LIPSCHITZ CONDITIONS FOR $\mathrm{Sub}_\varphi(\Omega)$-PROCESSES WITH APPLICATION TO WEAKLY SELF-SIMILAR STATIONARY INCREMENT PROCESSES

YURIY KOZACHENKO, TOMMI SOTTINEN, AND OLGA VASYLYK

ABSTRACT. We study the Lipschitz continuity of generalized sub-Gaussian processes, and provide estimates for the distribution of the norms of such processes. The results are applied to the case of weakly self-similar stationary-increment generalized sub-Gaussian processes (the fractional Brownian motions are special cases).

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1. Introduction

Let (T,ρ) be some pseudometric space. We consider the Lipschitz continuity of stochastic processes $X=(X(t),t\in T)$, and provide estimates for the distribution of norms of such processes. In particular, we provide function f, the modulus of continuity, such that

$$\limsup_{\varepsilon\to 0} \frac{\sup\limits_{\rho(t,s)<\varepsilon} |X(t)-X(s)|}{f(\varepsilon)} < 1$$

and estimates for the probabilities

$$\mathbf{P}\left\{ \sup_{0<\rho(t,s)\le v} \frac{|X(t)-X(s)|}{f(\rho(t,s))} > y \right\}.$$

The case when (T,ρ) is a subset of a d-dimensional Euclidean space is considered as an example.

Obtained results are applied then to the weakly self-similar stationary-increment processes (wsssi, for short) from the space $\mathrm{Sub}_\varphi(\Omega)$ of generalized sub-Gaussian processes.

For Gaussian processes the moduli of continuity f were found by Dudley [2]. These results were generalized for some classes of processes from Orlicz spaces in the paper by Kozachenko [4]. In [1] for random processes from some classes Δ of Orlicz spaces besides modula of continuity also there were found estimates for distributions of norms of such processes in Lipschitz spaces.

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2. Preliminaries

2.1. **Space** $\mathrm{Sub}_\varphi(\Omega)$**.** We recall briefly some basic facts about the generalized sub-Gaussian space $\mathrm{Sub}_\varphi(\Omega)$ [1, 3].

Definition 2.1. A continuous even convex function u is an *Orlicz N-function* if it is increasing for $x>0$, $\frac{u(x)}{x}\to 0$ as $x\to 0$ and $\frac{u(x)}{x}\to\infty$ as $x\to\infty$.

For details on convex functions in Orlicz spaces we refer to Krasnoselskii and Rutitskii [5].

Let $(\Omega,\mathscr{F},\mathbf{P})$ be a standard probability space.

Definition 2.2. Let φ be an Orlicz N-function such that

$$\liminf_{x\to 0}\frac{\varphi(x)}{x^2}=C>0$$

(*condition Q*). The constant C may be equal to $+\infty$. A zero mean random variable ξ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ if there exists a positive constant a such that the inequality

$$\mathbf{E}\exp(\lambda\xi)\le\exp(\varphi(a\lambda))$$

holds for all $\lambda\in\mathbb{R}$.

Example 2.3. The following functions are N-functions satisfying condition Q:

$$\varphi(x)=\frac{|x|^\alpha}{\alpha},\ 1<\alpha\le 2;$$

$$\varphi(x)=\begin{cases}\frac{|x|^2}{\alpha}, & |x|\le 1,\alpha>2;\\ \frac{|x|^\alpha}{\alpha}, & |x|>1.\end{cases}$$

The space $\mathrm{Sub}_\varphi(\Omega)$ is a Banach space with respect to the norm

$$\tau_\varphi(\xi)=\sup_{\lambda\neq 0}\frac{\varphi^{-1}(\ln\mathbf{E}\exp(\lambda\xi))}{|\lambda|}$$

and the inequalities

$$\begin{aligned}\mathbf{E}\exp(\lambda\xi) &\le \exp(\varphi(\lambda\tau_\varphi(\xi))), \qquad (2.1)\\ (\mathbf{E}\xi^2)^{\frac12} &\le C\tau_\varphi(\xi).\end{aligned}$$

hold for all $\lambda\in\mathbb{R}$, where $C>0$ is some constant.

Definition 2.4. Let (T,ρ) be a pseudometric space. The *metric entropy* is

$$H(u):=\ln N_{(T,\rho)}(u)$$

where $N_{(T,\rho)}(u)$ denotes the least number of closed ρ-balls whose diameter do not exceed $2u$ needed to cover T.

Definition 2.5. Let (T,ρ) be pseudometric separable space. A stochastic process $X=(X(t),t\in T)$ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ if $X(t)\in\mathrm{Sub}_\varphi(\Omega)$ for all $t\in T$.

2.2. **Auxiliary theorem.** Recall that the *Young-Fenchel transformation* φ^* of an Orlicz N-function φ is

$$\varphi^*(x) := \sup_{y>0} \left(xy - \varphi(y)\right), \quad x \geq 0.$$

The following theorem is rather technical, but it is needed to get our main results.

Theorem 2.6. *Let* $\{\xi_i\}_{i=1}^n \in \mathrm{Sub}_{\varphi}(\Omega)$, $x > 2$, M *and* b *be such numbers that* $b > 1$, $M \geq \frac{\varphi^*(2)}{\ln(2)}$, *then*

$$\begin{aligned} &\mathbf{P}\left\{\max_{j=\overline{1,n}} |\xi_j| > xb \max_{j=\overline{1,n}} \tau_{\varphi}(\xi_j) \cdot \varphi^{*(-1)}(M\ln(n))\right\} \\ &\leq n^{1-M} \frac{b+1}{b-1} \exp\{-\varphi^*(x)\}. \end{aligned} \tag{2.2}$$

Proof. Let $\eta\colon = \max_{j=\overline{1,n}} |\xi_j|$, $a\colon = b \max_{j=\overline{1,n}} \tau_{\varphi}(\xi_j)$, $u_n\colon = \varphi^{*(-1)}(M\ln(n))$.

$$\begin{aligned} \mathbf{P}\{\eta > xau_n\} &= \mathbf{E1}\{\omega\colon \eta > xau_n\} \\ &\leq \sum_{j=1}^n \mathbf{E1}\{\eta = |\xi_j|\} \cdot \mathbf{1}\{\omega\colon |\xi_j| > xau_n\} \\ &\leq n \max_{j=\overline{1,n}} \mathbf{E1}\{\omega\colon |\xi_j| > xau_n\} \qquad (2.3) \\ &\leq n^{1-M} n^M \max_{j=\overline{1,n}} \mathbf{E1}\{\omega\colon |\xi_j| > xau_n\} \cdot \frac{\exp\{\varphi^*(\frac{|\xi_j|}{au_n})\}}{\exp\{\varphi^*(x)\}}. \end{aligned}$$

Since if $\frac{|\xi_j|}{au_n} > x > 2$, $n \geq 2$ and $M\ln(n) \geq \varphi^*(2)$ (that is, $u_n \geq 2$) then

$$\begin{aligned} n^M \exp\left\{\varphi^*\left(\frac{|\xi_j|}{au_n}\right)\right\} &= \exp\left\{M\ln(n) + \varphi^*\left(\frac{|\xi_j|}{au_n}\right)\right\} \\ &= \exp\left\{\varphi^*(\varphi^{*(-1)}(M\ln(n)) + \varphi^*\left(\frac{|\xi_j|}{au_n}\right)\right\} \\ &\leq \exp\left\{\varphi^*(\varphi^{*(-1)}(M\ln(n)) + \frac{|\xi_j|}{au_n}\right\} \\ &= \exp\left\{\varphi^*\left(u_n + \frac{|\xi_j|}{au_n}\right)\right\} \leq \exp\left\{\varphi^*\left(\frac{|\xi_j|}{a}\right)\right\}. \end{aligned}$$

Therefore we have

$$\begin{aligned} \mathbf{P}\{\eta > xau_n\} &= n^{1-M} \exp\{-\varphi^*(x)\} \max_{j=\overline{1,n}} \mathbf{E} \exp\left\{\varphi^*\left(\frac{|\xi_i|}{a}\right)\right\} \\ &= n^{1-M} \exp\{-\varphi^*(x)\} \max_{j=\overline{1,n}} \mathbf{E} \exp\left\{\varphi^*\left(\frac{|\xi_i|}{b\tau_{\varphi}|\xi_i|}\right)\right\}. \qquad (2.4) \end{aligned}$$

In the book [1, Corollary 4.1] it is shown that if

$$\mathbf{P}\{|\xi| > x\} \leq C \exp\left\{\varphi^*\left(\frac{x}{D}\right)\right\},$$

where $C > 0$, $D > 0$ then for all $A > D$ we have

$$\mathbf{E}\exp\left\{\varphi^*\left(\frac{\xi}{A}\right)\right\} \le 1 + \frac{CD}{A-D}. \tag{2.5}$$

From [1, Lemma 4.3] we also have that

$$\mathbf{P}\{|\xi| > x\} \le 2\exp\left\{-\varphi^*\left(\frac{\xi}{\tau_\varphi(\xi)}\right)\right\}$$

then it follows from (2.5) that for $b > 1$

$$\mathbf{E}\exp\left\{\varphi^*\left(\frac{\xi_j}{b\tau_\varphi(\xi_j)}\right)\right\} \le \frac{b+1}{b-1}.$$

Therefore

$$\mathbf{P}\{\eta > xau_n\} \le n^{1-M}\frac{b+1}{b-1}\exp\{-\varphi^*(x)\}.$$

□

3. Main results

Let (T,ρ) be a metric (pseudometric) separable compact space, $X = \{X(t),\ t \in T\}$ be a separable random process from the space $\mathrm{Sub}_\varphi(\Omega)$.

Suppose that there exists a monotonically increasing continuous function $\sigma = \{\sigma(h), h \ge 0\}$ such that $\sigma(0) = 0$ and the following inequality holds

$$\sup_{\rho(t,s)\le h} \tau_\varphi(X(t) - X(s)) \le \sigma(h). \tag{3.1}$$

Let $N(u)$ be the least number of closed balls of radius u covering (T,ρ).

Theorem 3.1. *Let $N(u) \to \infty$ as $u \to 0$, $M \ge \max\bigl(1, \frac{\varphi^*(2)}{\ln(2)}\bigr)$,*

$$f_B(u) = \frac{1}{(11 - 2\sqrt{30})}\int_0^{\sigma(u)} \varphi^{*(-1)}(2M\ln(BN(\sigma^{(-1)}(v))))\,dv < \infty,$$

where $B > 1$, $b > 1$ are some numbers, and v is such a number that $N(v) > 2$. Then for $y > 2b$ the following inequality holds true

$$\mathbf{P}\left\{\sup_{0<\rho(t,s)\le v}\frac{|X(t)-X(s)|}{f_B(\rho(t,s))} > y\right\}$$
$$\le \frac{4B^{M-1}(b+1)}{(N(v))^{M-1}(B^{M-1}-1)(b-1)}\exp\left\{-\varphi^*\left(\frac{y}{b}\right)\right\}. \tag{3.2}$$

Proof. Let $r \in (0,1)$, $\{\nu_k, k = 0,1,2,\ldots\}$, be a such sequence that $\nu_0 = \inf_{s\in T}\sup_{t\in T}\rho(t,s)$, $\nu_{k+1} = \min\{r\nu_k, \delta_k\}$, where

$$\delta_k = A\inf\{\nu\colon N(\sigma^{(-1)}(\nu)) < BN(\sigma^{(-1)}(\nu_k))\}, \tag{3.3}$$

where $\sigma^{(-1)}(\nu)$ is the inverse function of the function σ, $B > 1$, A is such a number that $A > 1$ and $Ar < 1$. For sequence $\{\nu_k, k = 0,1,2,\ldots\}$ we have

$$\nu_{k+1} \le r\nu_k, \quad k = 0,1,2,\ldots \tag{3.4}$$

that is

$$\nu_k \le \frac{1}{1-r}(\nu_k - \nu_{k+1}). \tag{3.5}$$

From (3.3) and (3.4) we have that

$$\begin{aligned} N(\sigma^{(-1)}(\nu_{k+2})) &\ge N(\sigma^{(-1)}(r\nu_{k+1})) \\ &\ge N(\sigma^{(-1)}(r\delta_k)) \ge BN(\sigma^{(-1)}(\nu_k)). \end{aligned} \tag{3.6}$$

That is

$$N(\sigma^{(-1)}(\nu_k)) \ge BN(\sigma^{(-1)}(\nu_{k-2})) \ge B^2 N(\sigma^{(-1)}(\nu_{k-4})) \ge \dots \tag{3.7}$$

Let $\varepsilon_0 = \sigma^{(-1)}(\nu_0)$, ..., $\varepsilon_k = \sigma^{(-1)}(\nu_k)$. Let V_{ε_k}, $k = 0, 1, 2, \dots$, be a set of the centers of closed balls of radius ε_k that form a minimal covering of the space (T, ρ). The number of points in V_{ε_k} is equal to $N(\varepsilon_k) = N(\sigma^{(-1)}(\nu_k))$. Let $V_0 = \bigcup_{k=0}^{\infty} V_{\varepsilon_k}$. It follows from (3.1) using Chebyshev inequality that the process X is continuous in probability. Therefore the set V_0 is a set of separability of the process X. Let α_n be the mapping of the set V_0 into V_{ε_n}, where $\alpha_n(t) = t$, if $t \in V_{\varepsilon_n}$ and otherwise $\alpha_n(t)$ is a point in V_{ε_n} satisfying $\rho(t, \alpha_n(t)) < \varepsilon_n$. It follows from Chebyshev inequality, (3.1) and (3.4) that

$$\begin{aligned} \mathbf{P}\{|X(t) - X(\alpha_n(t))| > r^{\frac{n}{2}}\} &\le \frac{\mathbf{E}(X(t) - X(\alpha_n(t)))^2}{r^n} \\ &\le \frac{C\tau_\varphi^2(X(t) - X(\alpha_n(t)))}{r^n} \\ &\le \frac{C\sigma^2(\rho(t, \alpha_n(t)))}{r^n} \\ &\le \frac{C\sigma^2(\varepsilon_n)}{r^n} \le \frac{C\nu_n^2}{r^n} \le \frac{C\nu_0^2 r^{2n}}{r^n} = C\nu_0^2 r^n, \end{aligned}$$

where $C > 0$ is some constant.

Therefore

$$\sum_{n=1}^{\infty} \mathbf{P}\{|X(t) - X(\alpha_n(t))| > r^{\frac{n}{2}}\} < \infty.$$

Now it follows from Borel-Cantelli lemma that $X(\alpha_n(t)) \to X(t)$ with probability one as $n \to \infty$. Since the set V_0 is countable then $X(\alpha_n(t)) \to X(t)$ as $n \to \infty$ with probability one for all $t \in V_0$.

Take $0 < u \le \varepsilon_0$, and choose such m that $\varepsilon_{m+1} < u \le \varepsilon_m$. Since V_0 is a set of separability of the process X, then with probability one

$$\sup_{\substack{\rho(t,s)<u \\ t,s\in T}} |X(t) - X(s)| = \sup_{\substack{\rho(t,s)<u \\ t,s\in V_0}} |X(t) - X(s)|. \tag{3.8}$$

Let t and s belong to V_0 and $\rho(t, s) < u$. Let $k > m + 1$. Denote $t_k = \alpha_k(t)$, $t_{k-1} = \alpha_{k-1}(t_k)$, ..., $t_m = \alpha_m(t_{m+1})$; $s_k = \alpha_k(s)$ $s_{k-1} = \alpha_{k-1}(s_k), \dots,$ $s_m =$

$\alpha_m(t_{m+1})$. Then for any t, s such that $\rho(t,s) < u$ we have

$$\begin{aligned} X(t) - X(s) = (X(t) - X(t_k)) &+ \sum_{l=m+2}^{k} (X(t_l) - X(t_{l-1})) \\ &- (X(s) - X(s_k)) - \sum_{l=m+2}^{k} (X(s_l) - X(s_{l-1})) \\ &+ (X(t_{m+1}) - X(s_{m+1})). \end{aligned} \tag{3.9}$$

It follows from (3.9) that

$$\begin{aligned} X(t_{m+1}) - X(s_{m+1}) = (X(t) - X(s)) &- (X(t) - X(t_k)) \\ &+ (X(s) - X(s_k)) - \sum_{l=m+2}^{k} (X(t_l) - X(t_{l-1})) \\ &+ \sum_{l=m+2}^{k} (X(s_l) - X(s_{l-1})) \end{aligned}$$

and

$$\begin{aligned} &\tau_\varphi(X(t_{m+1}) - X(s_{m+1})) \\ &\le \tau_\varphi(X(t) - X(s)) + \tau_\varphi(X(t) - X(t_k)) + \tau_\varphi(X(t_s) - X(s_k)) \\ &+ \sum_{l=m+2}^{k} \tau_\varphi(X(t_l) - X(t_{l-1})) + \sum_{l=m+2}^{k} \tau_\varphi(X(s_l) - X(s_{l-1})) \\ &\le \sigma(\rho(t,s)) + \sigma(\rho(t,t_k)) + \sigma(\rho(s,s_k)) + \sum_{l=m+2}^{k} \sigma(\rho(t_l, t_{l-1})) \\ &+ \sum_{l=m+2}^{k} \sigma(\rho(s_l, s_{l-1})) \\ &\le \sigma(u) + 2\sigma(\varepsilon_k) + 2\sum_{l=m+2}^{k} \sigma(\varepsilon_{l-1}) \qquad (3.10) \\ &\le \sigma(u) + 2\sum_{l=m+2}^{\infty} \sigma(\varepsilon_{l-1}) = \sigma(u) + 2\sum_{l=m+2}^{\infty} \nu_{l-1} \\ &\le \sigma(u) + 2\sum_{l=1}^{\infty} \nu_{m+l} \le \sigma(u) + 2\sum_{l=1}^{\infty} \nu_{m+1} r^{l-1} \\ &= \sigma(u) + \nu_{m+1}\frac{2}{1-r} \le \sigma(u)\left(1 + \frac{2}{1-r}\right) \\ &= \sigma(u)\frac{3-r}{1-r}. \end{aligned}$$

It follows from (3.9) and (3.10) that for all $t,s\in T$ such that $\rho(t,s)<u$ we have

$$\begin{aligned}
&|X(t)-X(s)|\\
&\le \sum_{l=m+2}^{k}|X(t_l)-X(t_{l-1})|+\sum_{l=m+2}^{k}|X(s_l)-X(s_{l-1})|\\
&+|X(t)-X(t_k)|+|X(s)-X(s_k)|+|X(t_{m+1})-X(s_{m+1})|\\
&\le 2\sum_{l=m+2}^{k}\max_{w\in V_{\varepsilon_l}}|X(w)-X(\alpha_{l-1}(w))| \qquad (3.11)\\
&+\max_{\substack{w,v\in V_{\varepsilon_{m+1}}:\\ \tau_\varphi(X(w)-X(v))\le\sigma(u)\frac{3-r}{1-r}}}|X(w)-X(v)|\\
&+|X(t)-X(t_k)|+|X(s)-X(s_k)|.
\end{aligned}$$

Now making $k\to\infty$ in (3.11) we have that with probability one

$$\begin{aligned}
|X(t)-X(s)|\le 2&\sum_{l=m+2}^{k}\max_{w\in V_{\varepsilon_l}}|X(w)-X(\alpha_{l-1}(w))|\\
&+\max_{\substack{w,v\in V_{\varepsilon_{m+1}}:\\ \tau_\varphi(X(w)-X(v))\le\sigma(u)\frac{3-r}{1-r}}}|X(w)-X(v)|.
\end{aligned}$$

That is, it follows from (3.8) that

$$\begin{aligned}
\sup_{\substack{\rho(t,s)\le u\\ t,s\in T}}|X(t)-X(s)|=&\sup_{\substack{\rho(t,s)\le u\\ t,s\in V_0}}|X(t)-X(s)|\\
\le&2\sum_{k=m+2}^{\infty}\max_{w\in V_{\varepsilon_k}}|X(w)-X(\alpha_{l-1}(w))| \qquad (3.12)\\
&+\max_{\substack{w,v\in V_{\varepsilon_{m+1}}\\ \tau_\varphi(X(w)-X(v))\le\sigma(u)\frac{3-r}{1-r}}}|X(w)-X(v)|.
\end{aligned}$$

Let

$$\begin{aligned}
&c_l=b\sigma(\varepsilon_{l-1})\varphi^{*(-1)}(M\ln(N(\varepsilon_l))),\\
&b_m(u)=b\varphi^{*(-1)}(M\ln(N^2(\varepsilon_{m+1})))\sigma(u)\frac{3-r}{1-r},\\
&\varepsilon_{m+1}<u\le\varepsilon_m.
\end{aligned}$$

Let

$$\xi_l=\max_{t\in V_{\varepsilon_l}}|X(t)-X(\alpha_{l-1}(t))|$$

and for $\varepsilon_{m+1}<u\le\varepsilon_m$

$$\eta_m(u)=\max_{\substack{w,z\in V_{\varepsilon_{m+1}}\\ \tau_\varphi(X(w)-X(z))\le\sigma(u)\frac{3-r}{1-r}}}|X(w)-X(z)|.$$

Let $v > 0$ be such that $N(v) > 2$ and n be such a number that $\varepsilon_{n+1} < V \le \varepsilon_n$. Let $\{G(u), u \ge 0\}$, be such a function that $G(u)$ increases and

$$G(u) \ge 2 \sum_{l=m+2}^{\infty} c_l + b_m(u),$$

where m is such a number that $\varepsilon_{m+1} < u \le \varepsilon_m$. Then if $x > 2$, $N(v) > 2$

$$\begin{aligned}
&\mathbf{P}\Big\{ \sup_{0<\rho(t,s)\le v} \frac{|X(t)-X(s)|}{G(\rho(t,s))} > x \Big\} \\
&\le \mathbf{P}\Big\{ \max\Big[\sup_{m\ge n+1} \sup_{\varepsilon_{m+1}<\rho(t,s)\le\varepsilon_m} \frac{|X(t)-X(s)|}{G(\rho(t,s))}, \\
&\sup_{\varepsilon_{n+1}<\rho(t,s)\le v} \frac{|X(t)-X(s)|}{G(\rho(t,s))} \Big] > x \Big\} \qquad (3.13)\\
&\le \mathbf{P}\Big\{ \max\Big[\sup_{m\ge n+1} \sup_{\varepsilon_{m+1}<\rho(t,s)\le\varepsilon_m} \Big(2\sum_{l=m+2}^{\infty} \xi_l + \eta_m(\rho(t,s)) \Big) \times \\
&\times \Big(2\sum_{l=m+2}^{\infty} c_l + b_m(\rho(t,s)) \Big)^{-1}, \\
&\sup_{\varepsilon_{n+1}<\rho(t,s)\le v} \Big(2\sum_{l=n+2}^{\infty} \xi_l + \eta_n(\rho(t,s)) \Big) \Big(2\sum_{l=n+2}^{\infty} c_l + b_n(\rho(t,s)) \Big)^{-1} \Big] > x \Big\} \\
&\le \sum_{l=n+2}^{\infty} \mathbf{P}\Big\{ \frac{\xi_l}{c_l} > x \Big\} + \sum_{l=n+1}^{\infty} \mathbf{P}\Big\{ \sup_{\varepsilon_{l+1}<u\le\varepsilon_l} \frac{\eta_l(u)}{b_l(u)} > x \Big\} \\
&+ \mathbf{P}\Big\{ \sup_{\varepsilon_{n+1}<u\le v} \frac{\eta_n(u)}{b_n(u)} > x \Big\}.
\end{aligned}$$

Evaluate the probabilities in (3.13). It follows from Theorem 2.6 that

$$\begin{aligned}
&\mathbf{P}\Big\{ \sup_{\varepsilon_{l+1}<u\le\varepsilon_l} \frac{\eta_l(u)}{b_l(u)} > x \Big\} \\
&\le \mathbf{P}\Big\{ \sup_{\varepsilon_{l+1}<u\le\varepsilon_l} \max_{\substack{w,v\in V_{\varepsilon_{l+1}}, \\ \tau_\varphi(X(w)-X(v))\le\sigma(u)\frac{3-r}{1-r}}} \frac{|X(w)-X(v)|}{b_l(u)} > x \Big\} \\
&\le \mathbf{P}\Big\{ \sup_{\varepsilon_{l+1}<u<\varepsilon_l} \max_{\substack{w,v\in V_{\varepsilon_{l+1}}, \\ \tau_\varphi(X(w)-X(v))\ne 0, \\ \tau_\varphi(X(w)-X(v))\le\sigma(u)\frac{3-r}{1-r}}} \Big(\frac{|X(w)-X(v)|}{\tau_\varphi(X(w)-X(v))} \frac{\tau_\varphi(X(w)-X(v))}{\sigma(u)\frac{3-r}{1-r}} \times \\
&\times (b_l(u))^{-1}\sigma(u)\frac{3-r}{1-r} \Big) > x \Big\} \\
&\le \mathbf{P}\Big\{ \max_{\substack{w,v\in V_{\varepsilon_{l+1}} \\ \tau_\varphi(X(w)-X(v))\ne 0}} \frac{|X(w)-X(v)|}{\tau_\varphi(X(w)-X(v))} > xb\varphi^{*(-1)}(M\ln(N^2(\varepsilon_{l+1}))) \Big\} \qquad (3.14)\\
&\le \frac{b+1}{b-1} \left(N^2(\varepsilon_{l+1})\right)^{1-M} \exp\{-\varphi^*(x)\}.
\end{aligned}$$

Reasoning similarly we obtain that

$$\mathbf{P}\left\{\sup_{\varepsilon_{n+1}<u\le v}\frac{\eta_n(u)}{b_n(u)}>x\right\}\le\frac{b+1}{b-1}\left(N^2(\varepsilon_{n+1})\right)^{1-M}\exp\{-\varphi^*(x)\}. \tag{3.15}$$

It follows from the Theorem 2.6 also that

$$\begin{aligned}&\mathbf{P}\left\{\frac{\xi_l}{c_l}>x\right\}\\ &\le\mathbf{P}\left\{\max_{\substack{t\in V_{\varepsilon_l}:\\ \tau_\varphi(X(t)-X(\alpha_{l-1}(t)))\ne0}}\frac{(X(t)-X(\alpha_{l-1}(t)))}{\sigma(\varepsilon_{l-1})\varphi^{*(-1)}(M\ln(N(\varepsilon_l)))}>x\right\}\\ &\le\frac{b+1}{b-1}\left(N(\varepsilon_l)\right)^{1-M}\exp\{-\varphi^*(x)\}. \qquad (3.16)\end{aligned}$$

It follows from (3.14), (3.15), (3.16) and (3.6) that for $x>2$, $v>0$ such that $N(v)\ge2$

$$\begin{aligned}&\mathbf{P}\left\{\sup_{0<\rho(t,s)\le v}\frac{|X(t)-X(s)|}{G(\rho(t,s))}>x\right\}\\ &\le\left(\sum_{l=n+2}^{\infty}(N(\varepsilon_l))^{1-M}+\sum_{l=n+1}^{\infty}(N^2(\varepsilon_l))^{1-M}\right)\frac{b+1}{b-1}\exp\{-\varphi^*(x)\}\\ &\le2\sum_{l=n+1}^{\infty}(N(\varepsilon_l))^{1-M}\frac{b+1}{b-1}\exp\{-\varphi^*(x)\} \qquad (3.17)\\ &\le\frac{4}{(N(\varepsilon_{n+1}))^{M-1}}\sum_{l=0}^{\infty}\left(\frac{1}{B^{M-1}}\right)^l\frac{b+1}{b-1}\exp\{-\varphi^*(x)\}\\ &=\frac{4B^{M-1}(b+1)}{(N(\varepsilon_{n+1}))^{M-1}(B^{M-1}-1)(b-1)}\exp\{-\varphi^*(x)\}\\ &\le\frac{4B^{M-1}(b+1)}{(N(v))^{M-1}(B^{M-1}-1)(b-1)}\exp\{-\varphi^*(x)\}.\end{aligned}$$

Now we shall evaluate the sum $2\sum_{l=m+2}^{\infty}c_l+b_m(u)$. Set $Z(v)=b\varphi^{*(-1)}(Mv)$, then

$$\sum_{l=m+2}^{\infty}c_l=\sum_{l=m+2}^{\infty}\nu_{l-1}Z(\ln(N(\sigma^{(-1)}(\nu_l))))=A_1+A_2,$$

where

$$\begin{aligned}A_1&=\sum_{l\in D_1^{(m)}}\nu_{l-1}Z(\ln(N(\sigma^{(-1)}(\nu_l)))),\\ D_1^{(m)}&=\{l\ge m+2,\nu_l=r\nu_{l-1}\},\\ A_2&=\sum_{l\in D_2^{(m)}}\nu_{l-1}Z(\ln(N(\sigma^{(-1)}(\nu_l)))),\\ D_2^{(m)}&=\{l\ge m+2,\nu_l=\delta_{l-1}\},\end{aligned}$$

It follows from (3.5) that

$$\begin{aligned} A_1 &= \frac{1}{r} \sum_{l\in D_1^{(m)}} \nu_l Z(\ln(N(\sigma^{(-1)}(\nu_l)))) \\ &\le \frac{1}{r(1-r)} \sum_{l=m+2}^{\infty} (\nu_l - \nu_{l+1}) Z(\ln(N(\sigma^{(-1)}(\nu_l)))) \\ &\le \frac{1}{r(1-r)} \sum_{l=m+2}^{\infty} \int\limits_{\nu_{l+1}}^{\nu_l} Z(\ln(N(\sigma^{(-1)}(u))))\, du \\ &= \frac{1}{r(1-r)} \int\limits_0^{\nu_{m+2}} Z(\ln(N(\sigma^{(-1)}(u))))\, du. \end{aligned}$$

Therefore

$$A_1 \le \frac{1}{r(1-r)} \int\limits_0^{\nu_{m+2}} Z(\ln(N(\sigma^{(-1)}(u))))\, du. \tag{3.18}$$

Since $N(\sigma^{(-1)}(\delta_l)) < BN(\sigma^{(-1)}(\nu_l))$ then

$$\begin{aligned} A_2 &= \sum_{l\in D_2^{(m)}} \nu_{l-1} Z(\ln(N(\sigma^{(-1)}(\delta_{l-1})))) \\ &\le \sum_{l\in D_2^{(m)}} \nu_{l-1} Z(\ln(BN(\sigma^{(-1)}(\nu_{l-1})))) \\ &\le \frac{1}{1-r} \sum_{l=m+2}^{\infty} (\nu_{l-1} - \nu_l) Z(\ln(BN(\sigma^{(-1)}(\nu_{l-1})))) \qquad (3.19) \\ &\le \frac{1}{1-r} \int\limits_0^{\nu_{m+1}} Z(\ln(BN(\sigma^{(-1)}(u))))\, du. \end{aligned}$$

Since $\nu_{m+2} < \nu_{m+1} < \sigma(u)$ it follows from (3.18) and (3.19) then

$$2 \sum_{l=m+2}^{\infty} c_l \le \frac{2(1+r)}{r(1-r)} \int\limits_0^{\sigma(u)} Z(\ln(BN(\sigma^{(-1)}(u))))\, du. \tag{3.20}$$

For $\varepsilon_{m+1} < u \le \varepsilon_m$ $(\nu_{m+1} < \sigma(u) \le \nu_m)$

$$b_m(u) \le Z(2\ln(N(\sigma^{(-1)}(\nu_{m+1}))))\sigma(u)\frac{3-r}{1-r}.$$

Since $\nu_{m+1} = \min(r\nu_m, \delta_m)$ then let's consider two cases $\nu_{m+1} = \delta_m$ and $\nu_{m+1} = r\nu_m$. Let $\nu_{m+1} = \delta_m$ then it follows from (3.3)

$$\begin{aligned}\sigma(u)Z(2\ln(N(\sigma^{(-1)}(\nu_{m+1})))) &= \sigma(u)Z(2\ln(N(\sigma^{(-1)}(\delta_m))))\\ &\le \sigma(u)Z(2\ln(BN(\sigma^{(-1)}(\nu_m))))\\ &\le \sigma(u)Z(2\ln(BN(\sigma^{(-1)}(u))))\\ &\le \int_0^{\sigma(u)} Z(2\ln(BN(\sigma^{(-1)}(v))))\,dv.\end{aligned}$$

If $\nu_{m+1} = r\nu_m$ then

$$\begin{aligned}\sigma(u)Z(2\ln(N(\sigma^{(-1)}(\nu_{m+1})))) &= \sigma(u)Z(2\ln(N(\sigma^{(-1)}(r\nu_m))))\\ &\le \sigma(u)Z(2\ln(N(\sigma^{(-1)}(r\sigma(u)))))\\ &\le \int_0^{\sigma(u)} Z(2\ln(N(\sigma^{(-1)}(rv))))\,dv\\ &= \frac{1}{r}\int_0^{r\sigma(u)} Z(2\ln(N(\sigma^{(-1)}(t))))\,dt\\ &\le \frac{1}{r}\int_0^{\sigma(u)} Z(2\ln(BN(\sigma^{(-1)}(v))))\,dv.\end{aligned}$$

Therefore

$$b_m(u) \le \frac{3-r}{r(1-r)}\int_0^{\sigma(u)} Z(2\ln(BN(\sigma^{(-1)}(v))))\,dv.$$

So we have the following estimation

$$2\sum_{l=m+2}^{\infty} c_l + b_m(u) \le \frac{5+r}{r(1-r)} b \int_0^{\sigma(u)} \varphi^{*(-1)}(M2\ln(BN(\sigma^{(-1)}(v))))\,dv. \tag{3.21}$$

That is, it follows from (3.17) that for $x > 2$

$$\begin{aligned}&\mathbf{P}\left\{\sup_{0<\rho(t,s)\le v}\frac{|X(t)-X(s)|}{G_{r,b}(\rho(t,s))} > x\right\}\\ &\le \frac{4B^{M-1}(b+1)}{(N(v))^{M-1}(B^{M-1}-1)(b-1)}\exp\{-\varphi^*(x)\},\end{aligned} \tag{3.22}$$

where

$$G_{r,b}(u) = b\frac{5+r}{r(1-r)}\int_0^{\sigma(u)} \varphi^{*(-1)}(M2\ln(BN(\sigma^{(-1)}(v))))\,dv.$$

Since $\inf\limits_{0<r<1} \frac{5+r}{r(1-r)} = \frac{1}{11-2\sqrt{30}}$ then for $x > 2$

$$\mathbf{P}\left\{ \sup_{0<\rho(t,s)\leq v} \frac{|X(t)-X(s)|}{bf_B(\rho(t,s))} > x \right\}$$
$$\leq \frac{4B^{M-1}(b+1)}{(N(v))^{M-1}(B^{M-1}-1)(b-1)} \exp\{-\varphi^*(x)\}. \tag{3.23}$$

The inequality (3.2) follows from this inequality (for $y = xb > 2b$). □

Theorem 3.2. *Let the assumptions of the Theorem 3.1 hold true. Then with probability one*

$$\limsup_{\varepsilon\to 0} \frac{\sup\limits_{\rho(t,s)<\varepsilon} |X(t)-X(s)|}{2bf_B(\varepsilon)} < 1, \tag{3.24}$$

where

$$f_B(u) = \frac{1}{11-2\sqrt{30}} \int\limits_0^{\sigma(u)} \varphi^{*(-1)}(2M\ln(BN(\sigma^{(-1)}(v))))dv.$$

Proof. It follows from (3.12) that with probability one

$$\sup_{\rho(t,s)\leq u} |X(t)-X(s)| \leq 2\sum_{k=m+2}^{\infty} \xi_k + \eta_m(u) \tag{3.25}$$

It follows from (3.15) that for sufficiently large k $\eta_k(u) < 2b_k(u)$ with probability one. From (3.16) we have that for sufficiently large k $\xi_k < 2c_k$ with probability one. Therefore for sufficiently large k (or small enough u) we have

$$\sup_{\rho(t,s)\leq u} |X(t)-X(s)| \leq 2\left(2\sum_{k=m+2}^{\infty} c_k + b_m(u)\right). \tag{3.26}$$

Now it follows from (3.21) and (3.23) that for sufficiently small u

$$\sup_{0<\rho(t,s)\leq u} |X(t)-X(s)| \leq 2bf_B(u)$$

with probability one. □

The following corollary follows from the Theorem 3.2.

Corollary 3.3. *For small enough u*

$$\sup_{\rho(t,s)\leq u} |X(t)-X(s)| \leq 2bf_B(u)$$

with probability one.

4. Application to $\mathrm{Sub}_\varphi(\Omega)$ random processes in finite-dimensional spaces

Let T be a cube in finite-dimensional space, i.e., $T = \underbrace{[T_1, T_2] \times \ldots \times [T_1, T_2]}_{d\ times}$, $T_1 < T_2$, $\rho(t, s) = \max\limits_{1\le i\le d} |t_i - s_i|$, where $t = (t_i, i = \overline{1, d})$, $s = (s_i, i = \overline{1, d})$.

Theorem 4.1. *Let $X = \{X(t), t \in T\}$ be a separable random process from the space $\mathrm{Sub}_\varphi(\Omega)$. Suppose that there exists a monotonically increasing continuous function $\sigma = \{\sigma(h), h \ge 0\}$ such that $\sigma(0) = 0$ and the following inequality holds*

$$\sup_{\rho(t,s)\le h} \tau_\varphi(X(t) - X(s)) \le \sigma(h). \tag{4.1}$$

Let $M \ge \max\left(1, \frac{\varphi^*(2)}{\ln(2)}\right)$, $B > 1$, $b > 1$ be some numbers. Then for any $y > 2b$ and $v \le \frac{T_2 - T_1}{2\cdot 2^{1/d}}$ the following inequality holds true

$$\begin{aligned} &\mathbf{P}\left\{\sup_{0<\rho(t,s)\le v} \frac{|X(t) - X(s)|}{f_B^d(\rho(t, s))} > y\right\} \\ &\le \frac{4B^{M-1}(b + 1)}{(B^{M-1} - 1)(b - 1)} \left(\frac{2v}{T_2 - T_1}\right)^{d(M-1)} \exp\left\{-\varphi^*\left(\frac{y}{b}\right)\right\}, \end{aligned} \tag{4.2}$$

where

$$f_B^d(u) = \frac{1}{11 - 2\sqrt{30}} \int_0^{\sigma(u)} \varphi^{*(-1)}\left(2Md \ln\left(B^{1/d}\left(\frac{T_2 - T_1}{2\sigma^{(-1)}(s)} + 1\right)\right)\right) ds.$$

Moreover, with probability one

$$\limsup_{\varepsilon\to 0} \frac{\sup\limits_{\rho(t,s)\le\varepsilon} |X(t) - X(s)|}{2bf_B^d(\varepsilon)} < 1. \tag{4.3}$$

Proof. The theorem follows from the theorems 3.1 and 3.2 since in this case for all $z > 0$ the following inequalities hold true

$$\left(\frac{T_2 - T_1}{2z}\right)^d \le N(z) \le \left(\frac{T_2 - T_1}{2z} + 1\right)^d. \tag{4.4}$$

□

Remark 4.2. In (4.2) we have

$$f_B^d(u) \le \frac{1}{11 - 2\sqrt{30}} \int_0^{\sigma(u)} \varphi^{*(-1)}\left(2Md \ln\left(B^{1/d}\left(\frac{T_2 - T_1}{\sigma^{(-1)}(s)}\right)\right)\right) ds. \tag{4.5}$$

Indeed, in (4.2) $\sigma^{(-1)}(s) \le \sigma^{(-1)}(\sigma(v)) = v \le \frac{T_2 - T_1}{2^{1+1/d}}$. Therefore $\frac{T_2 - T_1}{2\sigma^{(-1)}(s)} \ge 2^{1/d} > 1$.

Example 4.3. Let $\varphi(x) = \frac{|x|^p}{p}, p > 1$, for sufficiently large $|x|$. In this case $\varphi^*(x) = \frac{|x|^q}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$, and $\varphi^{*(-1)}(x) = (qx)^{1/q}$. Suppose that $T_2 - T_1 > 1$

and $\sigma(h)=\frac{c}{\left(\ln\frac{1}{h}\right)^{\alpha}}$, $c>0$, $h\in(0,1)$, $\alpha>\frac{1}{q}$. Then $\sigma^{(-1)}(h)=\exp\left\{-\left(\frac{c}{h}\right)^{1/\alpha}\right\}$ and for sufficiently small u we have

$$\begin{aligned}
&f_B^d(u)\\
&\le \frac{1}{11-2\sqrt{30}}\int_0^{\sigma(u)} q^{1/q}\left(2Md\ln\left(B^{1/d}(T_2-T_1)\exp\left\{\left(\frac{c}{t}\right)^{1/\alpha}\right\}\right)\right)^{1/q}dt\\
&\le \frac{(2Mdq)^{1/q}}{11-2\sqrt{30}}\left(\int_0^{\sigma(u)}\left(\ln(B^{1/d}(T_2-T_1))\right)^{1/q}dt+\int_0^{\sigma(u)}\left(\frac{c}{t}\right)^{\frac{1}{\alpha q}}dt\right) \qquad (4.6)\\
&= \frac{(2Mdq)^{1/q}}{11-2\sqrt{30}}\left(\sigma(u)\left(\ln(B^{1/d}(T_2-T_1))\right)^{1/q}+\frac{c^{\frac{1}{\alpha q}}}{1-\frac{1}{\alpha q}}(\sigma(u))^{1-\frac{1}{\alpha q}}\right)\\
&\le A\cdot(\sigma(u))^{1-\frac{1}{\alpha q}}=\frac{Ac}{\left(\ln\frac{1}{u}\right)^{\alpha-\frac{1}{q}}},
\end{aligned}$$

where

$$A=\frac{(2Mdq)^{1/q}}{11-2\sqrt{30}}\left(\left(\ln B^{1/d}(T_2-T_1)\right)^{1/q}+\frac{c^{\frac{1}{\alpha q}}}{1-\frac{1}{\alpha q}}\right).$$

Example 4.4. Let $\varphi(x)$ be the same as in the Example 4.3, $\sigma(h)=Dh^{\alpha}$, $h>0$, $D>0$, $0<\alpha\le 1$, $T_2-T_1>1$. In this case $\sigma^{(-1)}(u)=\left(\frac{u}{D}\right)^{\frac{1}{\alpha}}$. Then

$$\begin{aligned}
&f_B^d(u)\\
&\le \frac{1}{11-2\sqrt{30}}\int_0^{Du^{\alpha}} q^{1/q}\left(2Md\ln\left(B^{1/d}(T_2-T_1)\left(\frac{D}{t}\right)^{1/\alpha}\right)\right)^{1/q}dt\\
&\le \frac{(2Mdq)^{1/q}}{11-2\sqrt{30}}\int_0^{Du^{\alpha}}\left[(\ln B^{1/d}(T_2-T_1))^{1/q}+\left(\frac{1}{\alpha}\ln\frac{D}{t}\right)^{1/q}\right]dt\\
&= \frac{(2Mdq)^{1/q}}{11-2\sqrt{30}}\left(Du^{\alpha}(\ln B^{1/d}(T_2-T_1))^{1/q}+\left(\frac{1}{\alpha}\right)^{1/q}\int_0^{Du^{\alpha}}\left(\ln\frac{D}{t}\right)^{1/q}dt\right),
\end{aligned}$$

$$\int_0^{Du^{\alpha}}\left(\ln\frac{D}{t}\right)^{1/q}dt=D\int_0^{u^{\alpha}}\left(\ln\frac{1}{t}\right)^{1/q}dt.$$

Since

$$\begin{aligned}
\int_0^{u^{\alpha}}\left(\ln\frac{1}{t}\right)^{1/q}dt &\le u^{\alpha}\left(\ln\frac{1}{u^{\alpha}}\right)^{1/q}\left(1+\frac{1}{q\ln\frac{1}{u^{\alpha}}}\right)\\
&\le u^{\alpha}\left(\ln\frac{1}{u}\right)^{1/q}\alpha^{1/q}\left(1+\frac{1}{q\alpha\ln\frac{1}{\varkappa}}\right),
\end{aligned}$$

$u < \varkappa < \frac{1}{e}$, then for sufficiently small u we have

$$f_B^d(u) \le C_1 u^\alpha + C_2 u^\alpha \left(\ln \frac{1}{u}\right)^{1/q} \le C_3 u^\alpha \left(\ln \frac{1}{u}\right)^{1/q},$$

where C_1, C_2, C_3 are some constants.

Let now $T = [T_1, T_2]$, $-\infty < T_1 < T_2 < \infty$, then $\frac{T_2 - T_1}{2u} \le N(u) \le \frac{T_2 - T_1}{2u} + 1$ and the next corollary holds.

Corollary 4.5. *Let $X = \{X(t), t \in [T_1, T_2]\}$ be a separable process from the space $\mathrm{Sub}_\varphi(\Omega)$. Suppose that there exists a monotonically increasing continuous function $\sigma = \{\sigma(h), h \ge 0\}$ such that $\sigma(0) = 0$ and the following inequality holds:*

$$\sup_{t,s \in [T_1, T_2]:\, \rho(t,s) \le h} \tau_\varphi(X(t) - X(s)) \le \sigma(h). \tag{4.7}$$

Let $M \ge \max\left(1, \frac{\varphi^*(2)}{\ln 2}\right)$, $B > 1$, $b > 1$ and u is such a number that $\frac{T_2 - T_1}{2u} > 2$, then for any $y > 2b$ the following inequality holds true

$$\begin{aligned} &\mathbf{P}\left\{ \sup_{0 < |t-s| \le u} \frac{|X(t) - X(s)|}{\tilde{f}_B(|t-s|)} > y \right\} \\ &\le \frac{4(b+1)(2u)^{M-1} B^{M-1}}{(b-1)(T_2 - T_1)^{M-1}(B^{M-1} - 1)} \exp\left\{-\varphi^*\left(\frac{y}{b}\right)\right\}, \end{aligned}$$

where

$$\begin{aligned} \tilde{f}_B(u) &= \frac{1}{(11 - 2\sqrt{30})} \int_0^{\sigma(u)} \varphi^{*(-1)}\left(2M \ln\left(B\left(\frac{T_2 - T_1}{2\sigma^{(-1)}(v)} + 1\right)\right)\right) dv \\ &\le \frac{1}{11 - 2\sqrt{30}} \int_0^{\sigma(u)} \varphi^{*(-1)}\left(2M \ln\left(\frac{B(T_2 - T_1)}{\sigma^{(-1)}(v)}\right)\right) dv. \end{aligned}$$

5. Lipschitz spaces

Definition 5.1. The function $q = \{q(t), t \in \mathbb{R}\}$ is called a modulus of continuity if $q(t) \ge 0$, $q(0) = 0$ and $q(t) < q(t+s) \le q(t) + q(s)$ for $t > 0, s > 0$.

Example 5.2. The function $q(t) = c|t|^\alpha$, $c > 0$, $0 < \alpha \le 1$, is a modulus of continuity.

Definition 5.3. Let (T, ρ) be a metric space and q be a modulus of continuity. The family of functions $\{x(t), t \in T\}$, for which

$$\sup_{\substack{t,s \in T \\ t \ne s}} \frac{|x(t) - x(s)|}{q(\rho(t,s))} < \infty \tag{5.1}$$

(or $\sup_{\rho(t,s) \le h} |x(t) - x(s)| = o(q(h))$ as $h \to 0$) is called a Lipschitz space $\Lambda_q(T, \rho)$ (or $\Lambda_q^o(T, \rho)$).

Theorem 5.4. *Let $X = \{X(t), t \in T\}$ be a random process, for which the assumptions of the Theorem 3.1 hold true. If $f_B(u) \le q(u)$ (or $f_B(u) = o(q(u))$) then X belongs to the space $\Lambda_q(T,\rho)$ (or $\Lambda_q^o(T,\rho)$) with probability one and the following inequality holds true*

$$\begin{aligned} &\mathbf{P}\left\{ \sup_{0<\rho(t,s)\le v} \frac{|X(t)-X(s)|}{q(\rho(t,s))} > y \right\} \\ &\le \frac{4B^{M-1}(b+1)}{(N(v))^{M-1}(B^{M-1}-1)(b-1)} \exp\left\{ -\varphi^*\left(\frac{y}{b}\right)\right\}. \end{aligned} \tag{5.2}$$

This theorem is a simple corollary of the Theorem 3.1.

Corollary 5.5. *Let $X = \{X(t), t \in T\}$ be a random process, for which the assumptions of the Theorem 3.1 hold true, q be a modulus of continuity. If*

$$f_B^q(u) = \int_0^{\sigma(u)} \frac{\varphi^{*(-1)}(2M\ln(BN(\sigma^{(-1)}(v))))}{q(v)} dv < \infty,$$

then X belongs to the space $\Lambda_q^o(T,\rho)$ with probability one.

Proof. In this case

$$\begin{aligned} f_B(u) &\le c\int_0^{\sigma(u)} \frac{q(u)\varphi^{*(-1)}(2M\ln(BN(\sigma^{(-1)}(v))))}{q(v)} dv \\ &\le q(u) c f_B^q(u) \\ &= o(q(u)), \end{aligned}$$

and assertion of this corollary follows from the Theorem 5.4. □

6. Application to weakly self-similar stationary increment processes from the space $\mathrm{Sub}_\varphi(\Omega)$

Consider a centred square integrable process $Z_H = (Z_H(t) : t \in [0,1])$, $H \in (0,1)$, that has the covariance function

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

and belongs to the space $\mathrm{Sub}_\varphi(\Omega)$. For short, we shall say that Z_H is wsssi-$\mathrm{Sub}_\varphi(\Omega)$ (weakly self-similar stationary increment processes from the space $\mathrm{Sub}_\varphi(\Omega)$).

Remark 6.1. Note that if a stationary-increment second-order process Z_H is self-similar, i.e., the finite-dimensional distributions of $Z_H(t)$ and $a^{-H}Z(at)$ coincide, then Z_H has necessarily the covariance function R_H.

Corollary 6.2. *Let Z_H be a wsssi-$\mathrm{Sub}_\varphi(\Omega)$-process, $M \geq \max\left(1, \frac{\varphi^*(2)}{\ln 2}\right)$, $B > 1$, $b > 1$ and $u \in (0, \frac{1}{4})$, then for any $y > 2b$ the following inequality holds true*

$$\mathbf{P}\left\{\sup_{0<|t-s|\leq u} \frac{|Z_H(t) - Z_H(s)|}{b\tilde{f}_B(|t-s|)} > y\right\} \leq \frac{4(b+1)B^{M-1}(2u)^{M-1}}{(b-1)(B^{M-1}-1)} \exp\left\{-\varphi^*\left(\frac{y}{b}\right)\right\},$$

where

$$\tilde{f}_B(u) = \frac{1}{(11-2\sqrt{30})} \int_0^{u^H} \varphi^{*(-1)}\left(2M \ln\left(B\left(\frac{1}{2v^{1/H}}+1\right)\right)\right) dv \leq \frac{1}{11-2\sqrt{30}} \int_0^{u^H} \varphi^{*(-1)}\left(2M \ln\left(\frac{B}{v^{1/H}}\right)\right) dv.$$

This result follows from the Corollary 4.5 for $\sigma(u) = u^H$, $u \geq 0$.

Example 6.3. Let $\varphi(x) = \frac{|x|^p}{p}$, $1 < p \leq 2$, for sufficiently large $|x|$. In this case $\varphi^*(x) = \frac{|x|^r}{r}$, where $\frac{1}{p} + \frac{1}{r} = 1$, and $\varphi^{*(-1)}(x) = (rx)^{1/r}$.

In case of the process Z_H we have $\sigma(u) = u^H$, $u > 0$, and $\sigma^{(-1)}(u) = (u)^{\frac{1}{H}}$. In accordance with Corollary 6.2 $u \in (0, \frac{1}{4})$. Then

$$\begin{aligned}
\tilde{f}_B(u) &\leq \frac{1}{11-2\sqrt{30}} \int_0^{u^H} r^{1/r}\left(2M\ln\left(B\left(\frac{1}{t}\right)^{1/H}\right)\right)^{1/r} dt \\
&\leq \frac{(2Mr)^{1/r}}{11-2\sqrt{30}} \int_0^{u^H}\left[(\ln B)^{1/r} + \left(\frac{1}{H}\ln\frac{1}{t}\right)^{1/r}\right] dt \\
&= \frac{(2Mr)^{1/r}}{11-2\sqrt{30}}\left(u^H(\ln B)^{1/r} + \left(\frac{1}{H}\right)^{1/r}\int_0^{u^H}\left(\ln\frac{1}{t}\right)^{1/r} dt\right),
\end{aligned}$$

Since

$$\int_0^{u^H}\left(\ln\frac{1}{t}\right)^{1/r} dt \leq u^H\left(\ln\frac{1}{u}\right)^{1/r} H^{1/r}\left(1 + \frac{1}{rH\ln\frac{1}{\varkappa}}\right),$$

$u < \varkappa < \frac{1}{e}$, then for sufficiently small u we have

$$\tilde{f}_B(u) \leq \left[\frac{(2Mr)^{1/r}}{11-2\sqrt{30}}(\ln B)^{1/r}\right] u^H + \left[\frac{(2Mr)^{1/r}}{11-2\sqrt{30}}\left(1+\frac{1}{rH\ln\frac{1}{\varkappa}}\right)\right] u^H\left(\ln\frac{1}{u}\right)^{1/r},$$

which implies that

$$\tilde{f}_B(u) \leq C_B\, u^H\left(\ln\frac{1}{u}\right)^{1/r},$$

where

$$C_B = \frac{(2Mr)^{1/r}}{11-2\sqrt{30}}\left((\ln B)^{1/r} + 1 + \frac{1}{rH\ln\frac{1}{\varkappa}}\right). \tag{6.1}$$

So, the following theorem holds true.

Theorem 6.4. *Let Z_H be wsssi-$\mathrm{Sub}_\varphi(\Omega)$ with $\varphi(x) = \frac{|x|^p}{p}$, $0 < p \le 1$. Then this random process belongs to the space $\Lambda_q(T,\rho)$ with probability one, where $T = [0,1]$, $\rho(t,s) = |t-s|$, $q(x) = C_B x^H \left(\ln\frac{1}{x}\right)^{\frac{1}{r}}$, C_B is given in (6.1). Besides that, for $u \in (0,\frac{1}{4})$ and $y > 2b$ the norm in this space satisfies the following inequality*

$$\mathbf{P}\left\{\sup_{0<|t-s|\le u}\frac{|Z_H(t)-Z_H(s)|}{C_B|t-s|^H\left(\ln\frac{1}{|t-s|}\right)^{\frac{1}{r}}} > y\right\} \tag{6.2}$$

$$\le \frac{4(b+1)B^{M-1}(2u)^{M-1}}{(b-1)(B^{M-1}-1)}\exp\left\{-\frac{y^r}{rb^r}\right\}$$

Remark 6.5. If Z_H is a Gaussian process, that is the process of fractional Brownian motion, then it satisfies theorem 6.4 with $p = 2$, $r = 2$ and $q(x) = \tilde{C}_B x^H\left(\ln\frac{1}{x}\right)^{\frac{1}{2}}$, $\tilde{C}_B = \frac{2\sqrt{M}}{11-2\sqrt{30}}\left((\ln B)^{1/2} + 1 + \frac{1}{2H\ln\frac{1}{\varkappa}}\right)$.

For $u \in (0,\frac{1}{4})$ and $y > 2b$

$$\mathbf{P}\left\{\sup_{0<|t-s|<u}\frac{|Z_H(t)-Z_H(s)|}{\tilde{C}_B|t-s|^H\left(\ln\frac{1}{|t-s|}\right)^{\frac{1}{2}}} > y\right\} \tag{6.3}$$

$$\le \frac{4(b+1)B^{M-1}(2u)^{M-1}}{(b-1)(B^{M-1}-1)}\exp\left\{-\frac{y^2}{2b^2}\right\}.$$

Remark 6.6. The constants b, B and M can be chosen in order to minimize the estimate in (6.2).

Example 6.7. Let

$$\varphi(x) = \begin{cases} \frac{|x|^2}{p}, & |x| < 1; \\ \frac{|x|^p}{p}, & |x| \ge 1. \end{cases}$$

In this case $\varphi^*(x) = \frac{|x|^r}{r}$ for $|x| \ge 1$, where $\frac{1}{p} + \frac{1}{r} = 1$, and $\varphi^{*(-1)}(x) = (rx)^{\frac{1}{r}}$ for $|x| \ge \frac{1}{r}$.

As in the previous example, under condition that

$$2M\ln\left(B\left(\frac{1}{u}\right)^{\frac{1}{H}}\right) \ge \frac{1}{r},$$

or

$$0 < u \le B^H\exp\left\{-\frac{H}{2Mr}\right\},$$

or for

$$u \in \left(0, \min\left(B^H \exp\left\{-\frac{H}{2Mr}\right\}, \frac{1}{4}\right)\right),$$

we have the same estimate as in the Example 6.3. Here

$$\tilde{f}_B(u) \le C_B u^H \left(\ln \frac{1}{u}\right)^{\frac{1}{r}}.$$

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Yuriy Kozachenko, Department of Probability Theory and Math. Statistics, Mechanics and Mathematics faculty, Taras Shevchenko Kyiv National University, Volodymyrska 64, Kyiv, Ukraine

E-mail address: `yvk@univ.kiev.ua`

Tommi Sottinen, University of Vaasa, Faculty of Technology, Department of Mathematics and Statistics, P.O.Box 700, FIN-65101 Vaasa, Finland

E-mail address: `tommi.sottinen@uwasa.fi`

Olga Vasylyk, Department of Probability Theory and Math. Statistics, Mechanics and Mathematics faculty, Taras Shevchenko Kyiv National University, Volodymyrska 64, Kyiv, Ukraine

E-mail address: `ovasylyk@univ.kiev.ua`