Tangent spaces to metric spaces

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Abstract. We introduce a tangent space at an arbitrary point of a general metric space. It is proved that all tangent spaces are complete. The conditions under which these spaces have a finite cardinality are found.

Key words: Metric spaces; Tangent spaces.

2000 Mathematics Subject Classification: 54E35.

1 Introduction. Main definitions.

The recent achievements in the metric space theory are closely related to some generalizations of differentiation. The concept of upper gradient [HeKo] and [Sh], Cheeger's notion of differentiability for Rademacher's theorem in certain metric measure spaces [Ch], the metric derivative in the studies of metric space valued functions of bounded variation [Am], [AmTi] and the Lipshitz type approach in [Ha] are interesting and important examples of such generalizations. These generalizations of the differentiability usually lead to nontrivial results only for assumption that metric spaces have "sufficiently many" rectifiable curves.

The our main goal is the introduction of the notion of "differentiable" functions from a metric space X to a metric space Y for arbitrary X and Y. We define "tangent" spaces at a point of a metric space as some quotient space of the sequences which converge to this point and after that introduce the "derivatives" of functions as corresponding quotient maps.

Let (X, d) be a metric space and let a be point of X. Fix a sequence $\tilde{r}$ of positive real numbers r_n which tend to zero. In what follows this sequence $\tilde{r}$ be called a *normalizing sequence.*

1.1. Definition. *Two sequences $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ and $\tilde{y} = \{y_n\}_{n\in\mathbb{N}}$, $x_n, y_n \in X$, are mutually stable (with respect to a normalizing sequence $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$) if there is a finite limit*

$$\lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} := \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y}). \tag{1.1}$$

We shall say that a family $\tilde{F}$ of sequences of points from X is *maximal mutually stable* (with respect to a normalizing sequence $\tilde{r}$) if every two $\tilde{x}, \tilde{y} \in \tilde{F}$ are mutually stable and for an arbitrary $\tilde{z} = \{z_n\}_{n\in\mathbb{N}}$ with $z_n \in X$ either $\tilde{z} \in \tilde{F}$ or there is $\tilde{x} \in \tilde{F}$ such that $\tilde{x}$ and $\tilde{z}$ are not mutually stable.

The standard application of Zorn's Lemma leads to the following

1.2. Proposition. *Let (X, d) be a metric space and let $a \in X$. Then for every normalizing sequence $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ there exists a maximal mutually stable family $\tilde{X}_a = \tilde{X}_{a,\tilde{r}}$ such that $\tilde{a} := \{a, a, ...\} \in \tilde{X}_a$.*

Note that the condition $\tilde{a} \in \tilde{X}_a$ implies the equality

$$\lim_{n\to\infty} d(x_n, a) = 0$$

for every $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ which belongs to $\tilde{X}_a$.

Consider a function $\tilde{d} : \tilde{X}_a \times \tilde{X}_a \to \mathbb{R}$ where $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y})$ is defined by (1.1). Obviously, $\tilde{d}$ is symmetric and nonnegative. Moreover, the triangle inequality for d implies

$$\tilde{d}(\tilde{x}, \tilde{y}) \le \tilde{d}(\tilde{x}, \tilde{z}) + \tilde{d}(\tilde{z}, \tilde{y})$$

for all $\tilde{x}, \tilde{y}, \tilde{z}$ from $\tilde{X}_a$. Hence $(\tilde{X}_a, \tilde{d})$ is a pseudometric space.

1.3. Definition. *The pretangent space to the space X at the point a with respect to normalizing sequence $\tilde{r}$ is the metric identification of the pseudometric space $(\tilde{X}_{a,\tilde{r}}, \tilde{d})$.*

Since the notion of pretangent space is an important step in the development of general machinery for the definition of tangent spaces and differentiability in arbitrary metric spaces, we recall this metric identification construction.

Define a relation $\sim$ on $\tilde{X}_a$ by $\tilde{x} \sim \tilde{y}$ if and only if $\tilde{d}(\tilde{x}, \tilde{y}) = 0$. Then $\sim$ is an equivalence relation and let $\Omega_{a,\tilde{r}}$ be the set of equivalence classes in $\tilde{X}_a$ under the equivalence relation $\sim$. It follows from general properties of pseudometric spaces, see for example [Kell, Chapter 4, Th.15], that if ρ is defined on Ω_a by

$$\rho(\alpha, \beta) := \tilde{d}(\tilde{x}, \tilde{y}) \tag{1.2}$$

for some $\tilde{x} \in \alpha$ and $\tilde{y} \in \beta$, then ρ is well-defined metric on Ω_a. By definition the metric space (Ω_a, ρ) is the metric identification of $(\tilde{X}_a, \tilde{d})$.

Remark that $\Omega_{a,\tilde{r}} \ne \emptyset$ for all $\tilde{r}$ because the constant sequence $\tilde{a}$ belongs to $\tilde{X}_{a,\tilde{r}}$ in accordance with Proposition 1.2.

Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers. Denote by $\tilde{r}'$ the subsequence $\{r_{n_k}\}_{k\in N}$ of the normalizing sequence $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$. It is clear that if $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$

and $\tilde{y} = \{y_n\}_{n\in\mathbb{N}}$ are mutually stable with respect to $\tilde{r}$, then $\tilde{x}' := \{x_{n_k}\}_{k\in\mathbb{N}}$ and $\tilde{y}' := \{y_{n_k}\}_{k\in\mathbb{N}}$ are mutually stable with respect to $\tilde{r}'$ and that

$$\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}'}(\tilde{x}', \tilde{y}'). \tag{1.3}$$

If $\tilde{X}_{a,\tilde{r}}$ is a maximal mutually stable (with respect $\tilde{r}$) family, then by Zorn's Lemma there exists a maximal mutually stable (with respect $\tilde{r}'$) family $\tilde{X}_{a,\tilde{r}'}$ such that

$$\{\tilde{x}' : \tilde{x} \in \tilde{X}_{a,\tilde{r}}\} \subseteq \tilde{X}_{a,\tilde{r}'}.$$

Denote by $\mathrm{in}_{\tilde{r}'}$ the mapping $\tilde{X}_{a,\tilde{r}} \to \tilde{X}_{a,\tilde{r}'}$ with $\mathrm{in}_{\tilde{r}'}(\tilde{x}) = \tilde{x}'$ for all $\tilde{x} \in \tilde{X}_{a,\tilde{r}}$. If follows from (1.3) that after metric identifications the mapping $\mathrm{in}_{\tilde{r}'}$ induces an isometric embedding in': $\Omega_{a,\tilde{r}} \to \Omega_{a,\tilde{r}'}$, i.e., the diagramm

$$\begin{array}{ccc} \tilde{X}_{a,\tilde{r}} & \xrightarrow{\mathrm{in}_{\tilde{r}'}} & \tilde{X}_{a,\tilde{r}'} \\ p\Big\downarrow & & \Big\downarrow p' \\ \Omega_{a,\tilde{r}} & \xrightarrow{\mathrm{in}'} & \Omega_{a,\tilde{r}'} \end{array} \tag{1.4}$$

is commutative. Here p, p' are metric identification mappings $p(\tilde{x}) = \{\tilde{y} \in \tilde{X}_{a,\tilde{r}} : \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0\}$ and $p'(\tilde{x}) = \{\tilde{y}' \in \tilde{X}_{a,\tilde{r}'} : \tilde{d}_{\tilde{r}'}(\tilde{x}', \tilde{y}') = 0\}$. Let X and Y be two metric spaces. Recall that a map $f : X \to Y$ is called an *isometry* if f is distance-preserving and onto.

1.4. Definition. *A pretangent $\Omega_{a,\tilde{r}}$ is tangent if for every $\tilde{r}'$ the mapping* in'*: $\Omega_{a,\tilde{r}} \to \Omega_{a,\tilde{r}'}$ is an isometry.*

1.5. Remark. As has been stated above, the function $\mathrm{in}' : \Omega_{a,\tilde{r}} \to \Omega_{a,\tilde{r}'}$ is an isometric embedding. Since every surjective, isometric embedding is an isometry, $\Omega_{a,\tilde{r}}$ is tangent if and only if in' is surjective for all $\tilde{r}'$.

The following question naturally arieses.

1.6. Problem. Let (X, d) be a metric space and let $a \in X$. Find the conditions under which all pretangent spaces $\Omega_{a,\tilde{r}}$ are tangent.

Let (X_1, d_1) and (X_2, d_2) be metric spaces, and $\tilde{r}_1$, $\tilde{r}_2$ normalizing sequences, and a_1, a_2 points of X_1 and X_2 respectively, and $\tilde{X}^1_{a_1,\tilde{r}_1}$, $\tilde{X}^2_{a_2,\tilde{r}_2}$ maximal mutually stable families of sequences of points from X_1 and X_2 respectively. Let $f : X_1 \to X_2$ be a function such that $f(a_1) = a_2$. For every $\tilde{x}_1 := \{x^1_n\}_{n\in\mathbb{N}} \in \tilde{X}^1_{a_1,\tilde{r}_1}$ write

$$\tilde{f}(\tilde{x}_1) := \{f(x^1_n)\}_{n\in\mathbb{N}}.$$

1.7. Definition. *The function f is differentiable with respect to the pair $(\tilde{X}^1_{a_1,\tilde{r}_1}, \tilde{X}^2_{a_2,\tilde{r}_2})$ if $f(\tilde{x}_1) \in \tilde{X}^2_{a_1,\tilde{r}_2}$ and $\tilde{d}_1(\tilde{x}_1, \tilde{y}_1) = 0$ implies $\tilde{d}_2(\tilde{f}(\tilde{x}_1), \tilde{f}(\tilde{y}_1)) = 0$ for all $\tilde{x}_1, \tilde{y}_1 \in \tilde{X}^1_{a_1,\tilde{r}_1}$.*

Let $p_i : \tilde{X}^i_{a_i \tilde{r}_i} \to \Omega_{a_i,\tilde{r}_i}$, $i = 1, 2$, be metric identification mappings.

1.8. Definition. *A function $Df : \Omega_{a_1,\tilde{r}_1} \to \Omega_{a_2,\tilde{r}_2}$ is a derivative of f with respect to pretangent spaces $\Omega_{a_i,\tilde{r}_i}$, $i = 1, 2$, if f is differentiable with respect to $(\tilde{X}^1_{a_1,\tilde{r}_1}, \tilde{X}^2_{a_2,\tilde{r}_2})$ and if the following diagramm*

$$\begin{array}{ccc} \tilde{X}^1_{a_1,\tilde{r}_1} & \xrightarrow{\quad \tilde{f} \quad} & \tilde{X}^2_{a_2,\tilde{r}_2} \\ p_1 \Big\downarrow & & \Big\downarrow p_2 \\ \Omega_{a_1,\tilde{r}_1} & \xrightarrow{\quad Df \quad} & \Omega_{a_2,\tilde{r}_2} \end{array}$$

is commutative.

Definitions 1.7 and 1.8 are a main goal of all previous constructions, so we present them here, although the properties of "metric space valued derivatives" Df are not discussed in this paper.

2 First examples. Finite tangent spaces.

It is clear that Ω_a is an one-point space if a is an isolated point of X. The converse proposition is also true.

2.1. Proposition. *Let (X, d) be a metric space and let $a \in X$. Then a is an isolated point of X if and only if the pretangent space $\Omega_{a,\tilde{r}}$ is one-point for every normalizing sequence $\tilde{r}$.*

Proof. If a is not an isolated point of X, then there is a sequence $\tilde{b} = \{b_n\}_{n \in N}$ of points in X such that $\lim\limits_{n\to\infty} d(a, b_n) = 0$ and $d(a, b_n) \neq 0$ for all $n \in \mathbb{N}$. Consider the normalizing sequence $\tilde{r} = \{r_n\}_{n\in N}$ with $r_n := d(a, b_n)$. It follows immediately from (1.1) that $\tilde{d}_{\tilde{r}}(\tilde{a}, \tilde{b}) = 1$ where $\tilde{a}$ is the constant sequence $\{a, a, ...\}$. The application of Zorn's Lemma shows that there is a maximal mutually stable family $\tilde{X}_{a,\tilde{r}}$ such that $\tilde{a}, \tilde{b} \in \tilde{X}_{a,\tilde{r}}$. Then the metric identification of the pseudometric space $(\tilde{X}_{a,\tilde{x}}, \tilde{d})$ has at least two points. ■

The following proposition characterizes the points $a \in X$ such that the pretangent spaces Ω_a have cardinality at most two.

Define the function $F : X \times X \to \mathbb{R}$ by the rule

$$F(x, y) = \begin{cases} \dfrac{d(x, y)(d(x, a) \wedge d(y, a))}{(d(x, a) \vee d(y, a))^2} & \text{if } \ (x, y) \neq (a, a) \\ 0 & \text{if } \ (x, y) = (a, a). \end{cases} \tag{2.1}$$

2.2. Theorem. *Let (X, d) be a metric space and let $a \in X$. Then the inequality*

$$\operatorname{card}(\Omega_{a,\tilde{r}}) \leq 2 \tag{2.2}$$

holds for every normalizing sequence $\tilde{r}$ if and only if

$$\lim_{\substack{x\to a \\ y\to a}} F(x, y) = 0 \tag{2.3}$$

where F is defined by (2.1).

Proof. Suppose that (2.3) does not hold. Then in X there are some sequences $\{x'_n\}_{n\in\mathbb{N}}$ and $\{y'_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} x'_n = \lim_{n\to\infty} y'_n = a$$

and $x'_n \neq a \neq y'_n$ for all $n \in \mathbb{N}$ and

$$\limsup_{n\to\infty} F(x'_n, y'_n) > 0. \tag{2.4}$$

Consider the normalizing sequence $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ with $r_n = d(x'_n, a) \vee d(y'_n, a)$. Write

$$(x_n, y_n) := \begin{cases} (x'_n, y'_n) & \text{if } \ d(x'_n, a) \geq d(y'_n, a) \\ (y'_n, x'_n) & \text{if } \ d(y'_n, a) > d(x'_n, a). \end{cases}$$

Then we obtain

$$F(x'_n, y'_n) = F(x_n, y_n) = \frac{d(x_n, y_n)}{r_n} \cdot \frac{d(y_n, a)}{r_n}. \tag{2.5}$$

Since

$$d(y_n, a) \leq r_n \tag{2.6}$$

and

$$d(x_n, y_n) \leq d(x_n, a) + d(y_n, a) \leq 2r_n, \tag{2.7}$$

we have

$$F(x_n, y_n) \leq 2 \tag{2.8}$$

for all $n \in \mathbb{N}$. Let $\{x_{n_k}, y_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\{x_n, y_n\}_{\mathbb{N}}$ for which

$$\limsup_{n\to\infty} F(x'_n, y'_n) = \lim_{k\to\infty} F(x_{n_k}, y_{n_k}).$$

The inequality (2.4) and (2.8) imply that

$$0 < \lim_{k\to\infty} F(x_{n_k}, y_{n_k}) \leq 2. \tag{2.9}$$

We may assume without loss of generality that $\{x_{n_k}, y_{n_k}\}_{k\in\mathbb{N}}$ and $\{x_n, y_n\}_{n\in\mathbb{N}}$ coincide. Conditions (2.6) and (2.7) imply the existence of finite limits

$$\lim_{m\to\infty} \frac{d(x_{n_m}, y_{n_m})}{r_{n_m}} \leq 2 \quad \text{and} \quad \lim_{m\to\infty} \frac{d(y_{n_m}, a)}{r_{n_m}} \leq 1$$

for some subsequence $\{x_{n_m}, y_{n_m}\}_{m\in\mathbb{N}}$ of $\{x_n, y_n\}_{n\in\mathbb{N}}$. Now we can, once again, take $n_m = n$. It follows from (2.5) and (2.9) that

$$0 < \lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} \cdot \lim_{n\to\infty} \frac{d(y_n, a)}{r_n} \leq 2.$$

Consequently, we have

$$\tilde{d}(\tilde{x}, \tilde{a}) = 1, \quad 0 < \tilde{d}(\tilde{y}, \tilde{a}) \leq 1, \quad 0 < \tilde{d}(\tilde{x}, \tilde{y}) \leq 2$$

for $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$, $\tilde{y} = \{\tilde{y}_n\}_{n\in\mathbb{N}}$. Hence, $\tilde{a}, \tilde{y}, \tilde{x}$ correspond to distinct points of the pretangent space $\Omega_{a,\tilde{r}}$, that contradicts (2.2). The implication (2.2)⇒(2.3) is established.

To prove that (2.3) implies (2.2), suppose that

$$\operatorname{card}(\Omega_{a,\tilde{r}}) \geq 3$$

for some normalizing sequence $\tilde{r}$. Then there exist sequences $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ and $\tilde{y} = \{y_n\}_{n\in\mathbb{N}}$ such that all three quantities

$$\tilde{d}(\tilde{x}, \tilde{a}) = \lim_{n\to\infty} \frac{d(x_n, a)}{r_n}, \quad \tilde{d}(\tilde{y}, \tilde{a}) = \lim_{n\to\infty} \frac{d(y_n, a)}{r_n}$$

and

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n}$$

are finite and positive. Consider the sequence $\{F(x_n, y_n)\}_{n\in\mathbb{N}}$ where F was defined by (2.1). Since

$$0 < \lim_{n\to\infty} \frac{d(x_n, a) \vee d(y_n, a)}{r_n} = \tilde{d}(\tilde{x}, \tilde{a}) \vee d(\tilde{y}, \tilde{a}) < \infty$$

and

$$0 < \lim_{n\to\infty} \frac{d(x_n, a) \wedge d(y_n, a)}{r_n} = \tilde{d}(\tilde{x}, \tilde{a}) \wedge d(\tilde{y}, \tilde{a}) < \infty,$$

we obtain

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} \frac{\frac{d(x_n,y_n)}{r_n} \cdot \frac{d(x_n,a)\wedge d(y_n,a)}{r_n}}{\left(\frac{d(x_n,a)\vee d(y_n,a)}{r_n}\right)^2} = \frac{\tilde{d}(\tilde{x}, \tilde{y})(\tilde{d}(\tilde{x}, \tilde{a}) \wedge \tilde{d}(\tilde{y}, \tilde{a}))}{\left(\tilde{d}(\tilde{x}, \tilde{a}) \vee \tilde{d}(\tilde{y}, \tilde{a})\right)^2} \in (0, \infty),$$

contrary to limit relation (2.3). Hence, (2.3) implies (2.2). ■

Theorem 2.2 can be generalized to the case of pretangent spaces with an arbitrary finite cardinality.

For every natural $n \geq 2$ let us denote by X^n the set of all n-tuples $x = (x_1, ..., x_n)$ with terms $x_k \in X$ for all $k = 1, ..., n$. Define the function $F_n : X^n \to \mathbb{R}$ by the rule

$$F_n(x_1, ..., x_n) := \begin{cases} \dfrac{\left(\prod\limits_{\substack{k,l=1\\k<l}}^{n} d(x_k, x_l)\right)\left(\bigwedge\limits_{k=1}^{n} d(x_k, a)\right)}{\left(\bigvee\limits_{k=1}^{n} d(x_k, a)\right)^{\frac{n(n-1)}{2}+1}} & \text{if } (x_1, ..., x_n) \neq (a, ..., a) \\ 0 & \text{if } (x_1, ..., x_n) = (a, ..., a) \end{cases} \tag{2.10}$$

where

$$\bigwedge_{k=1}^{n} d(x_k, a) := \min_{1\leq k\leq n} d(x_k, a) \quad \text{and} \quad \bigvee_{k=1}^{n} d(x_k, a) := \max_{1\leq k\leq n} d(x_k, a).$$

2.3. Theorem. *Let (X,d) be a metric space, $a \in X$ and let $n \geq 2$ be a natural number. Then the inequality*

$$\operatorname{card}(\Omega_{a,\tilde{r}}) \leq n \tag{2.11}$$

holds for every normalizing sequence $\tilde{r}$ and every pseudometric space $(\tilde{X}_{a,\tilde{r}}, \tilde{d})$ if and only if

$$\lim_{\substack{x_1 \to a \\ \vdots \\ x_n \to a}} F_n(x_1,...,x_n) = 0 \tag{2.12}$$

where F_n is defined by (2.10).

The proof of this theorem can be obtained as a direct generalization of the proof of Theorem 2.2 so we omit it.

2.4. Remarks. The value $\prod\limits_{\substack{k,l=1 \\ k<l}}^{n} d(x_k,x_l)$ in (2.10) can be regarded as a metric-space analog of the well-known Vandermonde determinant.

There is some other functions $\Phi_n : X^n \to \mathbb{R}$ which can be used similarly as the function F_n in Theorem 2.3. As an example consider the function

$$\Phi_n(x_1,...,x_n) := \begin{cases} \dfrac{\left(\bigwedge\limits_{\substack{k,l=1 \\ k<l}}^{n} d(x_k,x_l)\right)\left(\bigwedge\limits_{k=1}^{n} d(x_n,a)\right)}{\left(\bigvee\limits_{k=1}^{n} d(x_k,a)\right)^2} & \text{if} \quad (x_1,...,x_n) \neq (a,...,a) \\ 0 & \text{if} \quad (x_1,...,x_n) = (a,...,a). \end{cases} \tag{2.13}$$

Then (2.12) holds if and only if

$$\lim_{\substack{x_1 \to a \\ \vdots \\ x_n \to a}} \Phi_n(x_1,...,x_n) = 0. \tag{2.14}$$

Indeed, it follows from the simple inequality

$$d(x_k,x_l) \leq 2 \bigvee_{k=1}^{n} d(x_k,a), \quad k,l \in \{1,...,n\}$$

that

$$F_n(x_1,...,x_n) \leq 2^{\frac{n(n-1)}{2}-1} \Phi_n(x_1,...,x_n).$$

On the other hand we have

$$F_n(x_1,...,x_n) \geq \left(\frac{\bigwedge\limits_{\substack{k,l \\ k<l}}^{n} d(x_k,x_l)}{\bigvee\limits_{k=1}^{n} d(x_k,a)} \right)^{\frac{n(n-1)}{2}} \cdot \left(\frac{\bigwedge\limits_{k=1}^{n} d(x_k,a)}{\bigvee\limits_{k=1}^{n} d(x_k,a)} \right) \geq$$

$$\geq \left(\frac{\bigwedge\limits_{\substack{k,l \\ k<l}}^{n} d(x_k, x_l)}{\bigwedge\limits_{k=1}^{n} d(x_k, a)} \right)^{\frac{n(n-1)}{2}} \cdot \left(\frac{\bigwedge\limits_{k=1}^{n} d(x_k, a)}{\bigvee\limits_{k=1}^{n} d(x_k, a)} \right)^{\frac{n(n-1)}{2}} = (\Phi_n(x_1, ..., x_n))^{\frac{n(n-1)}{2}}.$$

Using Theorems 2.2, 2.3 we can easily construct metric spaces with finite tangent spaces. To this end, we first establish a lemma.

Let $\tilde{W}$ be a family of some sequences of points from X. Suppose that $\tilde{a} \in \tilde{W}$ and that every two $\tilde{x}, \tilde{y} \in \tilde{W}$ are mutually stable with respect to a fixed normalizing sequence $\tilde{r}$. Denote by $\tilde{W}_m$ a maximal mutually stable family such that $\tilde{W}_m \supseteq \tilde{W}$, by $\Omega_{a,\tilde{r}}$ the pretangent space corresponding to $(\tilde{W}_m, \tilde{d})$ and by $\Omega^w_{a,\tilde{r}}$ the metric identification of the pseudometric space $(\tilde{W}, \tilde{d})$. Clearly, there is an unique "natural", isometric embedding $E_m : \Omega^w_{a,\tilde{r}} \to \Omega_{a,\tilde{r}}$ for which the diagramm

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{\text{in}} & \tilde{W}_m \\ p \big\downarrow & & \big\downarrow p_m \\ \Omega^w_{a,\tilde{r}} & \xrightarrow{E_m} & \Omega_{a,\tilde{r}} \end{array} \tag{2.15}$$

is commutative. Here p and p_m are metric identification mappings and $\text{in}(\tilde{x}) = \tilde{x}$ for all $\tilde{x} \in \tilde{W}$.

2.5. Remark. E_m is bijection if and only if for each $\tilde{y} \in \tilde{W}_m$ there is $\tilde{x} \in \tilde{W}$ such that $\tilde{d}(\tilde{y}, \tilde{x}) = 0$. For a formal proof of this simple fact see Lemma 3.1 in Section 3.

2.6. Lemma. *Let (X, d) be a metric space, $a \in X$ and let $n \geq 2$ be a natural number. Suppose that limit relation (2.12) holds. Then the inequality*

$$\text{card}(\Omega^w_{a,\tilde{r}}) \geq n \tag{2.16}$$

implies that the embedding $E_m : \Omega^w_{a,\tilde{r}} \to \Omega_{a,\tilde{r}}$ is an isometry and that a pretangent space $\Omega_{a,\tilde{r}}$ is tangent.

Proof. By Theorem 2.3 limit relation (2.12) implies that

$$\text{card}(\Omega_{a,\tilde{r}}) \leq n. \tag{2.17}$$

Since E_m is an injective mapping, inequalities (2.16) and (2.17) imply

$$\text{card}(\Omega_{a,\tilde{r}}) = \text{card}(\Omega^w_{a,\tilde{r}}) = n.$$

These equalities show that each isometric embedding $\Omega^w_{a,\tilde{r}} \to \Omega_{a,\tilde{r}}$ is an isometry because n is a natural number.

Consider now commutative diagram (1.4) with $\tilde{X}_{a,\tilde{x}} = \tilde{W}_m$. In complete analogy with the above proof we can show that in', is an isometry for every subsequence $\tilde{r}'$ of $\tilde{r}$. Hence, $\Omega_{a,\tilde{r}}$ is tangent by Definition 1.4. ∎

2.7. Example. Let H be a Hilbert space with the norm $||\cdot||$ and

$$\dim\ H \geq m \tag{2.18}$$

where m is a positive integer. It follows from (2.18) that for every natural $k \leq m$ there are orthonormal vectors $e_1, ..., e_k$ in H. Let $\tilde{t} = \{t_j\}_{j\in\mathbb{N}}$ be a sequence of strictly decreasing positive numbers t_j for which

$$\lim_{j\to\infty} \frac{t_j}{t_{j+1}} = \infty. \tag{2.19}$$

Write

$$X := \{t_j e_i : \quad i = 1, ..., k, \quad j \in \mathbb{N}\} \cup \{0\}, \tag{2.20}$$

i.e., $0 \in X$, and

$$X \cap \{x \in H : ||x|| = t\} = \emptyset$$

if $t \neq r_j$ for all $j \in \mathbb{N}$, and

$$X \cap \{x \in H : ||x|| = t_j\} = \{t_j e_1, ..., t_j e_k\}.$$

Consider the following sequences of elements of X

$$\tilde{0} = \{0, ..., 0, ...\}, \quad \tilde{x}_1 = \{t_j e_1\}_{j\in\mathbb{N}}, ..., \tilde{x}_k = \{t_j e_k\}_{j\in\mathbb{N}}.$$

It is easy to see that all these sequences are pairwise mutually stable with respect to the normalizing sequence $\tilde{r} = \tilde{t}$ and that $\tilde{d}(\tilde{x}_p, \tilde{x}_q) := \lim\limits_{j\to\infty} \dfrac{||t_j e_p - t_j e_q||}{r_j} = \sqrt{2}$ for $p \neq q$, and that

$$d(\tilde{0}, \tilde{x}_p) := \lim_{j\to\infty} \frac{||t_j e_p||}{r_j} = 1$$

for all $p \in \{1, ..., k\}$.

Let $\tilde{W}_m$ be a maximal mutually stable family such that $\tilde{W}_m \supseteq \tilde{W} := \{\tilde{0}, \tilde{x}_1, ..., \tilde{x}_k\}$. We claim the following:

a) For each $\tilde{y} \in \tilde{W}_m$ there is $\tilde{x} \in \tilde{W}$ such that $\tilde{d}(\tilde{y}, \tilde{x}) = 0$;

b) The space $\Omega_{0,\tilde{r}}$, corresponding to $\tilde{W}_m$, is tangent.

Since $\operatorname{card}(\Omega^w_{0,\tilde{r}}) = k + 1$, for the proof of (a) and (b) it is enough to show that

$$\lim_{\substack{x_1\to 0\\ \cdots \\ x_{k+1}\to 0}} \Phi_{k+1}(x_1, ..., x_{k+1}) = 0 \tag{2.21}$$

where Φ_n is defined by (2.13), see Remarks 2.4 and Lemma 2.6.

Let $(x_1, ..., x_{k+1})$ be a $k + 1$-tuple with $x_i \in X$, $i \in \{1, ..., k + 1\}$. We may assume, without loss of generality, that $x_i \neq 0$ for all $i = 1, ..., k+1$ because $\Phi_{k+1}(x_1, ..., x_{k+1}) = 0$ for the opposite case. Using (2.20) we can define natural numbers $j = j(x_1, ..., x_{k+1})$ and $s = s(x_1, ..., x_{k+1})$ such that

$$t_j = \bigvee_{l=1}^{k+1} ||x_l||, \qquad t_{j+s} = \bigwedge_{l=1}^{k+1} ||x_l||.$$

Note that these numbers are well defined because the sequence $\tilde{t} = \{t_j\}_{j\in\mathbb{N}}$ is strictly decreasing. It follows from definition (2.20) that $s \geq 1$ and that $\lim\limits_{\substack{x_1\to 0\\ \dots\\ x_{k+1}\to 0}} j(x_1, ..., x_{k+1}) = \infty$. Now from (2.19) we obtain

$$0 \leq \limsup_{\substack{x_1\to 0\\ \dots\\ x_{k+1}\to 0}} \Phi(x_1, ..., x_{k+1}) \leq 2 \limsup_{\substack{x_1\to 0\\ \dots\\ x_{k+1}\to 0}} \frac{\bigwedge\limits_{l=1}^{k+1} ||x_l||}{\bigvee\limits_{l=1}^{k+1} ||x_l||} =$$

$$= 2 \limsup_{\substack{x_1\to 0\\ \dots\\ x_{k+1}\to 0}} \frac{t_{j+1}}{t_j} \cdot \frac{t_{j+2}}{t_{j+1}} ... \frac{t_{j+s-1}}{t_{j+s}} \leq 2 \lim_{j\to\infty} \frac{t_{j+1}}{t_j} = 0.$$

Relation(2.21) is proved.

Let (X, p) and (Y, d) be metric spaces. Recall that a homeomorphism $f : X \to Y$ is a *similarity* if there is a positive number k such that

$$d(f(x), f(y)) := kp(x, y)$$

for all $x, y \in X$. The number k is the *dilatation number* of f. Two metric spaces X and Y are *similar* if there exists a similarity of X onto Y.

2.8. Proposition. *Let H be the Hilbert space from Example 2.7. Then, each tangent space at the point $0 \in X$, where X was defined by (2.20), is either one-point or similar to the space $\{0, e_1, ..., e_k\} \subseteq H$*[1].

Proof. Let $\Omega_{0,\tilde{r}}$ be a tangent space to X at the point 0. Equality (2.21) implies that

$$\operatorname{card}(\Omega_{0,\tilde{r}}) := m \leq k + 1.$$

Let $\tilde{X}_{0,\tilde{r}}$ be a maximal mutually stable family for which $p(\tilde{X}_{0,\tilde{r}}) = \Omega_{0,\tilde{r}}$, see diagramm (1.4). If $\Omega_{0,\tilde{r}}$ is not one-point, then there are sequences $\tilde{x}_i = \{x_n^i\}_{n\in\mathbb{N}} \in \tilde{X}_{0,\tilde{r}}$ such that

$$\lim_{n\to\infty} \frac{||x_n^i||}{r_n} := \tilde{d}(\tilde{0}, \tilde{x}_i) \in (0, \infty) \tag{2.22}$$

for $i = 1, ..., m - 1$ and that

$$\lim_{n\to\infty} \frac{||x_n^i - x_n^j||}{r_n} := \tilde{d}(\tilde{x}_i, \tilde{x}_j) \in (0, \infty) \tag{2.23}$$

for $i, j \in \{1, ..., m - 1\}$ if $i \neq j$. Relations (2.22) imply

$$\lim_{n\to\infty} \frac{||x_n^j||}{||x_n^i||} = \frac{\tilde{d}(\tilde{0}, \tilde{x}_j)}{\tilde{d}(\tilde{0}, \tilde{x}_i)} \in (0, \infty). \tag{2.24}$$

[1]It is clear that a tangent space $\Omega_{0,\tilde{r}}$, in Example 2.7, is isometric with $\{0, e_1, ..., e_k\}$.

It follows from (2.20), that for every $x \in X \setminus \{0\}$ there is an unique $l = l(x) \in \mathbb{N}$ for which

$$||x|| = t_l.$$

Substituting $t_{l(x)}$ in (2.24) we obtain

$$\lim_{n\to\infty} \frac{t_{l(x_n^j)}}{t_{l(x_n^i)}} = \frac{\tilde{d}(\tilde{0}, \tilde{x}_j)}{d(\tilde{0}, \tilde{x}_j)} \in (0, \infty).$$

These relations and (2.19) imply the equality

$$||x_n^i|| = ||x_n^j||,$$

for all $i, j \in \{1, ..., m-1\}$, if n is taken large enough. Hence, using (2.23) and (2.20) for all sufficiently large n we see that

$$||x_n^i - x_n^j|| = t_l\sqrt{2}\delta_{ij}, \quad ||x_n^i|| = t_l \tag{2.25}$$

where δ_{ij} is Kronecker's delta, $i, j \in \{1, ..., m-1\}$ and $l = l(x_n^j) = l(x_n^i)$. If

$$\mathrm{card}(\Omega_{0,\tilde{r}}) < k + 1,$$

then there exists $\tilde{x}_m = \{x_n^m\}_{n\in\mathbb{N}}$ such that

$$||x_n^i - x_n^m|| = t_l\sqrt{2}, \quad ||x_n^m|| = t_l \tag{2.26}$$

for all $i = 1, ..., m-1$ and for all n, l which satisfy (2.25). Relations (2.22), (2.23), (2.25) and (2.26) imply the existence of positive finite limits

$$\lim_{n\to\infty} \frac{||x_n^m||}{r_n} \quad \text{and} \quad \lim_{n\to\infty} \frac{||x_n^i - x_n^m||}{r_n}$$

for all $i = 1, ..., m-1$ but it contradicts maximally of $\tilde{X}_{0,\tilde{r}}$. Consequently,

$$\mathrm{card}(\Omega_{0,\tilde{r}}) = k + 1.$$

It also follows from (2.25) that spaces $\Omega_{0,\tilde{r}}$ and $\{0, e_1, ..., e_k\} \subseteq H$ are similar.

To construct an one-point tangent space $\Omega_{0,\tilde{r}}$ consider the normalizing sequence $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ such that

$$r_n := \sqrt{t_n t_{n+1}} \tag{2.27}$$

where $\{t_j\}_{j\in\mathbb{N}}$ is the sequence from definition (2.20). For all $x \in X \setminus \{0\}$ this definition and (2.26) give either

$$\frac{||x||}{r_n} \geq \frac{t_n}{\sqrt{t_n t_{n+1}}} = \sqrt{\frac{t_n}{t_{n+1}}} \tag{2.28}$$

, if $||x|| \geq t_n$, or

$$\frac{||x||}{r_n} \leq \frac{t_{n+1}}{\sqrt{t_n t_{n+1}}} = \sqrt{\frac{t_{n+1}}{t_n}} \tag{2.29}$$

if $||x|| < t_n$. By (2.19) $\sqrt{\frac{t_n}{t_{n+1}}}$ tends to infinity with $n \to \infty$. Hence, if there is a finite $\lim\limits_{n\to\infty} \frac{||x_n||}{r_n}$, then $\tilde{d}(\tilde{0}, \tilde{x}) = 0$, i.e. the pretangent space $\Omega_{0,\tilde{r}}$ is one-point. To complete the proof, it suffices to observe that (2.28) and (2.29) imply

$$\frac{||x||}{r_{n_k}} \geq \sqrt{\frac{t_{n_k}}{t_{n_k+1}}}$$

or, respectively,

$$\frac{||x||}{r_{n_k}} \leq \sqrt{\frac{t_{n_k}}{t_{n_k+1}}}$$

for every subsequence $\tilde{r}' = \{r_{n_k}\}_{k\in\mathbb{N}}$ of $\tilde{r}$. Hence, $\Omega_{0,\tilde{r}'}$ is also one-point. Therefore the one-point pretangent space $\Omega_{0,\tilde{r}}$ is tangent. ■

The following problem seems to be interesting.

2.9. Problem. Let (X, d) be a metric space and let a be an accumulation point of X. Find the conditions under which there is an one-point tangent space $\Omega_{a,\tilde{r}}$.

2.10. Example. Let $\mathbb{C}$ be the complex plane with the usual distance function $d(z, w) = |z - w|$. Fix the following three sequences:

$$\{z_j\}_{j\in\mathbb{N}} \text{ with } z_j \in \mathbb{C} \text{ and } |z_j| = 1;$$

$$\{\alpha_j\}_{j\in\mathbb{N}} \text{ with } \alpha_j \in \mathbb{R} \text{ and } \lim_{j\to\infty} \alpha_j = 0;$$

$$\{t_j\}_{j\in\mathbb{N}} \text{ with } t_j \in \mathbb{R}^+, \ t_{j+1} < t_j \text{ and } \lim_{j\to\infty} \frac{t_{j+1}}{t_j} = 0.$$

Write $X := \{t_j z_j : j \in \mathbb{N}\} \cup \{t_j z_j e^{i\alpha_j} : j \in \mathbb{N}\} \cup \{0\}$. In the case under consideration the function (2.1) has the form

$$F(z, w) = \frac{|z - w|(|z| \wedge |w|)}{(|z| \vee |w|)^2} \tag{2.30}$$

if $(z, w) \neq (0, 0)$.

We claim that

$$\lim_{\substack{z\to 0\\ w\to 0}} F(z, w) = 0 \tag{2.31}$$

holds. Indeed, as in Example 2.7, for every $(w, z) \in (X \setminus \{0\})^2$ we can find j and s such that $j \leq s$ and

$$|w| \vee |z| = t_j \quad \text{and} \quad |w| \wedge |z| = t_s.$$

If $j < s$, then we have

$$F(z, w) \geq \frac{|t_j - t_s| t_s}{t_j^2} = \left|1 - \frac{t_s}{t_j}\right| \frac{t_s}{t_j} \geq \frac{t_s}{t_j} \geq \frac{t_{j+1}}{t_j}$$

but if $j = s$ we obtain

$$F(z,w) = \frac{|t_j z_j e^{i\alpha_j} - t_j z_j| t_j}{t_j^2} = |e^{i\alpha_j} - 1|.$$

Since

$$\lim_{j\to\infty} \frac{t_{j+1}}{t_j} = \lim_{j\to\infty} |1 - e^{i\alpha_j}| = 0,$$

(2.31) holds. Theorem 2.2 implies that each pretangent space at the point $0 \in X$ is either two-point or one-point. Note that two-point pretangent spaces at $0 \in X$ exist by Proposition 2.1 because 0 is not an isolated point of X. Moreover, all these pretangent spaces are tangent by Lemma 2.6. To construct an one-point tangent space it is enough to take a normalizing sequence $\tilde{r}$ with $r_j = \sqrt{t_j t_{j+1}}$, see the end of the proof of Proposition 2.8.

3 Some properties of pretangent and tangent spaces.

Let (X, d) be a metric space and let $a \in X$. If $\tilde{X}_{a,\tilde{r}}$ is a maximal mutually stable family of sequences $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$, $x_n \in X$ for $n \in \mathbb{N}$, and if $\tilde{Y}_{a,\tilde{r}}$ is a nonempty subset of $\tilde{X}_{a,\tilde{r}}$, then there is an unique isometric embedding $\text{in}_y : \Omega^y_{a,\tilde{r}} \to \Omega_{a,\tilde{r}}$ such that the following diagramm

$$\begin{array}{ccc} \tilde{Y}_{a,\tilde{r}} & \xrightarrow{\text{in}} & \tilde{X}_{a,\tilde{r}} \\ p_y \downarrow & & \downarrow p \\ \Omega^y_{a,\tilde{r}} & \xrightarrow{\text{in}_y} & \Omega_{a,\tilde{r}} \end{array} \tag{3.1}$$

is commutative. Here $\Omega_{a,\tilde{r}}$ is a pretangent space corresponding to $\tilde{X}_{a,r}$, $\Omega^y_{a,\tilde{r}}$ is a metric identification of $\tilde{Y}_{a,\tilde{r}}$, p_y and p are metric identification maps and $\text{in}(\tilde{y}) = \tilde{y}$ for all $\tilde{y} \in \tilde{Y}_{a,\tilde{r}}$.

3.1. Lemma. *The mapping* in_y *is an isometry if and only if for every* $\tilde{x} \in \tilde{X}_{a,\tilde{r}}$ *there is* $\tilde{y} \in \tilde{Y}_{a,\tilde{r}}$ *such that*

$$\tilde{d}(\tilde{x}, \tilde{y}) = 0. \tag{3.2}$$

Proof. The mapping in_y is an isometry if and only if this mapping is surjective, that is if

$$\text{in}_y^{-1}(\alpha) \neq \emptyset$$

for all $\alpha \in \Omega_{a,\tilde{r}}$. The last condition and

$$p_y^{-1}(\text{in}_y^{-1}(\alpha)) \neq \emptyset \tag{3.3}$$

are equivalent because p_y is surjective. Since $\text{in}_y \circ p_y = p \circ \text{in}$, we can rewrite (3.3) as

$$\emptyset \neq \text{in}^{-1}(p^{-1}(\alpha)). \tag{3.4}$$

If $\tilde{x}$ is an element of $\tilde{X}_{a,\tilde{r}}$ such that $p(\tilde{x}) = \alpha$, then $p^{-1}(\alpha) = \{\tilde{y} \in \tilde{X}_{a,\tilde{r}} : \tilde{d}(\tilde{x}, \tilde{y}) = 0\}$. Hence (3.4) holds if and only if (3.2) occurs for some $\tilde{y} \in \tilde{Y}_{a,\tilde{r}}$. ■

Suppose now that Y is a subspace of X. The closed spheres with center a and radius ρ, $0 < \rho < \infty$, are denoted by

$$S_\rho = S(a, \rho) := \{x \in X : d(a, x) = \rho\}.$$

Write

$$\varepsilon(\rho) := \rho^{-1} \sup_{x \in S_\rho} \inf_{y \in Y} d(x, y) \tag{3.5}$$

for $\rho > 0$.

We present some necessary and sufficient conditions under which the map in_y in (3.1) is an isometry. In the following theorem we denote by $\tilde{Y}_{a,\tilde{r}}$ a maximal mutually stable family of sequences $\tilde{y} = \{y_n\}_{n \in \mathbb{N}}$ for which all elements y_n belong to $Y \subseteq X$ and by $\Omega^y_{a,\tilde{r}}$ a pretangent space to Y at the point $a \in Y$.

3.2. Theorem. *Let (X, d) be a metric space, let Y be a subspace of X and let $a \in Y$. The following conditions are equivalent.*

(i) An embedding $\mathrm{in}_y : \Omega^y_{a,\tilde{r}} \to \Omega_{a,\tilde{r}}$ is an isometry for every normalizing sequence $\tilde{r}$ and all maximal mutually stable families $\tilde{Y}_{a,\tilde{r}}$ and $\tilde{X}_{a,\tilde{r}}$ for which $\tilde{Y}_{a,\tilde{r}} \subseteq \tilde{X}_{a,\tilde{r}}$.

(ii) The equality

$$\lim_{\rho \to 0} \varepsilon(\rho) = 0$$

holds.

Proof. Let $\tilde{X}_{a,\tilde{r}}$ and $\tilde{Y}_{a,\tilde{r}}$ be maximal mutually stable families and let $\tilde{Y}_{a,\tilde{r}} \subseteq \tilde{X}_{a,\tilde{r}}$. Suppose that condition (ii) holds true. Then for every $\tilde{x} = \{x_n\}_{n \in \mathbb{N}} \in \tilde{X}_{a,\tilde{r}}$ there is $\tilde{y} = \tilde{y}(\tilde{x}) = \{y_n\}_{n \in \mathbb{N}}$ such that $y_n \in Y$ for all $n \in \mathbb{N}$ and

$$\tilde{d}(\tilde{x}, \tilde{y}(\tilde{x})) = \lim_{n \to \infty} \frac{d(x_n, y_n)}{r_n} = 0. \tag{3.6}$$

Indeed, if $\tilde{d}(\tilde{x}, \tilde{a}) = 0$, then we can put $\tilde{y} = \tilde{a}$. If

$$\tilde{d}(\tilde{x}, \tilde{a}) = \lim_{n \to \infty} \frac{d(x_n, a)}{r_n} > 0,$$

then there is $\tilde{x}' = \{x'_n\}_{n \in \mathbb{N}} \in \tilde{X}_{a,\tilde{r}}$ such that $\tilde{d}(\tilde{x}, \tilde{x}') = 0$ and $d(a, x'_n) > 0$ for all $n \in \mathbb{N}$. It follows from (3.5) with $x = x'_n$ and $\rho = d(x'_n, a)$ that

$$d(x'_n, y_n) \leq \rho\varepsilon(\rho) + \rho^2$$

for some $y_n \in Y$. The last inequality and (ii) imply (3.6). Write

$$\tilde{F}_y = \{\tilde{y}(\tilde{x}) : \tilde{x} \in \tilde{X}_{a,\tilde{r}}\} \cup \tilde{Y}_{a,\tilde{r}}.$$

Since $\tilde{X}_{a,\tilde{r}}$ is mutually stable, (3.6) implies that $\tilde{F}_y$ is also *a* mutually stable family of sequences from Y. Maximality of $\tilde{Y}_{a,\tilde{r}}$ implies that $\tilde{Y}_{a,\tilde{r}} \supseteq \tilde{F}_y$. Hence, for every $\tilde{x} \in \tilde{X}_{a,\tilde{r}}$ there is $\tilde{y}(\tilde{x}) \in \tilde{Y}_{a,\tilde{r}}$ such that $\tilde{d}(\tilde{x}, \tilde{y}(\tilde{x})) = 0$. Condition (i) follows by Lemma 3.1.

Suppose now that (ii) does not hold. Then there is a sequence $\tilde{\rho}$ of positive numbers ρ_n with

$$\lim_{n\to\infty} \rho_n = 0$$

and there is a constant $c > 0$ such that for every $n \in \mathbb{N}$ there exists $x_n \in S(a, \rho_n)$ for which

$$\inf_{y\in Y} d(x_n, y) \geq c\rho_n. \tag{3.7}$$

Let us denote by $\tilde{x}$ the sequence of points x_n from X which satisfy (3.7). Take the sequence $\tilde{\rho} = \{\rho_n\}_{n\in\mathbb{N}}$ as a normalizing sequence. Let $\tilde{X}_{a,\tilde{\rho}}$ be a maximal mutually stable with respect to $\tilde{\rho}$ family such that $\tilde{a}, \tilde{x} \in \tilde{X}_{a,\tilde{\rho}}$. If in_y is an isometry for some $\tilde{Y}_{a,\tilde{\rho}} \subseteq \tilde{X}_{a,\tilde{\rho}}$, then there is $\tilde{y} \in \tilde{Y}_{a,\tilde{\rho}}$ such that $\tilde{d}(\tilde{x}, \tilde{y}) = 0$, see Lemma 3.1. It contradicts (3.7) because (3.7) implies

$$\limsup_{n\to\infty} \frac{d(x_n, y_n)}{\rho_n} \geq c > 0$$

for every $\tilde{y} = \{y_n\}_{n\in\mathbb{N}}$ with $y_n \in Y$. ■

Obviously, condition (ii) of Theorem 3.1 holds if Y is a dense subset in X. Therefore we have the following

3.3. Corollary. *Let (X, d) be a metric space, let Y be a dense subspace of X and let $a \in Y$. Then the pretangent spaces to X and Y at the point a are pairwise isometric for all normalizing sequences.*

The next our goal is to show that all tangent spaces to metric spaces are complete.

3.4. Lemma. *Let (X, d) be a metric space, let $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ be a normalizing sequence and let $\tilde{F}^1_{\tilde{r}}$, $\tilde{F}^2_{\tilde{r}}$, $\tilde{F}^3_{\tilde{r}}$ be mutually stable families of sequences $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ with $x_n \in X$ for $n \in \mathbb{N}$. Suppose that $\tilde{F}^1_{\tilde{r}}$ is maximal, $\tilde{F}^2_{\tilde{r}} \subseteq \tilde{F}^1_{\tilde{r}} \cap \tilde{F}^3_{\tilde{r}}$ and that pseudometric space $(\tilde{F}^2_{\tilde{r}}, \tilde{d})$ is a dense subspace of $(\tilde{F}^3_{\tilde{r}}, \tilde{d})$. Then the inclusion*

$$\tilde{F}^3_{\tilde{r}} \subseteq \tilde{F}^1_{\tilde{r}} \tag{3.8}$$

holds.

Proof. Suppose that (3.8) does not hold. Since $\tilde{F}^1_{\tilde{r}}$ is maximal mutually stable with respect to $\tilde{r}$, there are two sequences $\tilde{x} = \{x_n\}_{n\in\mathbb{N}} \in \tilde{F}^3_{\tilde{r}}$ and $\tilde{y} = \{y_n\}_{n\in\mathbb{N}} \in \tilde{F}^1_{\tilde{r}}$ such that either

$$\infty \geq \limsup_{n\to\infty} \frac{d(x_n, y_n)}{r_n} > \liminf_{n\to\infty} \frac{d(x_n, y_n)}{r_n} \geq 0 \tag{3.9}$$

or

$$\lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} = \infty. \tag{3.10}$$

For every $\varepsilon > 0$ we can find $\tilde{z} = \{z_n\}_{n\in\mathbb{N}} \in \tilde{F}^2_{\tilde{r}}$ such that

$$\tilde{d}(\tilde{x}, \tilde{y}) < \varepsilon$$

because $\tilde{F}^2_{\tilde{r}}$ is dense in $\tilde{F}^3_{\tilde{r}}$ by the hypothesis of the lemma. The triangle inequality implies

$$\left|\frac{d(x_n,y_n)}{r_n}-\frac{d(z_n,y_n)}{r_n}\right|\leq\frac{d(x_n,z_n)}{r_n}$$

for all $n\in\mathbb{N}$. Consequently we obtain

$$\left|\limsup_{n\to\infty}\frac{d(x_n,y_n)}{r_n}-\tilde{d}(\tilde{z},\tilde{y})\right|\leq\tilde{d}(\tilde{x},\tilde{y})<\varepsilon \tag{3.11}$$

and

$$\left|\liminf_{n\to\infty}\frac{d(x_n,y_n)}{r_n}-\tilde{d}(\tilde{z},\tilde{y})\right|<\varepsilon$$

Therefore, the equality

$$\left|\limsup_{n\to\infty}\frac{d(x_n,y_n)}{r_n}-\liminf_{n\to\infty}\frac{d(x_n,y_n)}{r_n}\right|<2\varepsilon$$

holds for all $\varepsilon>0$, this contradicts (3.9). To complete the proof, it suffices to observe that (3.11) contradicts (3.10). ∎

3.5. Lemma. *Let (X,d) be a metric space, let $a\in X$ and let $\Omega_{a,\tilde{r}}$, $\tilde{r}=\{r_n\}_{n\in\mathbb{N}}$, be a pretangent space with a corresponding maximal mutually stable family $\tilde{X}_{a,\tilde{r}}$. Suppose that Λ is a bounded subset of $\Omega_{a,\tilde{r}}$. Then there is a constant $c>0$ such that for every $\alpha\in\Lambda$ we can find $\tilde{x}=\{x_n\}_{n\in\mathbb{N}}\in\tilde{X}_{a,\tilde{r}}$ such that*

$$p(\tilde{x})=\alpha \quad \textit{and} \quad \frac{d(x_n,a)}{r_n}\leq c \tag{3.12}$$

for all $n\in\mathbb{N}$ where p is the metric identification of $\tilde{X}_{a,\tilde{r}}$.

Proof. Write

$$A:=\{\tilde{x}\in\tilde{X}_{a,\tilde{r}}:p(\tilde{x})\in\Lambda\}.$$

Since Λ is bounded in $\Omega_{a,\tilde{r}}$, we have

$$\sup\{\tilde{d}(\tilde{x},\tilde{a}):\tilde{x}\in A\}:=k<\infty.$$

For every $\tilde{x}=\{x_n\}_{n\in\mathbb{N}}\in A$ introduce the sequence $\tilde{x}^*=\{x^*_n\}_{n\in\mathbb{N}}$ by the rule

$$x^*_n=\begin{cases}x_n & \text{if} \quad d(x_na)<(k+1)r_n\\ a & \text{if} \quad d(x_n,a)\geq(k+1)r_n.\end{cases}$$

Note that for every $\tilde{x}\in A$ there is $n(\tilde{x})\in\mathbb{N}$ such that $x_n=x^*_n$ if $n\geq n(\tilde{x})$ because

$$\tilde{d}(\tilde{x},\tilde{a})=\lim_{n\to\infty}\frac{d(x_n,a)}{r_n}\leq k<k+1.$$

Consequently $\tilde{x}^*\in\tilde{X}_{a,\tilde{r}}$ for all $\tilde{x}\in A$. Moreover, relations (3.12) evidently hold with $\tilde{x}=\tilde{x}^*$, $x_n=x^*_n$ and $c=k+1$, which is what had to be proved. ∎

3.6. Theorem. *Let (X,d) be a metric space, $a \in X$ and let $\tilde{r} \in \{r_n\}_{n\in\mathbb{N}}$ be a normalizing sequence. Then a tangent space $\Omega_{a,\tilde{r}}$, if it exists, is complete.*

Proof. Let $\{\alpha^j\}_{j\in\mathbb{N}}$ be a fundamental sequence in the metric space $\Omega_{a,\tilde{r}}$. We must to show that $\{\alpha^j\}_{j\in\mathbb{N}}$ is convergent. Let $\tilde{X}_{a,\tilde{r}}$ be a maximal mutually stable family with the pretangent space $\Omega_{a,\tilde{r}}$ and let $p : \tilde{X}_{a,\tilde{r}} \to \Omega_{a,\tilde{r}}$ be a metric identification mapping. Every fundamental sequence of an arbitrary metric space is bounded. Hence, by Lemma 3.5, there is $c > 0$ such that for every $j \in \mathbb{N}$ there exists $\tilde{x}^j = \{x_n^j\}_{n\in\mathbb{N}} \in \tilde{X}_{a,\tilde{r}}$ for which

$$p(\tilde{x}^j) = \alpha^j \quad \text{and} \quad \frac{d(x_n^j, a)}{r_n} \le c \tag{3.13}$$

for all $n \in \mathbb{N}$.

In accordance with Definition 1.4 and Lemma 3.4 it suffices to prove that there exists a sequence $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ of points from X such that for some subsequence of natural numbers n_k and for all $j \in \mathbb{N}$ the sequences $\tilde{x}' = \{x_{n_k}\}_{k\in\mathbb{N}}$ and $\tilde{x}'^j = \{x_{n_k}^j\}_{k\in\mathbb{N}}$ are mutually stable with respect to $\tilde{r}' = \{r_{n_k}\}_{k\in\mathbb{N}}$ and

$$\lim_{j\to\infty}\lim_{k\to\infty} \frac{d(x_{n_k}, x_{n_k}^j)}{r_{n_k}} = 0, \tag{3.14}$$

see diagramm (1.4).

Suppose now that (3.14) holds and, in addition, we have the following condition: for every $k \in \mathbb{N}$ there is $j(k) \in \mathbb{N}$ such that

$$x_{n_k} = x_{n_k}^{j(k)} \tag{3.15}$$

where $x_{n_k}^{j(k)}$ is n_k-th element of $\tilde{x}^{j(k)}$. Then it follows from (3.13) that the inequality

$$\frac{d(x_{n_k}^j, x_{n_k})}{r_{n_k}} \le 2c$$

holds for all $j, k \in \mathbb{N}$. In particular, we obtain

$$\frac{d(x_{n_k}^1, x_{n_k})}{r_{n_k}} \le 2c$$

for all $k \in \mathbb{N}$. Since every bounded infinite sequence of real numbers has a subsequence that converges to a real number, there is a increasing infinite subsequence $\{n_k^{(1)}\}_{k\in\mathbb{N}}$ of the sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty} \dfrac{d(x_{n_k^{(1)}}, x_{n_k^{(1)}}^1)}{r_{n_k^{(1)}}}$ is finite. Hence, the sequences $\{x_{n_k^{(1)}}\}_{k\in\mathbb{N}}$ and $\{x_{n_k^{(1)}}^1\}_{k\in\mathbb{N}}$ are mutually stable with respect to $\{r_{n_k^{(1)}}\}_{k\in\mathbb{N}}$. Analogously, by induction, we can prove that for every integer $i \ge 2$ there is a subsequence $\{n_k^{(i)}\}_{k\in\mathbb{N}}$ of the sequence $\{n_k^{(i-1)}\}_{k\in\mathbb{N}}$ such that $\{x_{n_k^{(i)}}\}_{k\in\mathbb{N}}$ and $\{x_{n_k^{(i)}}^i\}_{k\in\mathbb{N}}$ are mutually stable with respect to $\{r_{n_k^{(i)}}\}_{k\in\mathbb{N}}$. Consider now the diagonal sequence $\{n_k^{(k)}\}_{k\in\mathbb{N}}$. For every $i \in \mathbb{N}$ the sequences

$\{x^i_{n_k^{(k)}}\}_{k\in\mathbb{N}}$ and $\{x_{n_k^{(k)}}\}_{k\in\mathbb{N}}$ are mutually stable with respect to $\{r_{n_k^{(k)}}\}_{k\in\mathbb{N}}$. Moreover, (3.14) evidently holds with $n_k = j(k) = n_k^{(k)}$. Consequently, for the proof of the theorem it suffices to construct $\tilde{x}$ such that (3.15) and (3.14) are satisfied. In order to construct this $\tilde{x}$, an estimate for $\dfrac{d(x^j_n, x^i_n)}{r_n}$ is needed.

3.7. Lemma. *Let $\{\tilde{x}_j\}_{j\in\mathbb{N}}$, $\tilde{x}_j = \{x^j_n\}_{n\in\mathbb{N}}$, be a fundamental sequence in a pseudometric space $(\tilde{X}_{a,\tilde{r}}, \tilde{d})$. Then there is a sequence $\{\tilde{x}^*_j\}_{j\in\mathbb{N}} \in \tilde{X}_{a,\tilde{r}}$ with the following properties:*

(i) The equality

$$\tilde{d}(\tilde{x}_j, \tilde{x}^*_j) = 0 \tag{3.16}$$

holds for all $j \in \mathbb{N}$;

(ii) For every $\varepsilon > 0$ there exists $j_0(\varepsilon) \in \mathbb{N}$ such that the inequality

$$\sup_{n\in\mathbb{N}} \frac{d(x^{*j}_n, x^{*i}_n)}{r_n} \le \varepsilon \tag{3.17}$$

holds if $i \wedge j \ge j_0(\varepsilon)$.

Proof. Let $\{\tilde{x}_{j_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\{\tilde{x}_j\}_{j\in\mathbb{N}}$ such that $j_1 < j_2 < ... < j_k...$ and

$$\tilde{d}(\tilde{x}_j, \tilde{x}_{j_k}) \le \left(\frac{1}{2}\right)^{k+1} \tag{3.18}$$

whenever $j \ge j_k$. For all $j, n \in \mathbb{N}$ write

$$^{(1)}x^j_n = \begin{cases} x^j_n & \text{if} \quad j \le j_1 \\ x^j_n & \text{if} \quad j > j_1 \quad \text{and} \quad \dfrac{d(x^{j_1}_n, x^j_n)}{r_n} \le \dfrac{1}{2} \\ x^{j_1}_n & \text{if} \quad j > j_1 \quad \text{and} \quad \dfrac{d(x^{j_1}_n, x^j_n)}{r_n} > \dfrac{1}{2} \end{cases} \tag{3.19}$$

and put

$$^{(1)}\tilde{x}_j := \{^{(1)}x^j_n\}_{n\in\mathbb{N}}.$$

Inequality (3.18) implies that for every $j \in \mathbb{N}$ there is $n_0(j) \in \mathbb{N}$ such that $^{(1)}x^j_n = x^j_n$ for all $n \ge n_0(j)$. Hence, we have the equality

$$\tilde{d}(\tilde{x}_j, {}^{(1)}\tilde{x}_j) = 0 \tag{3.20}$$

for all $j \in \mathbb{N}$. Note that (3.20) implies the inequality

$$\tilde{d}({}^{(1)}\tilde{x}_j, \tilde{x}_{j_k}) \le \left(\frac{1}{2}\right)^{k+1}$$

whenever $j \ge j_k$. Moreover, it follows from (3.19) that

$$\sup_{n\in\mathbb{N}} \frac{d({}^{(1)}x^j_n, {}^{(1)}x^i_n)}{r_n} \le \sup_{n\in\mathbb{N}} \frac{d({}^{(1)}x^j_n, x^{j_1}_n)}{r_n} + \sup_{n\in\mathbb{N}} \frac{d({}^{(1)}x^i_n, x^{j_1}_n)}{r_n} \le 1$$

whenever $i \wedge j > j_1$. Now we define ${}^{(k)}x_n^j$ by induction in k.

If $k \geq 2$ and $j, n \in \mathbb{N}$ write

$$ {}^{(k)}x_n^j = \begin{cases} {}^{(k-1)}x_n^j & \text{if} \quad j \leq j_k \\ {}^{(k-1)}x_n^j & \text{if} \quad j > j_k \quad \text{and} \quad \dfrac{d(x_n^{j_k}, {}^{(k-1)}x_n^j)}{r_n} \leq \left(\dfrac{1}{2}\right)^k \\ x_n^{j_k} \quad \text{if} \quad j > j_k \quad \text{and} \quad \dfrac{d(x_n^{j_k}, {}^{(k-1)}x_n^j)}{r_n} > \left(\dfrac{1}{2}\right)^k \end{cases} $$

and put

$$ {}^{(k)}\tilde{x}_j := \{{}^{(k)}x_n^j\}_{n \in \mathbb{N}}. $$

In the same manner as in the case $k = 1$ we obtain the equality

$$ d(\tilde{x}_j, {}^{(k)}\tilde{x}_j) = 0 \tag{3.21} $$

for all $j, k \in \mathbb{N}$ and the inequality

$$ \sup_{n \in \mathbb{N}} \frac{d({}^{(k)}x_n^j, {}^{(k)}x_n^i)}{r_n} \leq \left(\frac{1}{2}\right)^k \tag{3.22} $$

whenewer $i \wedge j > j_k$. Now set for all $n \in \mathbb{N}$

$$ x_n^{*j} = {}^{(1)}x_n^j \quad \text{if} \quad 1 \leq j \leq j_1, $$

$$ x_n^{*j} = {}^{(2)}x_n^j \quad \text{if} \quad j_1 < j \leq j_2, $$

....

$$ x_n^{*j} = {}^{(k)}x_n^j \quad \text{if} \quad j_{k-1} < j \leq j_k, $$

and so on. The sequence $\tilde{x}_j^* := \{x_n^{*j}\}_{n \in \mathbb{N}}$ has the properties (i) and (ii) because (3.21) implies (3.16) and, moreover, (3.17) follows from (3.22). ■

Continuation of the proof of Theorem 3.6. Using Lemma 3.7 we may assume that for every $\varepsilon > 0$ there is $j_0 = j_0(\varepsilon) \in \mathbb{N}$ such that

$$ \sup_{n \in \mathbb{N}} \frac{d(x_n^i, x_n^j)}{r_n} \leq \varepsilon \tag{3.23} $$

if $i \wedge j \geq j_0(\varepsilon)$. Set $x_n = x_n^n$ for all $n \in \mathbb{N}$. Then if follows from (3.23) that

$$ \limsup_{n \to \infty} \frac{d(x_n, x_n^j)}{r_n} \leq \varepsilon. \tag{3.24} $$

It was shown in the first part of the proof that there is a subsequence $\tilde{r}' = \{r_{n_k}\}_{k \in \mathbb{N}}$ of the sequence $\tilde{r}$ such that the sequences $\tilde{x}'^j = \{x_{n_k}^j\}_{k \in \mathbb{N}}$ and $\tilde{x}' = \{x_{n_k}\}_{k \in \mathbb{N}}$ are mutually stable with respect $\tilde{r}'$ for all $j \in \mathbb{N}$. Hence, by (3.24), we obtain

$$ \lim_{n \to \infty} \frac{d(x_{n_k}, x_{n_k}^j)}{r_{n_k}} \leq \varepsilon $$

if $j \geq j_0(\varepsilon)$. The last relation gives (3.14) when $j \to \infty$. To complete the proof, it suffices to observe that (3.15) holds with $j(k) = n_k$. ■

Recall that a map $f : X \to Y$ is called *closed* if the image of each set closed in X is closed in Y.

3.8. Corollary. *Let (X, d) be a metric space, let Y be a subspace of X and let $a \in Y$. If a pretangent space $\Omega^y_{a,\tilde{r}}$ is tangent, then the map $\mathrm{in}_y : \Omega^y_{a,\tilde{r}} \to \Omega_{a,\tilde{r}}$ is closed.*

Proof. The map in_y is an isometric embedding. Hence, in_y is closed if and only if the set $\mathrm{in}_y(\Omega^y_{a,\tilde{r}})$ is a closed subset of $\Omega_{a,\tilde{r}}$. The space $\Omega^y_{a,\tilde{r}}$ is complete by Theorem 3.6. Since a metric space is complete if and only if this space is closed in every its superspace, see for example [Sear, Th.10.2.1], $\mathrm{in}_y(\Omega^y_{a,\tilde{r}})$ is closed in $\Omega_{a,\tilde{r}}$. ■

Acknowledgment. The first author was partially supported by the State Foundations for Basic Research of Ukraine, Grant Φ25.1/055.

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