Dynamic Ehrenfeucht-Fraïssé Games for strong logics over linear orders and other practical results

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Abstract

We generalize the main theorem of [4] – that the quantifier-rank-k theory of a linear order is the 2^k-fold closure of a closure process which extends a finite set of marked elements and cuts of the linear order, by marking, for every local type ϕ and sets $b < a$ of marked elements, the elements or cuts: $\inf\{x : \phi(x) \wedge b < x < a\}$ and $\sup\{x : \phi(x) \wedge b < x < a\}$ – to infinitary languages, where the quantifier rank is not a finite number k, but either an ordinal or a general linear order. We write a program to enumerate the 82988077686330 $\equiv_4$ classes of linear orders as a proof of concept.

1 $\equiv_2$ and $\equiv_3$ classes of linear orders

The material of this section replicate enumerations of $\equiv_2$ and $\equiv_3$ classes of linear orders done in [2]. Our goal is to prove these results via a model existence game:

Proposition 1.1 *A set U of pairs of $\equiv_k$ classes is $\{(Th_k(\{a \in \lambda : a < b\}), Th_k(\{a \in \lambda : a > b\}))\}$ for some linear order λ just in case 1. there is an $\equiv_k$ class $\xi(U)$ such that for any $(\phi, \psi) \in U$, we have $\exists x(\phi^{<x} \wedge \psi^{>x}) \equiv_k \xi(U)$, and 2. there is a set W containing U and other sets of pairs of $\equiv_k$ classes such that every $V \in W$ has $\xi(V)$ as in part 1 and such that for any $V \in W$ and any element $(\phi, \psi) \in V$ there exist two elements V_0, V_1 of W such that $(\xi(V_0), \xi(V_1)) = (\phi, \psi)$ and $V_0 + \{(\emptyset, \emptyset)\} + V_1 = V$.*

The proposition can easily be proved by noting that the set W constructs a linear order λ such that $W = \{U_(Th_k(\lambda^{(a,b)})) : a < b \in \lambda\}$, where U_ϕ is the set of Th_{k-1}-types implied by ϕ.

Theorem 1.1 *Writing* 1 *for* $Th_1(1)$ *and* 0 *for* $Th_1(\emptyset)$*, the following sets of pairs of* $\equiv_1$ *classes of linear orders are consistent:*

$$\{(1,1)\}, \{(1,1),(1,0)\}, \{(0,1),(1,1)\}, \{(0,1),(1,1),(1,0)\}, \{(1,0),(0,1)\}, \{(0,0)\}, \emptyset,$$

and the following sets of pairs of $\equiv_1$ *classes of linear orders are inconsistent:*

$$\{(0,1)\}, \{(1,0)\}, \{(0,0)\} \cup U, \text{ if } U \not\subseteq \{(0,0)\}.$$

Let $W_Z = \{\{(1,1)\}, \{(1,1),(1,0)\}, \{(0,1),(1,1)\}, \{(0,1),(1,1),(1,0)\}, \emptyset\}$. The function ξ is uniformly 1 on W_Z, except that $\xi(\emptyset) = 0$.

- To $(1,1) \in \{(1,1)\}$, assign the pair $(\{(1,1),(1,0)\}, \{(0,1),(1,1)\})$. Writing 1 for $\xi(\{(1,0),(1,1)\})$, $\xi(\{(0,0)\})$, and $\xi(\{(0,1),(1,1)\})$ and applying the definition of addition for sets of pairs of $\equiv_1$ classes, we find: $\{(1,1),(1,0)\} + \{(0,0)\} + \{(0,1),(1,1)\} = \{(1,1+1+1),(1,0+1+1),(1+0,0+1),(1+1+0,1),(1+1+1,1)\} = \{(1,1)\}$ and the pair of ξ values $(\xi(\{(1,1),(1,0)\}), \xi(\{(0,1),(1,1)\})) = (1,1)$ is the chosen element.

- To $(1,1) \in \{(1,1),(1,0)\} \in W_Z$ we assign the pair in W_Z: $(\{(1,1),(1,0)\}, \{(0,1),(1,1),(1,0)\})$. The value of ξ on either element of that pair is 1, so the pair of values of ξ is $(\xi(\{(1,1),(1,0)\}), \xi(\{(0,1),(1,1),(1,0)\})) = (1,1)$ is the chosen element, and $\{(1,1),(1,0)\} + \{(0,0)\} + \{(0,1),(1,1),(1,0)\} = \{(1,1),(1,0)\}$ is the chosen set.

- To $(1,0) \in \{(1,1),(1,0)\}$ assign the pair $(\{(1,1),(1,0)\}, \emptyset)$. The value of ξ on this pair is $(1,0)$, the chosen element. The sets of pairs sum to: $\{(1,1),(1,0)\} + \{(0,0)\}$. On those summands, ξ has the value 1, so $= \{(1,1+1),(1,0+1),(1+0,0)\} = \{(1,1),(1,0)\}$ is the chosen set.

- The choices in $\{(0,1),(1,1)\}$ are symmetric.

- To $(0,1) \in \{(0,1),(1,1),(1,0)\}$, assign the pair $(\emptyset, \{(0,1),(1,1),(1,0)\})$. The value of ξ on this pair is $(0,1)$, the chosen element. The sets of pairs sum to: $\{(0,0)\} + \{(0,1),(1,1),(1,0)\} = \{(0,0+1),(1+0,1),(1+1,1),(1+1,0)\}$ which is the chosen set.

- To $(1,0) \in \{(0,1),(1,1),(1,0)\}$ we assign the pair $(\{(0,1),(1,1),(1,0)\}, \emptyset)$, symmetric to the previous item.

- To $(1,1) \in \{(0,1),(1,1),(1,0)\}$ we assign the pair $(\{(0,1),(1,1),(1,0)\}, \{(0,1),(1,1),(1,0)\})$. Since $\xi(\{(0,1),(1,1),(1,0)\}) = 1$, the pair of ξ values is $(1,1)$, the chosen element. Further, the the sum $\{(0,1),(1,1),(1,0)\} + \{(0,0)\} + \{(0,1),(1,1),(1,0)\} = \{(0,1),(1,1),(1,0)\}$ is the chosen set.

If that strategy is played against an exhaustive strategy of player I, with initial set $\{(1,1)\}$, the set $Z \times \eta$ is constructed: Every time player I plays $(1,1) \in \{(0,1),(1,1),(1,0)\}$, we begin a new copy of Z. Every time player

I plays $(1,0)$ or $(0,1)$, we add the immediate predecessor or successor to the greatest or least element of a copy of Z which is already being built. With the other elements of W_Z as initial sets, we get $Z \times \lambda$ for $\lambda = \eta+1, 1+\eta$, or $1+\eta+1$.

Next, let $W_2 = \{\{(1,0),(0,1)\}, \{(0,0)\}, \emptyset, \}$. Again, the value of ξ are 1, except $\xi(\emptyset) = 0$.

- To $(1,0) \in \{(1,0),(0,1)\}$ we assign the pair $(\{(0,0)\}, \emptyset)$. The pair of ξ values are the chosen element. The sum is: $\{(0,0)\} + \{(0,0)\} = \{(0,0+1),(1+0,0)\} = \{(1,0),(0,1)\}$, the chosen set.
- To $(0,1) \in \{(1,0),(0,1)\}$ we assign the pair $(\emptyset, \{(0,0)\})$. The pair of ξ values are the chosen element. The sum is: $\{(0,0)\} + \{(0,0)\} = \{(0,0+1),(1+0,0)\} = \{(1,0),(0,1)\}$, the chosen set.
- To $(0,0) \in \{(0,0)\}$ we assign the pair $(\emptyset, \emptyset)$. The pair of ξ values are the chosen element. The sum is $\emptyset + \{(0,0)\} + \emptyset = \{(0+0,0+0)\} = \{(0,0)\}$, the chosen set.

If that strategy is played against an exhaustive strategy of player I, with initial set $\{(1,0),(0,1)\}$, then we construct the model 2. If we play with initial condition $\{(0,0)\}$, then we construct the model 1.

To see that a set is inconsistent, we use first-order logic.

- Suppose $\{(1,0)\} = \{(\phi,\psi) : \lambda \models \exists x(\phi^{<x} \wedge \psi^{>x})\}$. Since 1, or $Th_1(1)$, is defined by $\exists y(y = y)$, we have: $\lambda \models \exists x(\exists y(y < x) \wedge \neg(\exists y(y > x)))$. But since $\{(1,0)\}$ is a singleton, we also have: $\lambda \models \forall x(\exists y((y < x) \wedge (y = y)) \wedge \neg(\exists y((y > x) \wedge (y = y))))$. Assign the variables of the first conjunct to $a \in \lambda, b \in \lambda$ so that $b < a$. Now the second conjunct does not hold if we assign x to b. That is: $\{(1,0)\}$ requires that some element is maximal and not minimal, and that every element is maximal and not minimal. But if a is not minimal, there exists $b < a$, and then b is not maximal.
- We treat $\{(0,1)\}$ symmetrically.
- $\{(0,0)\} \cup U$ where $U \not\subseteq \{(0,0)\}$. The element of $U \setminus \{(0,0)\}$ implies $\exists x(\exists y(y < x))$ or $\exists x(\exists y(y > x))$. But $(0,0)$ implies $\exists x(\neg\exists y(y < x) \wedge \neg\exists y(y > x))$. If the former is satisfied with x, y assigned to a, b and the latter is satisfied with x assigned to c, then by totality, a and b are related to c, violating the second formula. □

Applying a similar analysis to $\equiv_3$ is not profitable.

Definition 1.1 *If $\equiv$ respects addition, we define left equivalence: $\phi \equiv^{\text{left}} \psi$ if there is some $\equiv$ class γ such that for all $\equiv$ classes α and β and all $\equiv$ variations $\alpha_0 \equiv \alpha$ and $\beta_0 \equiv \beta$,*

$$\phi + \alpha + \gamma + \beta \equiv \psi + \alpha_0 + \gamma + \beta_0.$$

Likewise, $\phi \equiv^{\text{right}} \psi$ if there is some $\equiv$ class γ such that for all $\equiv$ and classes α and β and all $\equiv$ variations $\alpha_0 \equiv \alpha$ and $\beta_0 \equiv \beta$,

$$\alpha + \gamma + \beta + \phi \equiv \alpha_0 + \gamma + \beta_0 + \psi.$$

Finally, for any linear orders λ and μ and assignments r and s of a nonempty domain into λ and μ, we say $\lambda \equiv^{\text{loc}} \mu$ just in case

$$\lambda (\equiv^{\text{left}})^{\text{right}} \mu.$$

Theorem 1.2 *There is a normal form for $\equiv_k$ classes of linear orders – the 2^k-fold closure of the process of marking, for every local type ϕ and sets $b < a$ of marked elements, the elements or cuts:* $\inf\{x : \phi(x) \wedge b < x < a\}$ *and* $\sup\{x : \phi(x) \wedge b < x < a\}$ *which are* definable *in that*

- *the quantifier rank of ϕ is larger than the local types in some cofinal segment of b, or*
- *the elements satisfying ϕ are not cofinal in b, and above them all is some element of b which has quantifier rank greater than that of ϕ,*

This theorem is proved in [4].

Different $\equiv_3$ classes are separated by their labels $I_{\{0,1\}}(\lambda)$ or by the $\equiv_2^{\text{loc}}$ classes at labels and between labels, and further:

$$I_k(\lambda) = I_k^{\text{left}}(\lambda) \cup I_k^{\text{right}}(\lambda).$$

Theorem 1.3 *Labeling the $\equiv_0^{\text{loc}}$ class e_0, and the $\equiv_1^{\text{loc}}$ class e_1, if the linear order λ has at least five elements, then $I_{\{0,1\}}^{\text{left}}(\lambda)$ is one of the following:*

$$(de_0 \in (\emptyset, \emptyset)) = (de_1 \in ((de_0 \in (\emptyset, \emptyset)), \dots)) = (de_0 \in ((de_1 \in ((de_0 \in (\emptyset, \emptyset)), \dots)), \dots)),$$

$$(le_0 \in (\emptyset, \emptyset)) < (de_1 \in ((le_0 \in (\emptyset, \emptyset)), \dots)) = (de_0 \in ((de_1 \in ((le_0 \in (\emptyset, \emptyset)), \dots)), \dots)),$$

$$(le_0 \in (\emptyset, \emptyset)) < (le_1 \in ((le_0 \in (\emptyset, \emptyset)), \dots)) < (de_0 \in ((le_1 \in ((le_0 \in (\emptyset, \emptyset)), \dots)), \dots)),$$

$$(le_0 \in (\emptyset, \emptyset)) < (le_1 \in ((le_0 \in (\emptyset, \emptyset)), \dots)) < (le_0 \in ((le_1 \in ((le_0 \in (\emptyset, \emptyset)), \dots)), \dots)).$$

Proof: The label de_0 is assigned to $(\emptyset, \lambda)$ to indicate that λ has no least element. So for any k, there must be some $\equiv_k^{\mathrm{loc}}$ class such that λ has no least element of that type. There is only one $\equiv_1^{\mathrm{loc}}$ class, e_1. So there is no least element of $\equiv_1^{\mathrm{loc}}$ class e_1. Similarly, the label de_1 must be followed by de_0 again. If λ has at least five elements, then the four labels

$$(le_0 \in (\emptyset, \emptyset)) < (le_1 \in ((le_0 \in (\emptyset, \emptyset)), (ge_0 \in (\emptyset, \emptyset)))) <$$

$$(ge_1 \in ((le_0 \in (\emptyset, \emptyset)), (ge_0 \in (\emptyset, \emptyset)))) < (ge_0 \in (\emptyset, \emptyset))$$

don't exhaust λ. $\square$

Definition 1.2 *If τ_0 and τ_1 are $\equiv_{k-1}^{\mathrm{loc}}$ classes and if U is a set of $\equiv_{k-1}^{\mathrm{loc}}$ classes, then we call the triple (τ_0, U, τ_1) consistent just in case there is some linear order λ with elements a and b such that (λ, a) has $\equiv_{k-1}^{\mathrm{loc}}$ class τ_0 and (λ, b) has $\equiv_{k-1}^{\mathrm{loc}}$ class τ_1 and the set of $\equiv_{k-1}^{\mathrm{loc}}$ classes realized between a and b is U. If U is a set of $\equiv_{k-1}^{\mathrm{loc}}$ classes and τ_0 is an $\equiv_{k-1}^{\mathrm{loc}}$ class, then we call the triple $(\tau_0, U, -)$ consistent just in case there is some linear order λ and element $a \in \lambda$ such that (λ, a) has $\equiv_{k-1}^{\mathrm{loc}}$ class τ_0 and the set of $\equiv_{k-1}^{\mathrm{loc}}$ classes realized to the right of a is U. Similarly, we define consistency of $(-, U, \tau_1)$. $(-, U, -)$ is consistent just in case it is the set of $\equiv_{k-1}^{\mathrm{loc}}$ classes realized in some linear order.*

Theorem 1.4 *A triple as defined above is consistent just in case there is a set W of such triples, so that for any $(\tau_0, U, \tau_1) \in W$, the following holds:*

- *For any label described by τ_0, (or, symmetrically, τ_1) defining the least element of type τ', either 1. there is no element of U extending τ', and either 1a. τ_0 itself extends τ', and the union of what all elements of U imply about the the least element of type τ' is consistent with τ_1 being that least element, or 1b. the union of what τ_0 and all elements of U imply about the least element of type τ' is extended by τ_1's description of the least element of type τ' to the right of the triple, or 2. there is a pair $((\tau_0, U_0, \tau), (\tau, U_1, \tau_1))$ of triples in W such that τ extends τ' and U_0 contains no element extending τ', and $U_0 \cup \{\tau\} \cup U_1 = U$.*

- *For any label described by τ_0, (or, symmetrically, τ_1) defining a descending sequence of elements of type τ', either 1a. that sequence limits to τ_0, and $(-, U, \tau_1)$ is consistent and whenever the strategy splits $(-, U, \tau_1)$ into a pair (V_0, V_1) of triples in W, there is an element extending τ' in V_0, or 1b. there is no element of U extending τ', and conditions 1b from the previous item hold, or 2. there is a pair $((\tau_0, U_0, -), (-, U_1, \tau_1))$ of triples in W such that U_0 contains no element extending τ' and $U_0 \cup U_1 = U$, and, further, whenever the strategy splits $(-, U_1, \tau_1)$ into a pair (V_0, V_1) of triples in W, there is an element extending τ' in V_0.*

- *Symmetric conditions explain how each label described by τ_1 is realized at τ_0 or is realized below τ_0 or can be realized within U, splitting the triple into two triples, to be realized to its right and its left.*

- *For any element $\tau \in U$, there is a pair $((\tau_0, U_0, \tau), (\tau, U_1, \tau_1))$ of elements of W such that $U_0 \cup \{\tau\} \cup U_1 = U$.*

Proof: Given W, player II can last arbitrarily long in the linear consistency game. If player I plays an exhaustive strategy, then the result will be a linear order λ of elements between τ_0 and τ_1 in which τ_0 implies $Th^{\mathrm{left}}_{k-1}(\lambda)$, τ_1 implies $Th^{\mathrm{right}}_{k-1}(\lambda)$, and U is the set of $\equiv^{\mathrm{loc}}_{k-1}$ classes realized. If τ_1 exists, then the conditions in the first two items imply that the description of the linear order right of the triple given by τ_1 extends what all other played constants imply about any label, we can choose $\{c \in \lambda : c > b\}$ to be any element of the $\equiv^{\mathrm{left}}$ class that is the right part of τ_1; we can choose likewise $\{c \in \lambda : c < a\}$ and we have the linear order with constants (λ, a, b) that proves the triple to be consistent. □

Now we apply the theorem to enumerate $\equiv_3$ classes of linear orders. In $I^{\mathrm{left}}_{\{0,1\}}(\lambda)$, all of the labels either 1. label the same gap as the preceding label, or 2. label the element of λ which is the immediate successor of the preceding label. If $\lambda = 5$, then $I^{\mathrm{left}}_{\{0,1\}}(\lambda)$ and $I^{\mathrm{right}}_{\{0,1\}}(\lambda)$ overlap. If $|\lambda| > 5$, they don't.

This simplifies using the local linear consistency game: we hypothesize different combinations of $I^{\mathrm{left}}_{\{0,1\}}(\lambda)$ and $I^{\mathrm{right}}_{\{0,1\}}(\lambda)$, hypothesize different $\equiv^{\mathrm{loc}}_2$ classes for the greatest element of $I^{\mathrm{left}}_{\{0,1\}}(\lambda)$ and the least element of $I^{\mathrm{left}}_{\{0,1\}}(\lambda)$, hypothesize a set of $\equiv^{\mathrm{loc}}_2$ classes to be realized between them, and the determine the resulting set to be consistent (using the preceding theorem) or inconsistent (using first-order logic).

Theorem 1.5 *There are four inextensible $\equiv^{\mathrm{loc}}_2$ classes.*

Proof: $I^{\mathrm{left}}_{\{0\}}$ describes nothing (and is extensible) or describes either le or de, and nothing more. $I^{\mathrm{right}}_{\{0\}}$ likewise describes nothing (and is extensible) or describes ge or ae. The four formulas $ge^{>x} \wedge le^{<lx}, ge^{>x} \wedge de^{<lx}, ae^{>x} \wedge le^{<lx}, ae^{>x} \wedge de^{<lx}$ are $\not\equiv^{\mathrm{loc}}_2$ and exhaust the inextensible $\equiv^{\mathrm{loc}}_2$ classes. □

We abbreviate the $\equiv^{\mathrm{loc}}_2$ classes in U as $(ge, le), (ge, de), (ae, le), (ae, de)$ and as gl, gd, al, ad. We abbreviate the $\equiv^{\mathrm{loc}}_2$ class of the left element τ_0 as d or l (since there's no reason examining $\tau_0 = ad$ and $\tau_0 = gd$ independently) and we allow d to stand for d or $-$, the absence of any τ_0, since either one is consistent just in case the other is. We abbreviate the $\equiv^{\mathrm{loc}}_2$ class on the right as a or g and let a represent both a and $-$, likewise. We proceed to consider which triples are consistent:

Theorem 1.6 *The following lists all triples and which are consistent and inconsistent:*

1. *$(l, \emptyset, g)$ is consistent.*
2. *$(l, \emptyset, a)$, $(d, \emptyset, g)$, and $(d, \emptyset, a)$ are inconsistent.*
3. *$(d, \{ad\}, a)$ is consistent.*

4. *$(l, \{ad\}, g)$, $(l, \{ad\}, a)$, and $(d, \{ad\}, g)$ are inconsistent.*

5. *$(l, \{ad, gd\}, a)$ is consistent.*

6. *$(l, \{ad, gd\}, g)$, $(d, \{ad, gd\}, g)$, and $(d, \{ad, gd\}, a)$ are inconsistent.*

7. *$(d, \{ad, al\}, g)$ is consistent.*

8. *$(l, \{ad, al\}, g)$, $(l, \{ad, al\}, a)$, and $(d, \{ad, al\}, a)$ are consistent.*

9. *Any triple with $U = \{al, gd\}$, $U = \{al, gd, ad\}$, or U containing $\{gl\}$ is consistent.*

10. *Any triple with $U = \{al\}$ or $U = \{gd\}$ is inconsistent.*

Proof:

1. In the local linear consistency game, we only have to explain where l, the least element above the left end, is realized and where g, the greatest element below the right end, is realized. l is realized at τ_1 and g is realized at τ_0. Both of these satisfy condition 1a. in the theorem on the local linear consistency game.

2. If there is no element of λ between two cuts, those cuts are the same cut. We can't have different cuts separated by $\emptyset$. Consider, for instance, that the left end of the cut is $-$ or d, either because $I^{\text{left}}_{\{0,1\}}(\lambda)$'s last element is $de\ldots$, or because $I^{\text{left}}_{\{0,1\}}(\lambda) = le \in (\emptyset, \emptyset) < lf \in (le, \ldots) < le \in (lf, \ldots)$, and that last element has $\equiv_2^{\text{loc}}$ class (ge, de). These situations require an infinite descending sequence of elements, which is not supplied by $U = \emptyset$.

3. The label d describes a sequence which limits to τ_0; likewise, the label a describes a sequence which limits to τ_1. For the unique element of U, we split U into $(-, U, a), (d, U, -)$.

4. l describes the immediate successor to τ_0. If we are to realize both τ_0 and l in a linear order, then l must have an $\equiv_2^{\text{loc}}$ class consistent with its having an immediate predecessor.

5. For l, the immediate successor of τ_0, we choose $gd \in U$ and split U into the pair: $(l, \emptyset, gd), (gd, \{ad\}, a)$. Those are found to be consistent in previous items.

6. g describes the immediate predecessor to τ_1. So it must have an $\equiv_2^{\text{loc}}$ class consistent with having an immediate successor. To $gd \in U$ corresponds a cut, in which U is split into $(?, U_0, gd)$ and $(gd, U_1, ?)$. The g in the first of these triples requires an immediate predecessor, so either U_0 contains some $\equiv_2^{\text{loc}}$ class consistent with having an immediate successor (which doesn't exist in $U = \{gd, ad\}$) or $U_0 = \emptyset$. But $(d, \emptyset, gd)$ is inconsistent, by a previous item.

7. symmetric to item 5.

8. symmetric to item 6.

9. Let $U = \{al, gd\}$. In $(l, U, ?)$, we assign to l the pair $(l, \emptyset, gd), (gd, U, ?)$. Symmetrically, in $(?, U, g)$, we assign to g the pair $(?, U, al), (al, \emptyset, g)$. In $(-, U, -)$, to $gd \in U$ we assign the pair $(-, U, gd), (gd, U, -)$, and to $al \in U$ we assign the pair $(-, U, al), (al, U, -)$. Now the five triples: $\{(l, U, g), (l, U, a), (d, U, g), (d, U, a), (l, \emptyset, g)\}$ are all consistent, since we can pass from any information in any of the first four to a pair of them, satisfying consistency conditions, and the fifth is consistent. We can split $(?, \{al, gd, ad\}, ?)$ on ad into $(?, \{al, gd\}, ad), (ad, \{al, gd\}, ?)$, which we have just found to be consistent. If $gl \in U$, we have a simple procedure: answer any l in $(l, U, ?)$ with $(l, \emptyset, gl)(gl, U, ?)$; answer any g in $(?, U, g)$ with $(?, U, gl), (gl, \emptyset, g)$; split U always into $U_0 = U_1 = U$.

10. Realize $al \in \{al\}$ at x_0. x_0 requires a sequence of predecessors in with $\equiv_2^{\text{loc}}$ class in $\{al\}$. Realize one such at x_1. x_1 requires an immediate successor. Let x_2 be that successor. Now the $\equiv_2^{\text{loc}}$ class of (λ, x_2) recognizes that x_2 has an immediate predecessor, so it is not al. The case of $U = \{gd\}$ is symmetric. □

With that theorem it is easy to enumerate the $\equiv_3$ classes of linear orders:

- 1. The linear order 6, in which $I^{\text{left}}_{\{0,1\}}$ assigns its maximal element to $2 \in 6$ and $I^{\text{right}}_{\{0,1\}}$ assigns its minimal element to $3 \in 6$, and no element of 6 is realized between them: 1.
- 3. There are four ways $I^{\text{left}}_{\{0,1\}}$ could assign its maximal element to a cut, or assign its maximal element into λ, to an element with $\equiv_2^{\text{loc}}$ class (ge, de). Likewise, there are four ways $Th^{\text{right}}(\lambda)$ could require an ascending sequence (a), not a maximal element (g): +16.
- 5. Only one $I^{\text{left}}_{\{0,1\}}$ assigns its maximal element into λ, to an element of $\equiv_2^{\text{loc}}$ class (ge, le). There are four ways $I^{\text{right}}_{\{0,1\}}$ could assign its minimal element to a cut, or assign its minimal element into λ, to an element with $\equiv_2^{\text{loc}}$ class (ae, le): +4.
- 7. Symmetric to the previous item: +4.
- 9. These 10 sets U are consistent with all 5 possible $Th^{\text{left}}(\lambda)$ classes and all 5 possible $Th^{\text{right}}(\lambda)$ classes: $+10 \times 25$.
- There are five linear orders of size ≤ 4: +5.
- In the linear order 5, the greatest element of $I^{\text{left}}_{\{0,1\}}$ and the least element of $I^{\text{right}}_{\{0,1\}}$ are assigned to the same element $3 \in 5$: +1.

Definition 1.3 *An almost locally closed set is any nonempty $A \subseteq \lambda \cup \lambda^+$ such that for each $a \in A$ there is some $a_0 \in A$ such that $(\lambda, a) \equiv_{k-1} (\lambda, a_0)$ and there is a homomorphism h from the ordered set $I^{\text{loc}}_{\{i:i<k-1\}}(\lambda, a_0)$ into A sending $l\tau \in (b,c)$ to the least element between $h(b)$ and $h(c)$ of $\equiv^{\text{loc}}_{k-1}$ class τ, and sending $d\tau \in (b,c)$ to the greatest cut (e,f) in λ^+ such that f contains every element between $\sup b$ and $\inf c$ of $\equiv^{\text{loc}}_{k-1}$ class τ, and likewise for $g\tau \in (b,c)$ and $a\tau \in (b,c)$. For each label $d\tau \in (b,c)$ or $a\tau \in (b,c)$ of $I^{\text{loc}}_{\{i:i<k-1\}}(\lambda, a_0)$, A also contains an example: an element of type τ above $d\tau \in (b,c)$ (or an element of type τ below $a\tau \in (b,c)$) such that for any $g \in \lambda$ between the cut and the example, there is an $h \in A$ not between the example and the cut, such that $(\lambda, c, d, g) \equiv_{k-1} (\lambda, c, d, h)$. For each $\equiv^{\text{loc}}_{k-2}$ class τ which $Th^{\text{loc}}_{k-1}(\lambda, a_0)$ knows to exist between two labels, A contains an* example*: an element of type τ between h of those two labels.*

It was not profitable to form $\equiv^{\text{loc}}_2$-almost locally sets, because the $\equiv^{\text{loc}}_2$ classes are very independent. To enumerate $\equiv_4$, we will surely need to know $\equiv^{\text{loc}}_3$-almost locally sets.

2 Enumerating $\equiv_4$ classes of linear order

In this section we generate trees of labels that might be $I_{\{2,1,0\}}(\lambda)$ for some linear order λ and we will hypothesize $\equiv^{\text{loc}}_3$ classes that might complete $Th_4(\lambda)$. We'll use the local consistency game to find which of these are in fact consistent. The result is an effective algorithm for enumerating $\equiv_4$ classes of linear orders. We will write e for the unique $\equiv^{\text{loc}}_0$ class and f for the unique $\equiv^{\text{loc}}_1$ class. We first consider "short" assignments $I_{\{1,0\}}$, and then we consider assignments $I_{\{1,0\}} = I^{\text{left}}_{\{1,0\}} + I^{\text{right}}_{\{1,0\}}$ (by + we refer to the union of $I^{\text{left}}_{\{1,0\}}$ and $I^{\text{right}}_{\{1,0\}}$, with every element of $I^{\text{left}}_{\{1,0\}}$ preceding every element of $I^{\text{right}}_{\{1,0\}}$). There are many different $\equiv^{\text{loc}}_2$ refinements of $I^{\text{left}}_{\{1,0\}} + I^{\text{right}}_{\{1,0\}}$ because, unlike $\equiv^{\text{loc}}_0$ and $\equiv^{\text{loc}}_1$, $\equiv^{\text{loc}}_2$ is not a singleton. It has four inextensible elements and the intervals in which the $\equiv^{\text{loc}}_2$ classes exist can overlap in various ways. We refine each possible assignment $I_{\{2\}}$ with three singleton equivalence relations, in turn, and the results are all the possible assignments $I_{\{2,1,0\}}$. Then we assign an $\equiv^{\text{loc}}_3$ class to each element of $I_{\{2,1,0\}}$ and we assign sets of $\equiv^{\text{loc}}_3$ classes to each gap in $I_{\{2,1,0\}}$.

Definition 2.1 *This is our algorithm for enumerating $\equiv_4$ classes of linear orders:*

1. *Enumerate assignments $I_{\{0\}} = I_\emptyset$, $I_{\{1\}} = I_{\{0\}}$, and $I_{\{1,0\}} = I_{\{1\}}$ (where the last refinement adds no labels) and assignments $I_{\{0\}}$, $I_{\{1\}}$, and $I_{\{1,0\}}$ in which the last label in I^{left}_τ and some right label in I^{right}_τ are assigned to the same element.*[1] *Hereafter, assume that the first three refinements of $I_\emptyset$ are nontrivial and that $I_{\{1,0\}} = I^{\text{left}}_{\{1,0\}} + I^{\text{right}}_{\{1,0\}}$.*

[1] There are only five such assignments, and they are enumerated in the proof.

2. *Start with the* Stack $= \{I_{\emptyset}^{\text{left}}\} = \{\emptyset\}$.

3. *For each sequence A in the* Stack*, refine $(A, \emptyset)$ with respect to the singleton equivalence relation $\equiv_0^{\text{loc}}$ and append to A any labels defined in the refinement which are constant across an $\equiv_4^{\text{left}}$ class. Put the result in the* Stack.

4. *For each sequence A in the* Stack*, refine the left half of the final cut, $(A, \emptyset)$ with respect to the singleton equivalence relation $\equiv_1^{\text{loc}}$ and put the list of all possible refinements in the* Stack.[2]

5. *For each sequence A in the* Stack*, refine the left half of the final cut, $(A, \emptyset)$ with respect to the singleton equivalence relation $\equiv_0^{\text{loc}}$ and put the list of all possible refinements in the* Stack.[3]

6. *Remove each sequence A from the* Stack *and do the following:*

   (a) For each label m which could follow A, append m to A and put the result back onto the Stack.[4]

   (b) if A doesn't end $'de-'$ or $'df-'$, add A to the list of $I_{\{2\}}^{\text{left}}$ assignments.

7. *Group $I_{\{2\}}^{\text{left}}$ assignments into a common class if they agree on $I_{\{2\}}^{\text{left}} \setminus I_{\{1,0\}}^{\text{left}}$.*[5]

8. *Group $I_{\{2\}}^{\text{left}}$ assignments by the set of $\equiv_2^{\text{loc}}$ classes which are labeled.*

9. *Generate and group $I_{\{2\}}^{\text{right}}$ assignments, repeating steps* 2 *through* 8.

10. *For each $I_{\{2\}}^{\text{left}}$ and $I_{\{2\}}^{\text{right}}$ which label the same $\equiv_2^{\text{loc}}$ classes, we now determine all orderings of $I_{\{2\}}^{\text{left}} \cup I_{\{2\}}^{\text{right}} = I_{\{2\}}$ starting with the* Stack *containing one element – the assignment $I_{\{2\}}^{\text{left}} + I_{\{2\}}^{\text{right}}$. Remove each assignment I from the* Stack*, and do the following:*

    (a) Add I to the set of orderings of $I_{\{2\}}^{\text{left}} \cup I_{\{2\}}^{\text{right}}$.

    (b) For each ordered pair of labels $p < q$ in I, if p is "$d\ldots$" or "$l\ldots$" and q is "$a\ldots$" or "$g\ldots$" and p and q don't label the same $\equiv_2^{\text{loc}}$ type, switch them so that $q < p$ and push the result onto the Stack.

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[2] If the last label in A is $'de'$, the only left refinement of $(A, \emptyset)$ is A with a new label $'df'$ assigned to be equal to the last element of A. Otherwise, return the two assignments: A with the new greatest element $'lf'$, and A with the new greatest element $'df'$.

[3] If the last label in A is $'df'$, the only left refinement of $(A, \emptyset)$ is A with a new label $'de'$ assigned to be equal to the last element of A. Otherwise, return the two assignments: A with the new greatest element $'le'$, and A with the new greatest element $'de'$.

[4] The subroutine which finds the labels which could follow A is called the routine of *Left Legal Extensions* and is described at the end of this definition.

[5] This is similar to the classification of $\equiv_3^{\text{left}}$ classes into $\{de, le-df, le-lf-de, le-lf-le$ with the last label in $(ge, de)\}$ and $\{le-lf-le$ with the last label in $(ge, le)\}$ which was useful when enumerating $\equiv_3$ classes.

(c) *For each ordered pair of labels $p < q$ in I, if $p =$ "$l\alpha \in (I^{\text{left}}_{\{1,0\}}, \emptyset)$" and $q =$ "$g\alpha \in (\emptyset, I^{\text{right}}_{\{1,0\}})$", replace p and q by the single label "$o\alpha \in (I^{\text{left}}_{\{1,0\}}, I^{\text{right}}_{\{1,0\}})$"[6] and push the result onto the* Stack.

11. *Part* (*a*) *of the previous item has listed the set of possible $I_{\{2\}}$ assignments. For each assignment $I_{\{2\}}$, initialize a set* Available *of $\equiv^{\text{loc}}_2$ classes so that* Available $= \emptyset$. *Go through the ordered pairs of labels $p < q$ in $I_{\{2\}}$ in turn from least to greatest (from left to right) and:*

    (a) *if p is "$d \ldots$" or "$l \ldots$" and m labels $\equiv^{\text{loc}}_2$ class α, then add α to* Available.

    (b) *if q is "$a \ldots$" or "$g \ldots$" and m labels $\equiv^{\text{loc}}_2$ class α, then remove α from* Available.

    (c) *look up the triple* (p, Available, q) *in a* Branching Table, *which determines the possible refinements $I_{\{2,0\}}$ $I_{\{2,1\}}$ and $I_{\{2,1,0\}}$ in that cut of $I_{\{2\}}$ and, for each refinement, the possible $\equiv^{\text{loc}}_3$ class of each label and the set of $\equiv^{\text{loc}}_3$ classes realized in each cut.*

The Branching Table *constructs all possible assignments $I_{\{1,0\}}[p, q] = \{$labels $m \in I_{\{2,1,0\}}$ such that $p \leq m \leq q\}$ as the union $I^{\text{left}}_{\{1,0\}}[p, \emptyset) \cup I^{\text{right}}_{\{1,0\}}(\emptyset, q]$ so as to list the possible left and right assignments in the $\equiv^{\text{loc}}_0$ refinement of the $\equiv^{\text{loc}}_1$ refinement of the $\equiv^{\text{loc}}_0$ refinement of (p, q) separately. The $\equiv^{\text{loc}}_0$ and $\equiv^{\text{loc}}_1$ refinements label the least element above p just in case $p \neq$ 'a . . .'. In that case,*

1. *If (p, q) could be empty*[7] *then add that to the set of possible assignments $I_{\{1,0\}}[p, q]$.*

2. *If least elements above p and greatest elements below q are both labeled by $\equiv^{\text{loc}}_n$ refinements of $\{p, q\}$, for $n < 2$, consider that (p, q) may be exhausted before we refine it three times.*

3. *If the least elements above p are labeled, and the greatest elements below q are not, consider that (p, q) may be exhausted before we define three successors to p.*

4. *If the last elements below q are labeled, and the least elements above p are not, consider that (p, q) may be exhausted before we define three predecessors to q.*

5. *Hereafter, assume that $I_{\{1,0\}}[p, q] = I^{\text{left}}_{\{1,0\}}[p, \emptyset) + I^{\text{right}}_{\{1,0\}}(\emptyset, q]$, where $+$ refers to the ordering on the union in which the left ordered set precedes the right ordered set. We'll list possible assignments $I^{\text{left}}_{\{1,0\}}[p, \emptyset)$ first. If the least element(s) greater than p should be labeled in the $\equiv_n$ refinement of $\{p, q\}$ for $n < 2$, start with* Stack *the singleton containing only*

[6] This labels the "only" element of type α in the cut $(I^{\text{left}}_{\{1,0\}}, I^{\text{right}}_{\{1,0\}}) \in I^{+}_{\{1,0\}}$.

[7] That is, p could be the predecessor of q, and q could be the successor of p.

$I^{\text{left}}_{\emptyset}[p, \emptyset) = \{p\}$. *If the least element(s) greater than p should remain unlabeled in a low-rank refinement of* $\{p, q\}$*, let* Stack *the singleton containing only the word* unlabeled *and skip steps 6 through 9.* [8]

6. *For each sequence A in the* Stack*, refine the left half of the final cut,* $(A, \emptyset)$ *with respect to the singleton equivalence relation* $\equiv^{\text{loc}}_0$ *and put the list of all possible refinements in the* Stack.[9]

7. *For each sequence A in the* Stack*, refine the left half of the final cut,* $(A, \emptyset)$ *with respect to the singleton equivalence relation* $\equiv^{\text{loc}}_1$ *and put the list of all possible refinements in the* Stack.[10]

8. *For each sequence A in the* Stack*, refine the left half of the final cut,* $(A, \emptyset)$ *with respect to the singleton equivalence relation* $\equiv^{\text{loc}}_0$ *and put the list of all possible refinements in the* Stack.[11]

9. *For each sequence A in the* Stack*, choose the possible* $\equiv^{\text{loc}}_3$ *classes of the last element of A.*[12]

Symmetrically, construct possible assignments of labels $I^{\text{right}}_{\{1,0\}}(\emptyset, q]$ *and pick* $\equiv^{\text{loc}}_3$ *classes for each label. Then, for each assignment* $I^{\text{left}}_{\{1,0\}}[p, \emptyset)$ *and* $\equiv^{\text{loc}}_3$ *classes for those labels, for each assignment* $I^{\text{right}}_{\{1,0\}}(\emptyset, q]$ *and* $\equiv^{\text{loc}}_3$ *classes for those labels,*

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[8]We will pair this *Stack* to the list of possibilities for $I^{\text{right}}_{\{1,0\}}(\emptyset, q]$ and sets U of $\equiv^{\text{loc}}_3$ types and determine which sets U are consistent with which left and right ends.

[9]The possible refinements of the left half of $(A, \emptyset)$ are:

(a) If the last label in A is $'l\ldots'$ or $'g\ldots'$ or $'o\ldots'$ and labels the $\equiv^{\text{loc}}_2$ type (ge, de) or (ae, de) and *Available* is not $\emptyset$, then the only refinement of the left end of $(A, \emptyset)$ with respect to a singleton equivalence relation is the assignment which assigns the label $'de \in (A, \emptyset)'$ to immediately follow the last element of A (by the natural ordering on elements and cuts, $p < (\{x \in \lambda : x \leq p\}, \{x \in \lambda : x > p\})$, and no cut or element is between that element and that cut), and that assignment is a consistent refinement of the left end of $(A, \emptyset)$.

(b) If the last label in A is $'le'$ or $'lf'$ and *Available* contains (ge, de), then the label $'de'$ can follow A.

(c) If the last label in A is $'l\ldots'$ or $'g\ldots'$ or $'o\ldots'$ and labels the $\equiv^{\text{loc}}_2$ type (ge, le) or (ae, le), then the label $'le'$ can follow A.

(d) If the last label in A is $'le'$ and *Available* contains (ge, le), then the label $'le'$ can follow A.

[10]This is the same as the previous item, but with $\equiv^{\text{loc}}_1$ class f replacing $\equiv^{\text{loc}}_0$ class e.

[11]This is the same as footnote 10.

[12]If the last label of A is $'le\ldots'$ and if *Available* contains (ge, le) then assign the $\equiv^{\text{loc}}_3$ class (l, r) where l is determined by A (that is, l gives the names $ge - gf - ge$ to the elements $p - le - lf$ of A, and l assigns the $\equiv^{\text{loc}}_2$ class to ge or p which is the $\equiv^{\text{loc}}_2$ class which p labels) and r is the $\equiv^{\text{left}}_3$ class with $I^{\text{left}}_{\{1,0\}} = le - lf - le$ and the last element in $\equiv^{\text{loc}}_2$ class (ge, le). If the last label of A is $'le\ldots'$ and if *Available* contains (ge, de) then assign the $\equiv^{\text{loc}}_3$ class (l, r) where l is determined by A, and r is any of the four $\equiv^{\text{loc}}_3$ classes with $I^{\text{left}}_{\{1,0\}} = de, le - df, le - lf - de$, or $le - lf - le$ with the last element in $\equiv^{\text{loc}}_2$ class (ge, de).

construct all possible sets U of $\equiv_3^{\text{loc}}$ *types to be realized between* $(I^{\text{left}}_{\{1,0\}}[p,\emptyset)$ *and* $I^{\text{right}}_{\{1,0\}}(\emptyset,q])$. [13] *The* $\equiv_3^{\text{loc}}$ *types are always realized in groups, which describe*

1. *A sequence of n elements with a sequence of elements descending from above to the greatest of the n elements and a sequence of elements ascending from below to the least of the n elements, for* $1 \leq n \leq 7$.

2. *The* $\equiv_3^{\text{loc}}$ *class* (l,r) *such that l describes 3 elements* $ge - gf - ge$, *the least of which is in* $\equiv_2^{\text{loc}}$ *class* (ae,le) *indicating that it is the limit of a sequence of elements ascending from below, and r describes 3 elements* $le - lf - le$, *the greatest of which is in* $\equiv_2^{\text{loc}}$ *class* (le,ge) *indicating that it has an immediate successor.*

3. *The* $\equiv_3^{\text{loc}}$ *class* (l,r) *such that r describes 3 elements of which the greatest is the limit of a sequence of elements descending from above and l describes 3 elements of which the least has an immediate predecessor.*

4. *The* $\equiv_3^{\text{loc}}$ *class* (l,r) *such that r describes 3 elements, the third of which has an immediate successor and l describes 3 elements, the third of which has an immediate predecessor.*

We name these sets of $\equiv_3^{\text{loc}}$ *classes:*

1. $1 = \{(ae = af = ae, de = df = de)\}$ *contains the isolated element by itself.*

2. $2 = \{(ae = af - ge, de = df = de), (ae = af = ae, le - df = de)\}$ *contains a pair of 2 elements, one the immediate successor of the other, with sequences limiting to them from above and below.*

3. $N = \{3,4,5,6,7\}$, *where* $3 = \{(ae - gf - ge, de = df = de), (ae = af - ge, le - df = de), (ae = af = ae, le - lf - de)\}$ *and in general, n has n elements* $(2 < n < 8)$, *describes the finite set of between three and seven element elements, with sequences limiting to them from above and below.*

4. *The last three sets of a single* $\equiv_3^{\text{loc}}$ *class each, as enumerated above, we name:* W, X, *and* Z.

The subroutine Con(U) declares the triple $(I^{\text{left}}_{\{1,0\}}[p,\emptyset), U, I^{\text{right}}_{\{1,0\}}(\emptyset,q])$ *to be consistent just in case the following tests are met:* [14]

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[13] That is done by the subroutine *Con*(U).

[14] Footnote 7 carefully describes the case in which $(p,q) = \emptyset$ and the $\equiv_0^{\text{loc}}$ refinement of $[p,q]$ describes q as the immediate successor of p and describes p as the immediate predecessor of q. This leaves the case in which the $\equiv_0^{\text{loc}}$ refinement of $[p,q]$ doesn't describe the least element(s) above p or doesn't describe the greatest element(s) below q, so that the current subroutine will describe, for instance, $[p-$unknown left $-U = \emptyset-$unknown right$-q]$. Step 2 of the branching table describes $[p - le - df = de = q = (U = \emptyset)$ =unknown right= $q]$, since (p,q) is exhausted before three refinements can be made, with $I^{\text{left}}_{\{1,0\}}[p,\emptyset)$ preceding $I^{\text{right}}_{\{1,0\}}(\emptyset,q]$. However, $le - df = de$ is a complete $I^{\text{left}}_{\{1,0\}}[p,\emptyset)$ assignment, so we will consider that triple again in this subroutine. Rather than make arbitrary distinctions, when writing

1. *Each element of U implies the existence of a number of $\equiv_2^{\text{loc}}$ types. Check that all of those are in* Available.

2. *If $I_{\{1,0\}}[p,q]$ contains the label $'de'$ or $'ae'$, then $U \neq \emptyset$.*

3. *If p is $'dx'_0 = \ 'dx'_1 \ldots$ or q is $'ax'_0 = \ 'ax'_1 \ldots$, check that U contains an extension of each x_i into $\equiv_3^{\text{loc}}$.*

4. *If U contains W then it must contain X or Z or $I^{\text{right}}_{\{1,0\}}(\emptyset,q]$ has three predecessors to q, the least of which requires a predecessor.*

5. *If U contains X then it must contain W or Z or $I^{\text{left}}_{\{1,0\}}[p,\emptyset)$ has three successors to p, the last of which requires a successor.*

6. *If $I^{\text{left}}_{\{1,0\}}[p,\emptyset)$ has three successors to p, the last of which requires a successor, then U contains X or Z or U is empty and $I^{\text{right}}_{\{1,0\}}(\emptyset,q]$ has three predecessors to q, the least of which requires a predecessor.*

7. *If $I^{\text{right}}_{\{1,0\}}(\emptyset,q]$ has three predecessors to q, the least of which requires a predecessor, then U contains W or Z or U is empty and $I^{\text{left}}_{\{1,0\}}[p,\emptyset)$ has three successors to p, the last of which requires a successor.*

8. *If $I^{\text{left}}_{\{1,0\}}[p,\emptyset)$ requires a sequence limiting to p or to $p-le$ or to $p-le-lf$ or to $p-le-lf-le$, then U contains 1 or 2 or Z or an element of N or both W and X.*

9. *If $I^{\text{right}}_{\{1,0\}}(\emptyset,q]$ requires a sequence limiting to q or to $ge-q$ or to $gf-ge-q$ or to $ge-gf-ge-q$, then U contains 1 or 2 or Z or an element of N or both W and X.*

10. *If $W \in U$ then U contains 1 or 2 or Z or X or an element of N.*

11. *If $X \in U$ then U contains 1 or 2 or Z or W or an element of N.*

The subroutine Left Legal Extensions *accepts an assignment A and determines which labels might follow A as parts of an $\equiv_2^{\text{loc}}$ refinement of $I_{\{1,0\}}$.*

1. *initialize a set* Available

2. *If A labels the $\equiv_2^{\text{loc}}$ class (ae,le) or (ge,le), add the term* predecessor *to* Available.

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code we double-check the counting of $U = \emptyset$ by listing all short possibilities (for $I_{\{1,0\}}[p,q]$, $\equiv_3^{\text{loc}}$ classes of those labels, and the set $U = \emptyset$ of $\equiv_3^{\text{loc}}$ classes in the center) and then listing those which are accepted by Con(U), and then identifying and removing redundancy.

This is not to blur the theoretical distinction between assignments $I_{\{1,0\}}[p,q]$ which assign the least and greatest (sequences of, or single) elements of $\equiv_0^{\text{loc}}$ type e or declare that element to be unknown, and those which know the least and greatest elements but find the interval to be empty – either the $\equiv_0^{\text{loc}}$ refinement of $I_\emptyset(\{p,q\}) = \{p,q\}$, $I_{\{0\}}[p,q]$, or the next refinement, $I_{\{1\}}[p,q]$, or the next, $I_{\{1,0\}}[p,q]$, assign the least and greatest element to be equal, and be the unique element, or assign no labels at all.

3. *If A labels the* $\equiv_2^{\text{loc}}$ *class* (ge, de) *or* (ge, le)*, add the term* successor *to* Available*.*

4. *If A labels the* $\equiv_2^{\text{loc}}$ *class* (ge, le) *or* (ae, de) *or both* (ae, le) *and* (ge, de)*, add the term* repeatable *to* Available*.*

5. *if the last label in A is* $'le'$ *or* $'lx'$ *for* x *one of the* $\equiv_2^{\text{loc}}$ *classes* (ae, le) *or* (ge, le)*,*

    (a) *if* Available *contains the term* successor*,*

        i. *if* Available *contains the term* predecessor*, then* $'lx'$ *can follow A, for* x *either* (ge, de) *or* (ge, le)*.*

        ii. $'lx'$ *can follow A, for* x *either* (ae, le) *or* (ae, de)*.*

        iii. *A can be followed by* $dx_0 = dx_1 = \ldots$*, for* $\{x_i : i \leq n\}$ *any consistent descending sequence.*[15]

    (b) *if* Available *doesn't contain the term* successor*, then only* $'lx'$ *can follow A, for* x *either* (ge, de) *or* (ge, le)*.*

6. *if the last label in A is* $'le'$ *or* $'lx'$ *(for* x *one of the* $\equiv_2^{\text{loc}}$ *classes* (ae, de) *or* (ge, de)*) or* $'dx'$ *for any* $\equiv_2^{\text{loc}}$ *class* x*,*

    (a) *if* Available *contains the term* repeatable*,*

        i. *if* Available *contains the term* predecessor*, then* $'lx'$ *can follow A, for* x *either* (ge, de) *or* (ge, le)*.*

        ii. $'lx'$ *can follow A, for* x *either* (ae, le) *or* (ae, de)*.*

    (b) *A can be followed by* $dx_0 = dx_1 = \ldots$*, for* $\{x_i : i \leq n\}$ *any consistent descending sequence.*[16]

Theorem 2.1 *The program in the preceding definition enumerates* $\equiv_4$ *classes of linear orders.*

Proof:

- Steps 1 through 5 list all possible assignments $I_{\{1,0\}}$. Step 1. lists those assignments $I_{\{1,0\}}$ for which it does not hold that $I^{\text{left}}_{\{1,0\}} < I^{\text{right}}_{\{1,0\}}$. Those are $I_{\{0\}} = \emptyset, I_{\{0\}} = oe, I_{\{1\}} = le - \emptyset - ge, I_{\{1\}} = le - of - ge, I_{\{1,0\}} = le - lf - \emptyset - gf - ge, I_{\{1,0\}} = le - lf - oe - gf - ge$. This same list was constructed as an initial step in enumerating $\equiv_3$ theories of linear order. Steps 2 through 5 define $I^{\text{left}}_{\{1,0\}}$: Step 2 initializes $I^{\text{left}}_{\emptyset} = \emptyset$. Steps 3 through 5 add $'de'$ or $'le'$, $'df'$ or $'lf'$, and $'de'$ or $'le'$. When enumerating

[15] For each nonempty set U of $\equiv_2^{\text{loc}}$ types, if

A. $((ae, le) \in U \rightarrow ((ge, le) \in U$ or $(ge, de) \in U$ or *Available* contains the term *predecessor*)) and

B. $((ge, ae) \in U \rightarrow ((ge, le) \in U$ or $(ae, le) \in U$ or *Available* contains the term *successor*)),

then the labels $\{'dx' : x \in U\}$ can all be equal, and be the next distinct element of A.

[16] See the previous footnote.

$\equiv_3$ theories of linear order, we found that there are four possibilities for $I^{\text{left}}_{\{1,0\}}$. But at this point our program ceases to imitate the enumeration of $\equiv_3$ classes of linear orders.

- Step 6 defines the $\equiv^{\text{left}}_4$ equivalence class of the $\equiv^{\text{loc}}_2$ refinement of $I^{\text{left}}_{\{1,0\}}$ by adding a label to A, then putting the result back on the stack. Step 6 calls a subroutine to determine which labels can be next.

We claim that if A is ever on the stack (each label of an $\equiv^{\text{loc}}_2$ class mentioned in A is a *legal extension* of the preceding sequence) and the last label is not $'de'$, then then A is the assignment $I^{\text{left}}_{\{2\}}(\lambda)$ for some linear order λ. Conversely, we claim that if A is ever on the stack and its last label is $'de'$, then A is not the assignment $I^{\text{left}}_{\{2\}}(\lambda)$ for some linear order λ, and if m is a label which is not a *legal extension* of A, then appending m to A produces an inconsistent sequence.

First, suppose the last label of A is $'de'$. That label corresponds to a sequence of elements limiting to the left, each of which has some $\equiv^{\text{loc}}_2$ classes $x_0, x_1 \dots$, so if A were $I_{\{2\}}(\lambda)$ for some λ, A should label the least elements of type $x_0, x_1, \dots$, so A should end $'de' = \ 'dx'_0 = \ 'dx'_1 \dots$. To be precise, suppose $B = I^{\text{right}}_{\{2\}}(\lambda)$ so that $A < B$ is $I_{\{2\}}(\lambda)$. Now A and B should label the same $\equiv^{\text{loc}}_2$ classes. Since the last label in A is $'de'$, A labels no $\equiv^{\text{loc}}_2$ classes, so in the initial state of the local linear consistency game is $(de, \emptyset, B)$. As we argued in the enumeration of $\equiv_3$ classes, this is inconsistent, since $de < B$ and de requires a sequence of elements to descend from above, a terminal segment of which is $< B$, hence in the gap.

The other $A = I^{\text{left}}_{\{1,0\}}$ which can be on the stack when we first read step 6 is $le - lf - le$. This is consistent. Let $B = ge - gf - ge$, $A < B$, $U = \emptyset$; the triple $(le, \emptyset, ge)$ is consistent, since le can refer to the right end and ge can refer to the left end.

The subroutine *Legal Extension* announces in steps 5 and 6 which labels can follow A. For instance, $le - le - le - l(ge, de) - l(ae, de)$ is inconsistent, since the fourth label requires something to limit to it from the right, and nothing is available yet to make up that limiting set. On the other hand, $le - le - le - l(ge, le) - l(ae, de)$ is consistent, since the fourth label requires a successor, which can have $\equiv^{\text{loc}}_2$ type (ge, le) and the fifth label requires something to limit to it from the left, which can be an infinite sequence of elements in $\equiv^{\text{loc}}_2$ class (ge, le). If A labels no $\equiv^{\text{loc}}_2$ classes, and m is a label, then m can follow A just in case the triple $(A, \emptyset, m)$ is consistent. [17] Now suppose A contains already some labels in I^{left}_2. In the next triple (A, U, m), U will contain only $\equiv^{\text{loc}}_2$ types labeled in A. Steps 1 through 4 describe possibilities for U. Step 2: A *predecessor* is an element of U that player II can play, in response to player I's play at g in $(?, U, g)$. Step 3: A *successor* is an element of β that player II can play in response to l in $(l, U, ?)$. Step 4: A *repeatable* subset of U is a set U_0 of $\equiv^{\text{loc}}_2$ classes such that for each $\tau \in U_0$, player II can answer τ in $(?, U, \tau)$ or in $(\tau, U, ?)$ with another element

[17] For that reason, $'le'$ can be a final element of A and $'de'$ cannot be a final element of A –$\emptyset$ cannot follow $'de'$.

of U_0 and such that (d, U_0, a) is consistent – i.e., U_0 can form infinite sequences which ascend to a and descend to d. A repeatable element satisfies every need it creates during the local linear consistency game. Step 4 claims that $\{(ge, le)\}$, $\{(ae, de)\}$, and $\{(ae, le), (ge, de)\}$ have this property and that any repeatable set contains one of those. That these sets are repeatable is clear. Conversely, the only sets which contain none of $\{(ge, le)\}$ or $\{(ae, de)\}$ or $\{(ae, le), (ge, de)\}$ are: $\emptyset$, $\{(ae, le)\}$, and $\{(ge, de)\}$. Clearly, $\emptyset$ doesn't contain anything that can be repeated infinitely and descend to d or ascend to a. $\{(ae, le)\}$ requires an immediate successor, and doesn't contain one; $\{(ge, de)\}$ requires an immediate predecessor and does not contain one. Steps 5 and 6 of the subroutine *legal extension* match situations, in which player I can force a certain $\equiv_2^{\text{left}}$ or $\equiv_2^{\text{right}}$ class, to requirements on U having a successor or a predecessor or repeatable elements.

Suppose A is on the stack in Step 6 and the last label of A is not $'de'$ nor $'le'$. Then a nonempty set $x_0, x_1, \dots$ of $\equiv_2^{\text{loc}}$ types are labeled in A. Until the first $\equiv_2^{\text{loc}}$ type, it has sufficed to check whether two labels will label the same gap, and to check triples $(l, \emptyset, g)$. Only when we introduce the second, third, and final $\equiv_2^{\text{loc}}$ classes do we consider nonempty triples. We create a simple $Th^{\text{right}}(\lambda)$ to pair with A, in order to determine the consistency of A followed by a given next label. We choose: $ax_0 = ax_1 = \cdots = ae$ for the right end. If the last element of A is the $\equiv^2_{\text{loc}}$ class (x, y), e.g., $(x, y) = (ae, de)$, then $(y, \{x_0, x_1, \dots\}, ae)$ is the triple between A and $(\lambda, \emptyset)$, the common image of all labels in the right end. A is only consistent if this triple is consistent.

- Step 7 is justified because the initial labels of $I^{\text{left}}_{\{2\}}$ are irrelevant in every further situation where we use $I^{\text{left}}_{\{2\}}$.
- Step 8: We must only pair assignments $I^{\text{left}}_{\{2\}}$ and $I^{\text{right}}_{\{2\}}$ which label the same elements. For, in the definition of the $\equiv_2^{\text{loc}}$-refinement of $I_{\{1,0\}}$, we label every $\equiv_2^{\text{loc}}$ type which is realized in λ because 2 is greater than the indices of $I_{\{1,0\}}$.
- Step 10: Let's check that this process constructs all orderings of the union of the left assignment and the right assignment. For instance, given 12345 and abc, we'd initialize the stack with $12345abc$, then find the pair that can be switched, yielding the order $1234a5bc$, then find further pairs that can be switched, yielding the orders $123a45bc$ and $1234ab5c$, and again, yielding the orders $12a345bc$ and $123a4b5c$ (twice) and $1234abc5$, and so on. We claim that every ordering of two sets $A \cup B$ respecting an initial order on A and an initial order on B is found this way: if we want the least element of B to precede n_0-many elements of A, we switch it n_0-many times with the greatest elements of A; then, if we want the next-least element of B to precede $n_1 < n_0$-many elements of A, we switch it n_1-many times with the greatest elements of A; and so on. This routine is inefficient for finding all the orderings of $A \cup B$. But it is efficient for our purposes for two reasons: 1. each switch is illegal just in case it considers

switching an element of A and an element of B which label the least and greatest elements of the same $\equiv_n^{\mathrm{loc}}$ class (the least must precede the greatest), and 2. if we consider elements of A and B which label the least and greatest elements of an $\equiv_n^{\mathrm{loc}}$ class, we can enumerate the possibility that those labels are equal, labeling the unique element in that class.

- Step 11: We claim that this pass correctly finds the set of $\equiv_2^{\mathrm{loc}}$ classes α which can be realized in each cut $(b, c) \in I^+_{\{2\}}$, i.e., the set of α such that the least element(s) in α is/are labeled in b and the greatest element(s) in α is/are labeled in c. For each α, we put α in the set *available* when we see the least element(s) in that class, and we take α out of the set *available* when we see the last element(s) in that class.

Now we claim that the subroutine *branching table* correctly describes the $\equiv_4$-different possibilities for the interval $[p, q]$, where $(p, q) \in I^+_{\{2\}}$.

- Step 1: The interval $[p, q]$ could be just the pair $\{p, q\}$, if p is the least or greatest (or unique) element of type (ae, le) or (ge, le) and q is the least or greatest element of type (ge, le) or (ge, de). That happens, as in step 1, just in case p can be a predecessor and q can be a successor. In any other case, the interval must be nonempty – $p = \ 'd \ldots'$ or $p = \ 'lx'$ for $x = (ae, de)$ or $x = (ge, de)$, so that the triple for $[p, q]$ is $(-, U, ?)$.

- For $n < 2$, steps 2 through 4 consider whether an $\equiv_n^{\mathrm{loc}}$-refinement of $I_{\{2\}}$ labels the first element(s) in $\equiv_n^{\mathrm{loc}}$ class α above $p \in I_{\{2\}}$ (and below q). That occurs just in case: $p = \ 'd \ldots'$ or $p = \ 'l \ldots'$ or $p = \ 'g \ldots'$, since the condition $n > 2$ fails, and the final condition – that $p = \ 'a \ldots'$ and there exists a boundary $b < p$ such that α is not realized in (b, p) – fails because $\equiv_n^{\mathrm{loc}}$ is a singleton equivalence class, hence every element of (b, p) realizes α; further, (b, p) is not empty because $p = \ 'a \ldots'$.

- In step 2, if (p, q) is such that both the least elements and the greatest elements there are labeled in a $\equiv_0^{\mathrm{loc}}$ refinement, the interval (p, q) could have < 6 elements so that three refinements exhaust the interval and label some element twice just in case p is $'g \ldots'$ or $'l \ldots'$ or $'o \ldots'$ and labels $\equiv_2^{\mathrm{loc}}$ type (ae, le) or (ge, le), if q is $'g \ldots'$ or $'l \ldots'$ or $'o \ldots'$ (where o means that the least and greatest element are assigned to the same element, a unique element of the describe type) and labels $\equiv_2^{\mathrm{loc}}$ type (ge, de) or (ge, le), and if z is in *Available*. In this case, we add all possible $\equiv_0^{\mathrm{loc}}$ refinements in the cut (p, q) for which some labels are assigned to the same element, $\equiv_1^{\mathrm{loc}}$ refinements of $\equiv_0^{\mathrm{loc}}$ refinements in the cut (p, q) for which some labels are assigned to the same element, and $\equiv_0^{\mathrm{loc}}$ refinements or $\equiv_1^{\mathrm{loc}}$ refinements of $\equiv_0^{\mathrm{loc}}$ refinements in the cut (p, q) for which some labels are assigned to the same element: the least of this is the possibility that $'le > p'$ is assigned to q and $'ge < q'$ is assigned to p or that $'le > p'$ and $'ge < q'$ are assigned to the same element; next-least is: $'oe \in (p, q)'$; largest is possibility that the first two refinements occur without overlap, and then the final refinement assigns the least and greatest element to be equal. The full list is: $p - oe - q, p - le - ge - q, p - le - of - ge - q, p - le - lf - gf - ge - q, p - le - lf - oe - gf - ge - q$. The next-larger assignment, $p - le - lf - le - ge - gf - ge - q$, is a case in which the third element after p has 3 successors, the greatest of which has $\equiv_2^{\mathrm{loc}}$ type (ge, le) and in which the third element before q has 3 successors, the least of which has $\equiv_2^{\mathrm{loc}}$ type (ge, le), and in which the cut $(I_{\{1,0\}}^{\mathrm{left}}[p, \emptyset), I_{\{1,0\}}^{\mathrm{right}}(\emptyset, q])$ happens to contain no $\equiv_3^{\mathrm{loc}}$ classes at all.

- In step 3, if (p, q) is such that the least elements are labeled and the greatest elements are not labeled, then add all $\equiv_0^{\mathrm{loc}}$ refinements on the left of $\equiv_1^{\mathrm{loc}}$ refinements on the left of $\equiv_0^{\mathrm{loc}}$ refinements on the left in (p, q) for which $I_{\{1,0\}}^{\mathrm{left}}[p, \emptyset)$ defines every element between p and q, leaving no $\equiv_3^{\mathrm{loc}}$ elements in the gap $(I_{\{1,0\}}^{\mathrm{left}}[p, \emptyset), q)$, without defining all three refinements and assigning $\equiv_3^{\mathrm{loc}}$ types to them. This is possible just in case p is $'g\ldots'$ or $'l\ldots'$ or $'o\ldots'$ and labels $\equiv_2^{\mathrm{loc}}$ type (ae, le) or (ge, le). Then, if x is in *Available*, add $p - le - \emptyset - q$ to the list of possible $I_{\{1,0\}}^{\mathrm{left}}[p, q]$. The list has one or two elements: if $(ge, de) \in \mathit{Available}$, then $p - le - q$ could exhaust $[p, q]$; if if $(ge, de), (ge, le) \in \mathit{Available}$, then $p - le - le - q$ could exhaust $[p, q]$.
- Step 4 is symmetric.
- We have not considered all possibilities in which p leaves the least element unlabeled or q leaves the greatest element unlabeled– we have only considered the possibilities in which that happens and in which one side can be labeled but its labels don't get refined three times without exhausting the interval (p, q). We must still consider the $\not\equiv_4$ possibilities in which three refinements of the left side of (p, q) and and their $\equiv_3^{\mathrm{loc}}$ class exhaust the gap, while the right side is not labeled, and we also consider the possibility that both left and right ends are unlabeled. We achieve this by making *Stack* the singleton containing the word *unlabeled*, if that side is unlabeled, in Step 5.
- Steps 6,7,8 develop the possible labels on the left end of (p, q) if that end of the interval should be labeled. This completes $I_{\{2,1,0\}}$
- If we find $\equiv_3^{\mathrm{loc}}$ types for the elements discovered in step 8, then we will know $\equiv_3^{\mathrm{loc}}$ types for all elements of $I_{\{2,1,0\}}$.

Let's prove the grouping of $\equiv_3^{\mathrm{loc}}$ classes described in the definition.

Let's introduce a shorthand notation for inextensible $\equiv_3^{\mathrm{left}}$ classes: $'de', 'le - df', 'le - lf - de', 'le - lf - le'$ with the third element in $\equiv_2^{\mathrm{loc}}$ class (ge, de), and $'le - lf - le'$ with the third element in $\equiv_2^{\mathrm{loc}}$ class (ge, le) imply the existence of exactly $0, 1, 2$, or 3 least elements, or 4 or more least elements. So we'll refer to these $\equiv_3^{\mathrm{left}}$ classes as $0, 1, 2, 3, 4$. From this we obtain shorthand notations for $\equiv_3^{\mathrm{loc}}$ class: (n, m) where $n, m < 5$.

Whenever the $\equiv_3^{\text{left}}$ classes $(2,4)$ is realized, it has 4 or more immediate successors; the first of these is in $\equiv_3^{\text{left}}$ class $(3,4)$ or $(3,3)$. In this way, many types imply each other:

1. $(0,0)$ implies nothing.
2. $(0,1) \Leftrightarrow (1,0)$
3. $(0,2) \Leftrightarrow (1,1) \Leftrightarrow (2,0)$
4. $(0,3) \Leftrightarrow (1,2) \Leftrightarrow (2,1) \Leftrightarrow (3,0)$
5. $(0,4) \Leftarrow (1,3) \Leftrightarrow (2,2) \Leftrightarrow (3,1) \Rightarrow (4,0)$
6. $(0,4) \Leftarrow (1,4) \Leftarrow (2,3) \Leftrightarrow (3,2) \Rightarrow (4,1) \Rightarrow (4,0)$
7. $(0,4) \Leftarrow (1,4) \Leftarrow (2,4) \Leftarrow (3,3) \Rightarrow (4,2) \Rightarrow (4,1) \Rightarrow (4,0)$
8. $(3,4) \Rightarrow (4,3) \vee (4,4)$
9. $(4,3) \Rightarrow (3,4) \vee (4,4)$
10. $(4,4)$ implies nothing.

The lines of this enumeration are not an equivalence relation on $\equiv_3^{\text{loc}}$ classes, but they have the following properties: they are a set of sets of $\equiv_3^{\text{loc}}$ classes such that any $\equiv_3^{\text{loc}}$ class is in one of the given sets, the sets are closed under the implication between $\equiv_3^{\text{loc}}$ classes (l,r) and (p,q) which holds in case $\exists x(l^{<x} \wedge r^{>x}) \rightarrow \exists x(p^{<x} \wedge q^{>x})$, any any two sets A and B are distinguished by some $\equiv_3^{\text{loc}}$ class which is in one and not in the other.

If the $\equiv_3^{\text{loc}}$ class (n,m) for $n < m < 3$ exists, its existence is implied by the existence of its neighbor $(n+1, m-1)$. The $\equiv_3^{\text{loc}}$ classes are thereby grouped into between 1 and 7 elements in a short chain of immediate predecessors and successors, and the final three classes.

When the program asserts that N exists, that should be interpreted as saying that one of the groups of $2,3,4,5,6$ or 7 chains of $\equiv_3^{\text{loc}}$ classes in the list above exists.

- In Step 1, if a $\equiv_3^{\text{loc}}$ type γ exists in the gap $(b,c) \in I_{\{2,1,0\}}$ and implies the existence of an $\equiv_2^{\text{loc}}$ type γ' then (b,c) is a subinterval of $(e,f) \in I_{\{1,0\}}$. So the type γ' must exist in the interval (e,f) and the least element(s) and greatest element(s) of type γ' should be labeled in the $\equiv_2^{\text{loc}}$-refinement of $I_{\{1,0\}}$. Two $\equiv_3^{\text{loc}}$ classes in the same group imply the same sets of $\equiv_2^{\text{loc}}$ classes:
  - $\equiv_3^{\text{loc}}$ classes in 1 imply $\{(ae, de)\}$,
  - $\equiv_3^{\text{loc}}$ classes in 2 imply $\{(ae, le), (ge, de)\}$.
  - $\equiv_3^{\text{loc}}$ classes in N imply $\{(ae, le), (ge, le), (ge, de)\}$.
  - $\equiv_3^{\text{loc}}$ classes in W imply $\{(ae, le), (ge, le)\}$.

  - $\equiv_3^{\text{loc}}$ classes in X imply $\{(ge, le), (ge, de)\}$.
  - $\equiv_3^{\text{loc}}$ classes in Z imply $\{(ge, le)\}$.

- Step 2 is necessary, since if $U = \emptyset$, then in the labeled linear consistency game beginning with no labels ($A = \emptyset$) and $\alpha((\emptyset, \emptyset)) \equiv_3^{\text{left}}$ the $\equiv_3^{\text{left}}$ class that the the main program picked (in step 9.) for the last element of $I^{\text{left}}_{\{1,0\}}[p, \emptyset)$ and the $\alpha((\emptyset, \emptyset)) \equiv_3^{\text{right}}$ the $\equiv_3^{\text{right}}$ class that the the main program picked (in the paragraph after step 9.) for the least element of $I^{\text{left}}_{\{1,0\}}(\emptyset, p]$.

- In the labeled linear consistency game beginning with no labels ($A = \emptyset$) and $\alpha((\emptyset, \emptyset))$ an $\equiv_3$ class which implies the existence of $x_0, x_1, \ldots$, player II has lost unless there is an extension of each of those classes in $\beta((\emptyset, \emptyset)) = U$. So Step 3 is necessary.

- The remaining steps, 6 through 11, describe how W and Z and one $\equiv_3^{\text{left}}$ class require a successor, which require a predecessor, which requires a sequence limiting from below or above, and under what conditions these requirements are satisfied. The linear consistency game makes that sort of reasoning precise.

This notion of "grouping" can be extended to almost locally closed sets. For instance, each of the $\equiv_2^{\text{loc}}$ classes $\{ad, al, gd, gl\}$ requires either an ascending sequence of elements or a greatest predecessor and requires either a descending sequence of elements or a least successor. We can track when those requirements are satisfied. But we could consider, instead, a set of four $\equiv_2^{\text{loc}}$-almost locally closed sets, each of which is always consistent:

- $\{97, 98, 99\} \subseteq \omega$ is an almost-locally closed set in which every element has $\equiv_2^{\text{loc}}$ class gl.

- $\{97, (\omega, \omega \times 2 \setminus \omega), \omega, \omega + 1, \omega + 2\} \subseteq \omega \times 2 \cup (\omega \times 2)^+$ is an almost-locally closed set realizing the two $\equiv_2^{\text{loc}}$ classes gl and al.

- A symmetric subset of $(\omega \times 2)^*$ realizing the $\equiv_2^{\text{loc}}$ classes gl and gd.

- $\{(0, -97),\ (\{b \in 2 \times \eta : b < (0,0)\},\ \{b \in 2 \times \eta : b \geq (0,0)\}), (0,0), (1,0),$ $(\{b \in 2 \times \eta : b \leq (1,0)\},\ \{b \in 2 \times \eta : b > (1,0)\}),\ (0, 99)\} \subseteq 2 \times \eta \cup (2 \times \eta)^+$ is an almost-locally closed set realizing the two $\equiv_2^{\text{loc}}$ classes gd and al.

If we want to completely replace $\equiv_2^{\text{loc}}$ classes by $\equiv_2^{\text{loc}}$- almost locally closed sets, one further item is sufficient:

- $\{-97, (\{b \in \eta : b < 0\},\ \{b \in \eta : b \geq 0\}), 0, (\{b \in \eta : b \leq 0\},\ \{b \in \eta : b > 0\}), 99\} \subseteq \eta$ is an almost-locally closed set in which every element has $\equiv_2^{\text{loc}}$ class ad.

Similarly, for $\equiv_3$, we can realize all of the $\equiv_3^{\text{loc}}$ classes in each of the first 7 groups in an almost locally closed set of size $n + 4$, containing $n + 2$ elements of $n \times \eta$ and two cuts, where n is $1, 2, \ldots$, or 7. The remaining $\equiv$ classes are realized in the following almost-locally-closed sets:

- $\{(0,-97),(\{b\in 2\times\eta : b<(0,0)\},\ \{b\in 2\times\eta : b\geq(0,0)\}),\ (0,0),(1,0),\ (2,0),\ (3,0),\ (4,0),(5,0),(6,0),(7,0),(8,0),(\{b\in 2\times\eta : b\leq(8,0)\},\ \{b\in 2\times\eta : b>(8,0)\}),(0,99)\}$ is an almost-locally closed set found in the linear order $8\times\eta$, realizing the $\equiv_3^{\mathrm{loc}}$ classes $\{(0,4),(1,4),(2,4),\ (3,4),\ (4,3),\ (4,2),\ (4,1),\ (4,0)\}$.

- $\{95,(\omega,\omega\times 2\setminus\omega),\omega,\omega+1,\omega+2,\omega+3,\omega+4,\omega+5,\omega+6,\omega+7\}$ is an almost-locally closed set realizing in $\omega\times 2$ the $\equiv_3^{\mathrm{loc}}$ classes $\{(0,4),\ (1,4),\ (2,4),\ (3,4),\ (4,4)\}$, since $(\omega\times 2,\omega+4)\equiv_3(\omega\times 2,95)$.

- A symmetric subset of $(\omega\times 2)^*$ realizes the $\equiv_3^{\mathrm{loc}}$ classes $\{(4,0),\ (4,1),\ (4,2),\ (4,3),\ (4,4)\}$.

- $\{91,92,93,94,95,96,97\}\subseteq\omega$ is an almost-locally closed set realizing in ω the $\equiv_3^{\mathrm{loc}}$ class $(4,4)$.

Different sets of these almost locally closed sets realize the same set of $\equiv_3^{\mathrm{loc}}$ classes. For instance, the almost locally closed sets in $(\omega\times 2)^*$ and $\omega\times 2$ given as examples above together realize the same elements as the almost locally closed sets in $8\times\eta$ and ω. Thus, the power set of this set of almost locally closed sets counts certain consistent sets of $\equiv_3^{\mathrm{loc}}$ classes multiple times, and this should be discounted in an enumeration of $\equiv_4$ classes.

This concludes the proof that definition 2.1 defines an enumeration of the $\equiv_4$ classes of linear orders. $\square$

Implementing the program described in the definition in *Perl*, it ran on this laptop in about two minutes and counted 82988077686330 $\equiv_4$ classes of linear orders.

3 infinitary logic

For any linear order λ, let $EF_\lambda(\mu,\pi)$ be the game in which player I chooses an element $a\in\lambda$, players I and II choose elements $m\in\mu$ and $p\in\pi$ (a normal turn of the EF game) and then the players play $EF_{\{b\in\lambda:b<a\}}((\mu,m),(\pi,p))$. After this process has been repeated, and elements $a_i : i\in I$ of λ have been chosen by player I, and pairs $m_i\in\mu, p_i\in\pi$ have been played by both players, then the players play $EF_{\{b\in\lambda:\forall i\in I(b<a_i)\}}((\mu,m_i)_{i\in I},(\pi,p_i)_{i\in I})$. We call λ the *clock* in this game. If player II wins $EF_\lambda(\mu,\pi)$, then we say $\mu\equiv_\lambda\pi$.

For the rest of this section, suppose that the clock λ is an ordinal. We want to express $\equiv_\lambda$ as a tree of labels of local classes; now that tree will have a rank for every descending sequence $A\in\lambda$ and have infinitely wide ranks.

Definition 3.1 *If α is an ordinal and μ is a linear order and I is an assignment of labels into $\mu\cup\mu^+$ then the $\equiv_\alpha^{\mathrm{loc}}$-*refinement *of I is the assignment I' which contains I and in addition, for each cut (b,c) in I and for each inextensible $\equiv_\alpha^{\mathrm{loc}}$-equivalence class τ realized in λ between b and c, a label for the least element or elements of type τ if one of the following conditions hold:*

1. $b = \emptyset$ *or*

2. *there is a maximal* $b' \in b$ *(for all* $e \in b, b' \geq e$*) such that the label of* b' *begins "*$g \ldots$*" or "*$l \ldots$*" or "*$d \ldots$*" or* b' *is "*$a\tau' \in (e,f)$*" for* τ' *an* $\equiv^{\text{loc}}_{\beta}$ *class such that* $\beta < \alpha$ *or*

3. *labels* $x\tau' \ldots$ *where* τ' *has quantifier rank* $\geq \alpha$ *are bounded in* b *by some element of* μ *or*

4. *elements* $n \in \mu$ *such that* (μ, n) *is in* $\equiv^{\text{loc}}_{\alpha}$ *class* τ *are bounded in* b *by some element of* μ*.* [18]

In those cases, I' *labels the least element in class* τ *above* b *with the label "*$l\tau \in (b,c)$*", or* I' *labels the cut below the least decreasing sequence of elements in class* τ *above* b *with the label "*$d\tau \in (b,c)$*". 2. Similar conditions determine whether* I' *labels the greatest element or elements of type* τ *below* c*.*

The item which allows τ to be defined above $\sup b$ if b is ultimately of low rank (the 3rd item in the preceding definition) does not pass over limits: suppose β is the least element of A and $\alpha < \beta$ and α is a limit and τ is an $\equiv^{\text{loc}}_{\beta}$ class and "$a\tau \ldots$" is a label in I_A assigned to the cut (b,c) in λ^+. Suppose that for each $\gamma < \alpha$ no $\equiv^{\text{loc}}_{\gamma}$ class which is realized in λ is realized in b is not bounded in b. It does not follow that the $\equiv^{\text{loc}}_{\alpha}$ type τ is realized in b. Nor does it follow that, if τ is realized in b, then it is not bounded in b. Thus, the least α for which some $\equiv^{\text{loc}}_{\alpha}$ type's least occurrence above b is defined in the $\equiv^{\text{loc}}_{\alpha}$ refinement of I_A can be any $\alpha < \beta$.

Let's see how condition 3 (in the preceding enumeration) works in an example.

- For any $n < \omega$ there is an $m < \omega$ such that $\omega^m \equiv_n \omega^\omega$. In particular, the equivalence holds if $n \leq 2m$.
- $\omega^\omega \equiv_\omega \omega^\omega + \omega^\omega$ (So for any ordinal α, $\omega^\omega + \alpha \equiv_\omega \omega^\omega + \omega^\omega + \alpha$).
- $\omega^\omega \times 2 \equiv_{\omega+1} \omega^\omega \times 3$ (So for any ordinal α, $\omega^\omega \times 2 + \alpha \equiv_{\omega+1} \omega^\omega \times 3 + \alpha$).
- $\omega^\omega \times 4 \equiv_{\omega+2} \omega^\omega \times 5$.

Now let's compute the labelings of those last two linear orders. For $n = 4$ or $n = 5$, the assignment $\cup_{n\in\omega} \cup_{A\subseteq n} I_A(\omega^\omega \times n)$ labels every element of the least copy of ω^ω and labels the gap at the right end, $(\omega^\omega \times n, \emptyset)$. The elements of $\omega^\omega \times n$ which are not labeled by $\cup_{n\in\omega} \cup_{A\subseteq n} I_A(\omega^\omega \times n)$ form an interval. Let's name the interval of unlabeled elements in each model $\lambda_0 \subseteq \omega^\omega \times 4$ and $\mu_0 \subseteq \omega^\omega \times 5$ so that $\omega^\omega \times 4 = \omega^\omega + \lambda_0$ and $\omega^\omega \times 5 = \omega^\omega + \mu_0$. The assignment $I_{\{\omega\}}(\omega^\omega \times n)$ labels the least and greatest element of each $\equiv_\omega$ type in λ_0 and μ_0. This labels every element of the second copy of ω^ω in $\omega^\omega \times n$. The elements of $\omega^\omega \times n$ which are not labeled by $I_{\{\omega\}}(\omega^\omega \times n)$ are an interval, too. Let's call that interval λ_1 in

[18] That is, there is some bounding element m: $\exists m \in \mu((m < \sup b) \wedge \neg\exists n(n \in \mu^{(m,\sup b)} \wedge (\mu, n) \in \tau))$.

one model and μ_1 in the other model, so that $\omega^\omega \times 4 = \omega^\omega + \lambda_0 = \omega^\omega + \omega^\omega + \lambda_1$ and likewise $\omega^\omega \times 5 = \omega^\omega + \omega^\omega + \mu_1$.

For any finite set $A \subset \omega$, for $n = 4$ or $n = 5$, the assignment $I_{\{\omega\}\cup A}(\omega^\omega \times n)$ labels nothing; if the least ordinal not in A is n, no $\equiv_n^{\mathrm{loc}}$ type can be labeled in the interval $(b, c) = (\omega^\omega + \omega^\omega, \emptyset)$ between the elements of $\omega^\omega \times n$ which have been defined by $I_{\{\omega\}}(\omega^\omega \times n)$, because b is an unbounded sequence of labels of quantifier rank ω and $\omega > n$. For each label in $I_{\{\omega\}}(\omega^\omega \times n)$, the $\equiv_{\omega+1}^{\mathrm{loc}}$ class of that label is the same in both models. All intervals (b, c) are empty, since every element is defined, except for the interval $(b, c) = (\omega^\omega + \omega^\omega, \emptyset)$. There, the $\equiv_{\omega+1}^{\mathrm{loc}}$ classes $(Th_{\omega+1}^{\mathrm{right}}(\omega^\omega + \alpha), Th_{\omega+1}^{\mathrm{left}}(\omega^\omega))$ and $(Th_{\omega+1}^{\mathrm{right}}(\omega^\omega + \alpha), Th_{\omega+1}^{\mathrm{left}}(\omega^\omega + \omega^\omega))$ are realized, for all ordinals $\alpha < \omega^\omega$, and nothing else is realized.

The reader who has patiently followed this description of the labels relevant to $\equiv_{\omega+2}$ will now see the role of condition 3: if condition 3 were not required and $I_{\{\omega\}\cup A}(\omega^\omega \times n)$ were to define every element of the third copy of ω^ω, then the interval of elements not defined in $\omega^\omega \times 4$ would not contain the $\equiv_{\omega+1}^{\mathrm{loc}}$ class $(\omega^\omega \times 2 + \alpha, \omega^\omega \times 2)$, whereas the interval of undefined elements of $\omega^\omega \times 5$ would indeed contain that $\equiv_{\omega+1}^{\mathrm{loc}}$ class, indicating that these linear orders would be $\not\equiv_{\omega+2}$.

The four conditions enumerated in definition 3.1 prevent the labeling of many $\equiv_\alpha^{\mathrm{loc}}$ classes in many intervals. In the case of finite k, these conditions make the assignment of labels which is relevant to $\equiv_k$ seem to me like an unusual subset of the set of all first and last elements of all types in all intervals. But when the quantifier rank is infinite, defining τ only above b which are not $a\tau$ and which are ultimately of low rank eliminates many labels from the assignment, often reducing the cardinality of the assignment which labels the first and last element of each $\equiv_\alpha^{\mathrm{loc}}$ class in all intervals.

Definition 3.2 *For any linear order μ, for any finite sets A and B of ordinals in λ, we define $I_A(\lambda)$ before $I_B(\lambda)$ just in case $\sum_{\alpha\in A} 2^\alpha < \sum_{\alpha\in B} 2^\alpha$.*[19] *Let $I_\emptyset(\lambda) = \emptyset$.*

For each ordinal β, for each finite (possibly empty) set A of ordinals all greater than β, let $I_{A\cup\{\beta\}}(\lambda)$ be the $\equiv_\beta^{\mathrm{loc}}$ refinement of $\cup\{I_{A\cup B}(\lambda) : B$ is a finite subset of $\beta\}$.

The reader should notice that if $\beta \neq 0$, $I_{\{\beta\}}(\lambda)$ is the $\equiv_\beta^{\mathrm{loc}}$ refinement – not of $\emptyset$, but of an already rich set of labels, indeed, the union of labels in the tree up to the rank indexed by $\{\beta\}$.

Lemma 3.1 *Let $\alpha+1$ be any successor ordinal. Player I has a winning strategy in the game $EF_{\alpha+1}(\mu_0, \mu_1)$ after the first move has identified $a_i \in \mu_i$ if there is a finite subset $A \subseteq \alpha$ such that conditions $1 \wedge 2 \wedge 3$ or $4 \wedge 5 \wedge 6$ hold:*

1. *for some $\beta < \alpha$, $A \subseteq \beta$, and*

[19] This is the usual lexicographical ordering on decreasing sequences of ordinals where $A < B$ holds just in case the greatest element of B is larger than any element of A or the greatest elements of A and B are both α, and $A \setminus \{\alpha\} < B \setminus \{\alpha\}$.

2. *$I_A(\mu_0)$ and $I_A(\mu_1)$ induce the same ordering $\leq$ on the same labels and the elements a_0 and a_1 are in the same cut $(b, c) \in (I_A(\mu_i))^+$, and*

3. *some $\equiv^{\text{loc}}_{\beta}$ type δ (the discrepancy) is realized between* $\sup b$ *and a_i in μ_i and δ is not realized between* $\sup b$ *and a_{1-i} in μ, and the least occurrences of δ are definable above* $\sup b$ *in the sense of definition 3.1, or*

4. *1 A contains the predecessor, $\alpha - 1$, of α and*

5. *for each $B \subseteq \alpha - 1$, $I_B(\mu_0)$ and $I_B(\mu_0)$ induce the same ordering $\leq$ on the same labels and there is some cut $(b, c) \in (\cup_{B \subseteq \alpha-1} I_B)^+$ such that a_i and a_{1-i} are between b and c, and the same $\equiv^{\text{loc}}_{\alpha-1}$ classes are realized between b and a_i as are realized between b and a_{1-i}, and*

6. *for some $\equiv_{\alpha-1}$ class ρ, for some $i < 2$ and $p_i \in \mu_i$ and for all $p_{1-i} \in \mu_{1-i}$ if $I_{A\setminus\{\alpha-1\}}(\mu_0^{>p_0})$ and $I_{A\setminus\{\alpha-1\}}(\mu_1^{>p_1})$ assign the same labels in the same order, then conditions $(1 \wedge 2 \wedge 3) \vee (4 \wedge 5 \wedge 6)$ hold at rank $A \setminus \{\alpha - 1\}$ in the tree of labels after $a_0 \in \mu_0^{>p_0}$ and $a_1 \in \mu_1^{>p_1}$ are played on the first move in the game $EF_\alpha(\mu_0^{>p_0}, \mu_1^{>p_1})$.*

Proof: Suppose there exists a finite set $A \subseteq \alpha$ such that conditions $1 \wedge 2 \wedge 3$ hold. Player I plays the element of $\equiv^{\text{loc}}_{\beta}$ type δ (the discrepancy) between $\sup b$ and a_i in μ_i. Player II must respond with an element of type δ, since β-many moves will remain after this turn is completed (player I could begin the next turn by choosing $\beta < \alpha$ to be the "number of moves" remaining). By assumption, player II will only find such an element below $\leq b$ or above $\geq c$ in μ. But if this was a winning second move in $EF_{\alpha+1}$, then it was a winning first move in $EF_\alpha(\{a \in \mu_i : a < a_i\}, \{a \in \mu_{1-i} : a < a_{1-i}\})$. This contradicts theorem 3.3.

Suppose there exists a finite set $A \subseteq \alpha$ such that conditions $4 \wedge 5 \wedge 6$ hold. Player I plays so that on the second move the players have identified $p_0 \in \mu_0$ and $p_1 \in \mu_1$ with the properties described in condition 6. If the antecedent of condition 6 fails, then by theorem 3.3, player I has a winning strategy in the game in which (a_0, a_1) was the first move in the game $EF_\alpha(\{a \in \mu_i : a > p_i\}, \{a \in \mu_{1-i} : a > p_{1-i}\})$, since that move did not respect the labels which are known to $\equiv_\alpha$. So the antecedent of condition 6 holds, and by its conclusion we find that the lemma is repeated in $EF_{\alpha-1}(\{a \in \mu_0 : a > p_0\}, \{a \in \mu_1 : a > p_1\})$, i.e., we can repeatedly drop quantifier rank using conditions $4\wedge5\wedge6$ and preserve the discrepancy of condition 3. $\square$

Theorem 3.1 *If player II has a winning strategy in $EF_{\alpha+1}(\mu_0, \mu_1)$, then for each finite $A \subseteq \alpha$,*

- *$I_A(\mu_i)$ is the same ordered set of labels assigned into $\mu_i \cup (\mu_i)^+$, and*
- *if player I plays the first move at the image of a label in one model, either player II plays the image of that label in the other model or player I has a winning strategy in the remainder of the game, and*

- *if* $(b,c) \in (I_A(\mu_i))^+$ *and player I plays the first move in* μ_i *between* $\sup b$ *and* $\inf c$*, then player II plays in* μ_i *between* $\sup b$ *and* $\inf c$*, or player I has a winning strategy in the remainder of the game.*

Proof by induction on $\sum_{\beta \in A} 2^\beta$.[20]

Suppose that for all $B < A$ that player II must respect I_B or lose. So we can define a cut (b,c) in $\cup\{I_B(\mu_i) : B < A\}$ such that the first move (a_0, a_1) is played between $\sup b$ and $\inf c$ in either model. If I_A labels the least element in $\equiv^{\text{loc}}_\beta$ class τ in (b,c), then by the definition of 2^β, there is a cofinal set of B such that $B < A$ and β is not in any of these B. If some $B < A$ is the immediate predecessor of A, then $A = B \cup \{n\} \setminus n$, for n the least natural number not in B.[21] However, it is very possible that no I_B, for B in that set, have added any labels to b. By definition 3.1, that happens just in case $b \cap I_{A\setminus\{\beta\}}$ is nonempty (condition 1) and has no greatest element labeled "$x\tau$" for τ an $\equiv^{\text{loc}}_\delta$ class, $\delta \geq \beta$ (condition 3) or $b \cap I_{A\setminus\{\beta\}}$ is nonempty and has a last element labeled "$a\ldots$" in which elements of every $\equiv^{\text{loc}}_\delta$ class are not bounded (condition 2). In one of those cases, we see that I_A would *not* label the least element in $\equiv^{\text{loc}}_\beta$ class τ in (b,c). If condition 3 were violated, then that same cofinal subset of b of quantifier rank $\geq \beta$ means that condition 3 is violated in the $\equiv^{\text{loc}}_\beta$ refinement of $\cup_{B<A} I_B$. If condition 2 were violated in the $\equiv^{\text{loc}}_\delta$ refinement of $I_{A\setminus\{\beta\}}$ for every $\delta < \beta$, it does not hold that each element of every $\equiv^{\text{loc}}_\beta$ type τ is unbounded below $\sup b$.

We say player I has a *winning condition* if after the players have identified a_0 and a_1 the preceding lemma has held, with conditions $4 \wedge 5 \wedge 6$ so that pairs (p_0, p_1) have been played on each turn, preserving the lemma, as though in the game $EF_{\beta+1}(\mu_0^{p_0}, \mu_1^{p_1})$, for some $\beta < \alpha$, the first move had been to identify $a_0 \in \mu_0$ and $a_1 \in \mu_1$. This lemma identifies a *discrepancy* between μ_0 and μ_1 in its 3rd condition, which player I tries to exploit. When the discrepancy, an $\equiv^{\text{loc}}_\beta$ class τ, exists in μ_i in (b,c), player I plays in μ_{1-i} in answer to $g\tau_0 \in (b', c')$ or in answer to $a\tau_0 \in (b', c')$ or if b has no greatest element. In the latter two cases, when player I is to play in b_0, an initial segment of b, player I needs to play above a possibly infinite number of upper bounds in b. Let β_{b_0} be such that $\equiv^{\text{loc}}_{\beta_{b_0}}$ classes are defined confinally in b_0. All $\equiv^{\text{loc}}_\gamma$ classes for $\gamma > \beta$ are definable above b_0. Further, $\equiv^{\text{loc}}_\gamma$ classes ρ for $\gamma \leq \beta$ are definable above b_0 if occurrences of ρ are bounded below b_0. Player I will be unable to exploit the discrepancy just in case the following occurs: b_0 is $I_{A\setminus\{\alpha-1\}} \cap b$. There is a discrepancy, an $\equiv^{\text{loc}}_\beta$ class δ between $\sup b$ and a_i, and no element of $\equiv^{\text{loc}}_\beta$ class δ between $\sup b$ and a_{1-i} in μ_{1-i}. Player I plays $q_{1-i} \in b_0$ in μ_{1-i}, and player II answers in μ_i, and the set of labels corresponding to $b \setminus b_0$ in $I_{A\setminus\{\alpha-1\}}(\mu_{1-i}^{>q})$ is now bounded by an element of $\equiv^{\text{loc}}_\beta$ class δ between $\sup b$ and a_{1-i} in μ_{1-i}. As we increase q in b_0 and it passes over various upper bounds defining elements of $b \setminus b_0$, we find that $I_{A\setminus\{\alpha-1\}}(\mu_{1-i}^{>q})$ assigns various labels to the same element as in $b \setminus b_0$, including

[20] We use that sum to define the function 2^α for infinite ordinals α: 2^α is the least ordinal number greater than $\sum_{\beta \in A} 2^\beta$ for all $A \subset \alpha$.

[21] $\{B : B < A, B = A \setminus \{\beta\} \cup B', B' \subseteq \beta$ is finite $\}$ are the final sets B such that $B < A$.

– all high-rank elements (as soon as we pass the last high-rank element of b_0) and an increasing number of low quantifier-rank classes. But if $q >$ the upper bound of some $\equiv_\beta^{\rm loc}$ class, then it's least realization above q is the same as it's least realization above b_0. All further labels are the same, too. So if player I intends to play in a series of cuts $b_0 < b_1 < \ldots b$, then it is enough to choose some

To determine which of these player I's move p_i must exceed, player I looks ahead to defining the anomaly δ. Player I finds a finite set of

So, some aspects of the induction, such as the claim that *players can play arbitrarily close to a cut* and the argument for player I playing labels "d" and "l" in π and labels "a" and "g" in μ, go through just as in the case where the clock is finite.

We have supposed that $\cup\{I_B : B < A\}$ assigns the same labels into $\pi \cup \pi^+$ and $\mu \cup \mu^+$ and that β is the least element of A. Now suppose that the (b, c) is a cut, $(b, c) \in (\cup\{I_B : B < A\})^+$ in which the $\equiv_\beta^{\rm loc}$ refinement of $\cup\{I_B : B < A\}$ will differ between π and μ. We follow the definition 3.1 to see how this could happen. First, suppose $\equiv_\beta^{\rm loc}$ class τ exists in π between $\sup b$ and $\inf c$, but not between $\sup b$ and $\inf c$ in μ. Then player I plays the first move at the realization of τ. For player II to play a realization of τ, player II must play outside the interval (b, c). Player II therefore plays below some element of b or above some element of c. That label and element exist in I_B for some $B < A$. So player II has not respected I_B, and by the induction hypothesis, loses.

Next, suppose that τ is bounded below $\sup b$ by $m \in \pi$ but τ is not bounded below $\sup b$ in μ. Player I plays the first move between m and $\sup b$ in π. By assumption, the second player respects $\cup\{I_B : B < A\}$. Now a winning condition exists, since the type τ exists between the first move and $\sup b$.

That labels of quantifier rank greater than that of τ are cofinal in $\sup b$ in μ but not in π is a property of the order of labels agreed on by $\cup\{I_B : B < A\}(\pi)$ and $\cup\{I_B : B < A\}(\mu)$.

If there is a least element of type τ in (b, c) in π but not in μ, then player I plays that least element, player II plays an element of type τ in (b, c) in μ by the inductive assumption, and now a discrepancy exists – the elements of type τ above $\sup b$ and below the second player's first move in μ, which are absent in the analogous interval of π.

Now to see that I_A has the same ordering on labels in both models, we follow the argument in Theorem 2.1 in *Ehrenfeucht–Fraïssé Games on Linear Orders*, specifically, the itemized list on the last page of that proof. $\square$

Lemma 2.2 of *Ehrenfeucht–Fraïssé Games on Linear Orders* also holds for infinitary assignments: For any linear order π and for any finite set A of ordinals, for every label m assigned by $I_A(\pi)$ to $a \in \pi$, for any order types α, β, and γ, the assignment $I_A(\pi + \alpha + \gamma + \beta)$ assigns m to $a \in \pi$. On the other hand, every label m in the domain of $I_A(\pi + \alpha + \gamma + \beta)$ which is fixed under varying α and β is, in fact, assigned to π.

Theorem 3.2 *For any linear order π and any element $a \in \pi$, for any ordinals $\alpha > \beta$, from*

- *$\cup\{I_A(\pi) : A \subseteq \alpha\}$,*

- *$Th^{\text{right}}_\beta(\{b \in \pi : b < a\})$ and $Th^{\text{left}}_\beta(\{b \in \pi : b > a\})$, i.e., $Th^{\text{loc}}_\beta(\pi, a)$,*

we can construct $\cup\{I_B(\{b \in \pi : b < a\}) : B \subseteq \beta\}$.

As in the case when k is finite, we must truncate local types, i.e., we can't say that every label in $I_B(\{b \in \pi : b < a\})$ appears in $I_A(\pi)$ or the assignment of labels into $Th^{\text{left}}_\beta(\{b \in \pi : b > a\})$, but that any label in $I_B(\{b \in \pi : b < a\})$ is the truncation of some label appearing in the latter assignments. The proof goes through as for finite k. □

Theorem 3.3 *If $\alpha + 1$ is a successor ordinal, $\pi \equiv_{\alpha+1} \mu$ holds just in case*

- *for all finite sets $A \subseteq \alpha$, $I_A(\pi)$ and $I_A(\mu)$ assign the same labels to elements and cuts in π and μ in the same order, and*

- *for each finite set $A \subseteq \alpha$ and for each label e, if $(e, f) \in I_A(\pi)$ and $(e, m) \in I_B(\mu)$ then $f \in \pi$ just in case $m \in \mu$; if both those conditions hold, then for some $\equiv^{\text{loc}}_\alpha$ class τ, $(\pi, f) \in \tau$ and $(\mu, m) \in \tau$, and*

- *for each cut $(b, c) \in (I_A(\pi))^+$ and cut $(b', c') \in (I_A(\mu))^+$ such that b and b' contain the same labels, for each $\equiv^{\text{loc}}_\alpha$ class τ, there is some element $n \in \pi^{(b,c)}$ such that $(\pi, n) \in \tau$ just in case there is some element $m \in \mu^{(b',c')}$ such that $(\mu, m) \in \tau$.*

If B is an unbounded set of ordinals, $\pi \equiv_{\cup B} \mu$ holds just in case, for all $\beta \in B$, $\pi \equiv_{\cup\beta} \mu$.

Proof: This is a corollary of the preceding two theorems. □

Theorem 3.4 *For any finite $k \geq 1$,*

- *$\omega + Z \times (2^k - 2) + \omega^* \not\equiv^{\text{left}}_{\omega+k} \omega + Z \times (2^k - 1) + \omega^*$ and*

- *$Z \times (2^k - 1) \equiv_{\omega+k} Z \times 2^k$.*

Proof: The following establish the base case, $k = 1$:

- For any finite number k, a unique $\equiv^{\text{loc}}_k$ class is realized in Z (or $Z \times \lambda$ for any nonempty λ). Call it $\zeta_k = (\phi_k, \psi_k)$.[22]

- For any finite number k, any $\equiv^{\text{loc}}_k$ class realized in ω, Z, or ω^* is either (ϕ, ψ_k) for some ϕ such that $\phi_k \subseteq \phi$ or (ϕ_k, ψ) for some ψ such that $\psi_k \subseteq \psi$.

- For any finite set $A \subset \omega$ and for any linear order λ, $I^{\text{left}}_A(\omega + Z \times \lambda)$ labels a finite subset of ω, since the least element of type ζ_k above any finite subset of ω occurs within ω.

[22] Translation is an automorphism of the integers.

- Let I be the union of $I_A^{\text{left}}(\omega + Z \times \lambda)$ over all finite sets $A \subset \omega$, for some linear order λ. I is independent of λ. I labels every element of ω, since if the element $a \in \omega$ is the least element not labeled and if every element below a is labeled in $I_A(\omega + \lambda)$ and if k is the least number not in A and if the $\equiv_k^{\text{loc}}$ class of a is (ϕ, ψ_k), then $l(\phi, \psi_k) \in (\{b \in \omega : b < a\}, \emptyset)$ labels a in $I_{A \cup \{k\} \setminus k}(\omega + \lambda)$.

- Only one $\equiv_\omega^{\text{loc}}$ type is realized in Z (or $Z \times \lambda$ for any nonempty λ). Call it ζ_ω.[23]

- $\omega + Z + \omega^* \not\equiv_{\omega+1} \omega + \omega^*$ because ζ_ω is realized in $(I, \emptyset)$ in the former model and not in the latter.[24] This inequivalence is $\not\equiv_{\omega+1}^{\text{left}}$ or left-invariant: for any $\equiv_k$ class γ there is some $\equiv_k$ class α and a label m[25] in the domain of both assignments $I_\omega^{\text{left}}(\omega + Z + \omega^* + \alpha)$ and $I_\omega^{\text{left}}(\omega + \omega^* + \alpha)$ such that $\omega + Z + \omega^* + \alpha + \gamma + \beta \not\equiv_{\omega+1} \omega + \omega^* + \alpha + \gamma + \beta$ because ζ_ω is realized in (I, m) in the former model and not in the latter.

- On the other hand, $Z + Z \equiv_{\omega+1} Z$ because all labels are assigned to the cuts at the left and right ends of the linear order, and ζ_ω occurs in both models.

Now we proceed by induction: Suppose $\omega + Z \times (2^k - 2) + \omega^* \not\equiv_{\omega+k}^{\text{left}} \omega + Z \times (2^k - 1) + \omega^*$ has been proven for some k. Let δ be the $\equiv_{\omega+k}$ class of $\omega + Z \times (2^k - 1) + \omega^*$. Now $(Th^{\text{right}}(\delta), Th^{\text{left}}(\delta))$ is an $\equiv_{\omega+k}^{\text{loc}}$ class. Call it $\zeta_{\omega+k}$. Now $\zeta_{\omega+k}$ is not realized in $\omega + Z \times (2^{k+1} - 2) + \omega^*$ because no element of that model has $(2^k - 1)$-many copies of Z to its left and to its right. On the other hand, $\zeta_{\omega+k}$ is realized in $\omega + Z \times (2^{k+1} - 1) + \omega^*$ because $2^{k+1} - 1 = (2^k - 1) + 1 + (2^k - 1)$. If we let $\alpha = Z$, then the greatest element of ω^* is labeled "$l(ge, de) \in (\emptyset, \emptyset)$" in $\omega + Z \times n + \omega^* + \alpha + \gamma$, where (ge, de) is an $\equiv_2^{\text{loc}}$ equivalence class, so that we have proved $\omega + Z \times (2^{k+1} - 2) + \omega^* \not\equiv_{\omega+k+1}^{\text{left}} \omega + Z \times (2^{k+1} - 1) + \omega^*$.

Let's define a shorthand: Write n for the $\equiv_{\omega+k}^{\text{left}}$ class of $\omega + Z \times n + \omega^*$, if $n < 2^k$ and write (m, n) for the $\equiv_{\omega+k}^{\text{loc}}$ class of pairs (λ, a) such that $\lambda = Z \times \mu$ for some μ and there are m-many copies of Z left of a and n-many copies of Z right of a. Now the $\equiv_{\omega+k}^{\text{loc}}$ classes imply one another in the following way:

- $(m, n) \to (m + 1, n - 1)$ and $(m, n) \to (m - 1, n + 1)$ if $m < 2^k - 1$ and $n < 2^k - 1$,

- $(m, n) \to (m, n - 1)$ if $m = 2^k - 1$ and $n < 2^k - 1$,

- $(m, n) \to (m - 1, n)$ if $n = 2^k - 1$ and $m < 2^k - 1$,

- $(m, n) \to (m + 1, n) \vee (m + 1, n - 1)$ if $n = 2^k - 1$ and $m < 2^k - 1$,

[23] $\equiv_\omega^{\text{loc}}$ classes are *types* in the traditional sense of that word – infinite sets of formulas which we hope to realize at a single element of some model – and they are diverse, in general.

[24] We are using theorem 3.3.

[25] If we let $\alpha = 1 + Z$, then $m =$ "$l(ge, de) \in (I, \emptyset)$" labels the least element of α. If we let $\alpha = Z$, then $m =$ "$l(ge, de) \in (I, \emptyset)$" labels the greatest element of ω^*.

- $(m, n) \to (m, n+1) \vee (m-1, n+1)$ if $m = 2^k - 1$ and $n < 2^k - 1$.

If $n = 2^k - 1$, then $Th_{\omega+k+1}(Z \times (2^k - 2))$ assigns the labels

$$d(0,n) < d(1,n) < d(2,n) < \dots d(2^{k-1}-3, n) < d(2^{k-1}-2, n) <$$

$$a(n, 2^{k-1}-2) < a(n, 2^{k-1}-3) < \dots a(n,2) < a(n,1) < a(n,0)$$

and doesn't realize $\zeta_{\omega+k}$ in the central gap $(\{d(2^{k-1}-2,n)\}, \{a(n, 2^{k-1}-2)\})$. For $m \geq n$, $Th_{\omega+k+1}(Z \times m)$ assigns the same labels, and does realize $\zeta_{\omega+k}$ in the central gap. By theorem 3.3, nothing but assignments of labels and the realization of different $\equiv^{\text{loc}}_{\omega+k}$ classes separate $\equiv_{\omega+k+1}$ classes. So in particular, that $Z \times m$ for $m \geq n$ label the same elements and realize the same $\equiv_{\omega+k+1}$ classes between labels implies that $Z \times n \equiv_{\omega+k} Z \times (n+1)$. $\square$

This proof applies theorem 3.3. We now sketch a simpler proof: During the Ehrenfeucht–Fraïssé game of length $\omega + k$, if the first k-many moves identify $a_0 \dots a_{k-1}$ in λ and $b_0 \dots b_{k-1}$ in μ, then player I has a winning strategy if for some $i < k-1$, $\{a \in \lambda : a_i < a < a_{i+1}\}$ is finite and $\{b \in \mu : b_i < b < b_{i+1}\}$ is not finite, or visa versa. On the other hand, if the interval between a_i and a_{i+1} is not finite, then its $\equiv_\omega$ theory is $Th_\omega(\omega+\omega^*)$, so player II has a winning strategy just in case for all $i < k-1$, $\{a \in \lambda : a_i < a < a_{i+1}\}$ and $\{b \in \mu : b_i < b < b_{i+1}\}$ are finite and equal, or infinite. It is possible to give a proof which is simpler than applying theorem 3.3 because it is easy to say in this case what $\equiv_\omega$ classes of intervals exist in the model and how those classes limit the first k-many moves of either player.

4 quantifier ranks $\equiv_\alpha$ on wellorders

For any linear order μ, let $D(\mu)$ be the set of elements $a \in \mu$ such that a is the limit of an infinite sequence of elements tending towards a from the left. I.e., $D(\mu) = \{a \in \mu : (\exists b \in \mu(b < a)) \wedge (\forall b \in \mu(b < a) \to (\exists c \in \mu(b < c < a)))\}$. For a limit ordinal $\delta < \lambda$, we define the δ-th iterate of D, D^δ, to be $\cap_{\gamma<\delta} D^\gamma(\mu)$. For a successor ordinal $\delta = \gamma + 1$, D^δ is the compound function which computes D of $D^\gamma(\mu)$.

$D^\delta(\lambda)$ is definable by the δ-fold iteration of the preceding definition: The sentences ϕ_δ, defined as

$$\phi_{\gamma+1} = (\exists y \phi_\gamma^{<y}) \wedge (\forall y \exists z (y < z \wedge \phi^{<z})),$$

$$\phi_{\sup B} = \wedge_{\beta \in B} \phi_\beta$$

have quantifier rank $2 \times \delta$ and for any ordinal α, $\alpha \models \phi_\delta$ just in case for all ordinals $\beta > \alpha$, $\alpha \in D^\delta(\beta)$.

For δ any ordinal, consider $Th_{2\times\delta}(\omega^\delta)$ and $Th_{2\times\delta+1}(\omega^\delta)$.

Theorem 4.1 *Two minimal almost locally closed sets – a minimal almost locally closed set which contains* 0 *and a minimal almost locally closed set* A_δ *which contains* $\omega^\delta \times 99$ *– realize all* $\equiv^{\text{loc}}_{2\times\delta}$ *and* $\equiv^{\text{loc}}_{2\times\delta+1}$ *classes which are realized* *in $\omega^{\delta+1}$. Further, if we let A_γ be the minimal almost locally closed set which contains a typical element of $D^\gamma(\omega^{\delta+1}) \setminus D^{\gamma+1}(\omega^{\delta+1})$ (an element a of D^γ is typical if each of its minimal almost locally closed sets is $\equiv_{2\times\delta}$ to a minimal almost locally closed set of $a+\omega^\gamma$), then for any interval in $\omega^{\delta+1}$ there is an initial segment $(A_\gamma : \gamma < \delta_0)$ of $(A_\gamma : \gamma < \delta)$ realizing exactly the set of $\equiv^{\mathrm{loc}}_{2\times\delta}$ classes realized in the interval.*

Proof, by induction on δ. The $\equiv^{\mathrm{loc}}_{2\times\delta+2}$ class of $a \in \omega^{\delta+2}$ is determined by $Th_{2\times\delta+2}(\omega^{\delta+1}, a)$. This is determined, in turn, by the tree of labels $I_{\leq 2\times\delta}$ and $\equiv^{\mathrm{loc}}_{2\times\delta+2+1}$ classes at labels and between labels. By induction on this theorem, for any $j \leq 2\times\delta$, we can replace each $\equiv^{\mathrm{loc}}_j$ class by the almost locally closed class A_γ which realizes it and has least γ, and we can replace any set U of $\equiv^{\mathrm{loc}}_j$ classes by initial segments of the sequence $(A_\gamma : \gamma < \delta)$. We can't easily list the $\equiv^{\mathrm{loc}}_{2\times\delta+2}$ classes because there is a variety of possibilities. But for any a, copies of A_δ either: 0. don't occur below a, or occur below a and 1. there is a last copy of A_δ below a, or 2. are unbounded below a. In those three cases, 0. the $\equiv^{\mathrm{loc}}_{2\times\delta+2}$ class of a is realized in any minimal almost locally closed set which contains 0, 1. the $\equiv^{\mathrm{loc}}_{2\times\delta+2}$ class of a follows from its $\equiv^{\mathrm{loc}}_{2\times\delta}$ class, from the fact that $\omega^{\delta+2}$ is an ordinal, and from the fact that there is a last copy of A_δ below a, and 2. the $\equiv^{\mathrm{loc}}_{2\times\delta+2}$ class of a is realized in $A_{\delta+1}$. The theory of ordinals and the fact of whether A_δ occurs below a and then unboundedly below a or not isn't enough to determine the $\equiv^{\mathrm{loc}}_{2\times\delta+2}$ class of a, but it is enough to determine the $\equiv^{\mathrm{loc}}_{2\times\delta+2}$- almost locally closed set around a. For in case 1, this set must contain that last copy of A_δ below a. In case 2, this set contains the cut which is the supremum of the copies of A_δ below a. Adding the least ordinal above this cut gives $A_{\delta+1}$. In the end, the result is an almost locally closed set from which either 1. a cannot be eliminated, without giving up almost closure, so that a is realized in $A_{\delta+1}$, or 2. a can be given up, in which case its $\equiv^{\mathrm{loc}}_{2\times\delta+2}$ class is realized in $A_{\delta+1}$. Similarly, we induct quantifier rank $2\times\delta+3$ from quantifier rank $2\times\delta+1$. □

The following theorem can be proved directly, by showing that for any clock $\beta < \alpha$, for any element of one model there is an element of the other model such that the same theorem holds in either direction with the clock β. The details seem to be inevitably gory.

By the preceding theorem, on the first move in $EF_{2\times\beta+1}$ or $EF_{2\times\beta+2}$, player I can distinguish locally a variety of $\equiv^{\mathrm{loc}}_{2\times\beta}$ classes or $\equiv^{\mathrm{loc}}_{2\times\beta+1}$ classes, but the minimal almost locally closed set containing that local class is $\equiv_{2\times\beta+1}$ or $\equiv_{2\times\beta+2}$ to A_γ, for some $\gamma < \beta$. These are the only definable sets of ordinals that player I can use to choose a first move. Player I can 1. show that D^δ is nonempty in one model and not in the other, or 2. show that D^δ has a greatest element in one model and not in the other, or 3. show that the finite part of $D^\delta(\mu_i) = \omega\times\pi_i + n_i$ is different in the two models ($n_0 \neq n_1$).

Theorem 4.2 *For any ordinals δ and x, let*

$$\pi_{i,\delta} = \{y \in \mu_i : y + \omega^\delta > \mu_i\},$$

which is empty or has as its least element $\omega^\delta \times x_\delta$*.*

For any ordinals α, μ_0, μ_1*,* $\mu_0 \equiv_\alpha \mu_1$ *holds just in case, for each* $\delta < \alpha$*, the following hold in both* μ_i *or fail in both* μ_i*:*

1. $\exists\beta(2 \times \delta \leq \beta < \alpha)$ *and* $(\delta > 0 \wedge \omega^\delta < \mu_i) \vee (\delta = 0 \wedge 0 < \mu_i)$*,*
2. $\exists\beta_0\exists\beta_1(2 \times \delta \leq \beta_0 < \beta_1 < \alpha)$ *and* $\exists x < \mu_i(\mu_i = \omega^{\delta+1} \times x + \pi_{i,\delta})$ *and*
$$(\pi_{i,\delta} = \emptyset) \vee (\pi_{i,\delta} \setminus \{\omega^{\delta+1} \times x\}) \not\equiv_{\alpha-2} \omega^{\delta+1} + \pi_{i,\delta}, \text{and}$$
3. $\chi_\delta(\mu_0) = \chi_\delta(\mu_1)$ *or* $\chi_\delta(\mu_0)$ *and* $\chi_\delta(\mu_1)$ *are both* $\geq 2^{\alpha-(2\times\delta)} - 1$*,*

where $\chi_\delta(\mu_i)$ *is:*

- *The number of* x *such that* $\omega^\delta \times x < \mu_i$ *and* $\omega^\delta \times x + \omega^{d+1} > \mu_i$*,*
- $+3$ *if* $\omega^{d+1} < \mu_i$*,*
- -1 *if* $\delta > 0$ *and* $\mu_i < \omega^{\delta+1}$*,*
- -1 *if* $\pi_{i,\delta} \neq \emptyset$ *and* $(\pi_{i,\delta} \setminus \{\omega^\delta \times x_\delta\}) \not\equiv_{2\times\delta} \omega^\delta + \pi_{i,\delta}$*.*

Further, if one of those final three summands applies to μ_i *and not to* μ_{1-i} *and this difference renders* $\xi_\delta(\mu_i)$ *small and equal, then there is another* $\delta' \neq \delta$ *which witnesses* $\mu_0 \not\equiv_\alpha \mu_1$*.*

Proof: The three conditions correspond to properties of μ_i of quantifier-rank α, describing three properties of the elements of $D^\delta(\mu_i)$ which are not individually definable in $\equiv_\alpha$:

1. $D^\delta(\mu_i) \neq \emptyset$. The formula $\phi_\delta^{<x}$ has quantifier rank $2 \times \delta = \beta < \alpha$, so the sentence $\exists x(\phi_\delta^{<x})$ has quantifier rank α. In the game $EF_\alpha(\mu_0, \mu_1)$, player I plays $\omega^\delta \in \mu_i$, then proceeds to verify $\omega^\delta \models \phi_\delta$.
2. If this condition holds, the Cantor normal form of μ_i has no term with exponent δ, and $\pi_{i,\delta}$ is small enough that we can express this fact with a formula ϕ_δ^+ which is similar to ϕ_δ and says $\forall x \in D^\delta(\mu_i)\exists y \in D^\delta(\mu_i) y > x$ and $y \notin \pi_{i,\delta}$.
3. The set of elements of $D^\delta(\mu_i)$ which are not individually definable in $\equiv_\alpha$ are pseudo-finite – i.e., we replace all of $D^\delta(\mu_i)$ before and including the last element of $D^{\delta+1}(\mu_i)$ with 3 elements of $D^\delta(\mu_i)$ and play EF as though $D^\delta(\mu_i)$ were finite sets. Hence the upper bound $2^{\alpha-(2\times\delta)}$ on the definably-sized sets $D^\delta(\mu_i)$.

We should explain the summands of χ_δ.

- If δ is not the largest exponent in the Cantor normal form of μ, the infinite set of elements of $D^\delta(\mu_i)$ less than some element of $D^{\delta+1}(\mu_i)$ are equivalent to 3 elements of $D^\delta(\mu_i)$, since when $2 \times \delta + 2$ moves are left, this infinite subset of $D^\delta(\mu_i)$ can be confused with $le_0 < le_1 < le_2$, the three elements that $I_{<2}^{\text{left}}(D^\delta(\mu_i))$ defines.

- The least multiple of ω^δ with no multiple of $\omega^{\delta+1}$ beyond it is, in fact, the last multiple of $\omega^{\delta+1}$, if that exists. This point is available to be played, unless there are absolutely no multiples of $\omega^{\delta+1}$ in μ_i at all, i.e., δ is maximal. On the other hand, if $\delta = 0$, then the element 0 is playable. Just in case $\delta > 0$, the least multiple $\omega^\delta \times 0$ of ω^δ is not playable as an element of $D^\delta(\mu_i)$.

- If this condition holds, then the last element of $D^\delta(\mu_i)$ is individually definable because it is close to the right end.

If these three conditions fail, then for the same δ ($2 \times \delta < \alpha$), $D^\delta(\mu_i)$ is nonempty (condition 1); for the same δ ($2 \times \delta < \beta_0 < \alpha$) $D^\delta(\mu_i)$ has a final element; and the elements of $D^\delta(\mu_i)$ which are not individually definable in $Th_{2\times\delta}(\mu_i)$ as 0 or as the final element of $D^\delta(\mu_i)$ which is definably close to the right are either equinumerous or numerous enough that $\equiv_{\alpha-(2\times\delta)}$ cannot count them. By the preceding theorem, player I's first move will be locally trivial, in the sense that $\mu_i \in \omega^{\mu_i}$, and interpreting player I's move in ω^{μ_i}, it will be equivalent to something in the minimal almost locally closed set of 0 or of the minimal almost locally closed set containing a typical element of $D^\delta(\omega^{\mu_i}) \setminus D^{\delta+1}(\omega^{\mu_i})$. Player I in fact has a variety of first moves just in case the ordinal μ_i has a complicated Cantor normal form, so that different extensible local types occur close to the right end of μ_i. The deficiencies of those local types are revealed by adding ω^δ to the $\equiv^{\mathrm{loc}}_\beta$ class of the first move of player I, for various δ, and checking whether a smaller $\equiv^{\mathrm{loc}}_\beta$ class results. Thus, the only difference that player I can find between μ_0 and μ_1 is a different number of elements in D^δ after the last limit element of that set (if it is infinite). In summary: the final three summands of χ_δ should be intuitive: 1. the infinite initial part of $D^\delta(\mu)$ will be encountered when there are $\equiv_{2\times\delta+2}$ moves left, at which time it plays the same as three undefinable elements of $D^\delta(\mu)$. 2. If $\delta > 0$ is maximal in the Cantor Normal form of μ, then the first copy of ω^δ contributes to $D^\delta(\mu)$ the element 0, which is $\equiv_1$-definable, and hence useless when counting $D^\delta(\mu)$. 3. A final element of $D^\delta(\mu)$ – the initial element of $\pi_{i,\delta}$ is useless when counting $D^\delta(\mu)$ if the set of elements of $\pi_{i,\delta}$ greater than it is $\equiv_{2\times\delta}$-definable. □

A corollary of the preceding theorem is an enumeration of $\equiv_k$ classes, for finite k: For any ordinal μ, let $D(\mu)\setminus\{$the least element of $D(\mu)$, if $D(\mu)$ contains any element$\} \setminus \{$a definable greatest element, if there is one$\} = D^+(\mu)$.

Theorem 4.3 *For any ordinal α, the $\equiv_{2+\alpha}$ classes of ordinals can be enumerated by enumerating the $Th_\alpha(D^+(\mu))$ and the following:*

- *If $Th_\alpha(D^+(\mu)) = 0$ then $\mu \in \{0 \dots 2^{2+\alpha} - 2\}$ or $\mu \equiv_{2+\alpha} 2^{2+\alpha} - 1$ is finite or $\mu \in \{\omega, \omega+1, \omega+2, \omega+3\}$.*

- *If $\exists\pi(Th_\alpha(D^+(\mu)) = \pi + 1)$ then $\mu = \omega \times \pi + n$ for $n \in 4 \dots 2^{2+\alpha} - 5\}$ or $\mu \equiv_{2+\alpha} \omega \times \pi + 2^{2+\alpha} - 4$ or $\mu = \omega \times \pi + \omega + n$ for $n \in \{0, 1, 2, 3\}$.*

- *If $Th_\alpha(D^+(\mu))$ is a limit ordinal π, then $\mu = \omega \times \pi + n$ for $n \in \{0, 1, 2, 3\}$.*

Furthermore, each $\equiv_\alpha$ class has one extension into an $\equiv_{2+\alpha}$ class which describes a limit ordinal, and these exhaust the $\equiv_{2+\alpha}$ classes which describes a limit ordinal.

Proof: Apply the previous theorem with $\delta = 1$ throughout. When we add $n \in \{0, 1, 2, 3\}$, then either $n = 0$, in which case $\pi_{i,1} = \emptyset$, or $n > 0$, in which case $\pi_{i,1}$ is definable. In these four cases, $\pi_{i,1}$ adds no $\equiv_2$-undefinable final element to $D(\mu)$. When we add $n \in \{4 \dots 2^\alpha - 2\}$, then we do add a final $\equiv_2$-undefinable final element to $D(\mu)$, and this is taken into account in the enumeration. □

So, writing $e(k, WO)$ for the number of $\equiv_k$ classes of wellorders,

$$e(k, WO) = e(k-2, WO) \times (2^k - 3) - e(k-4, WO) \times (2^k - 7) + 7.$$

For instance, there are two $\equiv_1$ classes (let $\alpha = 1$). We enumerate the $\equiv_3$ classes of ordinals as:

- $\{0 \dots 7, \omega, \omega+1, \omega+2, \omega+3\}$ extending $Th_\alpha(D^+(\mu)) = 0$ and
- $\omega + 4$ and $\omega + \omega + n$ for $n \in \{0, 1, 2, 3\}$ extending $Th_\alpha(D^+(\mu)) = 1$.

Similarly, there is a single $\equiv_0$ class of linear orders and there are five $\equiv_2$ classes of ordinals, so the formula indicates $5 \times 13 - 2 = 63$ $\equiv_4$ classes of ordinals. 20 of them have $D^+(\mu) = 0$, 13 of them have $D^+(\mu) = 1$, 13 of them have $D^+(\mu) = 2$, 13 of them have $D^+(\mu) = 3$, and 4 of them have $D^+(\mu) = \omega$. The $\equiv_4$ classes of ordinals are $\{\omega \times \alpha + n : \alpha \in A, n \in N\}$ as (A, N) range over the following set:

- $A = \{0\}, N = \{n\}$, for $n = 0..14$;
- $A = \{0\}, N = \{n : n \geq 15\}$;
- $A = \{1\}, N = \{n\}$, for $n = 0..11$;
- $A = \{1\}, N = \{n : n \geq 12\}$;
- $A = \{2\}, N = \{n\}$, for $n = 0..11$;
- $A = \{2\}, N = \{n : n \geq 12\}$;
- $A = \{3\}, N = \{n\}$, for $n = 0..3$;
- $A = \{\alpha : \alpha \geq 3\}, N = \{n\}$, for $n = 4..11$;
- $A = \{\alpha : \alpha \geq 3\}, N = \{n : n \geq 12\}$;
- $A = \{\alpha : \exists\beta : \alpha = 3 + \beta + 1\}, N = \{n\}$, for $n = 0..3$;
- $A = \{$ limit ordinals $\}, N = \{n\}$, for $n = 0..3$;

The $\equiv_3$ class containing most random Cantor normal form polynomials has as its smallest member $\omega + 4$. So we could say that $\omega + 4$ is $\equiv_3$-typical. A more precise definition is that $\omega + 4$ is the smallest Cantor normal form polynomial such that increasing any coefficient (even adding a term which is not there) leaves it $\equiv_3$. The typical $\equiv_4$ ordinal is $\omega \times 3+12$, and the typical $\equiv_5$ polynomial is $\omega^2 + \omega \times 4 + 28$.

Theorem 4.4 *For $k \geq 6$, $e(k, WO) = 2^{q(k)-\epsilon_k}$, where $q(k) = (k+1)\times(k+1)/4$ if k is even, $q(k) = (k + 2) \times k/4$ if k is odd, and $\epsilon_k \in (0.20, 0.37)$.*

Proof: This follows from iterating the formula above – $e(3, WO) = 2\times5-0+7$ (if we set $e(-1) = 0$), and $e(4, WO) = 5 \times 13 - 1 \times 9 + 7 = 63$, and $e(5, WO) = 17 \times 29 - 2 \times 25 + 7 = 450$; $e(6, WO) = 63 \times 61 - 5 \times 57 + 7 = 3565$; $e(7, WO) = 450 \times 125 - 17 \times 121 + 7 = 54200$. For $k = 0, 1, 2, 3, 4, 5, 6, 7$, $\log_2(e(k, WO)) = 0, 1, 2.3, 4.1, 6.0, 8.8, 11.800, 15.726$. Thereafter, $\log_2 e(k, WO) \leq k + \log_2 e(k - 2, W =)$, and $\log_2 e(k, WO) \geq \log_2 e(k - 2, WO) + k - e(k - 4)/e(k - 2, WO) \times \log_e / \log_2$. The sums of all numbers below k and of the same parity is $q(k)$. Finally, $\epsilon < \sum_{k<\infty,k \text{ even}} e(k - 4, WO)/e(k - 2, WO)$ is bounded by a geometric series $\sum(2^{-1/4})^k = 0.841^k$ (so every time k increases by 4, another bit of ϵ is determined), which is bounded by 6.3 times its first term, and $6.3 \times e(4, WO)/e(6, WO) < 0.1604; 6.3 \times e(5, WO)/e(7, WO) < 0.077$ give the bound. □

We compute the limiting ϵ_{even} and ϵ_{odd} by numerically iterating the computation of $e(k, WO)$ and inverting the formula in the preceding theorem:

$$2^{\epsilon_{\text{even}}} = 1.19411673235052; 2^{\epsilon_{\text{odd}}} = 1.23201682615002.$$

Computing $e(k, WO)$ for $k \leq 52$ gives these two values of 2^ϵ to 14 digits of accuracy. i.e., we get one more bit in one of the ϵ_i with each increase in k.

5 undecidable linear orders

In [1] we find a linear order λ for which $Th(\lambda)$ is undecidable, though the set of all Σ_n-formulas is computable over λ, for all finite n. These formulas have nested sequences of quantifiers which alternate at most n times between $\exists$ to $\forall$. Counting the number of alternations is different than counting the quantifier rank of a formula – Σ_n corresponds to a game in which player I plays a finite sequence of elements in one model – rather than a single element. The authors of that paper ask whether a simple construction for such a λ exists. Their construction, using iterated dense shuffles, is intuitive. Ordinals are another way to hide some information from Σ_n, and we make a construction on that basis:

Definition 5.1 *Let $(U_i : i \in \omega)$ be a nested sequence of sets of natural numbers: $U_0 \supseteq U_1 \supseteq U_2 \ldots$. For any $\beta < \omega^\omega$, choose δ maximal such that $\exists x \beta = \omega^\delta \times x$.*

Now for some ordinal y and some finite number n, $x = \omega \times y + n$. That n is the last Cantor normal form coefficient of β. Let $f(\beta)$ be the nth element of U_δ.

$$\text{Let } \lambda = \sum_{\beta < \omega^\omega} \omega + ((\eta + Z) \times f(\beta)).$$

Now η and Z both have finite axiomatizations in $\equiv_3$, since they realized only one $\equiv^{\text{loc}}_{k-1}$ class, for each $k \geq 3$. The $\equiv^{\text{loc}}_2$ class in η is $\forall\exists$, and the $\equiv^{\text{loc}}_2$ class in Z is $\exists\forall$. We will capture this unfortunate difference in 5, a constant error term:

The following is a $\Sigma_{5+\delta}$ formula: A copy of $((\eta+Z)\times m)$ occurs immediately above the β-th copy of ω for some $\beta \in D^\delta(\omega^\omega)$ just in case $m \in U_\delta$. On the other hand, if we rendered the sequence $(U_i : i < \omega)$ eventually constant, this would not affect $\Sigma_n(\lambda)$ for low n, for Σ_n cannot define $D^{n+1}(\omega^\omega)$, since that set is definable only with $n+1$-many quantifier alternations. If the sequence U_i is eventually constant and each set U_i is periodic, then $\Sigma_n(\lambda)$ is finitely axiomatizable – For "whenever β is divisible by ω^δ but not $\omega^{\delta+1}$ and there exist n such β' below β and above the last element divisible by $\omega^{\delta+1}$ then there are $f(\beta)$-many copies of $\eta + Z$ before the next copy of ω" is a different sentence of Σ_n for each $f(\beta)$, unless $f(\beta)$ is simply $f(\beta - 1) + k$ for some constant k, in which case we can express the periodic part of U_i with a single formula. If some U_i is not periodic, then any Σ_n which defines $\{0 \in \omega_\beta : \beta \in D^i(\lambda)\}$ is not finitely axiomatizable.

If we choose U_i to be undecidable, then that Σ_n which defines $\{0 \in \omega_\beta : \beta \in D^i(\lambda)\}$ is undecidable. On the other hand, if we choose each U_i to be decidable but the sequence $(U_i : i < \omega)$ to be undecidable (e.g., by diagonalizing that sequence against all programs), then all Σ_n are decidable, but $Th(\lambda)$ is not.

6 $\equiv_\lambda$ for λ not wellordered

The sequence of elements $A = (a_i : i \in I)$ of λ which will be chosen during the game EF_λ will decrease in λ as the game progresses; the reversed ordering A^* will be a wellorder. Suppose λ_0 is an initial segment of λ. The same player wins $EF_\lambda(\mu, \pi)$ as wins $EF_{\lambda_0}(\mu, \pi)$, for all linear orders μ and π, if player II wins the *clock-comparison game* between λ and λ_0. Player I plays $a \in \lambda$ and player II responds with $b \in \lambda_0$. After player I has played $a_i : i \in I$ in λ and player II has played $b_i : i \in I$ in λ_0, player I plays an element $a \in \lambda$ such that $\forall i \in I(a < a_i)$, and player II plays an element $b \in \lambda_0$ such that $\forall i \in I(b < b_i)$. The first player who cannot play loses. Now λ_0 may replace λ as a clock if player II have a winning strategy in the clock-comparison game, for then we can translate clock moves in λ into clock moves in λ_0 until λ is exhausted. Informally, player II can survive just as long using λ_0 for a clock as player II can survive using λ for a clock. On initial segments of λ we form equivalence classes: For all $b < a \in \lambda$, we say $b \equiv^{\text{clock}} a$ just in case player II has a winning strategy in the clock-comparison game between $\{c \in \lambda : c < a\}$ and $\{c \in \lambda : c < b\}$.

Lemma 6.1 *The $\equiv^{\text{clock}}$ classes of λ are wellordered.*

Proof: We expand the notion of $\equiv^{\text{clock}}$ to a quasi-ordering on linear orders: $\mu <^{\text{clock}} \pi$ holds just in case the first player wins the clock-comparison game in which the first player plays in π and the second player plays in μ. Now $\mu \equiv^{\text{clock}} \pi$ if neither $\mu <^{\text{clock}} \pi$ nor $\pi <^{\text{clock}} \mu$ – i.e., the second player wins the clock-comparison games of μ versus π and π versus μ. That λ_0 is an initial segment of λ shows $\lambda \not<^{\text{clock}} \mu$.

Suppose $e_i : i \in \omega$ is a descending sequence of $\equiv^{\text{clock}}$ classes of λ. If for some $i \in \omega$, $|e_i| < |e_{i+1}|$, then for any $a \in e_i$ and $b \in e_{i+1}$, player II can win the clock-comparison game between $\{c \in \lambda : c < a\}$ and $\{c \in \lambda : c < b\}$ with the following strategy: Play slowly in e_{i+1}, until player I has exhausted e_i. Then play again in e_{i+1}. Player I will now play in e_j, for $j > i$, a move $\equiv^{\text{clock}}$ or ,$^{\text{clock}}$ to the move player II has just played. That equivalence shows how to play the remaining moves in the game. This contradicts the idea that e_i and e_{i+1} are equivalence classes. So for all $i \in \omega$, $|e_{i+1}| \not> |e_i|$. Suppose that there is some $i \in \omega$ such that for all $j \geq i$ $|e_j| = |e_{j+1}|$. Then for any $a \in e_i$ and $b \in e_{i+1}$, player II can win the clock-comparison game between $\{c \in \lambda : c < a\}$ and $\{c \in \lambda : c < b\}$ with the following strategy: play in e_{j+1} while player I is playing in e_j. The players will reach $inf \cup_{i\in\omega} e_i$ together. Then it will be player I's turn to play, and player II can copy and remaining moves. This contradicts the idea that e_i and e_{i+1} are separate equivalence classes. So for infinitely many $i \in \omega$ it holds that $|e_{i-1}| > |e_i|$. Now playing the clock-comparison game in which the smaller clock is an $\equiv^{\text{clock}}$ class is simpler than the general clock-comparison game since on any move after $\{a_i : i \in I\}$ has been played, if player II has not yet lost, then $\{c \in e_i : \forall i \in I c < a_i\} \equiv^{\text{clock}} \{c \in e_i : c < a\}$ for any single element $a \in e_i$. If player I can win this game, the winning strategy cannot depend finely on what element player II plays in e_i, since all of those elements are $\equiv^{\text{clock}}$. That is, player I never finds that $b \in e_{i-1}$ is a winning response to $a \in e_i$, but $b \in e_{i-1}$ would lose as a response to $a' \in e_i$, since a and a' are $\equiv^{\text{clock}}$. A winning strategy which is, in this sense, "blind" to the choices of player II, can only exist if there is an ordinal α such that α^* injects into e_{i-1} and not into e_i. The set U_i of ordinals which inject into e_i is closed under embedding, i.e, if α embeds into β and β embeds into e_i, then α embeds into e_i. So U_i is in fact the set of all ordinals less than α_i, for $\alpha_i = \sup U_i$. Now if U_i is a decreasing function of i, then $(\alpha_i : i < \omega)$ is a sequence of ordinals which decreases infinitely often. $\square$

Now we turn our attention to the expressive power of play within an $\equiv^{\text{clock}}$ class. For instance, if $a \in \lambda$ and $b \in \lambda$ and $a > b$ and $(\lambda, a) \cong (\lambda, b)$ and f is the isomorphism mapping (λ, a) onto (λ, b), then $b = f(a)$ and $(\lambda, b) \cong (\lambda, f(b))$ and likewise $(\lambda, f^n(a)) \cong (\lambda, f^{n+1}(a))$ for all finite numbers n. So the game $EF_{\{b\in\lambda:b<a\}}$ presents player I with, at least, a string of ω-many initial moves, each of which has the same descriptive power. Indeed, if $a \in \lambda$ and $b \in \lambda$ and $c \in \lambda$ and $a > b > c$ and $\equiv_{\{d\in\lambda:d<a\}}$ is the same as $\equiv_{\{d\in\lambda:d<c\}}$, then the following hold: First, by monotonicity, $\equiv_{\{d\in\lambda:d<b\}}$ is equal to those two equivalence classes. Second, $\equiv_{\{d\in\lambda:d<a\}}$ describes the first and last elements of each equivalence class in $\equiv_{\{d\in\lambda:d<c\}}$, which is $\equiv_{\{d\in\lambda:d<a\}}$ again.

Rather than define $\equiv^{\text{loc}}_{\alpha^*}$ sets, we directly define which $A \subseteq \lambda \cup \lambda^+$ are *locally closed*; we call $Th_{\alpha^*}(\lambda, x)_{x\in A}$ of such a set its $\equiv^{\text{loc}-\text{closed}}_{\alpha^*}$ class. We then call an $\equiv^{\text{loc}}_{\alpha^*}$ class the pair (A, a) where A is an $\equiv^{\text{loc}-\text{closed}}_{\alpha^*}$ class and $a \in A$ is a chosen element. We don't insist on A being minimal, lest this reduce A to the empty set. The *locally closed* sets of λ is the smallest set LC of subsets of $\lambda \cup \lambda^+$ such that each element of λ is in one, and such that for each element $A \in LC$, each $\beta < \alpha$, each set $A_0 = (a_i : i \in \beta) \subseteq A$, each cut $(b, c) \in (A_0)^+$ and each $\equiv^{\text{loc}}_{(\alpha\setminus\beta)^*}$ class τ 1. if $Th^{\text{loc}}_{\alpha^*}((\lambda, a_i)_{i\in\beta})$ implies the existence of an element of type τ in (b, c) then there is an element of type τ in A, 2. if $Th^{\text{loc}}_{\alpha^*}((\lambda, a_i)_{i\in\beta})$ implies there is a least element of type τ in A between b and c then that element is in A, and 3. if $Th^{\text{loc}}_{\alpha^*}((\lambda, a_i)_{i\in\beta})$ implies there are elements of type τ descending towards b without bound then A contains both 2a. the cut describing the limit of those least elements of type τ in (b, c), and 2b. a descending sequence of elements of $\equiv^{\text{loc}}_{(\alpha\setminus\beta)^*}$ class τ in (b, c), indexed by $(\alpha - \beta)$.

The minimal set $A_0 = \emptyset$ requires an $\equiv^{\text{loc}}_{\alpha^*}$-closed set A to contain, for $(\emptyset, \emptyset) \in \emptyset^+$, an element of each type τ which is implied by $\equiv^{\text{loc}}_{(\alpha\setminus\beta)^*}$. But as that equivalence class is trivial, it implies nothing. So $A_0 = \emptyset$ requires nothing of a local closure. Indeed, the empty set is locally closed. Let A_0 be a singleton, containing $a \in \lambda$ of $\equiv^{\text{loc}}_{\alpha^*}$ class τ_0. If $\alpha \leq 1$, then nothing is required of A so that A_0 is locally closed – indeed, A_0 is its own $\equiv^{\text{loc}}_1$-closure. If $\alpha > 1$, then assign $0 \in 1 = \beta$ to $a \in A_0$ and consider any $\equiv_{(\alpha\setminus 1)^*}$ class τ such that $Th^{\text{loc}}_{\alpha^*}((\lambda, a))$ implies the existence of an element of type τ near a. For instance, if $\alpha = 2$ and we analyze the linear order $\eta + \omega + 3 + Z + Z$ and consider $a = 1 \in 3$, then the $\equiv^{\text{loc}}_2$ class of a determines that a has an immediate successor and an immediate predecessor. So a locally closed set containing a must contain all of 3. $Th^{\text{loc}}_2(0 \in 3)$ determines that $0 \in 3$ is a limit of elements from below, so a locally closed set contains $(\eta + \omega, 3 + Z + Z)$ and contains an element of ω. Closing under immediate predecessors and successors brings us to $0 \in \omega$, so an element of η is in the locally closed set. That element of η has limits from above and below.

So the minimal locally closed set containing $1 \in 3$ contains, for ϕ some homomorphism of Z into η, $\phi \cup \{(\eta, \omega + 3 + Z + Z)\} \cup \omega \cup \{(\eta + \omega, 3 + Z + Z)\} \cup 3 \cup \{(\eta + \omega + 3, Z + Z)\} \cup$ the first copy of Z. This might seem unnecessarily large, especially to a reader who, like me, enjoys the notion of almost locally closed sets – sets that are cut off when they become repetitive. A locally closed set, on the other hand, must continue propagating until it is closed under its own local Skolem functions. We define an almost-locally closed set in the same way, except that we do not add to A the closure of each $A_0 = (a_i : i < \beta) \subseteq A$, but only close A under one example of each $Th_{(\alpha\setminus\beta)^*}(\lambda, a_i)_{i<\beta}$ class.

If λ is a single $\equiv^{\text{clock}}$ class, then $\mu \equiv_\lambda \pi$ holds just in case the same $\equiv^{\text{loc}}_{\alpha^*}$-closed sets exist in both μ and π, for each ordinal α such that α^* injects into λ.

More generally, we construct $I_\lambda(\mu)$, a tree of labeled elements of μ, by induction on the $\equiv^{\text{clock}}$ classes $(e_i : i < \gamma)$ of λ. For each class e_i in turn except the last one, with $a \in e_i$, we create the tree at rank e_i: above each branch

B of the tree passing through ranks $(e_j : j < i)$, for each $\equiv_{\{b\in\lambda:b<a\}}$-class of $\equiv^{\text{loc}}_{\{b\in\lambda:b<a\}}$-closed sets in μ with a single starting element $b \in \mu$ (for a much narrower tree, use almost-locally closed classes), we label the least realization of τ above B – either the least copy of τ in μ (ordered by where they realize the element equivalent to $b \in \mu$, or the gap below which no copy of τ in μ realizes an element equivalent to $b \in \mu$ above B, and above which every element exceeds the element equivalent to $b \in \mu$ in a copy of τ in μ.

Theorem 6.1 *For any linear orders λ, μ, π, $\mu \equiv_\lambda \pi$ holds just in case I_λ injects the same tree of labels into μ and π, and if for all $a \in \lambda$, the same $\equiv^{\text{loc}-\text{closed}}_{\{b\in\lambda:b<a\}}$ classes are realized in μ and π at each label of I_λ and in each gap between labels of I_λ.*

Proof: With the notion of $\equiv^{\text{loc}}_\lambda$ as the $\equiv_\lambda$ type of a $\equiv_\lambda$ locally closed set with a chosen element given just before this theorem, we can define for which $\equiv^{\text{loc}}_\lambda$ classes τ the least occurrence(s) of τ can be defined in an interval (b, c) in the set of labels already defined, using the conditions that definition 3.1 gives. Likewise, the tree of labels for a nonwellordered λ can be defined as in definition 3.2, of the tree of labels for a wellordered clock λ, since the $\equiv^{\text{clock}}$ classes in λ are wellordered. With those definitions, theorem 3.1 holds even if the clock λ is not wellordered, since for every finite (descending sequence) $A = (a_i : i < n) \subseteq \lambda$ the labels which are definable in $I_{\{a\in\lambda:a<a_i\}}$ can in fact be played by player I in a descending sequence, leading eventually to the $\equiv^{\text{clock}}$ class of a_{n-1}, at which we wish to define I_A. If there is a discrepancy in the models at this level, player I can play down the sequence of elements of A, and play a sequence of nested intervals, eventually exploiting the discrepancy as in lemma 3.1. Theorem 3.2 can be carried out as in the case of finite k – we can reverse the indices of trees and labels, so that they depend on the leftmost label and describe the rightmost, or so that they depend on the rightmost and describe the leftmost. □

If $\lambda < \kappa$ are infinite cardinals, is $\equiv_{\kappa^*}$ a strict refinement of $\equiv_{\lambda^*}$? Of course, $\equiv_{\kappa^*}$ can express the sentence $\phi_\kappa =$ "there exists a decreasing sequence of κ-many elements." In [3] we find a construction of large, $\equiv_\lambda$ linear orders, one of which satisfies ϕ_κ and one of which does not, for λ any linear order which does not satisfy ϕ_κ. These can be constructed by iterated application of a rule like that creating the "surreal" numbers, or by the exponentiation of linear orders. Our criterion can guide the construction of linear orders which are $\equiv_\lambda$, even though they are not highly homogeneous, and of linear orders close to the "watershed" between linear orders which are $\equiv_\lambda$ and those which are not.

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