4}$$

is a Gaussian probability distribution function on R^3, called the *microcanonical p.d.f.*.

Thus, according to Gibbs conditioning principle, the momenta θ_i of the particles in the ideal gas are i.i.d. random variables obeying the Gaussian microcanonical p.d.f..

3.2 Gibbs conditioning principle in stochastic equilibrium economics

Gibbs conditioning principle has an analogy in stochastic equilibrium economics:

Suppose that we observe the random equilibrium price p^* at the price p (see 2.3.1). Due to the law of large numbers the observed value p is necessarily equal (or at least "near") to the expected equilibrium price under the unknown "true" aposteriori macroeconomic probability law $P(d\omega|\exists p^* : |p^* - p| < \delta)$. Thus at the ideal limit $n = \infty,\ \delta = 0$ we ought to have

$$E(Z(p)|\exists p^* : |p^* - p| < \delta) \doteq \int Z(\omega; p) P(d\omega|\exists p^* : |p^* - p| < \delta) = 0.$$

This implies that the *aposteriori macroeconomic probability distribution function* $g(\omega|\exists p^* : |p^* - p| < \delta)$ defined as the *density* of the aposteriori macroeconomic probability law w.r.t. the apriori law,

$$g(\omega|\exists p^* : |p^* - p| < \delta) P(d\omega) \doteq P(d\omega|\exists p^* : |p^* - p| < \delta)$$

ought to satisfy the relation

$$\int Z(\omega; p) g(\omega|\exists p^* : |p^* - p| < \delta) P(d\omega) = 0. \tag{3.2.1}$$

Although condition (3.2.1) is a necessary condition for the aposteriori p.d.f., it alone is not sufficient for its unique characterization. The economic analog of Gibbs conditioning principle will characterize the socalled *canonical macroeconomic p.d.f.* as the aposteriori p.d.f.. As it should, the canonical macroeconomic p.d.f. satisfies (3.2.1), see (3.2.5).

Suggested by condition (3.2.1) we make the following definition:

Definition 2. A price p will be called an *(apriori) possible equilibrium price (p.e.p)*, if there is a strictly positive probability distribution function $g(\omega; p) > 0$, $\int g(\omega; p) P(d\omega) = 1$, such that p is an expected equilibrium price under the transformed macroeconomic probability law $P_g(d\omega; p) \doteq g(\omega; p) P(d\omega)$, viz.

$$E_g Z(p) \doteq \int Z(\omega; p) g(\omega; p) P(d\omega) = 0. \tag{3.2.2}$$

A probability distribution function $g(\omega; p)$ which satisfies (3.2.2) can be regarded as a *candidate* for the aposteriori macroeconomic p.d.f. $g(\omega|\exists p^* : |p^* - p| < \delta)$.

With any price p, for which the conjugate parameter $\alpha(p)$ satisfying the equation (2.3.5) exists, we can associate a p.d.f., which, due to the thermodynamic analogy, will be called the *canonical macroeconomic p.d.f.*:

$$g(\omega|p) \doteq \Lambda(p)^{-1} e^{\alpha(p) \cdot Z(\omega; p)} \tag{3.2.3}$$

We will prove later in Theorem 1 that the existence of a conjugate parameter $\alpha(p)$ is equivalent to p being a possible equilibrium price. (The notation for the canonical macroeconomic p.d.f anticipates its role as the aposteriori macroeconomic p.d.f..)

The probability law

$$P(d\omega|p) \doteq g(\omega|p) P(d\omega) = \Lambda(p)^{-1} e^{\alpha(p) \cdot Z(\omega; p)} P(d\omega) \tag{3.2.4}$$

will be called the *canonical (macroeconomic) probability law*.

In analogy with the thermodynamic formula (3.1.3) , the price p is an expected equilibrium price under the associated canonical probability law :

Namely, in view of relations (2.3.4), (2.3.5) and (2.3.6) we have

$$\begin{aligned} E(Z(p)|p) &\doteq \int Z(\omega;p)g(\omega|p)P(d\omega) \\ &= \Lambda(p)^{-1}\int Z(\omega;p)e^{\alpha(p)\cdot Z(\omega;p)}P(d\omega) \\ &= \Lambda(\alpha(p);p)^{-1}\frac{\partial}{\partial\alpha}\Lambda(\alpha(p);p) \\ &= \frac{\partial}{\partial\alpha}\log\Lambda(\alpha(p);p) \\ &= 0. \end{aligned} \tag{3.2.5}$$

The result of (3.2.5) means that the canonical p.d.f. is a candidate for the aposteriori macroeconomic p.d.f..

Now, in fact, according to the economic analog of Gibbs conditioning principle, conditionally on the observation of the random equilibrium price at a possible equilibrium price, the ensuing aposteriori macroeconomic probability distribution function is given by the canonical macroeconomic p.d.f.:

Gibbs Conditioning Principle in Stochastic Equilibrium Economics:

$$g(\omega|\exists p^*:|p^*-p|<\delta) = g(\omega|p). \tag{3.2.6}$$

Remarks:

(i) Of course, equivalently with (3.2.6), the aposteriori macroeconomic probability law is given by the canonical macroeconomic probability law:

$$P(d\omega|\exists p^*:|p^*-p|<\delta) = P(d\omega|p).$$

In what follows we shall often formulate GCP in terms of probability laws rather than probability distribution functions.

(ii) Geometrically, the definition of the possible equilibrium price means that 0 belongs to the topological interior of the convex hull of the support of the distribution of the total excess demand, see Theorem 1 in Section 4.2.

(iii) As a mathematical theorem GCP is a *conditional law of large numbers* concerning macroeconomic random variables (Theorem 3 in Section 5.2). According to the conditional LLN, conditionally on the observation of the random equilibrium price, macroeconomic random variables are centered at their canonical expectations.

(iv) According to the *principle of minimum entropy (PME),* conditionally on a partial observation of the random equilibrium price (see Remark 2.3(iv)), the aposteriori r.e.p. is equal to the entropy minimizing price which is compatible with the observation:

$$\exists p^* \in B \Rightarrow p^* = p^*_B,$$

cf. [27] : Theorem 3.7. Combining this with GCP it follows that, conditionally on a partial observation of the equilibrium price, the aposteriori macroeconomic probability law is given by the canonical law associated with the entropy-minimizing price:

$$P(d\omega|\exists p^* \in B) = P(d\omega|p^*_B),$$

cf. [27]: Theorem 4.7.

In the forthcoming study [28] we apply this principle in the prediction of asset prices in a dynamical asset market.

(v) In analogy with the characterization of a possible equilibrium price as a zero of the derivative of the c.g.f. of the total excess demand, one may characterize a *possible composite equilibrium (p.c.e)* as such a pair $(p, x) \in S^l \times R^d$ with which it is possible to associate the conjugate variables $\alpha(p, x)$ and $\beta(p, x)$ satisfying the equations (2.3.10).

The *canonical macroeconomic probability law* associated with a p.c.e. (p, x) is defined by

$$P(d\omega|p, x) \doteq \Lambda(p, x)^{-1} e^{\alpha(p,x)\cdot Z(\omega;p)+\beta(p,x)\cdot X(\omega;p)} P(d\omega). \tag{3.2.7}$$

According to a *generalized Gibbs conditioning principle*, conditionally on the observation of a random composite equilibrium at the price-variable pair (p, x), the governing aposteriori macroeconomic probability law is given by the canonical macroeconomic probability law (3.2.7) associated with the observation:

$$P(d\omega|\exists p^* : |p^* - p| < \delta, |x^* - x| < \delta) = P(d\omega|p, x),$$

see [29].

3.3 Gibbs conditioning principle for ideal random economies

Consider an ideal random economy as described in Section 2.4. In view of (2.4.1), (2.4.3), (2.4.5) and (3.2.4) the canonical macroeconomic probability law has the product form

$$\begin{aligned} P(d\omega|p) &= \Lambda(p)^{-1} e^{\sum_{i=1}^{n} \alpha(p)\cdot z(\theta_i;p)} f(\theta_1)\cdots f(\theta_n) d\theta_1 \cdots d\theta_n \\ &= f(\theta_1|p)\cdots f(\theta_n|p) d\theta_1 \cdots d\theta_n. \end{aligned} \tag{3.3.1}$$

where

$$f(\theta|p) \doteq \lambda(p)^{-1} e^{\alpha(p)\cdot z(\theta;p)} f(\theta) \tag{3.3.2}$$

is a probability distribution function on the set Θ of economic characteristics. It will be called the *canonical microeconomic p.d.f.* (associated with the price p). Thus, under the canonical macroeconomic probability law the economic characteristics are i.i.d. random variables obeying the canonical microeconomic p.d.f..

Now it follows, according to Gibbs conditioning principle (3.2.6) , that in an ideal random economy, conditionally on observing a r.e.p. at p, the economy is still ideal, i.e., the economic characteristics are i.i.d., and the *aposteriori microeconomic p.d.f.* is given by the canonical microeconomic p.d.f.:

$$f(\theta|\exists p^* : |p^* - p| < \delta) = f(\theta|p). \tag{3.3.3}$$

Examples:

(i) Consider again an ideal random CD economy. According to GCP, conditionally on the observation of an equilibrium price, the aposteriori economy is still an ideal CD economy having the canonical microeconomic p.d.f. as the aposteriori microeconomic p.d.f.:

$$\begin{aligned} f(a,e|\exists p^* : |p^*-p|<\delta) &= f(a,e|p) \\ &\doteq \lambda(p)^{-1} e^{\alpha(p)\cdot z(a,e;p)} f(a,e) \\ &= \lambda(p)^{-1} e^{\Sigma_{j=1}^{l} \alpha^j(p)((p^j)^{-1} a^j p\cdot e - e^j)} f(a,e) \end{aligned} \quad (3.3.4)$$

cf. (3.3.2) and (2.4.9).

(ii) Consider again the survival model of Example 2.2(iii) and assume that the underlying Cobb-Douglas economy is ideal. Suppose that the random equilibrium price p^* is observed at the price p and the proportion x^* of non-surviving agents at x (cf. Remark 2.3(iv)). The ensuing aposteriori microeconomic p.d.f. is now given by the associated canonical microeconomic p.d.f. $f(a,e|p,x)$:

$$f(a,e|\exists p^* : |p^*-p|<\delta, |x^*-x|<\delta) = f(a,e|p,x).$$

If only the proportion x^* of non-surviving agents is observed, i.e., we have only a partial observation of the random composite equilibrium (p^*,x^*), then, according to the PME and GCP, the aposteriori microeconomic p.d.f. is given by

$$f(a,e||x^*-x|<\delta) = f(a,e|x) \doteq f(a,e|p(x),x),$$

where $p(x)$ denotes the price which minimizes the bivariate entropy (2.3.11) over the price variable:

$$I(p(x),x) = \min_p I(p,x) = \min_p(-n\log\lambda(p,x) + n\beta(p,x)\cdot x),$$

see [29].

PART II: EXACT RESULTS

4 Theorems of Large Deviations

4.1 The second law of thermodynamics as a theorem of large deviations

Let $\beta > 0$ be an arbitrary fixed inverse temperature.

Recall from Section 3.1 that the energy $U = \{U(\omega); \omega \in \Omega\}$ of a thermodynamic system can be regarded as a random variable on the ensemble Ω under the canonical law $P(d\omega|\beta)$ (Section 3.1). Also recall that then the internal energy $E(\beta)$ equals the expectation of U.

Suppose that the energy satisfies the *(weak) law of large numbers* under the canonical probability law $P(d\omega|\beta)$; namely,

$$\lim_{n\to\infty} P(n^{-1}|U - E(\beta)| < \varepsilon|\beta) = 1 \text{ for all } \varepsilon > 0. \tag{4.1.1}$$

Under this hypothesis the thermodynamic second law can be formulated as a *theorem of large deviations* concerning the energy.

To this end, let $\delta > 0$ be an arbitrary fixed constant. Then one can prove that

$$|\log Vol(|U - E(\beta)| < n\delta) - (\log \Lambda(\beta) + \beta E(\beta))| < n\delta \text{ eventually.} \tag{4.1.2}$$

(The phrase "eventually" means the same as "for all sufficiently big n".) Clearly this implies also the following:

$$\lim_{\delta\to 0} \limsup_{n\to\infty} |n^{-1}[\log Vol(|U - E(\beta)| < n\delta) - (\log \Lambda(\beta) + \beta E(\beta))]| = 0. \tag{4.1.3}$$

The (integral form of the) thermodynamic second law (2.1.8) is obtained from (4.1.3) at the ideal limit $n = \infty,\ \delta = 0$.

In order to prove (4.1.2) note first that, due to the hypothesis (4.1.1),

$$\gamma_n \doteq \log P(|U - E(\beta)| < \frac{n\delta}{2\beta}) \to 0 \text{ as } n \to \infty.$$

On the other hand, in view of the definition (3.1.1) of the canonical probability law, we can also write

$$\gamma_n = \log \int\limits_{\{|U-E(\beta)|<\frac{n\delta}{2\beta}|\beta\}} e^{-\beta U(\omega)} d\omega - \log \Lambda(\beta). \tag{4.1.4}$$

Since clearly,

$$|\log \int\limits_{\{|U-E(\beta)|<\frac{n\delta}{2\beta}\}} e^{-\beta U(\omega)} d\omega - \log Vol(|U - E(\beta)| < \frac{n\delta}{2\beta}) + \beta E(\beta)| \le \frac{n\delta}{2}, \tag{4.1.5}$$

we obtain by combining (4.1.4) and (4.1.5) :

$$|\log Vol(|U - E(\beta)| < \frac{n\delta}{2\beta}) - (\log \Lambda(\beta) + \beta E(\beta))| \le \frac{n\delta}{2} + \gamma_n,$$

from which (4.1.3) clearly follows.

Example: The ideal gas

The energies $u_i = \frac{|\theta_i|^2}{2m}$ of the particles of the ideal gas are i.i.d. random variables under the canonical laws. Therefore, due to the classical LLN concerning i.i.d. random variables, the hypothesis (4.1.1) for the TLD is automatically satisfied:

$$\lim_{n\to\infty} P(|n^{-1}U - \frac{3}{2\beta}| < \delta|\beta) = 1 \text{ for all } \delta > 0.$$

(Recall from (2.1.11) and (2.1.12) the formula for the internal energy.)

In view of the formula (2.1.13) for the entropy of the ideal gas, the TLD (4.1.2) obtains the form

$$\lim_{\delta\to 0} \lim_{n\to\infty} |n^{-1} \log Vol(|n^{-1}U - \frac{3}{2\beta}| < \delta) + \frac{3}{2}\log\beta - \frac{3}{2}\log 2\pi e m| < n\delta \text{ eventually.}$$

4.2 Characterization of possible equilibrium prices

In order to be able to formulate the economic second law as a theorem of large deviations we need the following characterization of possible equilibrium prices.

We say that a random variable $X \in R^l$ is *degenerate*, if there is a lower dimensional affine hyperplane $H \subset R^{l'}$ with $l' < l$ such that $P(X \in H) = 1$. Otherwise we call X *non-degenerate.*

The *support* of a random variable X is defined as the minimal topologically closed set F such that $P(X \in F) = 1$.

Theorem 1. Let p be an arbitrary price belonging the interior

$$\mathring{S}^l \doteq \{p \in S^l : \; p^j > 0 \text{ for all } j = 1, ..., l+1\}$$

of the price simplex S^l. Suppose that the excess demand $Z(p)$ is non-degenerate. Then the following four conditions are equivalent:

(i) p is a possible equilibrium price;

(ii) 0 belongs to the topological interior of the convex hull of the support of the distribution of the total excess demand;

(iii) there exists a conjugate parameter $\alpha(p) \in R^l$ satisfying (2.3.5);

(iv) the price p is an expected equilibrium price under the canonical probability law:

$$E(Z(p)|p) \doteq \int Z(\omega; p) P(d\omega|p) = \int Z(\omega; p) g(\omega|p) P(d\omega) = 0.$$

Proof: Suppose that p is a possible equilibrium price, i.e., there exists a strictly positive probability density function $g(\omega; p)$ such that p is an expected equilibrim price under the transformed probability law $P_g(d\omega; p) \doteq g(\omega; p)P(d\omega)$. Since P and P_g are mutually absolutely continuous (as measures, cf. [8]: p.422), it follows that $Z(p)$ is non-degenerate under P_g, too. Now, it is a general fact that the expectation of a non-degenerate random variable belongs to the topological interior of the convex hull of the support of the distribution of the random variable. This proves (ii).

Suppose now that (ii) holds true. It is known that the derivative of the cumulant generating function ($\doteq$ the logarithm of the Laplace transform) of a random variable defines a bijection between the domain of the c.g.f. and the interior of the convex hull of the support of the random variable ([31]: Theorem 26.5, [5]: Proposition 9.7, [23]). Thus it follows that 0 belongs to the range of the derivative of the c.g.f. $\log \Lambda(\alpha; p)$, i.e., a conjugate parameter $\alpha(p) \in R^l$ exists, indeed.

That (iv) follows from (iii) was proved already in (3.2.5).

The implication (iv) $\Rightarrow$ (i) is trivial.

4.3 The second law of stochastic equilibrium economics as a theorem of large deviations

We are now able to formulate the second law as a a *theorem of large deviations concerning the random equilibrium prices.*

To this end, let p be an arbitrary possible equilibrium price belonging to the interior $\mathring{S}^l$ of the price simplex S^l. We postulate three hypotheses:

(i) The random total excess demand $Z(p)$ satisfies the (weak) law of large numbers under the canonical probability law $P(d\omega|p)$, i.e.,

$$\lim_{n\to\infty} P(|n^{-1}Z(p)| < \varepsilon|p) = 1 \text{ for all } \varepsilon > 0.$$

(Recall from Theorem 1 that $E(Z(p)|p) = 0$.)

(ii) The second derivative of the mean total excess demand $n^{-1}Z(q)$ is bounded on some closed neighborhood $\overline{U}$ of the price p; namely, there is a constant $A_2 < \infty$ such that

$$|n^{-1}Z''(q)| < A_2 \text{ for } \; q \in \overline{U}.$$

(iii) The derivative (matrix) $Z'(p) \in R^{l\times l}$ is invertible, and moreover, the inverse of the derivative of the mean total excess demand is bounded; namely, there is a constant $A_{-1} < \infty$ such that

$$|nZ'(p)^{-1}| < A_{-1}.$$

Theorem 2. Under the stated hypotheses, for any $\delta > 0$,

$$|\mathcal{I}(\exists p^* : |p^* - p| < \delta|) - \log \Lambda(p)| < \varepsilon(\delta)n \text{ eventually},$$

where $\varepsilon(\delta) \to 0$ as $\delta \to 0$.

Remark. Clearly, the statement of Theorem 1 implies that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} |n^{-1}[\mathcal{I}(\exists p^* : |p^* - p| < \delta) + \log \Lambda(p)]| = 0.$$

Proof of Theorem 2. The first half of the proof is analogous to the proof of the thermodynamic TLD, cf. Section 4.1.

First note that, due to the hypothesis (i),

$$\gamma_n \doteq \log P(|Z(p)| < n\varepsilon|p) \to 0 \text{ as } n \to \infty.$$

On the other hand, in view of the definition (3.2.4) of the canonical macroeconomic probability law we can also write

$$\gamma_n = \log \int_{\{|Z(p)|<n\varepsilon\}} e^{\alpha\cdot Z(\omega;p)} P(d\omega) - \log \Lambda(p). \tag{4.3.1}$$

Since clearly,

$$|\log \int_{\{|Z(p)|<n\varepsilon\}} e^{\alpha\cdot Z(\omega;p)} P(d\omega) - \log P(|Z(p)| < n\varepsilon)| \le |\alpha|n\varepsilon, \tag{4.3.2}$$

we obtain by combining (4.3.1) and (4.3.2):

$$|\log P(|n^{-1}Z(p)| < \varepsilon) - \log \Lambda(p)| \le |\alpha|n\varepsilon + \gamma_n. \tag{4.3.3}$$

Now, it can be proved by using the standard *mean value theorem* and a special *inverse function theorem* (see [27]: Lemma 2.5) that the hypotheses (ii) and (iii) imply that the mean total excess demand $n^{-1}Z(p)$ is in a neighborhood of 0 if and only there is a random equilibrium price in a neighborhood of p, see [27]: the proof of Theorem 2.2. Therefore it is possible to deduce from (4.3.3) that

$$|\log P(\exists p^* : |p^* - p| < \delta|) - \log \Lambda(p)| < \varepsilon(\delta)n \text{ eventually,}$$

where $\varepsilon(\delta) \to 0$ as $\delta \to 0$. This proves the assertion the TLD.

Remarks:

(i) Hypothesis (i) is the analog of hypothesis (4.1.1) for the thermodynamic TLD.

(ii) In terms of probabilities the statement of the TLD can be written as:

$$\Lambda(p)e^{-n\varepsilon(\delta)} < P(\exists p^* : |p^* - p| < \delta) < \Lambda(p)e^{n\varepsilon(\delta)} \text{ eventually.}$$

If only the right hand inequality holds true, viz.

$$P(\exists p^* : |p^* - p| < \delta) < \Lambda(p)e^{n\varepsilon(\delta)} \text{ eventually,}$$

then the *LD upper bound* is said to hold true at p. The LD upper bound alone implies the *law of large numbers* for the random equilibrium prices ([27]:Theorem 3.5).

(iii) For an alternative weaker hypothesis for hypothesis (iii), see [16]) .

(iv) According to the TLD concerning *partial observations*, under appropriate regularity conditions, we have

$$\lim_{n\to\infty} n^{-1}|\mathcal{I}(\exists p^* \in B) + \log \Lambda(p_B^*)| = 0,$$

where $B \subset S^l$ is convex and p_B^* denotes the entropy-minimizing price in B, cf. [27]: Theorem 3.4. The principle of minimum entropy for partial observations (cf. Remark 2.3(iii)) is obtained from this at the ideal limit $n = \infty$.

4.4 The theorem of large deviations for ideal random economies

The following corollary of Theorem 1 gives sufficient conditions for a price p to be a possible equilibrium price in an *ideal random economy.*

Let Θ denote the set of parameter values at which the microeconomic p.d.f. is strictly positive:

$$\Theta \doteq \{\theta \in R^m;\ f(\theta) > 0\}.$$

Its topological closure $\overline{\Theta}$ equals the *support* of the economic characteristics θ_i of the economic agents, cf. Section 4.2.

Corollary 1. Let p be an arbitrary price belonging to the interior $\mathring{S}^l$ of the price simplex. Suppose that

(i) the economic characteristics θ_i are bounded random variables with dimension $m \geq l$;

(ii) the microeconomic p.d.f. $f(\theta)$ is continuous on its support $\overline{\Theta}$;

(iii) the derivative $\frac{\partial z(\theta;p)}{\partial\theta}$ of the structure function is continuous on $\overline{\Theta}\times\mathring{S}^l$; and

(iv) there exists a parameter $\theta=\theta(p)\in\Theta$ such that $z(\theta(p);p)=0$ and the matrix $\frac{\partial z(\theta;p)}{\partial\theta}\in R^{m\times l}$ has full rank ($=l$) at $\theta=\theta(p)$.

Then p is a possible equilibrium price.

Proof. Due to the hypotheses (i) and (ii), the support $\overline{\Theta}$ is compact, and Θ is its (relatively) open subset.

Due to the hypotheses (iii) and (iv), and due to the *implicit function theorem* (see e.g. [17]), the structure function function $z(\theta;p)$ is invertible in a neighborhood of $\theta(p)$, i.e., there is a constant $\delta>0$ such that for $|z|<\delta$ there exists $\theta=\theta(p,z)\in\Theta$ such that

$$z(\theta(p,z);p)\equiv z.$$

Moreover the inverse $\theta(p,z)$ is a continuous function of its variables p and z. Let $\varepsilon>0$ be an arbitrary positive constant. Due to the continuity of the structure function $z(\theta;p)$ (implied by the hypothesis (iii)), there is a constant $\eta_\varepsilon>0$ such that $|z(\theta;p)-z|<\varepsilon$ whenever $|\theta-\theta(p,z)|<\eta_\varepsilon$. In view of the hypothesis (ii) it follows that $f(\theta)$ is strictly positive in a neighborhood of $\theta(p,z)\in\Theta$, whence for any $|z|<\delta$,

$$P(|z(\theta_i;p)-z|<\varepsilon)\geq P(|\theta_i-\theta(p;z)|<\eta_\varepsilon)=\int\limits_{\{|\theta-\theta(p;z)|<\eta_\varepsilon\}} f(\theta)d\theta>0,$$

i.e., 0 belongs to the topological interior of the support of the distribution of the individual excess demand $\zeta_i(p)=z(\theta_i;p)$. Recalling the argument used in the proof of Theorem 1, it follows that 0 belongs to the range of the derivative $\frac{\partial}{\partial\alpha}\log\lambda(\alpha;p)$. In view of the same theorem and the relation (2.4.7), p is a possible equilibrium price.

In the following corollary we formulate a set of sufficient conditions for the TLD to hold true for an ideal random economy:

Corollary 2. Let $p\in\mathring{S}^l$ be arbitrary. Suppose that the hypotheses (i)-(iv) for Corollary 1 hold true (so that p is a possible equilibrium price).

Moreover, suppose that

(v) the derivative $\frac{\partial^2 z(\theta;p)}{\partial p^2}$ is continuous on $\overline{\Theta}\times\overline{U}$ for some closed neighborhood $\overline{U}\subset\mathring{S}^l$ of p; and

(vi) $z(\theta;p)$ is a bounded perturbation of a deterministic function $z(p)$ (i.e., $z(p)$ does not depend on the parameter θ) in the following sense:

$$\rho\doteq\max_{\theta\in\overline{\Theta}}|\frac{\partial z}{\partial p}(\theta;p)z'(p)^{-1}-I|<1.$$

Then the TLD holds true at the price p.

Proof. We show that the hypotheses (i)-(iii) for the general TLD (Theorem 2) hold true.

Recall that under the canonical macroeconomic probability law the economic characteristics are i.i.d. and obey the canonical p.d.f. $f(\theta|p)$. It follows that as their deterministic transforms the microeconomic random variables $\zeta_i(p) = z(\theta_i; p)$ are also i.i.d.. Thus the hypothesis (i) for the general TLD follows directly from the LLN for i.i.d. random variables.

Due to the hypothesis (v), $\frac{\partial^2 z(\theta;q)}{\partial q^2}$ is bounded by some constant A_2 on the compact set $\overline{\Theta} \times \overline{U}$. Therefore

$$|Z''(q)| = |\sum_{i=1}^{n} \frac{\partial^2 z(\theta_i; q)}{\partial q^2}| \leq A_2 n \text{ on } \overline{U}.$$

Thus the hypothesis (ii) for the TLD is satisfied.

In view of the hypothesis (vi) we have

$$|n^{-1} Z'(p) z'(p)^{-1} - I| = |n^{-1} \sum_{i=1}^{n} (\frac{\partial z}{\partial p}(\theta_i; p) z'(p)^{-1} - I| \leq \rho,$$

whence

$$|Z'(p)^{-1}| < \frac{|z'(p)^{-1}|}{n(1-\rho)},$$

i.e., condition (iii) for the TLD holds true with $A_{-1} = \frac{|z'(p)^{-1}|}{1-\rho}$.

Remarks:

(i) Condition (vi) can be considerably weakened ([16]). Namely, it suffices to assume the the derivative $\mu'(p)$ of the individual expected excess demand is non-singular. Clearly, in the Cobb-Douglas case Det $\mu'(p) = 0$ is a polynomial equation, and therefore $\mu'(p)$ is non-singular except for a set of prices p having Lebesgue measure zero.

(ii) Suppose that $l = 1$ so that p and $z(\theta; p)$ are scalars. It is natural to assume that $z'(p) < 0$. In this case the hypothesis (vi) becomes

(vi') $$(2-\delta) z'(p) \leq \frac{\partial z}{\partial p}(\theta; p) \leq \delta z'(p) \text{ for some } \delta > 0.$$

Example:

The following corollary of Corollary 1 gives sufficient conditions for a price $p \in \mathring{S}^l$ to be a possible equilibrium price in an ideal CD economy:

Let

$$\Theta \doteq \{\theta = (a, e) \in S^l \times R^{l+1} : \; f(a, e) > 0\}.$$

Corollary 3. Suppose that

(i) the initial endowments e_i are bounded;

(ii) the microeconomic p.d.f. $f(a, e)$ is continuous on the support $\overline{\Theta}$; and

(iii) the microeconomic p.d.f. $f(a, e)$ is strictly positive at $a = p, \; e \equiv 1$, (i.e., $\theta(p) \doteq (p, 1) \in \Theta$).

Then p is a possible equilibrium price.

Proof. We verify the hypotheses (i)-(iv) for Corollary 1:

For an ideal random CD economy, the dimension m of the economic characteristics $\theta = (a, e)$ is $2l + 1$, which is $> l$ as required for Corollary 1. Since the share parameters belong to the (bounded) simplex S^l, it follows that the economic characteristics $\theta_i = (a_i, e_i)$ are bounded as required.

Clearly, the structure function (2.4.9) of a CD agent is infinitely many times differentiable w.r.t. its variables $a \in S^l$, $e \in R^{l+1}$ and $p \in \mathring{S}^l$. Thus the hypothesis (iii) for Corollary 1 is automatically satisfied.

A direct calculation shows that

$$z(\theta(p); p) = z(p, 1; p) = 0.$$

Moreover, as is easy to see, already the derivative

$$\frac{\partial z(a, e; p)}{\partial a} \in R^{(l+1)\times l})$$

has the (full) rank $(= l)$ at $\theta(p) = (p, 1)$. Thus also the hypothesis (iv) for Corollary 1 holds true.

In the following corollary we formulate a set of sufficient conditions for the TLD to hold true for an ideal random CD economy:

For simplicity we assume that $l = 1$.

Corollary 4. Suppose that $l = 1$ and that the hypotheses (i)-(iii) for Corollary 3 hold true. In addition, suppose that

(iv) the initial endowment of the commodity 2 is bounded away from zero and bounded from above, and that the share parameter of commodity 1 is bounded away from zero, i.e., $\underline{e}^2 \leq e^2 \leq \overline{e}^2$, and $a^1 \leq \underline{a}^1$ for some constants $\underline{e}^2 > 0$, $\overline{e}^2 < \infty$, $\underline{a}^1 > 0$. Then the TLD holds true.

Proof:

We check that the conditions (i)-(v) and (vi') for Corollary 2 are satisfied.

The conditions (i)-(iv) for Corollary 2 were verified already in the proof of Corollary 3. Condition (v) follows from the smoothness of the structure function of a CD agent in the interior of the price simplex, cf. the proof of Corollary 3.

In order to see that condition (vi') is satisfied note first that

$$\frac{\partial z}{\partial p}(a, e; p) = -p^{-2} a^1 e^2.$$

Let $z(p)$ denote the individual excess demand by a deterministic CD agent having parameters $a = (\frac{1}{2}, \frac{1}{2})$, $e = (1, 2\overline{e}^2)$ so that

$$z(p) = \frac{1-p}{p}\overline{e}^2 - \frac{1}{2},$$

and,

$$z'(p) = -p^{-2}\overline{e}^2.$$

Now, a straightforward calculation shows that condition (vi') is true with

$$\delta = \frac{\underline{a}^1 \underline{e}^2}{\overline{e}^2}.$$

5 Conditional Laws of Large Numbers

5.1 Thermodynamic Gibbs conditioning principle as a conditional law of large numbers

Let $\beta > 0$ be a fixed inverse temperature.

Recall that the *measurement* of the inverse temperature at β means that the particle configuration ω belongs to the energy shell $\{|U - E(\beta)| < \Delta\}$, where $E(\beta)$ denotes the *internal energy* at β, and Δ denotes the thickness of the energy shell. In the exact formulation of the conditional LLN $\Delta = n\delta$, where n is the total number of the particles and $\delta \to 0$.

A variable $X = \{X(\omega); \omega \in \Omega\}$ which depends on the particle configuration $\omega \in \Omega$ will be called a *thermodynamic variable.* Such a variable can be regarded as a *random variable* under the canonical probability laws $P(d\omega|\beta),\ \beta > 0$.

As a mathematical theorem thermodynamic Gibbs conditioning principle is a *conditional law of large numbers* concerning thermodynamic variables.

We will call a thermodynamic variable X *regular* (at β), if the *(weak) law of large numbers* holds true for X under the canonical probability law $P(d\omega|\beta)$ with *geometric rate,* i.e., for all $\varepsilon > 0$ there exists a constant $\eta = \eta(\varepsilon; \beta) > 0$ such that

$$P(n^{-1}|X - E(X|\beta)| < \varepsilon|\beta)) > 1 - e^{-n\eta} \text{ eventually.} \tag{5.1.1}$$

According to the conditional LLN, conditionally on the measurement of the inverse temperature at β, regular thermodynamic variables are *centered* at their canonical means; namely, the proportion of the total volume of those particle configutarions ω in the energy shell $\{|U-E(\beta)| < \Delta\}$ where the thermodynamic variable $X(\omega)$ is near to its canonical expectation $E(X|\beta)$ is near to 1.

In exact terms, the conditional LLN is as follows:

Suppose that the statement (4.1.3) of the thermodynamic TLD holds true at β, i.e.,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} |n^{-1}[\log \mathrm{Vol}(|U - E(\beta)| < n\delta) - (\log \Lambda(\beta) + \beta E(\beta))]| = 0. \tag{5.1.2}$$

Let X be an arbitrary regular thermodynamic variable and let $\varepsilon > 0$ be an arbitrary constant. Then for all sufficiently small $\delta > 0$,

$$\lim_{n\to\infty} \frac{\mathrm{Vol}(|X - E(X|\beta)| < n\varepsilon, |U - E(\beta)| < n\delta)}{\mathrm{Vol}(|U - E(\beta)| < n\delta)} = 1. \tag{5.1.3}$$

In order to prove this, note first that by inversing the defining formula (3.1.1) of the canonical probability law, we can express the volume of any (Borel) subset $A \subset \Omega$ with the aid of the canonical probability law:

$$\mathrm{Vol}(A) = \int_A d\omega = \Lambda(\beta) \int_A e^{\beta U(\omega)} P(d\omega|\beta).$$

Let $\varepsilon > 0$ be an arbitary constant, and let $\eta = \eta(\varepsilon; \beta)$ be such that (5.1.1) holds true. Moreover, let $0 < \delta < \frac{\eta}{2\beta}$ be arbitrary. We can now write

$$\begin{aligned}
\mathrm{Vol}(|X - E(X|\beta)| \geq n\varepsilon, \quad & | \quad U - E(\beta)| < n\delta) \\
&= \Lambda(\beta) \int\limits_{\{|X-E(X|\beta)|\geq n\varepsilon,\ |U-E(\beta)|<n\delta\}} e^{\beta U(\omega)} P(d\omega|\beta) \\
&\leq \Lambda(\beta) e^{\beta E(\beta)+\beta\delta n} P(|X - E(X|\beta)| \geq n\varepsilon|\beta) \\
&< \Lambda(\beta) e^{\beta E(\beta)+\beta\delta n} e^{-n\eta} \text{ eventually, by (5.1.1),} \\
&< \Lambda(\beta) e^{\beta E(\beta)} e^{-n\frac{\eta}{2}} \text{ because } \delta < \frac{\eta}{2\beta}.
\end{aligned}$$

Now, according to the hypothesis (5.1.2) :

$$\mathrm{Vol}(|U - E(\beta)| < n\delta) > \Lambda(\beta) e^{\beta E(\beta)} e^{-n\frac{\eta}{3}} \text{ eventually,}$$

if δ is sufficiently small. Therefore, for sufficiently small δ we have eventually

$$\frac{\mathrm{Vol}(|X - E(X|\beta)| \geq n\varepsilon, |U - E(\beta)| < n\delta)}{\mathrm{Vol}(|U - E(\beta)| < n\delta)} < \frac{\Lambda(\beta) e^{\beta E(\beta)} e^{-n\frac{\eta}{2}}}{\Lambda(\beta) e^{\beta E(\beta)} e^{-n\frac{\eta}{3}}} = e^{-n\frac{\eta}{6}}.$$

This proves that (5.1.3) holds true (the convergence having geometric rate).

Example: The ideal gas

Recall that the TLD is true for the ideal gas (Section 4.1). Therefore the conditional LLN holds true automatically for any regular thermodynamic variable.

There is a natural class of regular thermodynamical variables for the ideal gas:

To this end, let $A \subset R^3$ be an arbitrary (Borel-measurable) subset of R^3, and let $\chi_A(\theta) \doteq 1$ or 0, according as $\theta \in A$ or $\theta \in A^c$, denote the *indicator function* of A.

Recall that under the canonical probability law the momenta θ_i of the particles are i.i.d. random variables. Therefore their (deterministic) transforms $\chi_A(\theta_i)$ form also an i.i.d. sequence.

Let us define the thermodynamic variable N_A as the *number* of particles i having momentum $\theta_i \in A$:

$$N_A(\omega) \doteq \sum_{i=1}^{n} \chi_A(\theta_i).$$

Now it follows that N_A is automatically regular. This is due to a general result, according to which for bounded i.i.d. random variables the convergence in the LLN is always geometric (see e.g. [14]: Section 2.3). Clearly

$$E(N_A|\beta) = nP(\theta_i \in A|\beta) = n \int\limits_A f(\theta|\beta),$$

where $f(\theta|\beta)$ denotes the Gaussian canonical p.d.f. given by the formula (3.1.4).

Let $\hat{n}_A \doteq n^{-1} N_A$ denote the *proportion* of particles i having momentum θ_i. It follows that the conditional LLN for the variable N_A obtains the form

$$\lim_{n\to\infty} \frac{\mathrm{Vol}(|\hat{n}_A - \int\limits_A f(\theta|\beta)| < \varepsilon, |U - ne(\beta)| < n\delta)}{\mathrm{Vol}(|U - ne(\beta)| < n\delta)} = 1, \qquad (5.1.4)$$

where $e(\beta) = \frac{3}{2\beta}$ denotes the internal energy of a single particle (see (2.1.12)) and ε and δ are small in the same sense as in (5.1.3). Thus, at the ideal limit $\varepsilon = 0$, in accordance with Gibbs conditioning principle, at a measured inverse temperature, for "most" particle configurations, the proportion of particles with momentum in A equals the canonical probability of A.

In the standard terminology of probability and statistics, the proportions $\hat{n}_A$, $A \subset R^3$, form the *empirical probability distribution* of the momenta θ_i. The result (5.1.4) of the conditional LLN means that at the thermodynamical limit $n \to \infty$ the empirical probability distribution $\hat{n}$ converges to the canonical probability distribution associated with the measured inverse temperature.

5.2 The economic Gibbs conditioning principle as a conditional law of large numbers

As a mathematical theorem the economic Gibbs conditioning principle is a *conditional law of large numbers* concerning *macroeconomic random variables.*

Let $p \in \mathring{S}^l$ be a fixed possible equilibrium price (see Definition 2 in Section 3.2). We postulate three hypotheses (i)-(iii):

(i) The statement of the theorem of large deviations holds true at p, i.e.,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} |n^{-1}[\mathcal{I}(\exists p^* : |p^* - p| < \delta) + \log \Lambda(p)]| = 0,$$

cf. Theorem 2.

Let $X(p) = \{X(\omega; p); \omega \in \Omega\}$ be a *macroeconomic random variable*, viz. a random variable which depends on the price p. We will call it *regular (at the price p)* if the following condition is satisfied (cf. (5.1.1)):

(ii) The *law of large numbers* holds true for $X(p)$ under the canonical probability law $P(d\omega|p)$ with *geometric rate:* for all $\varepsilon > 0$ there exists a constant $\eta = \eta(\varepsilon; p) > 0$ such that

$$P(n^{-1}|X(p) - E(X(p)|p)| < \varepsilon|p)) > 1 - e^{-n\eta} \text{ eventually; and}$$

Moreover, we assume the following:

(iii) The derivatives of the mean total excess demand and of the mean of the macroeconomic variable are bounded on some closed neighborhood $\overline{U}$ of the price p; namely, there is a constant $A < \infty$ such that

$$|n^{-1} Z'(q)| \leq A \text{ on } \overline{U} \text{ and } |n^{-1} X'(q)| \leq A \text{ on } \overline{U}.$$

Let $X^* = X(p^*)$ denote the *value* of the macroeconomic random variable $X(p)$ *at the equilibrium.*

According to the conditional LLN, conditionally on the observation of a random equilibrium price at the price p, regular macroeconomic random variables are centered at values at which they would be centered if they obeyed the canonical probability law associated with the observed equilibrium:

Theorem 3. Under the hypotheses (i)-(iii), for any fixed $\varepsilon > 0$, all sufficiently small $\delta > 0$:

$$\lim_{n\to\infty} P(n^{-1}|X^* - E(X^*|p)| < \varepsilon | \exists p^* : |p^* - p| < \delta) = 1.$$

Proof. By inversing the defining formula (3.2.4) of the canonical probability law we can express the apriori macroeconomic probability law with the aid of the canonical probability law:

$$P(d\omega) = \Lambda(p) e^{-\alpha(p)\cdot Z(\omega;p)} P(d\omega|p).$$

Let ε and $\gamma > 0$ be arbitary constants. (γ will be fixed soon.) We can now write

$$\begin{aligned} P(n^{-1}|X(p) &- E(X(p)|p)| \geq \varepsilon,\ n^{-1}|Z(p)| < \gamma) \\ &= \Lambda(p) \int_{\{n^{-1}|X(p)-E(X(p)|p)|\geq\varepsilon,\ n^{-1}|Z(p)|<\gamma\}} e^{-\alpha(p)\cdot Z(\omega;p)} P(d\omega|p) \\ &\leq \Lambda(p) e^{n|\alpha(p)|\gamma} P(n^{-1}|X(p) - E(X(p)|p)| \geq \varepsilon | p) \\ &< \Lambda(p) e^{n|\alpha(p)|\gamma} e^{-n\eta} \text{ eventually, by (iii),} \\ &= \Lambda(p) e^{-n\frac{\eta}{2}} \text{ if we choose } \gamma = \frac{\eta}{2|\alpha(p)|}. \qquad (5.2.1) \end{aligned}$$

Now, due to the hypothesis (iii) and the standard *mean value theorem* we can conclude that, if $\delta > 0$ is sufficiently small, then the event $\exists p^* : |p^* - p| < \delta$ implies the event

$$|n^{-1} Z(p)| < \gamma.$$

Therefore, for sufficiently small δ we have also

$$P(n^{-1}|X(p) - E(X(p)|p)| \geq \varepsilon,\ \exists p^* : |p^* - p| < \delta) < \Lambda(p) e^{-n\frac{\eta}{2}} \text{ eventually.}$$

Now, according to the hypothesis (i):

$$P(\exists p^* : |p^* - p| < \delta) > \Lambda(p) e^{-n\frac{\eta}{3}} \text{ eventually.}$$

Therefore, for sufficiently small δ we have eventually

$$\begin{aligned} P(n^{-1}|X(p) &- E(X(p)|p)| \geq \varepsilon |\ \exists p^* : |p^* - p| < \delta) \\ &= \frac{P(n^{-1}|X(p) - E(X(p)|p)| \geq \varepsilon,\ \exists p^* : |p^* - p| < \delta)}{P(\exists p^* : |p^* - p| < \delta)} \\ &< \frac{\Lambda(p) e^{-n\frac{\eta}{2}}}{\Lambda(p) e^{-n\frac{\eta}{3}}} \\ &= e^{-n\frac{\eta}{6}}. \qquad (5.2.2) \end{aligned}$$

In order to complete the proof note first that, due to (iii) and the mean value theorem, if $|p^* - p| < \delta$, then

$$|n^{-1}X(p^*) - n^{-1}X(p)| \le A\delta.$$

Therefore, if $\delta < \frac{\varepsilon}{A}$ is sufficently small, then

$$P(n^{-1}|X^* - E(X^*|p)| \ge 3\varepsilon|\,\exists p^* : |p^* - p| < \delta) \le P(n^{-1}|X(p) - E(X(p)|p)| \ge \varepsilon|\,\exists p^* : |p^* - p| < \delta),$$

which $\to 0$ as $n \to \infty$ as proved above.

Remarks:

(i) For a sufficient condition for hypothesis (ii) cf. Lemma 4.2 in [27].

(ii) Note that if $X(\omega; p) \equiv X(\omega)$ is a macroeconomic random variable, which does not depend on the price, then hypothesis (iii) is trivially true for it. In this case the conditional LLN obtains the form

$$\lim_{n\to\infty} P(n^{-1}|X - E(X|p)| < \varepsilon|\exists p^* : |p^* - p| < \delta) = 1.$$

5.3 The conditional law of large numbers for ideal random economies

Consider an ideal random economy as described in Section 2.4. A random variable $\xi_i(p) = x(\theta_i; p)$ which depends via a deterministic *structure function* $x(\theta; p)$ on the economic characteristics θ_i and on the price p will be called a *microeconomic random variable* (associated with the agent i). Examples are, e.g., the agent's individual demand, supply, production or type (cf. the example below). Recall that statistical indepedence is preserved under deterministic transformations. Therefore the microeconomic r.v.'s $\xi_i(p)$ are i.i.d., too.

Let us define the macroeconomic random variable $X(p)$ as the sum

$$X(p) \doteq \sum_{i=1}^{n} \xi_i(p),$$

and let $\hat{\xi}(p) \doteq n^{-1}X(p)$ denote its mean.

Let $X^* \doteq X(p^*)$ and $\hat{\xi}^* \doteq \hat{\xi}(p^*)$ denote the value and the mean, respectively, of the macroeconomic random value $X(p)$ at the equilibrium.

Clearly

$$E(X^*|p) = nE(\xi_i(p^*)|p) = n \int x(\theta; p^*) f(\theta|p) d\theta.$$

The conditional LLN of Theorem 3 obtains now the following form:

Corollary 1. Suppose that conditions (i) and (ii) for Corollary 1 in Section 4.4 hold true. In addition, suppose that

(iii) the TLD holds true at the price p (cf. Corollary 2 in Section 4.4); and

(iv) the derivatives $\frac{\partial z(\theta;q)}{\partial q}$ and $\frac{\partial x(\theta;q)}{\partial q}$ are continuous on $\overline{\Theta} \times \overline{U}$ for some closed neighborhood $\overline{U} \subset \mathring{S}^l$ of p.

Then

$$\lim_{n\to\infty} P(|\hat{\xi}^* - \int x(\theta; p^*) f(\theta|p) d\theta| < \varepsilon | \exists p^* : |p^* - p| < \delta) = 1.$$

Proof. We show that the hypotheses (i)-(iii) for Theorem 3 hold true.

Recall that under the hypotheses (i) and (ii) for Corollary 1 in Section 4.4, the economic characteristics are bounded and their support $\overline{\Theta}$ is compact. In view of hypothesis (iv) the structure function $x(\theta; p)$ of the microeconomic r.v.'s $\xi_i(p)$ is continuous on $\overline{\Theta}$ rendering $\xi_i(p),\ i = 1, 2, ...$ bounded. Also recall that under the canonical macroeconomic probability law the economic characteristics are i.i.d. and obey the canonical p.d.f. $f(\theta|p)$. The hypothesis (ii) for Theorem 3 now follows from a general result, according to which the convergence in the LLN for bounded i.i.d. random variables is always geometric.

Due to the hypothesis (iv), $\frac{\partial z(\theta;q)}{\partial q}$ and $\frac{\partial x(\theta;q)}{\partial q}$ are bounded on the compact set $\overline{\Theta} \times \overline{U}$. Therefore also

$$|n^{-1}Z'(q)| = |n^{-1}\sum_{i=1}^{n} \frac{\partial z(\theta_i; q)}{\partial q}|$$

and

$$|n^{-1}X'(q)| = |n^{-1}\sum_{i=1}^{n} \frac{\partial x(\theta_i; q)}{\partial q}|$$

are bounded by the same constant on $\overline{U}$. Thus the hypothesis (iii) for Theorem 3 is satisfied.

Examples:

(i) Consider the survival model of Example 3.2(ii), and let us define the microeconomic random variable $\xi_i(p) \doteq \chi_{\{p \cdot e_i < w(p)\}}$ as the indicator of non-survival.

Let us define the macroeconomic variable $N(p)$ as the *number* of non-surviving agents i at the price p:

$$N(p) \doteq \sum_{i=1}^{n} \chi_{\{p \cdot e_i < w(p)\}},$$

and let $\hat{n}(p) \doteq n^{-1}N(p)$ denote the *proportion* of nonsurviving agents at the price p.

The proportion $\hat{n}(p) \doteq n^{-1}N(p)$ can be interpreted as the *empirical probability* of non-survival at the the price p. As a sum of i.i.d. random variables the macroeconomic variable $N(p)$ is automatically regular.

Let $N^* \doteq N(p^*)$ and $\hat{n}^* \doteq \hat{n}(p^*)$ denote the *equilibrium values* of $N(p)$ and $\hat{n}(p)$, respectively. Clearly

$$E(N^*|p) = nP(p^* \cdot e_i < w(p^*)|p) = n \int\limits_{\{p^* \cdot e_i < w(p^*)\}} f(\theta|p) d\theta.$$

It follows that the conditional LLN for the variable N_A obtains the form

$$\lim_{n\to\infty} P(|\hat{n}^* - \int\limits_{\{p^* \cdot e_i < w(p^*)\}} f(\theta|p) d\theta| < \varepsilon | \exists p^* : |p^* - p| < \delta) = 1.$$

This means that, conditionally on the observation of an equilibrium, the empirical probability of non-survival converges to its canonical probability.

Thus, at the ideal limit $\varepsilon = 0$, in accordance with Gibbs conditioning principle, at an observed equilibrium price, for "most" macroeconomic configurations ω, the proportion of non-surviving agents equals the canonical probability of non-survival.

The conditional LLN implies also the convergence of the whole *empirical probability distribution* of the economic characteristics (cf. the ideal gas as the analogy, Section 5.1).

To this end, let $A \subset \Theta$ be an arbitrary (Borel) subset of the support $\Theta \subset R^m$ of the microeconomic p.d.f. $f(\theta)$, and let $\chi_A(\theta)$ denote the indicator function of A.

Let us define the macroeconomic variable N_A as the *number* of agents i with characteristics $\theta_i \in A$:

$$N_A(\omega) \doteq \sum_{i=1}^{n} \chi_A(\theta_i).$$

As a sum of i.i.d. bounded random variables the macroeconomic variable N_A is automatically regular. Clearly

$$E(N_A|p) = nP(\theta_i \in A|p) = n \int_A f(\theta|p) d\theta.$$

Let $\hat{n}_A \doteq n^{-1} N_A$ denote the *proportion* of agents i with characteristics $\theta_i \in A$. The proportions $\hat{n}_A,\ A \subset \Theta$, form the *empirical probability distribution* of the economic characteristics θ_i.

According to the conditional LLN, the empirical probability distribution converges to the canonical probability distribution associated with the observed equilibrium:

$$\lim_{n \to \infty} P(|\hat{n}_A - \int_A f(\theta|p) d\theta| < \varepsilon | \exists p^* : |p^* - p| < \delta) = 1,$$

cf. (5.1.4). Thus, at the ideal limit $\varepsilon = 0$, in accordance with Gibbs conditioning principle, at an observed equilibrium price, for "most" macroeconomic configurations ω, the proportion of agents having their economic characteristics in A equals the canonical probability of A.

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