AHLFORS-DAVID REGULAR SETS AND BILIPSCHITZ MAPS

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Abstract. Given two Ahlfors-David regular sets in metric spaces, we study the question whether one of them has a subset bilipschitz equivalent with the other.

1. Introduction

In this paper we shall study Ahlfors-David regular subsets of metric spaces. Throughout (X, d) and (Y, d) will be metric spaces. For $E, F \subset X$ and $x \in X$ we shall denote by $d(E)$ the diameter of E, by $d(E, F)$ the distance between E and F, and by $d(x, E)$ the distance from x to E. The closed ball with center x and radius r is denoted by $B(x, r)$.

1.1. **Definition.** Let $E \subset X$ and $0 < s < \infty$. We say that E is s-regular if it is closed and if there exists a Borel (outer) measure μ on X and a constant $C_E, 1 \leq C_E < \infty$, such that $\mu(X \setminus E) = 0$ and

$$r^s \leq \mu(B(x,r)) \leq C_E r^s \text{ for all } x \in E, 0 < r \leq d(E), r < \infty.$$

Observe that this implies that the right hand inequality holds for all $x \in E, r > 0$, and

$$\mu(B(x,r)) \leq 2^s C_E r^s \text{ for all } x \in X, r > 0.$$

We would get an equivalent definition (up to the value of C_E), if we would use the restriction of the s-dimensional Hausdorff measure on E, with r^s on the left hand side replaced by r^s/C_E. When we shall speak about a regular set E, μ will always stand for a measure as above.

We remark that closed and bounded subsets of regular sets are compact, see Corollary 5.2 in [DS2]. Self similar subsets of $\mathbb{R}^n$ satisfying the open set condition are standard examples of regular sets, see [H].

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A map $f : X \to Y$ is said to be bilipschitz if it is onto and there is a positive number L, called a bilipschitz constant of f, such that

$$d(x,y)/L \leq d(f(x),f(y)) \leq Ld(x,y) \text{ for all } x,y \in X.$$

The smallest such L is denoted by $\operatorname{bilip}(f)$. Evidently any bilipschitz image of an s-regular set is s-regular. But two regular sets of the same dimension s need not be bilipschitz equivalent. This is so even for very simple Cantor sets in $\mathbb{R}$, see [FM], [RRX] and [RRY] for results on the bilipschitz equivalence of such Cantor sets, and for [DS2] for extensive analysis of bilipschitz invariance properties of fractal type sets.

The main content of this paper is devoted to the following question: suppose E is s-regular and F is t-regular. If $s < t$, does F have a subset which is bilipschitz equivalent to E? In this generality the answer is obviously no due to topological reasons; E could be connected and F totally disconnected. We shall prove in Theorem 3.1 that the answer is yes for any $0 < s < t$ if E is a standard s-dimensional Cantor set in some $\mathbb{R}^n$ with $s < n$. We shall also prove in Theorem 3.3 that the answer is always yes if $s < 1$. In Section 4 we show that if E and F as above are subsets of $\mathbb{R}^n$ and s is sufficiently small, then a bilipschitz map f with $f(E) \subset F$ can be defined in the whole of $\mathbb{R}^n$. We don't know if this holds always when $0 < s < 1$.

In the last section of the paper we shall discuss sub- and supersets of regular sets. It follows from Theorem 3.1 that an s-regular set contains a t-regular subset for any $0 < t < s$. In the other direction we shall show that if $E \subset X$ is s-regular and X is u-regular, then for any $s < t < u$ there is a t-regular set F such that $E \subset F \subset X$. On the other hand there are rather nice sets which do not contain any regular subsets: we shall construct a compact subset of $\mathbb{R}$ with positive Lebesgue measure which does not contain any s-regular subset for any $s > 0$.

Regular sets in connection of various topics of analysis are discussed for example in [DS1] and [JW].

2. Some lemmas on regular sets

In this section we shall prove some simple lemmas on regular sets.

2.1. **Lemma.** *Let $0 < s < \infty$ and let $E \subset X$ be s-regular. For every $0 < r < R \leq d(E), R < \infty$, and $p \in E$ there exist disjoint closed balls $B(x_i, r), i = 1, \ldots, m$, such that $x_i \in E \cap B(p,R)$,*

$$(5^s C_E)^{-1}(R/r)^s \leq m \leq 2^s C_E (R/r)^s$$

and

$$E \cap B(p,R) \subset \bigcup_{i=1}^{m} B(x_i, 5r).$$

Proof. By a standard covering theorem, see, e.g., Theorem 2.1 in [M], we can find disjoint balls $B(x_i,r), i = 1,2,\ldots,$ such that $x_i \in E \cap B(p,R)$ and the balls $B(x_i,5r)$ cover $E \cap B(p,R)$. There are only finitely many, say m, of these balls, since the disjoint sets $B(x_i,r)$ have all μ measure at least r^s, they are contained in $B(p,2R)$ which has measure at most $C_E(2R)^s$. More precisely, we have

$$mr^s \leq \sum_{i=1}^{m} \mu(B(x_i,r)) \leq \mu(B(p,2R)) \leq C_E(2R)^s,$$

whence $m \leq 2^s C_E(R/r)^s$, and

$$mC_E 5^s r^s \geq \sum_{i=1}^{m} \mu(B(x_i,5r)) \geq \mu(B(p,R)) \geq R^s,$$

whence $m \geq (5^s C_E)^{-1}(R/r)^s$. □

For less than one-dimensional sets we can get more information:

2.2. **Lemma.** *Let $0 < s < 1, C \geq 1, R > 0$, let $E \subset X$ be closed and bounded and let μ be a Borel measure on X such that $\mu(X \setminus E) = 0$ and that*

$$\mu(B(x,r)) \leq Cr^s \text{ for all } x \in E, r > 0,$$

and

$$\mu(B(x,r)) \geq r^s \text{ for all } x \in E, 0 < r < R.$$

Let $D = (3C2^s)^{1/(1-s)} + 1$. For every $0 < r < R/(2D)$ there exist disjoint closed balls $B(x_i,r), i = 1,\ldots,m$, and positive numbers $\rho_i, r \leq \rho_i \leq Dr$, such that $m \leq Cd(E)^s/r^s, x_i \in E, x_j \notin B(x_i,\rho_i)$ for $i < j$,

$$E \subset \bigcup_{i=1}^{m} B(x_i,\rho_i) \text{ and } E \cap B(x_i,\rho_i + r) \setminus B(x_i,\rho_i) = \emptyset.$$

Proof. Let $x_1 \in E$. Denote

$$A_0 = B(x_1,r), A_i = B(x_1,(i+1)r) \setminus B(x_1,ir), i = 1,2,\ldots.$$

If $E \cap A_1 = \emptyset$, denote $\rho_1 = r$. Otherwise, let l be the largest positive integer such that $2lr < R$ and $E \cap A_i \neq \emptyset$ for $i = 1,\ldots,l$, say $y_i \in E \cap A_i$.

Then $B(y_i,r)\subset A_{i-1}\cup A_i\cup A_{i+1}\subset B(x_1,2lr)$. Therefore

$$lr^s\le\sum_{i=1}^{l}\mu(B(y_i,r))\le\sum_{i=1}^{l}\mu(A_{i-1}\cup A_i\cup A_{i+1})\le$$
$$3\mu(B(x_1,2lr))\le 3C2^sl^sr^s,$$

whence $l^{1-s}\le 3C2^s$ and, since $s<1$, $l\le(3C2^s)^{1/(1-s)}=D-1$. As $2(l+1)\le 2D<R/r$, we conclude that $E\cap A_{l+1}=\emptyset$ by the maximality of l. Let $\rho_1=(l+1)r$. Then $r\le\rho_1\le Dr$ and $E\cap B(x_1,\rho_1+r)\setminus B(x_1,\rho_1)=\emptyset$. Let $x_2\in E\setminus B(x_1,\rho_1)=E\setminus B(x_1,\rho_1+r)$. Then the balls $B(x_1,r)$ and $B(x_2,r)$ are disjoint. Repeating the same argument as above with x_1 replaced by x_2 we find ρ_2 such that $r\le\rho_2\le Dr$ and $E\cap B(x_2,\rho_2+r)\setminus B(x_2,\rho_2)=\emptyset$. After $k-1$ steps we choose

$$x_k\in E\setminus\bigcup_{i=1}^{k-1}B(x_i,\rho_i),$$

if this set is non-empty. As in the proof of Lemma 2.1 this process ends after some m steps when E is covered by the balls $B(x_i,\rho_i), i=1,\dots,m$. Also, as before, m satisfies the required estimate $m\le Cd(E)^s/r^s$. □

The following lemma will be needed to get bilipschitz maps in the whole $\mathbb{R}^n$.

2.3. **Lemma.** *Let $C\ge 1$ and $\lambda\ge 9$. There are positive numbers $s_0=s_0(C,\lambda), 0<s_0<1$, and $D=D(C,\lambda)>1$, depending only on C and λ, with the following property.*

Let $0<s<s_0$, let $E\subset X$ be closed and bounded, let $R>0$ and let μ be a Borel measure on X such that $\mu(X\setminus E)=0$ and that

$$\mu(B(x,r))\le Cr^s \text{ for all } x\in E, r>0,$$

and

$$\mu(B(x,r))\ge r^s \text{ for all } x\in E, 0<r<R.$$

For every $0<r<R/D$ there exist disjoint closed balls $B(x_i,\lambda\rho_i/3), i=1,\dots,m$, such that $x_i\in E$, $r\le\rho_i\le Dr$, $m\le Cd(E)^s/r^s$,

$$E\subset\bigcup_{i=1}^{m}B(x_i,\rho_i) \text{ and } E\cap B(x_i,\lambda\rho_i)\setminus B(x_i,\rho_i)=\emptyset.$$

Proof. The function $s\mapsto(1-3C\lambda^{2s}(\lambda^s-1))^{-1/s}$ is positive and increasing in some interval $(0,s_1)$, so it is bounded in some interval $(0,s_0)$. We choose s_0 and D so that

$$\lambda(1-3C\lambda^{2s}(\lambda^s-1))^{-1/s}\le D \text{ for } 0<s<s_0.$$

Set $c = \log D / \log \lambda$.

Let $x \in E$ and denote

$$A_0 = B(x,r), A_i = B(x,\lambda^i r) \setminus B(x,\lambda^{i-1} r), i = 1,2,\dots.$$

If $E \cap A_1 = \emptyset$, denote $r(x) = r$. Otherwise, let l be the largest positive integer such that $l \le c$ and that $E \cap A_i \neq \emptyset$ for $1 = 1,\dots,l$. Then for $i = 1,\dots,l$ there is $y_i \in E \cap A_i$ with $B(y_i, \lambda^{i-2} r) \subset A_{i-1} \cup A_i \cup A_{i+1}$. By the choice of c, $\lambda^{l-2} r < Dr < R$. Hence

$$\begin{aligned}
&r^s \lambda^{-s} \frac{\lambda^{sl} - 1}{\lambda^s - 1} = r^s \sum_{i=1}^{l} \lambda^{s(i-2)} \le \\
&\sum_{i=1}^{l} \mu(B(y_i, \lambda^{i-2} r)) \le \sum_{i=1}^{l} \mu(A_{i-1} \cup A_i \cup A_{i+1}) \le \\
&3\mu(E \cap B(x, \lambda^{l+1} r)) \le 3C\lambda^{s(l+1)} r^s.
\end{aligned}$$

This gives

$$(1 - 3C\lambda^{2s}(\lambda^s - 1))\lambda^{sl} \le 1,$$

whence

$$\lambda^{l+1} \le \lambda(1 - 3C\lambda^{2s}(\lambda^s - 1))^{-1/s} \le D.$$

Thus $l + 1 \le c$ and we conclude that $E \cap A_{l+1} = \emptyset$. Let $r(x) = \lambda^l r$. We have now shown that for any $x \in E$ there is $r(x), r \le r(x) \le Dr$, such that $E \cap B(x, \lambda r(x)) \setminus B(x, r(x)) = \emptyset$.

Let $M_1 = \sup\{r(x) : x \in E\}$. Choose $x_1 \in E$ with $r(x) > M_1/2$, and then inductively

$$x_{j+1} \in E \setminus \bigcup_{i=1}^{j} B(x_i, r(x_i)) \text{ with } r(x_{j+1}) > M_1/2$$

as long as possible. Thus we get points $x_i \in E$ and radii $r(x_i), r \le r(x_i) \le Dr$, for $i = 1,\dots,k_1$ such that $r(x_i)/2 \le r(x_j) \le 2r(x_i)$, $x_j \notin B(x_i, r(x_i))$ for $i < j$, and

$$\{x \in E : r(x) > M_1/2\} \subset \bigcup_{i=1}^{k_1} B(x_i, r(x_i)).$$

If for some $l = 1,2,\dots$ the points $x_1,\dots,x_{k_l}$ have been selected and $E \setminus \bigcup_{i=1}^{k_l} B(x_i, r(x_i)) \neq \emptyset$, let

$$M_{l+1} = \sup\{r(x) : x \in E \setminus \bigcup_{i=1}^{k_l} B(x_i, r(x_i))\},$$

choose $x_{k_{l+1}} \in E \setminus \bigcup_{i=1}^{k_l} B(x_i, r(x_i))$ with $r(x_{k_{l+1}}) > M_{l+1}/2$, and so on. This process will end for some $l = p$. Thus we get points $x_1,\dots,x_m \in$

$E, m = k_p$, such that, with $\rho_i = r(x_i), r \leq \rho_i \leq Dr$, for $i < j, x_j \notin B(x_i, \rho_i)$ and $r_j \leq 2\rho_i$,

$$E \subset \bigcup_{i=1}^{m} B(x_i, \rho_i) \text{ and } E \cap B(x_i, \lambda\rho_i) \setminus B(x_i, \rho_i) = \emptyset.$$

To show that the balls $B(x_i, \lambda\rho_i/3)$ are disjoint, let $i < j$. Then $\rho_j \leq 2\rho_i$ and $x_j \in E \cap (\mathbb{R}^n \setminus B(x_i, \rho_i)) = E \cap (\mathbb{R}^n \setminus B(x_i, \lambda\rho_i))$. So $d(x_i, x_j) > \lambda\rho_i$ and $(\lambda/3)(\rho_i + \rho_j) \leq \lambda\rho_i < d(x_i, x_j)$, which implies that $B(x_i, \lambda\rho_i/3) \cap B(x_j, \lambda\rho_j/3) = \emptyset$. The required estimate $m \leq Cd(E)^s/r^s$ follows as before. $\square$

3. Bilipschitz maps

In this section we begin to prove the bilipschitz equivalences mentioned in the introduction. It is easy to get explicit bounds for the bilipschitz constants of the maps from the proofs. In Theorem 3.1 $\text{bilip}(f)$ is bounded by a constant depending only on s, t, n and C_E. In Theorems 3.3 and 4.2, if C_E, C_F and $d(E)/d(F)$ (interpreted as 0 if F is unbounded) are all $\leq C$, then $\text{bilip}(f) \leq L$ where L depends only on s, t and C, and also on n in Theorem 4.2. If E and F are bounded, this dependence on the diameters is seen by first observing that we may assume that $d(F) \leq d(E)$; otherwise F can be replaced in the proofs by $F \cap B(p, d(E)/2)$ for any $p \in F$. Secondly, changing the metrics to $d_E(x, y) = d(x, y)/d(E)$ and $d_F(x, y) = d(x, y)/d(F)$, we have $d(E) = d(F) = 1$, the regularity constants don't change and a bilipschitz constant L changes to $Ld(E)/d(F)$.

For any $0 < t < n$ we shall define some standard t-dimensional Cantor sets in $\mathbb{R}^n$. Define $0 < d < 1/2$ by $2^n d^t = 1$. Let $Q \subset \mathbb{R}^n$ be a closed cube of side-length a. Let $Q_1, \dots, Q_{2^n} \subset Q$ be the closed cubes of side-length da in the corners of Q. Continue this process. Then $C(t, a)$ is defined as

$$C(t, a) = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \dots i_k} Q_{i_1 \dots i_k},$$

where $i_j = 1, \dots 2^n$ and each $Q_{i_1 \dots i_k}$ is a closed cube of sidelength $d^k a$ such that $Q_{i_1 \dots i_k i}, i = 1, \dots, 2^n$, are contained in the corners of $Q_{i_1 \dots i_k}$. It is well known and easy to prove that $C(t, a)$ is t-regular, it is also a particular case of a self similar set satisfying the open set as considered in [H].

3.1. **Theorem.** *Let $E \subset X$ be a bounded s-regular set and $0 < t < s$. Then there is a t-regular subset F of E and a bilipschitz map $f : F \to$*

$C(t, d(E))$ where $C(t, d(E))$ is a Cantor subset of $\mathbb{R}^n$ with $t < n$ as above. Moreover, $C_F \leq C$ where C depends only s, t, n and C_E.

Proof. We may assume that $d(E) = 1$. Choose a sufficiently large integer N so that denoting $d = 2^{-Nn/t}$, i.e., $2^{Nn}d^t = 1$, we have $d < 1/3$ and $d^{s-t} < (15^s C_E)^{-1} =: c$. Then we can write $C(t, 1)$ as

$$C(t, 1) = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \dots i_k} Q_{i_1 \dots i_k}$$

where each $Q_{i_1 \dots i_k}, 1 \leq i_j \leq 2^{Nn}$, is a closed cube of side-length d^k such that $Q_{i_1 \dots i_k i_{k+1}} \subset Q_{i_1 \dots i_k}$. By Lemma 2.1 we can find disjoint balls $B(x_i, 3d), x_i \in E, i = 1, \dots, m$, such that $m \geq cd^{-s} > d^{-t} = 2^{Nn}$. Now we keep the first 2^{Nn} points x_i and forget about the others. Repeating this argument with E replaced by $E \cap B(x_i, d)$ and so on, we can choose points

$$x_{i_1 \dots i_k i_{k+1}} \in E \cap B(x_{i_1 \dots i_k}, d^k), 1 \leq i_j \leq 2^{Nn},$$

such that the balls $B(x_{i_1 \dots i_k i}, 3d^{k+1}), i = 1, \dots, 2^{Nn}$, are disjoint subsets of $B(x_{i_1 \dots i_k}, 3d^k)$. Then for $1 \leq l < k$,

$$d(x_{i_1 \dots i_l}, x_{i_1 \dots i_k}) \leq \sum_{j=l}^{k-1} d(x_{i_1 \dots i_j}, x_{i_1 \dots i_{j+1}}) \leq \sum_{j=l}^{k-1} d^j < 2d^l \tag{3.2}$$

as $d < 1/2$. Denote

$$F = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \dots i_k} B(x_{i_1 \dots i_k}, 3d^k).$$

Then $F \subset E$. Let $y_{i_1 \dots i_k}$ be the center of $Q_{i_1 \dots i_k}$ and denote

$$F_k = \{x_{i_1 \dots i_k} : i_j = 1, \dots, 2^{Nn}, j = 1, \dots, k\}$$

and

$$C_k = \{y_{i_1 \dots i_k} : i_j = 1, \dots, 2^{Nn}, j = 1, \dots, k\}.$$

Define the maps

$$f_k : F_k \to C_k \text{ by } f(x_{i_1 \dots i_k}) = y_{i_1 \dots i_k}.$$

We check now that f_k is bilipschitz with a constant depending only on s, t, n and C_E. Let $x = x_{i_1 \dots i_k}, x' = x_{j_1 \dots j_k} \in F_k$ with $x \neq x'$. Let $l \geq 1$ be such that $i_1 = j_1, \dots, i_l = j_l$ and $i_{l+1} \neq j_{l+1}$; if $i_1 \neq j_1$ the argument is similar. Then by (3.2) $x \in B(x_{i_1 \dots i_l i_{l+1}}, 2d^{l+1}) \cap B(x_{i_1 \dots i_l}, 2d^l)$ and $x' \in B(x_{j_1 \dots j_l j_{l+1}}, 2d^{l+1}) \cap B(x_{i_1 \dots i_l}, 2d^l)$. Since the balls $B(x_{i_1 \dots i_l i_{l+1}}, 3d^{l+1})$ and $B(x_{j_1 \dots j_l j_{l+1}}, 3d^{l+1})$ are disjoint, we get that $d^{l+1} \leq d(x, x') \leq 4d^l$.

Letting $y = y_{i_1\dots i_k}$ and $y' = y_{j_1\dots j_k}$ we see from the construction of $C(t,1)$ that $(1-2d)d^l \le |y-y'| \le \sqrt{n}d^l$. Hence

$$d(f_k(x), f_k(x')) = |y-y'| \le (\sqrt{n}/d)d(x,x')$$

and

$$d(f_k(x), f_k(x')) = |y-y'| \ge ((1-2d)/4)d(x,x').$$

Denote $L = \max\{\sqrt{n}/d, 4/(1-2d)\}$.

If $x \in F$ there is a unique sequence $(i_1, i_2, \dots)$ such that $x \in B(x_{i_1\dots i_k}, 3d^k)$ for all $k = 1, 2, \dots$. Let $y \in C(t,1)$ be the point for which $y \in Q_{i_1\dots i_k}$ for all $k = 1, 2, \dots$. Then $y = \lim_{k\to\infty} y_{i_1\dots i_k} = \lim_{k\to\infty} f_k(x_{i_1\dots i_k})$. We define the map $f : F \to C(t,1)$ by setting $f(x) = y$. If also $x' = \lim_{k\to\infty} x_{j_1\dots j_k}$ and $y' = \lim_{k\to\infty} y_{j_1\dots j_k}$ we have

$$\begin{aligned} d(f(x), f(x')) &= \lim_{k\to\infty}(f_k(x_{i_1\dots i_k}), f_k(x_{j_1\dots j_k})) \\ &\le \lim_{k\to\infty} Ld(x_{i_1\dots i_k}, x_{j_1\dots j_k}) = Ld(x,x') \end{aligned}$$

and similarly $d(f(x), f(x')) \ge d(x,x')/L$. Obviously, $f(F) = C(t,1)$. The last statement, $C_F \le C$, of the theorem follows immediately from the fact that L depends only s, t, n and C_E. □

Next we turn to study less than one-dimensional sets.

3.3. **Theorem.** *Let $E \subset X$ be s-regular and $F \subset Y$ t-regular with $0 < s < 1$ and $s < t$. Suppose that either E is bounded or both E and F are unbounded. Then there is a bilipschitz map $f : E \to f(E) \subset F$.*

Proof. We shall first consider the case where both E and F are bounded. By the remarks in the beginning of this section, we then may assume that $d(E) = d(F) = 1$. Let $D = (3C_E 2^s)^{1/(1-s)} + 1$. Choose d so small that

$$0 < d^{t-s} < (2^s 15^t D^s C_E C_F)^{-1} \text{ and } 2Dd < 1.$$

We shall show that there exist s-regular sets $E_{i_1\dots i_k}$, points $x_{i_1\dots i_k} \in E, y_{i_1\dots i_k} \in F$ and radii $\rho_{i_1\dots i_k}$ where

$$1 \le i_j \le m_{i_0\dots i_{j-1}}, j = 1, \dots, k, \text{ with } m_{i_0\dots i_{j-1}} \le C_E 2^s D^s/d^s, i_0 = 0,$$

such that for all $k = 1, 2, \dots,$

$$
\begin{aligned}
&E = \bigcup_{i_1 \dots i_k} E_{i_1 \dots i_k}, \\
&E_{i_1 \dots i_k i_{k+1}} \subset E_{i_1 \dots i_k}, \\
&d^k \leq \rho_{i_1 \dots i_k} \leq D d^k, \\
&x_{i_1 \dots i_k} \in E_{i_1 \dots i_k} \subset B(x_{i_1 \dots i_k}, \rho_{i_1 \dots i_k}), \\
&E \cap B(x_{i_1 \dots i_k}, \rho_{i_1 \dots i_k} + d^k) \setminus B(x_{i_1 \dots i_k}, \rho_{i_1 \dots i_k}) = \emptyset, \\
&d(E_{i_1 \dots i_k}, E_{j_1 \dots j_k}) \geq d^k \text{ if } i_k \neq j_k, \\
&y_{i_1 \dots i_k i_{k+1}} \in F \cap B(y_{i_1 \dots i_k}, d^k), \\
&B(y_{i_1 \dots i_k i_{k+1}}, 2d^{k+1}) \subset B(y_{i_1 \dots i_k}, 2d^k), \\
&B(y_{i_1 \dots i_k}, 3d^k) \cap B(y_{j_1 \dots j_k}, 3d^k) = \emptyset \text{ if } i_k \neq j_k.
\end{aligned}
$$

By Lemma 2.2 we find $x_i \in E$ and $\rho_i, d \leq \rho_i \leq Dd$, with $i = 1, \dots, m_0, m_0 \leq C_E/d^s$, such that the balls $B(x_i, d)$ are disjoint, $x_j \notin B(x_i, \rho_i)$ for $i < j$,

$$E \subset \bigcup_{i=1}^{m_0} B(x_i, \rho_i)$$

and

$$E \cap B(x_i, \rho_i + d) \setminus B(x_i, \rho_i) = \emptyset.$$

By Lemma 2.1 we find $y_i \in F$ with $i = 1, \dots, n_0, n_0 \geq (15^t C_F d^t)^{-1} \geq C_E/d^s \geq m_0$ such that the balls $B(y_i, 3d)$ are disjoint. We define

$$E_1 = E \cap B(x_1, \rho_1) \text{ and } E_i = E \cap B(x_i, \rho_i) \setminus \bigcup_{j=1}^{i-1} E_j \text{ for } i \geq 2.$$

Then the required properties for $k = 1$ are readily checked.

Suppose then that for some $k \geq 1, E_{i_1 \dots i_k}, x_{i_1 \dots i_k} \in E, y_{i_1 \dots i_k} \in F$ and $\rho_{i_1 \dots i_k}$ have been found with the asserted properties. Fix $i_1 \dots i_k$. We shall apply Lemma 2.2 with $E = E_{i_1 \dots i_k}, R = d^k$, $r = d^{k+1}$ and $C = C_E$, recall that $2Dd < 1$. Since $d(E_{i_1 \dots i_k}, E \setminus E_{i_1 \dots i_k}) \geq d^k$, we have $E \cap B(x, r) = E_{i_1 \dots i_k} \cap B(x, r)$ for $x \in E_{i_1 \dots i_k}$ and $0 < r < d^k$, so this is possible. Thus we obtain $x_{i_1 \dots i_k i} \in E_{i_1 \dots i_k}$ and $\rho_{i_1 \dots i_k i}, i = 1, \dots, m_{i_0 \dots i_k}$, such that $m_{i_0 \dots i_k} \leq C_E d(E_{i_1 \dots i_k})^s / d^{(k+1)s} \leq C_E 2^s D^s / d^s$, the balls $B(x_{i_1 \dots i_k i}, d^{k+1})$ are disjoint, $x_{i_1 \dots i_k j} \notin B(x_{i_1 \dots i_k i}, \rho_{i_1 \cdots_k i})$ for $i < j$,

$$d^{k+1} \leq \rho_{i_1 \dots i_k i} \leq D d^{k+1},$$

$$E_{i_1 \dots i_k} \subset \bigcup_{i=1}^{m_{i_0 \dots i_k}} B(x_{i_1 \dots i_k i}, \rho_{i_1 \dots i_k i})$$

and

$$E \cap B(x_{i_1 \ldots i_k i}, \rho_{i_1 \ldots i_k i} + d^{k+1}) \setminus B(x_{i_1 \ldots i_k i}, \rho_{i_1 \ldots i_k i}) = \emptyset.$$

Define

$$E_{i_1 \ldots i_k 1} = E_{i_1 \ldots i_k} \cap B(x_{i_1 \ldots i_k 1}, \rho_{i_1 \ldots i_k 1})$$

and

$$E_{i_1 \ldots i_k i} = E_{i_1 \ldots i_k} \cap B(x_{i_1 \ldots i_k i}, \rho_{i_1 \ldots i_k i}) \setminus \bigcup_{j=1}^{i-1} E_{i_1 \ldots i_k j} \text{ for } i \geq 2.$$

Applying Lemma 2.1 we find points $y_{i_1 \ldots i_k i} \in F \cap B(y_{i_1 \ldots i_k}, d^k), i = 1, \ldots, n_{i_0 \ldots i_k}$, with $n_{i_0 \ldots i_k} \geq (15^t C_F d^t)^{-1} \geq C_E 2^s D^s / d^s \geq m_{i_0 \ldots i_k}$ such that the balls $B(y_{i_1 \ldots i_k i}, 3d^{k+1}), i = 1, \ldots, n_{i_0 \ldots i_k}$, are disjoint. Then the required properties are easily checked.

Set

$$A_k = \{x_{i_1 \ldots i_k} : i_j = 1, \ldots, m_{i_0 \ldots i_{j-1}}, j = 1, \ldots, k\}$$

and

$$B_k = \{y_{i_1 \ldots i_k} : i_j = 1, \ldots, m_{i_0 \ldots i_{j-1}}, j = 1, \ldots, k\}.$$

Define the maps

$$f_k : A_k \to B_k \text{ by } f(x_{i_1 \ldots i_k}) = y_{i_1 \ldots i_k}.$$

We check now that f_k is bilipschitz with a constant depending only on s, t, C_E and C_F. Let $x = x_{i_1 \ldots i_k}, x' = x_{j_1 \ldots j_k} \in A_k$ with $x \neq x'$. Let $l \geq 1$ be such that $i_1 = j_1, \ldots, i_l = j_l$ and $i_{l+1} \neq j_{l+1}$; if $i_1 \neq j_1$ the argument is similar. Then, as in (3.2) in the proof of Theorem 3.1, $x \in E_{i_1 \ldots i_l i_{l+1}} \cap B(x_{i_1 \ldots i_l}, 2Dd^l)$ and $x' \in E_{j_1 \ldots j_l j_{l+1}} \cap B(x_{i_1 \ldots i_l}, 2Dd^l)$. Since $d(E_{i_1 \ldots i_l i_{l+1}}, E_{j_1 \ldots j_l j_{l+1}}) \geq d^{l+1}$, we get that $d^{l+1} \leq d(x, x') \leq 4Dd^l$. Letting $y = y_{i_1 \ldots i_k}$ and $y' = y_{j_1 \ldots j_k}$, we have $y \in B(y_{i_1 \ldots i_l i_{l+1}}, 2d^{l+1}) \cap B(y_{i_1 \ldots i_l}, 2d^l)$ and $y' \in B(y_{j_1 \ldots j_l j_{l+1}}, 2d^{l+1}) \cap B(y_{i_1 \ldots i_l}, 2d^l)$. Hence, as $B(y_{i_1 \ldots i_l i_{l+1}}, 3d^{l+1}) \cap B(y_{j_1 \ldots j_l j_{l+1}}, 3d^{l+1}) = \emptyset$, $d^{l+1} \leq d(y, y') \leq 4d^l$,

$$d(f_k(x), f_k(x')) = d(y, y') \leq (4/d) d(x, x')$$

and

$$d(f_k(x), f_k(x')) = d(y, y') \geq (d/(4D)) d(x, x').$$

Denote $L = 4D/d > 4/d$.

As in the proof of Theorem 3.1 we define the map $f : E \to f(E) \subset F$ by

$$f(x) = \lim_{k \to \infty} f_k(x_{i_1 \ldots i_k})$$

when $x = \lim_{k \to \infty} x_{i_1 \ldots i_k}$. Then $\mathrm{bilip}(f) \leq L$.

If E is bounded and F unbounded, the same proof works with F replaced by $F \cap B(p, 1)$ for any $p \in F$. Suppose E and F are unbounded, and let $p \in E$. Using the proof of Lemma 2.2 we find R_k, $(2D)^k \leq R_k \leq D(2D)^k$, $k = 1, 2, \ldots$, such that

$$E \cap B(p, R_k + (2D)^k) \setminus B(p, R_k) = \emptyset.$$

Let $E_k = E \cap B(p, R_k)$. We check that E_k is s-regular with $C_{E_k} \leq (2D)^s C_E$. To see this, let $x \in E_k$ and $0 < r \leq d(E_k) \leq (2D)^{k+1}$. If $r \leq (2D)^k$, then $E_k \cap B(x, r) = E \cap B(x, r)$, so $\mu(E_k \cap B(x, r)) \geq r^s$. If $r > (2D)^k$, we have $\mu(E_k \cap B(x, r)) \geq (2D)^{ks} \geq (2D)^{-s} r^s$. These facts imply that $C_{E_k} \leq (2D)^s C_E$. Since the sets E_k are bounded we can find bilipschitz maps $f_k : E_k \to f(E_k) \subset F$ with $\text{bilip}(f_k) \leq L$ where L depends only on s, t, C_E and C_F. Using Arzela-Ascoli theorem we can extract a subsequence (f_{k_i}) such that the sequence $(f_{k_i})_{k_i \geq k}$ converges on E_k for every $k = 1, 2, \ldots$. Then $f = \lim_{i \to \infty} f_{k_i} : E \to f(E) \subset F$ is bilipschitz with $\text{bilip}(f) \leq L$.

□

4. MAPPINGS IN $\mathbb{R}^n$

In this section we prove for small dimensional sets in $\mathbb{R}^n$ that we can find bilipschitz mappings of the whole $\mathbb{R}^n$. The following lemma may be well known, but we have not found a suitable reference in literature.

4.1. **Lemma.** *Let $0 < \delta < c(n)$, where $c(n) < 1/2$ is a positive constant depending only on n and determined later. Let $p, q \in \mathbb{R}^n$ and $R > 0$. For $i = 1, \ldots, m$ let $\delta R \leq r_i \leq R/3$ and $x_i \in B(p, R)$ and $y_i \in B(q, R)$ with $B(x_i, 3r_i) \cap B(x_j, 3r_j) = \emptyset$ and $B(y_i, 3r_i) \cap B(y_j, 3r_j) = \emptyset$ for $i \neq j$. Then there is a bilipschitz map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x) = x - p + q$ for $x \in \mathbb{R}^n \setminus B(p, 2R)$ and $f(x) = x - x_i + y_i$ for $x \in B(x_i, r_i)$. Moreover, bilip$(f) \leq L$ where L depends only on n and δ.*

Proof. We may assume that $p = q = 0$ and $R = 1$. Let $\epsilon = \delta^{2n+3}$. It is enough to construct a bilipschitz map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $\text{bilip}(f) \leq L$, L depending only on n and δ, such that $f(x) = x$ for $|x| > 3\sqrt{n}$ and $f(x) = x - x_i + y_i$ for $x \in B(x_i, \epsilon)$. To see this, consider bilipschitz maps $g, h : \mathbb{R}^n \to \mathbb{R}^n$ with bilipschitz constants depending only on n and δ such that $g(x) = (\epsilon/r_i)(x - x_i) + x_i$ for $x \in B(x_i, r_i)$, $g(x) = x$ for $x \in B(0, 3/2) \setminus \bigcup_{i=1}^m B(x_i, 2r_i)$, $h(y) = (\epsilon/r_i)(y - y_i) + y_i$ for $y \in B(y_i, r_i)$, $h(y) = y$ for $y \in B(0, 3/2) \setminus \bigcup_{i=1}^m B(y_i, 2r_i)$, $g(x) = h(x)$ for $|x| > 2$ and $g(B(0, 2)) = h(B(0, 2)) = B(0, 3\sqrt{n})$. Then $h^{-1} \circ f \circ g$ has the required properties.

For the rest of the proof we assume that $n \geq 2$, for $n = 1$ a much simpler argument works. Denote $Q = [-2, 2]^{n-1}$. Let $a, b \in B(0, 1) \subset \mathbb{R}^{n-1}$. For $v \in \partial B(a, \epsilon)$ denote by v' the single point in $\partial Q \cap \{t(v -$

$a) + a : t \geq 1\}$. Let $g(a,b) : Q \to Q$ be the bilipschitz map such that

$$g(a,b)(x) = x - a + b \text{ for } x \in B(a,\epsilon)$$

and for $v \in \partial B(a,\epsilon)$ $g(a,b)$ maps the line segment $[v, v']$, affinely onto the line segment $[v - a + b, v']$. Then $g(a,b)(x) = x$ for $x \in \partial Q$ and $g(a,a)$ is the identity map. Moreover, $g(a,b)$ has a bilipschitz constant depending only on n.

Now we show that there exists a unit vector $\theta \in S^{n-1}$ such that $|\theta \cdot (x_i - x_j)| > 5\epsilon$ and $|\theta \cdot (y_i - y_j)| > 5\epsilon$ for $i \neq j$. To see this, let σ denote the surface measure on S^{n-1}. We have by some simple geometry (or one can consult [M], Lemma 3.11)

$$\sigma(\{\theta \in S^{n-1} : |\theta \cdot (x_i - x_j)| \leq 5\epsilon\}) \leq C_1(n)|x_i - x_j|^{-1}\epsilon \leq C_1(n)\delta^{2n+2},$$

and similarly for y_i, y_j. There are less than $C_2(n)\delta^{-2n}$ pairs (x_i, x_j) and (y_i, y_j), whence

$$\sigma(\{\theta \in S^{n-1} : |\theta \cdot (x_i - x_j)| \leq 5\epsilon \text{ or } |\theta \cdot (y_i - y_j)| \leq 5\epsilon \text{ for some } i \neq j\}) < \delta,$$

if $C_1(n)C_2(n)\delta < 1$, which we have taking $c(n) \leq (C_1(n)C_2(n))^{-1}$ in the statement of the theorem. Taking also $c(n) \leq \sigma(S^{n-1})$ our θ exists. We may assume that $\theta = (0, \ldots, 0, 1)$.

Let t_i and $u_i, i = 1, \ldots, m$, be the n'th coordinates of x_i and y_i, respectively, and let $t_0 = u_0 = -2, t_{m+1} = u_{m+1} = 2$. We may assume that $t_i < t_{i+1}$ and $u_i < u_{i+1}$ for $i = 0, \ldots, m$. Then $|t_i - t_j| > 5\epsilon$ and $|u_i - u_j| > 5\epsilon$ for $i \neq j, i, j = 0, \ldots, m+1$. For $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, let $\tilde{x} = (x^1, \ldots, x^{n-1})$. Let $Q_0 = [-2, 2]^n$ and for $i = 1, \ldots, m$,

$$R_i = \{x \in Q_0 : |x^n - t_i| \leq \epsilon\},$$

$$S_i = \{y \in Q_0 : |y^n - u_i| \leq \epsilon\}.$$

We shall define f in Q_0 with the help of the maps $g(a,b)$ in such a way that it maps R_i onto S_i translating $B(x_i, \epsilon)$ onto $B(y_i, \epsilon)$. Between R_i and R_{i+1} f is defined by simple homotopies changing $f|R_i$ to $f|R_{i+1}$, and similarly in Q_0 'below' R_1 and 'above' R_m. Finally f can be extended from Q_0 to all of $\mathbb{R}^n$ rather trivially. We do this now more precisely.

Let $x \in Q_0$ and $1 \le i \le m+1$. We set

$f(x) = (g(\tilde{x}_i, \tilde{y}_i)(\tilde{x}), x^n - t_i + u_i)$ if $|x^n - t_i| \le \epsilon$ and $i \le m$,

$f(x) = (g((2\epsilon - |x^n - t_i|)/\epsilon)\tilde{x}_i, (2\epsilon - |x^n - t_i|)/\epsilon)\tilde{y}_i)(\tilde{x}), x^n + u_i - t_i)$
if $\epsilon \le |x^n - t_i| \le 2\epsilon$ and $i \le m$,

$f(x) = (\tilde{x}, \dfrac{x^n - t_{i-1} - 2\epsilon}{t_i - t_{i-1} - 4\epsilon}(u_i - 2\epsilon) + \dfrac{t_i - 2\epsilon - x^n}{t_i - t_{i-1} - 4\epsilon}(u_{i-1} + 2\epsilon))$
if $t_{i-1} + 2\epsilon \le x^n \le t_i - 2\epsilon$,

$f(x) = x$ if $-2 \le x^n \le -2 + 2\epsilon$ or $2 - 2\epsilon \le x^n \le 2$.

Then $f : Q_0 \to Q_0$ is bilipschitz with a constant depending only on n and δ, $f(x) = x - x_i + y_i$ for $x \in B(x_i, \epsilon)$, $f(x) = x$ for $x \in Q_0$ with $x_n = -2$ or $x_n = 2$, and at the other parts of the boundary of Q_0 f is of the form $f(x) = (\tilde{x}, \phi(x^n))$ where $\phi : [-2, 2] \to [-2, 2]$ is strictly increasing and piecewise affine. It is an easy matter to extend f to a bilipschitz mapping of $\mathbb{R}^n$ with a bilipschitz constant depending only on n and δ and with $f(x) = x$ for $x \in \mathbb{R}^n \setminus B(0, 3\sqrt{n})$. For example, setting $||\tilde{x}||_\infty = \max\{|x^1|, \dots, |x^{n-1}|\}$, we can take

$$f(x) = (\tilde{x}, (3 - ||x||_\infty)\phi(x^n) + (||x||_\infty - 2)x^n)$$

when $2 \le ||\tilde{x}||_\infty \le 3$ and $|x^n| \le 2$, and $f(x) = x$ when $||\tilde{x}||_\infty > 3$ or $|x^n| > 2$. $\square$

4.2. **Theorem.** *Let $C \ge 1$ and let $s_0 = s_0(C, 18), 0 < s_0 < 1/6$, be the constant of Lemma 2.3. Let $0 < s < s_0$ and $s < t < n$, let $E \subset \mathbb{R}^n$ be s-regular and $F \subset \mathbb{R}^n$ t-regular with $C_E, C_F \le C$. Suppose that either E is bounded or both E and F are unbounded. Then there is a bilipschitz map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(E) \subset F$.*

Proof. We assume that E and F are bounded. The remaining case can be dealt with as at the end of the proof of Theorem 3.3. We can then assume that $E, F \subset B(0, 1)$ with $d(E) = d(F) = 1/2$. Let $c(n)$ and $D = D(C, 18)$ be as in Lemma 2.3, and choose d such that

$$d < c(n), 12Dd < 1 \text{ and } 0 < d^{t-s} < (2^s 60^t C_E C_F D^t)^{-1}.$$

By Lemma 2.3 we find $x_i \in E$ and $\rho_i, d \le \rho_i \le Dd$, with $i = 1, \dots, m_0, m_0 \le C_E/d^s$, such that the balls $B(x_i, 6\rho_i)$ are disjoint,

$$E \subset \bigcup_{i=1}^{m_0} B(x_i, \rho_i)$$

and

$$E \cap B(x_i, 18\rho_i) \setminus B(x_i, \rho_i) = \emptyset.$$

By Lemma 2.1 we find $y_i \in F$ with $i = 1, \dots, n_0$, $n_0 \geq (5^t C_F)^{-1}(1/(12Dd))^t \geq C_E/d^s \geq m_0$ such that the balls $B(y_i, 6Dd)$, $j = 1, \dots, n_0$, are disjoint. Next applying Lemma 2.3 with E replaced by $E \cap B(x_i, \rho_i), R = d, r = d^2$ and $C = C_E$, we find for every $i = 1, \dots, m_0, x_{ij} \in E \cap B(x_i, \rho_i)$ and $\rho_{ij}, d^2 \leq \rho_{ij} \leq Dd^2$, with $j = 1, \dots, m_i, m_i \leq C_E d(E \cap B(x_i, \rho_i))^s/d^{2s} \leq C_E 2^s D^s/d^s$, such that the balls $B(x_{ij}, 6\rho_{ij})$ are disjoint,

$$E \cap B(x_i, \rho_i) \subset \bigcup_{j=1}^{m_i} B(x_{ij}, \rho_{ij})$$

and

$$E \cap B(x_{ij}, 18\rho_{ij}) \setminus B(x_{ij}, \rho_{ij}) = \emptyset,$$

and by Lemma 2.1 we find $y_{ij} \in F \cap B(y_i, d), j = 1, \dots, n_i,\ n_i \geq (5^t C_F)^{-1}(d/(6Dd^2))^t \geq 2^s C_E/d^s \geq m_i$ such that the balls $B(y_{ij}, 6Dd^2)$ are disjoint. Continuing this we find for all $k = 1, 2, \dots,\ x_{i_1 \dots i_k}, \rho_{i_1 \dots i_k}$ and $y_{i_1 \dots i_k}$ such that for all $i_j = m_{i_0 \dots i_{j-1}}, j = 1, \dots, k, k = 1, 2, \dots,$ with $i_0 = 0$,

$$\begin{aligned}
&E \subset \bigcup_{i_1 \dots i_k} B(x_{i_1 \dots i_k}, \rho_{i_1, \dots, i_k}),\\
&B(x_{i_1 \dots i_k}, 6\rho_{i_1 \dots i_k}) \cap B(x_{j_1 \dots j_k}, 6\rho_{i_1 \dots i_k}) = \emptyset \text{ if } i_k \neq j_k,\\
&x_{i_1 \dots i_k i_{k+1}} \in E \cap B(x_{i_1 \dots i_k}, \rho_{i_1 \dots i_k}),\\
&d^k \leq \rho_{i_1 \dots i_k} \leq Dd^k,\\
&B(x_{i_1 \dots i_k i_{k+1}}, 4\rho_{i_1 \dots i_k i_{k+1}}) \subset B(x_{i_1 \dots i_k}, 2\rho_{i_1 \dots i_k}),\\
&E \cap B(x_{i_1 \dots i_k}, 18\rho_{i_1 \dots i_k}) \setminus B(x_{i_1 \dots i_k}, \rho_{i_1 \dots i_k}) = \emptyset,\\
&y_{i_1 \dots i_k i_{k+1}} \in F \cap B(y_{i_1 \dots i_k}, d^k),\\
&B(y_{i_1 \dots i_k}, 6Dd^k) \cap B(y_{j_1 \dots j_k}, 6Dd^k) = \emptyset \text{ if } i_k \neq j_k.
\end{aligned}$$

Using Lemma 4.1 we find a bilipschitz map $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_1(x) = x$ for $|x| > 2$ and $f_1(x) = x - x_i + y_i$ for $x \in B(x_i, 2\rho_i)$, and $\mathrm{bilip}(f) \leq L$ where L depends only on s, t, n and C. Let

$$B_k = \bigcup_{i_1 \dots i_k} B(x_{i_1 \dots i_k}, 2\rho_{i_1, \dots, i_k}).$$

Then $B_{k+1} \subset B_k$ for all k and $E = \bigcap_{k=1}^{\infty} B_k$. We use Lemma 4.1 to define inductively $f_k : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_{k+1}(x) = f_k(x)$ for $x \in \mathbb{R}^n \setminus B_k^o$, where B_k^o is the interior of B_k, $f_{k+1} | B(x_{i_1 \dots i_k}, 2\rho_{i_1, \dots, i_k})$ is L-bilipschitz and $f_{k+1}(x) = x - x_{i_1, \dots, i_{k+1}} + y_{i_1, \dots, i_{k+1}}$ for $x \in B(x_{i_1 \dots i_{k+1}}, 2\rho_{i_1, \dots, i_{k+1}})$. We check now by induction that

$$|x - y|/L \leq |f_k(x) - f_k(y)| \leq L|x - y| \text{ for all } x, y \in \mathbb{R}^n. \tag{4.3}$$

For $k=1$ this was already stated. Suppose this is true for $k-1$ for some $k \geq 2$ and let $x, y \in \mathbb{R}^n$. If $x, y \in \mathbb{R}^n \setminus B_k^o$, (4.3) follows from the definition of f_k and the induction hypothesis. If $x, y \in B(x_{i_1\dots i_k}, 2\rho_{i_1,\dots,i_k})$ for some $i_1 \dots i_k$, then (4.3) follows from the fact that f_k is a translation in $B(x_{i_1\dots i_k}, 2\rho_{i_1,\dots,i_k})$. Finally, let $x \in B(x_{i_1\dots i_k}, 2\rho_{i_1,\dots,i_k})$ and $y \in \mathbb{R}^n \setminus B(x_{i_1\dots i_k}, 2\rho_{i_1,\dots,i_k})$. Let $z \in \partial B(x_{i_1\dots i_k}, 2\rho_{i_1,\dots,i_k})$ be the point on the line segment with end points x and y. Then, by the two previous cases,

$$|f_k(x) - f_k(y)| \leq |f_k(x) - f_k(z)| + |f_k(z) - f_k(y)| \leq$$
$$L|x-z| + L|z-y| = L|x-z|.$$

This proves the right hand inequality of (4.3). A similar argument for f_k^{-1} with the balls $B(y_{i_1\dots i_k}, 2\rho_{i_1,\dots,i_k})$ gives the left hand inequality.

We have left to show that the limit $\lim_{k\to\infty} f_k(x) = f(x)$ exists for all $x \in \mathbb{R}^n$. Then also f satisfies (4.3) and $f(E) \subset F$. First, if $x \in \mathbb{R}^n \setminus E$, then $x \in \mathbb{R}^n \setminus B_l$ for some l, and so $f_k(x) = f_l(x)$ for $k \geq l$. If $x \in E$, there are $i_1, i_2, \dots,$ such that $x \in B(x_{i,\dots i_k}, 2\rho_{i_1\dots i_k})$ for all k. Then $f_k(x) \in B(y_{i_1\dots i_k}, 2Dd^k)$ and $\lim_{k\to\infty} f_k(x) = y = f(x)$ where $y = \lim_{k\to\infty} y_{i_1\dots i_k}$.

□

5. SUB- AND SUPERSETS

In this section we shall consider the question whether a given regular set contains regular subsets of smaller dimension and whether it is contained in higher dimensional regular sets.

5.1. **Theorem.** *Let $E \subset X$ be s-regular and $0 < t < s$. For every $x \in E$ and $0 < r < d(E)$, $E \cap B(x,r)$ contains a t-regular subset F such that $C_F \leq C$ and $d(F) \geq cr$ where C and c are positive constants depending only on s, t and C_E.*

This can be proven with the same method as Theorem 3.1. In fact, that method gives that $E \cap B(x,r)$ has a t-regular subset which is bilipschitz equivalent with $C(t,r)$ with a bilipschitz constant depending only on s, t and C_E. Observe that the regularity of E implies that $d(E \cap B(x,r)) \geq C_E^{-1/s} r$.

5.2. **Theorem.** *Let $0 < s < t < u$. Suppose that $E \subset X$ is s-regular and that X is u-regular. Then there is a t-regular set F with $E \subset F \subset X$. Moreover, $C_F \leq C$ where C depends only on s, t, C_E and C_X.*

Proof. We shall only consider the case where X and E are bounded. A slight modification of the proof works if X or both X and E are unbounded. Recalling the remarks at the beginning of Section 3, we may assume that $d(E) = 1$. Let $0 < d < 1/30$ be such that $d^{u-s} < 4^{-s}30^{-u}C_E^{-1}C_X^{-1}$. By Lemma 2.1 there are for every $k = 1, 2\dots,$ disjoint balls $B(x_{k,i}, 6d^k), i = 1, \dots, m_k$, such that $x_{k,i} \in E$ and the balls $B(x_{k,i}, 30d^k)$ cover E. Further, there are disjoint balls $B(x_{k,i}, 6d^k), i = m_k + 1, \dots, n_k$, such that $x_{k,i} \in X \setminus \cup_{i=1}^{m_k} B(x_{k,i}, 30d^k)$ and the balls $B(x_{k,i}, 30d^k), i = 1, \dots, n_k$, cover X.

Fix k and $i, 1 \le i \le m_k$. Denote

$$J = \{j \in \{1, \dots, n_k\} : B(x_{k+1,j}, d^{k+1}) \subset B(x_{k,i}, 3d^k)\},$$
$$J' = \{j \in \{1, \dots, n_k\} : B(x_{k+1,j}, 30d^{k+1}) \cap B(x_{k,i}, d^k) \neq \emptyset\},$$
$$I = \{j \in J : E \cap B(x_{k+1,j}, 6d^{k+1}) \neq \emptyset\},$$

and let n, n' and m be the number of indices in J, J' and I, respectively. Then, as $d < 2/31$, $J' \subset J$ and so $n' \le n$. Since $B(x_{k,i}, d^k) \subset \cup_{j \in J'} B(x_{k+1,j}, 30d^{k+1})$, we have, comparing measures as in the proof of Lemma 2.1, that $n \ge n' \ge 30^{-u}C_X^{-1}d^{-u}$. If $j \in I$ there is $z_j \in E \cap B(x_{k+1,j}, 6d^{k+1})$ and then, as also $j \in J$ and $d < 1/7$,

$$B(z_j, d^{k+1}) \subset B(x_{k+1,j}, 7d^{k+1}) \subset B(x_{k,i}, 4d^k).$$

Then the balls $B(z_j, d^{k+1}), j \in I$, are disjoint and

$$md^{(k+1)s} \le \sum_{j \in I} \mu(B(z_j, d^{k+1})) \le \mu(B(x_{k,i}, 4d^k)) \le 4^s C_E d^{ks},$$

whence $m \le 4^s C_E d^{-s}$. Combining these inequalities and recalling the choice of d, we find that

$$m \le 4^s C_E d^{-s} < 30^{-u} C_X^{-1} d^{-u} \le n.$$

Thus we can choose some $j \in J \setminus I$. Let $y_{k,i} = x_{k+1,j}$ and $B_{k,i} = B(y_{k,i}, d^{k+1})$. Denote also $2B_{k,i} = B(y_{k,i}, 2d^{k+1})$. Then for a fixed k the balls $2B_{k,i}, i = 1, \dots, m_k$, are disjoint. If $x \in 2B_{k,i}$, then, as $j \notin I$, $d(x, E) \ge 4d^{k+1}$. On the other hand, as $j \in J$, $d(x, E) \le d(x, x_{k,i}) \le d(x, y_{k,i}) + d(y_{k,i}, x_{k,i}) \le 2d^{k+1} + 3d^k < d^{k-1}$. It follows that the balls $2B_{k,i}$ and $2B_{l,j}$ with $|k - l| \ge 2$ are always disjoint. Hence any point of X can belong to at most two balls $2B_{k,i}, i = 1, \dots, m_k, k = 1, 2, \dots$.

By Theorem 5.1 we can choose for every $k, i, 1 \le i \le m_k$, t-regular sets $F_{k,i} \subset B_{k,i}$ such that $C_{F_{k,i}} \le C$ and $d(F_{k,i}) \ge cd^k$ with C and c depending only on t, u and C_X. Let $\nu_{k,i}$ be the Borel measure related to $F_{k,i}$ as in Definition 1.1. We define

$$F = E \cup \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{m_k} F_{k,i}$$

and

$$\nu = \sum_{k=1}^{\infty} \sum_{i=1}^{m_k} \nu_{k,i}.$$

Then F is a closed and bounded subset of X containing E.

We check now that F is t-regular. Let $x \in F$ and $0 < r \leq d(F)$. It is enough to verify the required inequalities for $r < d$, so we assume this. Let l be the positive integer for which $d^{l+1} \leq r < d^l$. Denote

$$K = \{(k,i) : i = 1, \dots, m_k, k < l \text{ and } B_{k,i} \cap B(x,r) \neq \emptyset\}$$

and

$$L = \{(k,i) : i = 1, \dots, m_k, k \geq l \text{ and } B_{k,i} \cap B(x,r) \neq \emptyset\}.$$

We have

$$\nu(B(x,r)) \leq \sum_{(k,i)\in K} \nu_{k,i}(B_{k,i} \cap B(x,r)) + \sum_{(k,i)\in L} \nu_{k,i}(B_{k,i} \cap B(x,r)).$$

If $(k,i) \in K$, then $r < d^{k+1}$ and $B(x,r) \subset 2B_{k,i}$. Since this can happen for at most two balls $2B_{k,i}$, K can contain at most two elements and the first sum above is bounded by $2^{t+1}Cr^t$. To estimate the second sum, let p_k be the number of indices in $I_k = \{i : (k,i) \in L\}$. Let $(k,i) \in L$. Then $B_{k,i} \cap B(x,r) \neq \emptyset$, and so $d(x_{k,i}, x) \leq d(x_{k,i}, y_{k,i}) + d(y_{k,i}, x) \leq 3d^k + 2d^{k+1} + r < 5d^l$, which gives $B(x_{k,i}, d^k) \subset B(x, 6d^l)$. Consequently,

$$p_k d^{ks} \leq \sum_{i \in I_k} \mu(B(x_{k,i}, d^k)) \leq \mu(B(x, 6d^l)) \leq C_E 12^s d^{ls},$$

and so $p_k \leq 12^s C_E d^{(l-k)s}$. Hence

$$\sum_{(k,i)\in L} \nu_{k,i}(B_{k,i} \cap B(x,r)) \leq \sum_{k=l}^{\infty} 12^s C_E d^{(l-k)s} C 4^t d^{(k+1)t} \leq$$

$$12^s 4^t C_E C d^{ls} \sum_{k=l}^{\infty} d^{(t-s)k} = 12^s 4^t C_E C d^{lt} \frac{1}{1 - d^{(t-s)}} \leq$$

$$12^s 4^t C_E C d^{-t} \frac{1}{1 - d^{(t-s)}} r^t.$$

This proves the upper regularity of ν.

To prove the opposite inequality, suppose first that $x \in E$. Let k be the positive integer for which $33d^k \leq r < 33d^{k-1}$. Then for some

$i, 1 \leq i \leq m_k, x \in B(x_{k,i}, 30d^k)$. Since $B_{k,i} \subset B(x_{k,i}, 3d^k)$ we have that $B_{k,i} \subset B(x, 33d^k) \subset B(x, r)$. Thus

$$\nu(B(x,r)) \geq \nu_{k,i}(B_{k,i}) \geq d(F_{k,i})^t \geq c^t d^{kt} \geq c^t d^t 33^{-t} r^t.$$

Suppose finally that $x \in F_{k,i}$ for some k and i. If $r \leq 9d^k$, then $d(F_{k,i}) \geq cd^k \geq (c/9)r$, whence

$$\nu(B(x,r)) \geq \nu(B(x,(c/9)r) \geq (c/9)^t r^t.$$

If $r > 9d^k$, then $d(x, x_{k,i}) \leq 3d^k < r/3$, so $B(x_{k,i}, r/2) \subset B(x, r)$. Since $x_{k,i} \in E$, the required inequality follows from the case $x \in E$.

□

In the next example note that $\lim_{r\to 0} \mathcal{L}^1(F \cap B(x,r))/(2r) = 1$ for $\mathcal{L}^1$ almost all $x \in F$ by the Lebesgue density theorem. However, F has no subset E with $\mathcal{L}^1(E) > 0$ for which $\mathcal{L}^1(F \cap B(x,r))/(2r)$ would be bounded below with a positive number uniformly for small $r > 0$.

5.3**. Example.** There exists a compact set $F \subset \mathbb{R}$ with Lebesgue measure $\mathcal{L}^1(F) > 0$ such that it contains no non-empty s-regular subset for any $s > 0$.

Proof. Let $a < b, 0 < \lambda < 1/2$ and $0 < t < 1$. We shall construct a family $\mathcal{I}([a,b], \lambda, t)$ of closed disjoint subintervals of $[a,b]$. We do this for $[0,1]$ and then define $\mathcal{I}([a,b], \lambda, t) = \{f(I) : I \in \mathcal{I}([0,1], \lambda, t)\}$ where $f(x) = (b-a)x + a$.

Let

$$I_{1,1} = [(1-\lambda)/2, (1+\lambda)/2].$$

Then $[0,1] \setminus I_{1,1}$ consists of two intervals $J_{1,1}$ and $J_{1,2}$ of length $(1-\lambda)/2$. We select closed intervals $I_{2,1}$ and $I_{2,2}$ of length $\lambda(1-\lambda)/2$ in the middle of them (that is, the center of $I_{2,i}$ is the center of $J_{1,i}$). Continuing this we get intervals $I_{k,i}, i = 1, \ldots, 2^{k-1}$, and $J_{k,i}, i = 1, \ldots, 2^k$, such that $d(I_{k,i}) = 2^{1-k}\lambda(1-\lambda)^{k-1}$ and $d(J_{k,i}) = 2^{-k}(1-\lambda)^k$. Moreover, each $I_{k,i}$ is the mid-interval of some $J_{k-1,j}$ and $J_{k-1,j} \setminus I_{k,i}$ consists of two intervals J_{k,j_1} and J_{k,j_2}. Then

$$\begin{aligned}\sum_{k=1}^{l} \sum_{i=1}^{2^{k-1}} d(I_{k,i}) &= \sum_{k=1}^{l} \lambda(1-\lambda)^{k-1} \\ &= 1 - (1-\lambda)^l \to 1 \text{ as } l \to \infty.\end{aligned}$$

We choose l such that

$$\sum_{k=1}^{l} \sum_{i=1}^{2^{k-1}} d(I_{k,i}) > t$$

and denote

$$\mathcal{I}([0,1],\lambda,t) = \{I_{k,i} : i = 1,\dots,2^{k-1}, k = 1,\dots,l\}.$$

Then for any compact interval $I \subset \mathbb{R}$,

$$\sum_{J\in\mathcal{I}(I,\lambda,t)} d(J) > td(I).$$

Let $0 < \lambda_k < 1/2, 0 < t_k < 1, k = 1,2,\dots,$ such that $\lim_{k\to\infty}\lambda_k = 0$ and $t = \prod_{k=1}^{\infty} t_k > 0$. Define

$$\mathcal{I}_1 = \mathcal{I}([0,1],\lambda_1,t_1),$$

and inductively for $m = 1,2,\dots,$

$$\mathcal{I}_{m+1} = \{J : J \in \mathcal{I}(I,\lambda_{m+1},t_{m+1}), I \in \mathcal{I}_m\}.$$

The compact set F is now defined as

$$F = \bigcap_{m=1}^{\infty}\bigcup_{I\in\mathcal{I}_m} I.$$

For every $m = 1,2,\dots$ we have

$$\sum_{I\in\mathcal{I}_m} d(I) > t_1\cdot\dots\cdot t_m > t,$$

whence $\mathcal{L}^1(F) \geq t$.

Suppose that $s > 0$ and that E is an s-regular subset of F. Choose m so large that $\lambda_m < C_E^{-s}/4$. Let $x \in E$. Then for every $m = 1,2,\dots, x \in I$ for some $I \in \mathcal{I}_m$. Suppose that I would be one of the shortest intervals in the family $\mathcal{I}_m$. Then by our construction there is an interval J such that I is in the middle of $J, I\cap E = J\cap E$ and $d(I) = \lambda_m d(J)$. As $B(x,d(J)/4) \subset J$ we have by the regularity of E,

$$\begin{aligned}&4^{-s}d(J)^s \leq \mu(B(x,d(J)/4)) =\\ &\mu(B(x,d(I))) \leq C_E d(I)^s = C_E(\lambda_m d(J))^s.\end{aligned}$$

Thus $\lambda_m \geq C_E^{-1/s}/4$. This contradicts with the choice of m. So E contains no points in the shortest intervals of $\mathcal{I}_m$. But then we can repeat the same argument with the second shortest intervals of $\mathcal{I}_m$ concluding that neither can they contain any points of E. Continuing this we see that $E = \emptyset$.

□

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