

Conditional full support of Gaussian processes with stationary increments

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Abstract

We investigate the conditional full support (CFS) property, introduced by Guasoni et al. (2008a), for Gaussian processes with stationary increments. We give integrability conditions on the spectral measure of such a process that ensure that the process has CFS or not. In particular, the general results imply that for a process with spectral density f such that $f(\lambda) \sim c_1 \lambda^p e^{-c_2 \lambda^q}$ for $\lambda \rightarrow \infty$ (with necessarily $p < 1$ if $q = 0$), the CFS property holds if and only if $q < 1$.

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1 Introduction

In mathematical finance there has lately been considerable interest in pushing beyond the by now classical setting of asset prices modelled by semimartingales, cf. e.g. Delbaen and Schachermayer (2006) and the references therein. New approaches that either restrict the class of allowed trading strategies or introduce transaction costs allow for instance for the use of

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fractional Brownian motion to model (log-)prices, cf. Bender et al. (2008), Guasoni et al. (2008a), Jarrow et al. (2008). The results of Guasoni et al. (2008a,b) show that when transaction costs are introduced, a satisfactory arbitrage theory can be build up using continuous price processes that only satisfy a mild condition on their conditional support (we will give a precise definition below).

To understand the scope of the newly developed theories it has become important to investigate the conditional support properties for classes of stochastic processes. Guasoni et al. (2008a) proved that the fractional Brownian motion with arbitrary Hurst parameter has the desired property. This was recently generalized by Cherny (2008), who proved the conditional full support property for a class of Gaussian processes that can be written as a moving average with respect to ordinary Brownian motion.

In the present paper we focus on the class of continuous, centered Gaussian processes X with stationary increments, starting in 0, i.e. $X_0 = 0$. We will briefly call such a process a *continuous Gaussian si-process*. It is a classical fact that with each such process we can associate a *spectral measure*. This is a symmetric Borel measure μ on the line such that

$$\mathbb{E}X_s X_t = \int_{\mathbb{R}} \frac{(e^{i\lambda s} - 1)(e^{-i\lambda t} - 1)}{\lambda^2} \mu(d\lambda) \quad (1.1)$$

for all $s, t \geq 0$ (cf. Doob (1953)). If μ admits a Lebesgue density we call this the *spectral density* of the process. The prime example is the fractional Brownian motion with Hurst index $H \in (0, 1)$, which has spectral density

$$f(\lambda) = \frac{\sin(\pi H)\Gamma(1 + 2H)}{2\pi} |\lambda|^{1-2H} d\lambda,$$

see for instance Samorodnitsky and Taqqu (1994). The continuous moving average processes considered by Cherny (2008) are particular examples of continuous Gaussian si-processes. Included in this class are the continuous Gaussian si-processes with spectral densities is of the form

$$f(\lambda) = \lambda^2 |\hat{K}(\lambda)|^2,$$

where \hat{K} the Fourier transform of a function $K \in L^2[0, \infty)$. We use this connection in the proof of our main theorem.

The conditional full support property of a process indexed by a time-interval $[0, T]$ roughly means that given the information up to some time $t \in [0, T]$, the process can go anywhere after time t with positive probability. To give the precise definition, let $X = (X_t : t \in [0, T])$ be a continuous stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let (\mathcal{F}_t) be its natural filtration. The process is said to have *conditional full support*

(CFS) if for all $t \in [0, T]$, the conditional law of $(X_u : u \in [t, T])$ given \mathcal{F}_t almost surely has support $C_{X_t}[t, T]$, where $C_x[t, T]$ is the space of continuous functions f on $[t, T]$ satisfying $f(t) = x$. Equivalently, this means that for all $t \in [0, T]$, $f \in C_0[t, T]$ and $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{u \in [t, T]} |X_u - X_t - f(u)| < \varepsilon \mid \mathcal{F}_t\right) > 0$$

almost surely. According to Lemma 2.9 of Guasoni et al. (2008a) the CFS condition is in fact equivalent to the seemingly stronger condition that is obtained by replacing the deterministic times t in the definition by stopping times.

The goal of this paper is to give spectral conditions for a Gaussian si-process to have conditional full support. It turns out that we can give rather precise conditions on the tails of the spectral density. The results quantify the intuitively reasonable statement that a Gaussian si-process has CFS if and only if the tails of its spectral measure are heavy enough. The sufficient tail condition for CFS is very mild and easy to verify in concrete cases, thus providing a wide class of processes potentially suitable for asset price modelling in the context of Guasoni et al. (2008a,b).

2 Results

Our main result gives integrability conditions on the spectral measure of a Gaussian si-process that ensure that the process has the CFS property or not.

Being a symmetric Borel measure on the line, the spectral measure μ of a Gaussian si-process can always be written as $\mu(d\lambda) = f(\lambda) d\lambda + \mu_s(d\lambda)$ with f a nonnegative, symmetric, Borel measurable function and μ_s a symmetric Borel measure orthogonal to Lebesgue measure. As usual we call f the density of the absolutely continuous part of μ .

Theorem 2.1. *Let $X = (X_t : t \in [0, T])$ be a continuous Gaussian si-process with spectral measure μ . Let f be the density of the absolutely continuous part of μ .*

(i) *If for some $\lambda_0 > 0$*

$$\int_{\lambda_0}^{\infty} \frac{\log f(\lambda)}{\lambda^2} d\lambda > -\infty,$$

then X has CFS.

(ii) If for some $a, \lambda_0 > 0$

$$\int_{\lambda_0}^{\infty} e^{a\lambda} \mu(d\lambda) < \infty,$$

then X does not have CFS.

To illustrate the sharpness of the result, suppose for instance that for $c_1, c_2 > 0, p \in \mathbb{R}$ and $q \geq 0$, the spectral density f of the continuous Gaussian si-process X satisfies

$$f(\lambda) \sim c_1 \lambda^p e^{-c_2 \lambda^q}$$

as $\lambda \rightarrow \infty$ (with necessarily $p < 1$ if $q = 0$). Then by combining parts (i) and (ii) of the theorem, we see that the CFS property holds if and only if $q < 1$. Taking $q = 0$ and $p = 1 - 2H$ for $H \in (0, 1)$, this confirms the result of Guasoni et al. (2008a) that the fractional Brownian motion with Hurst parameter $H \in (0, 1)$ has CFS. In fact, any Gaussian si-process with a spectral density with a power tail has CFS. Another concrete example is the integrated Ornstein-Uhlenbeck process, which corresponds to $q = 0$ and $p = -2$.

Processes with $q \in (0, 1)$ are in some sense peculiar. On the one hand they are “very predictable” in the sense that their sample paths are infinitely often differentiable. On the other hand they satisfy CFS. So in the context of Guasoni et al. (2008a,b), modeling log-prices with such a smooth process does not introduce arbitrage opportunities. This is of course completely different from what happens the classical theory.

The proof of Theorem 2.1 employs the result of Cherny (2008) on Brownian moving averages. Conversely, the theorem also includes a conditional full support result for a class of moving averages. To show this, let $K \in L^2[0, \infty)$ be non-trivial, i.e. in particular K vanishes on the negative half-line, and consider the moving average process

$$X_t = \int_{-\infty}^t \left(K(t-s) - K(-s) \right) dB_s, \quad (2.1)$$

where B is a standard Brownian motion. Let $\hat{g}(\lambda) = \int g(x) \exp(i\lambda x) dx$ denote the Fourier transform of g . By the Parseval relation we have

$$\int_{\mathbb{R}} K(t-s) e^{i\lambda s} ds = e^{i\lambda t} \hat{K}(-\lambda) = e^{i\lambda t} \overline{\hat{K}(\lambda)}.$$

It follows that

$$\mathbb{E} X_s X_t = \frac{1}{2\pi} \int_{\mathbb{R}} (e^{i\lambda s} - 1)(e^{-i\lambda t} - 1) |\hat{K}(\lambda)|^2 d\lambda.$$

Hence, X is a Gaussian si-process with spectral density $f(\lambda) = \lambda^2 |\hat{K}(\lambda)|^2 / (2\pi)$. Since \hat{K} is the Fourier transform of the function $K \in L^2[0, \infty)$ it is a non-trivial Hardy function in \mathbb{H}^{2+} (in the notation of Dym and McKean (1976)), and hence

$$\int_{\mathbb{R}} \frac{\log |\hat{K}(\lambda)|}{1 + \lambda^2} d\lambda > -\infty,$$

see for instance Dym and McKean (1976), Section 2.6. It follows that the spectral density f satisfies the condition of part (i) of Theorem 2.1. This proves the following statement.

Corollary 2.2. *A continuous Brownian moving average process $X = (X_t : t \in [0, T])$ as in (2.1), with $K \in L^2[0, \infty)$ a non-trivial kernel, has CFS.*

It should be noted that Theorem 1.1 of Cherny (2008) is more general than Corollary 2.2. Cherny considers non-trivial kernels K that satisfy the less restrictive condition that $\int (K(t-s) - K(-s))^2 ds < \infty$ for all $t \geq 0$. Statement (i) of Theorem 2.1 above and Cherny's result should be seen as complementing each other. Some processes have tractable spectral measures, or are in fact defined through their spectrum, while a moving-average representation might not exist or is untractable. In such cases Theorem 2.1 can be used to investigate the CFS property. Cherny's result is appropriate for situations where a moving-average representation is known to exist, while the spectral measure may be untractable.

3 Proof of the main result

3.1 Auxiliary lemmas

The two lemmas presented in this subsection are used in the proof of Theorem 2.1, but are also useful in their own right. They state that the CFS property is preserved by an equivalent change of measure and by adding an independent continuous process.

Lemma 3.1. *Let $X = (X_t : t \in [0, T])$ and $Y = (Y_t : t \in [0, T])$ be two continuous processes. If the laws of X and Y are equivalent, then X has CFS if and only if Y has CFS.*

Proof. Consider the canonical space (Ω, \mathcal{F}) of continuous functions on $[0, T]$. On (Ω, \mathcal{F}) , let \mathbb{P}_X and \mathbb{P}_Y be the distributions of the processes X and Y , respectively. It suffices to show that for every sub- σ -field $\mathcal{G} \subseteq \mathcal{F}$ and $A \in \mathcal{F}$, it holds that $\mathbb{P}_X(A | \mathcal{G}) > 0$, \mathbb{P}_X -a.s. if and only if $\mathbb{P}_Y(A | \mathcal{G}) > 0$, \mathbb{P}_Y -a.s..

So suppose that $\mathbb{P}_X(A | \mathcal{G}) > 0$, \mathbb{P}_X -almost surely and consider the set $B = \{\mathbb{P}_Y(A | \mathcal{G}) = 0\}$. Clearly we have $B \in \mathcal{G}$. Hence, by the definition of conditional expectation and the definition of B ,

$$\mathbb{P}_Y(A \cap B) = \int_B \mathbb{P}_Y(A | \mathcal{G}) d\mathbb{P}_Y = 0.$$

But \mathbb{P}_X and \mathbb{P}_Y are equivalent by assumption, so we also have $\mathbb{P}_X(A \cap B) = 0$. It follows that

$$\int_B \mathbb{P}_X(A | \mathcal{G}) d\mathbb{P}_X = \mathbb{P}_X(A \cap B) = 0.$$

Since $\mathbb{P}_X(A | \mathcal{G}) > 0$, \mathbb{P}_X -a.s., this implies that $\mathbb{P}_X(B) = 0$. By equivalence we then also have $\mathbb{P}_Y(B) = 0$, which means that $\mathbb{P}_Y(A | \mathcal{G}) > 0$, \mathbb{P}_Y -a.s..

Repeating the argument with the roles of X and Y reversed completes the proof. \square

Lemma 3.2. *Let $X = (X_t : t \in [0, T])$ and $Y = (Y_t : t \in [0, T])$ be two independent continuous processes. If X has CFS, then $X + Y$ has CFS as well.*

Proof. Define $Z = X + Y$ and let (\mathcal{F}_t) be the natural filtration of Z . Let (\mathcal{F}_t^X) (resp. (\mathcal{F}_t^Y)) be the filtration generated by the process X (resp. Y).

Fix $t \in [0, T]$, $f \in C_0[t, T]$ and $\varepsilon > 0$. By the tower property of conditional expectations,

$$\begin{aligned} & \mathbb{P}\left(\sup_{u \in [t, T]} |Z_u - Z_t - f(u)| < \varepsilon \mid \mathcal{F}_t\right) \\ &= \mathbb{E}\left(\mathbb{P}\left(\sup_{u \in [t, T]} |Z_u - Z_t - f(u)| < \varepsilon \mid \mathcal{F}_t^X \vee \mathcal{F}_t^Y\right) \mid \mathcal{F}_t\right). \end{aligned}$$

By independence the inner conditional probability equals $\mathbb{E}((f(u) - (Y_u - Y_t))_{u \in [t, T]}, t)$, where, for $g \in C_0[t, T]$,

$$\mathbb{E}(g, t) = \mathbb{P}\left(\sup_{u \in [t, T]} |X_u - X_t - g(u)| < \varepsilon \mid \mathcal{F}_t^X\right).$$

Hence, since X has CFS, the inner probability is strictly positive almost surely. By the strict positivity of the conditional expectation operator, it follows that also

$$\mathbb{P}\left(\sup_{u \in [t, T]} |Z_u - Z_t - f(u)| < \varepsilon \mid \mathcal{F}_t\right) > 0$$

almost surely. □

3.2 Proof of Theorem 2.1

We begin with the proof of part (i). First observe that X can be written as the sum of two independent Gaussian si-processes, one with spectral density f and one with a singular spectral measure μ_s . In view of Lemma 3.2 it suffices to show that the process with spectral density f has CFS. Therefore, we proceed without loss of generality under the assumption that μ is absolutely continuous, i.e. that $\mu_s = 0$.

The condition on f implies that $f(\lambda) > 0$ for almost every λ outside $[-\lambda_0, \lambda_0]$. Since changing f on a Lebesgue null set does not change the law of the si-process, we may assume that f is strictly positive outside $[-\lambda_0, \lambda_0]$. Now define a new spectral density $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(\lambda) = \begin{cases} f(\lambda), & |\lambda| > \lambda_0 \\ \lambda^2, & |\lambda| \leq \lambda_0. \end{cases}$$

Let Y be a Gaussian si-process with spectral density g . By construction it holds that

$$\int_{\mathbb{R}} \frac{\log g(\lambda)}{1 + \lambda^2} d\lambda > -\infty,$$

i.e. Y is purely non-deterministic, in the terminology of Van Zanten (2007). Since the spectral densities f and g are equal outside a bounded set, Theorem 5.1 of Van Zanten (2007) then implies that the laws of $(X_t : t \in [0, T])$ and $(Y_t : t \in [0, T])$ are equivalent. Hence, by Lemma 3.1, it suffices to prove that the process $(Y_t : t \in [0, T])$ with spectral density g has CFS.

It follows from the assumption on f and the construction of g that

$$\int_{\mathbb{R}} \frac{g(\lambda)}{\lambda^2} d\lambda < \infty$$

and

$$\int_{\mathbb{R}} \frac{\log g(\lambda)/\lambda^2}{1 + \lambda^2} d\lambda > -\infty.$$

Hence, we have the representation $g(\lambda)/\lambda^2 = |\psi(\lambda)|^2$ for an outer Hardy function $\psi \in \mathbb{H}^{2+}$ satisfying $\overline{\psi(\lambda)} = \psi(-\lambda)$, cf. Dym and McKean (1976), p. 38. By the Paley-Wiener theorem for Hardy functions, see Section 2.3 of Dym and McKean (1976) or Rudin (1987), Theorem 19.2, we have $\psi = \hat{K}$ for some real-valued $K \in L^2[0, \infty)$. We conclude that $g(\lambda) = \lambda^2 |\hat{K}(\lambda)|^2$ and hence, by the computations carried out in the preceding section before Corollary 2.2, Y is a moving average process of the type (2.1) considered by Cherny (2008). By Theorem 1.1 if Cherny (2008) the process $(Y_t : t \in [0, T])$ then has CFS, which completes the proof of part (i).

For the proof of statement (ii) we note that by (the proof of) Lemma 2.1 of Van der Vaart and Van Zanten (2007), the condition on μ implies that the natural filtration (\mathcal{F}_t) of the process X satisfies $\mathcal{F}_s = \mathcal{F}_t$ for all $0 < s < t$. In other words, the process is completely determined by what happens immediately after time 0. It follows that for $t \in (0, T)$, the support of $(X_u : u \in [t, T])$ given \mathcal{F}_t equals the support of $(X_u : u \in [t, T])$ given \mathcal{F}_T . The latter is obviously almost surely degenerate, which implies that the process does not have CFS. This completes the proof of part (ii).

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