BOUNDEDNESS AND CONVERGENCE FOR SINGULAR INTEGRALS OF MEASURES SEPARATED BY LIPSCHITZ GRAPHS

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Abstract. We shall consider the truncated singular integral operators

$$T^{\varepsilon}_{\mu,K}f(x)=\int_{\mathbb{R}^n\setminus B(x,\varepsilon)}K(x-y)f(y)d\mu y$$

and related maximal operators $T^{*}_{\mu,K}f(x)=\sup_{\varepsilon>0}\left|T^{\varepsilon}_{\mu,K}f(x)\right|$. We shall prove for a large class of kernels K and measures μ and ν that if μ and ν are separated by a Lipschitz graph, then $T^{*}_{\nu,K}:L^p(\nu)\to L^p(\mu)$ is bounded for $1<p<\infty$. We shall also show that the truncated operators $T^{\varepsilon}_{\mu,K}$ converge weakly in some dense subspaces of $L^2(\mu)$ under mild assumptions for the measures and the kernels.

1. Introduction

Let $K:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ be some continuously differentiable function and μ some finite Radon measure in $\mathbb{R}^n$. The truncated singular integral operators associated with μ and K are given for $f\in L^1(\mu)$ by

$$T^{\varepsilon}_{\mu,K}f(x)=\int_{\mathbb{R}^n\setminus B(x,\varepsilon)}K(x-y)f(y)d\mu y.$$

Here $B(x,\varepsilon)$ is the closed ball centered at x with radius ε. Since the kernels we are interested in will remain fixed in the proofs, although the measures might vary, we will use the notation T^{ε}_{μ} instead of $T^{\varepsilon}_{\mu,K}$. Following this convention, the maximal singular integral operator is defined as

$$T^{*}_{\mu}f(x)=\sup_{\varepsilon>0}\left|T^{\varepsilon}_{\mu}f(x)\right|.$$

One of the key concepts in the theory of singular integral operators is L^2 boundedness. It is well known that even with very nice kernels the boundedness of $T^{*}_{\mu}:L^2(\mu)\to L^2(\mu)$ requires strong regularity properties of μ. In this paper we consider two measures μ and ν which live on different sides of some $(n-1)$-dimensional Lipschitz graph. We shall prove that then $T^{*}_{\nu}:L^2(\nu)\to L^2(\mu)$ is

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bounded very generally. The case where $\nu = \mathcal{H}^{n-1}\lfloor S$, the restriction of the $(n-1)$-dimensional Hausdorff measure to a Lipschitz graph S, was proved by David in [D1] and our proof relies on this result. We shall apply our boundedness theorem to show that the truncated operators T_μ^ε converge weakly in some dense subspaces of $L^2(\mu)$.

Before stating our main results we give some basic definitions that determine our setting.

Definition 1.1. The class Δ will contain all finite Radon measures μ on $\mathbb{R}^n$ such that

$$\mu(B(x,r)) \leq C_\mu r^{n-1} \text{ for } x \in \mathbb{R}^n \text{ and } r > 0, \tag{1.1}$$

where C_μ is some constant depending on μ.

We restrict to finite Radon measures only for convenience. Since by definition Radon measures are always locally finite, all our results easily extend to general Radon measures.

Definition 1.2. The class $\mathcal{K}$ will contain all continuously differentiable kernels $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ satisfying for all $x \in \mathbb{R}^n \setminus \{0\}$,

(i) $K(-x) = -K(x)$ (Antisymmetry),
(ii) $|K(x)| \leq C_0^K |x|^{-(n-1)}$,
(iii) $|\nabla K(x)| \leq C_1^K \, |x|^{-n}$,

where the constants C_0^K and C_1^K depend on K.

The classes $\mathcal{K}$ and Δ have been studied widely, see e.g. [D2] and the references therein. Notice also that both $\mathcal{K}$ and Δ are quite broad. For example, the class Δ contains measures supported on $(n-1)$-dimensional planes and Lipschitz graphs but it also contains measures whose support is some fractal set like the 1-dimensional four corners Cantor set in $\mathbb{R}^2$. Moreover Riesz kernels $|x|^{-n}x, x \in \mathbb{R}^n$, belong to $\mathcal{K}$, as well as stranger kernels like the ones appearing in [D3].

Denote the graph of a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$C_f = \left\{(x, f(x)) : x \in \mathbb{R}^{n-1}\right\}$$

and the corresponding half spaces by

$$H_f^+ = \{(x,y) : x \in \mathbb{R}^{n-1}, y > f(x)\} \text{ and } H_f^- = \{(x,y) : x \in \mathbb{R}^{n-1}, y < f(x)\}.$$

Our first main result reads as follows.

Theorem 1.3. Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be some Lipschitz function and μ and ν measures in $\mathbb{R}^n$ such that

(i) $\mu(H_f^-) = \nu(H_f^+) = 0$,
(ii) $\mu, \nu \in \Delta$.

There exist constants $C_p, 1 \leq p < \infty$, depending only on p, n, C_μ, C_ν and $\mathrm{Lip}(f)$ such that for all $g \in L^1(\nu)$,

$$\int (T_\nu^* g)^p d\mu \leq C_p \int |g|^p d\nu \text{ for } 1 < p < \infty, \tag{1.2}$$

and

$$\mu(\{x \in \mathbb{R}^n : T_\nu^* g(x) > t\}) \leq \frac{C_1}{t} \int |g| d\nu \text{ for } t > 0. \tag{1.3}$$

The proof is based on the following two theorems. The first one is a special case of a classical result, for related discussion and references see [DS], p.13. The second was proved by David in [D1]. Although David worked only in the plane, his proof generalizes without any essential changes.

Theorem 1.4. Let $S \subset \mathbb{R}^n$ be some $(n-1)$-dimensional Lipschitz graph and let $\sigma = \mathcal{H}^{n-1} \lfloor S$. Then if $K \in \mathcal{K}$ the corresponding maximal operator

$$T_\sigma^* : L^p(\sigma) \to L^p(\sigma)$$

is bounded for $1 < p < \infty$.

Theorem 1.5. Let $K \in \mathcal{K}$ and $\mu, \sigma \in \Delta$. Suppose that there exists a positive constant c_σ such that $\sigma(B(x,r)) \geq c_\sigma r^{n-1}$ for x in the support of σ and for $0 < r < 1$, and that

$$T_\sigma^* : L^p(\sigma) \to L^p(\sigma)$$

is bounded for $1 < p < \infty$. Then

$$T_\sigma^* : L^p(\sigma) \to L^p(\mu) \text{ and } T_\mu^* : L^p(\mu) \to L^p(\sigma)$$

are also bounded for $1 < p < \infty$.

We shall apply Theorem 1.3 to obtain certain weak convergence results. Recently it was shown in [MV] that for general measures and kernels the $L^2(\mu)$-boundedness of the operators $T_{\mu,K}^\varepsilon$ forces them to converge weakly in $L^2(\mu)$. This means that there exists a bounded linear operator $T_{\mu,K} : L^2(\mu) \to L^2(\mu)$ such that for all $f, g \in L^2(\mu)$,

$$\lim_{\varepsilon \to 0} \int T_{\mu,K}^\varepsilon(f) g d\mu = \int T(f) g d\mu.$$

Motivated by this recent development it is natural to ask if limits of this type might exist if we remove the very strong L^2-boundedness assumption. But, as it was remarked in [MV], by the Banach-Steinhaus theorem the converse also holds often; weak convergence implies L^2-boundedness. And L^2-boundedness is known to fail very often, for example, by [MeV] and [L], if K is the Cauchy kernel, $K(z) = 1/z, z \in \mathbb{C}$, and μ has positive and finite 1-upper density, i.e,

$$0 < \limsup_{r \to 0} \frac{\mu(B(x,r))}{r} < \infty \ \mu \text{ a.e},$$

and is purely unrectifiable, that is, $\mu(\Gamma)=0$ for every rectifiable curve Γ. Hence we cannot hope for the full weak convergence in $L^2(\mu)$ in such cases. However, we shall prove that the operators $T^{\varepsilon}_{\mu,K}$ converge weakly in a restricted sense, see Theorem 1.8, under some mild assumptions for the measures and the kernels, including also many purely unrectifiable measures.

For these convergence results we shall also use the following theorem. It was first proved in [MM] for the Cauchy transform in the plane, and then by a different method by Verdera in [V]. Verdera's proof easily extends to the present setting, one can also consult [M], Section 20.

Theorem 1.6. Let $S \subset \mathbb{R}^n$ be some $(n-1)$-dimensional Lipschitz graph. Then if $K \in \mathcal{K}$ and ν is any finite Radon measure in $\mathbb{R}^n$, the principal values

$$\lim_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} K(x-y)\,d\nu y$$

exist and are finite for $\mathcal{H}^{n-1}$ almost all $x \in S$.

Using Theorems 1.3 and 1.6 we are able to prove rather easily the following fact.

Theorem 1.7. Let $\mu \in \Delta$ and $K \in \mathcal{K}$. Then for any Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ the finite limit

$$\lim_{\varepsilon \to 0} \int_{\substack{\mathbb{R}^n \setminus H_f^- \\ |x-y|>\varepsilon}} \int_{H_f^-} K(x-y)\,d\mu y\,d\mu x \tag{1.4}$$

exists.

Theorem 1.7 is the main tool used to establish weak convergence. Consider the following function spaces, which are dense subsets of $L^2(\mu)$ for $\mu \in \Delta$,

$$\mathcal{X}_Q(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R}, f \text{ is a finite linear combination of characteristic functions of rectangles in } \mathbb{R}^n\}$$

and

$$\mathcal{X}_B(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R}, f \text{ is a finite linear combination of characteristic functions of balls in } \mathbb{R}^n\}.$$

Rectangles in $\mathcal{X}_Q$ need not have their sides parallel to the axis.

Theorem 1.8. If $\mu \in \Delta$ and $K \in \mathcal{K}$, the finite limit

$$\lim_{\varepsilon \to 0} \int T^{\varepsilon}_{\mu}(f)(x) g(x)\,d\mu x$$

exists for $f, g \in \mathcal{X}_B(\mathbb{R}^n)$ and $f, g \in \mathcal{X}_Q(\mathbb{R}^n)$.

Theorem 1.8 was proved in [C2] for more general kernels K but under more restrictive porosity conditions on the measure μ. Further discussions on boundedness and convergence properties of singular integrals with general measures can be found for example in [M], [MV], [T], [D3] and [C1].

Throughout this paper $A \lesssim B$ means $A \leq CB$ for some constant C depending only on the appropriate structural constants, that is, the dimension n, the exponent p, the Lipschitz constants of the Lipschitz graphs and the regularity constants C_μ of the measures.

2. $L^p(\nu) \to L^p(\mu)$ BOUNDEDNESS

In this section we prove Theorems 1.3 and 1.7.

Proof of Theorem 1.3. By standard Calderón-Zygmund techniques it suffices to prove (1.2). Let $C > 0$ be some constant such that

$$\mu(B(x,r)) \leq Cr^{n-1} \text{ and } \nu(B(x,r)) \leq Cr^{n-1} \text{ for } x \in \mathbb{R}^n \text{ and } r > 0.$$

Write $\mu = \mu_1 + \mu_2$ and $\nu = \nu_1 + \nu_2$ where $\mu_1 = \mu\lfloor C_f$ and $\nu_1 = \nu\lfloor C_f$. By standard differentiation theory of measures, see, e.g., [M], Section 2, the measures μ_1 and ν_1 are absolutely continuous with respect to $\sigma = \mathcal{H}^{n-1}\lfloor S$ with bounded Radon-Nikodym derivatives. Hence there exist Borel functions h_μ and h_ν such that $0 \leq h_\mu \lesssim 1$ and $0 \leq h_\nu \lesssim 1$ and that

$$d\mu_1 = h_\mu d\sigma \text{ and } d\nu_1 = h_\nu d\sigma.$$

By Theorems 1.4 and 1.5 we have for $g \in L^p(\nu)$,

$$\begin{aligned}\int (T^*_{\nu_1} g)^p d\mu_1 = \int (T^*_\sigma(gh_\nu))^p h_\mu d\sigma &\lesssim \int |gh_\nu|^p d\sigma \\ &\lesssim \int |g|^p h_\nu d\sigma = \int |g|^p d\nu_1 \leq \int |g|^p d\nu,\end{aligned}$$

$$\int (T^*_{\nu_1} g)^p d\mu_2 = \int (T^*_\sigma(gh_\nu))^p d\mu_2 \lesssim \int |gh_\nu|^p d\sigma \leq \int |g|^p d\nu,$$

and

$$\int (T^*_{\nu_2} g)^p d\mu_1 \lesssim \int (T^*_{\nu_2}(g))^p d\sigma \lesssim \int |g|^p d\nu_2 \leq \int |g|^p d\nu.$$

As $T^*_\nu \leq T^*_{\nu_1} + T^*_{\nu_2}$ we may thus assume that $\mu = \mu_2$ and $\nu = \nu_2$, that is, $\mu(H^-_f \cup C_f) = \nu(H^+_f \cup C_f) = 0$, and also that $g(x) = 0$ for $x \in H^+_f \cup C_f$

Let $L > \max\{1, \mathrm{Lip}(f)\}$. For $x_0 = (u_0, f(u_0)) \in C_f$ define the cone

$$\Gamma(x_0) = \{(u,t) \in \mathbb{R}^n : t - f(u_0) > 4L|u - u_0|\},$$

and observe that

$$|y - x| \geq \frac{1}{8L}|y - x_0| \text{ for } y \in \Gamma(x_0), x \in H^-_f. \tag{2.1}$$

We define the non-tangential maximal function $N(g)$ for any function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ by

$$N(g)(x) = \sup\{|g(y)| : y \in \Gamma(x)\}.$$

For the maximal function $N(g)$, the following L^p estimate holds.

Lemma 2.1. For any $0 < p < \infty$, and any μ measurable function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$,

$$\int |g|^p d\mu \lesssim \int_{C_f} N(g)^p d\mathcal{H}^{n-1}.$$

This follows from the fact that μ is a Carleson measure in H_f^+, i.e.,

$$\mu(B(x,r)) \leq C\mathcal{H}^{n-1}(C_f \cap B(x,r)) \text{ for } x \in C_f, r > 0.$$

A simple proof is given in [Tor] for the case where $C_f = \mathbb{R}^{n-1}$ but the same argument holds for general C_f.

Lemma 2.2. For any $g \in L^1(\nu)$ and any $x \in C_f$,

$$N(T^*_\nu g)(x) \leq C_N(T^*_\nu g(x) + M_\nu g(x))$$

where

$$M_\nu g(x) = \sup_{r>0} r^{1-n} \int_{B(x,r)} |f| d\nu$$

and C_N depends only on n, L and C.

Proof. Let $y \in \Gamma(x)$ and $\varepsilon > 0$. We will estimate $|T^\varepsilon_\nu g(y)|$ by dividing the argument to two cases. Let $r = |x - y|$ and assume first that $\varepsilon < r$. Then

$$\begin{aligned}
|T^\varepsilon_\nu g(x) - T^\varepsilon_\nu g(y)| &= \left| \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} K(x-z)g(z)d\nu z - \int_{\mathbb{R}^n \setminus B(y,\varepsilon)} K(y-z)g(z)d\nu z \right| \\
&\leq \int_{\mathbb{R}^n \setminus B(x,2r)} |K(x-z) - K(y-z)|\,|g(z)|d\nu z \\
&\quad + \int_{H_f^- \cap B(x,2r)} |K(y-z)||g(z)|d\nu z \\
&\quad + \left| \int_{B(x,2r)\setminus B(x,\varepsilon)} K(x-z)g(z)d\nu z \right|
\end{aligned}$$

We estimate the first integral by integrating over the annuli $B(x, 2^i r)\setminus B(x, 2^{i-1}r), i \in \mathbb{N}, i \geq 2$. By the Mean Value Theorem we derive that

$$\begin{aligned}
|K(x-z) - K(y-z)| &\leq |\nabla K(\xi(z))|\,|x-y| \\
&\leq \frac{C_1^K\,|x-y|}{|\xi(z)|^n}
\end{aligned}$$

where $\xi(z)$ lies in the line segment joining $y - z$ to $x - z$. Furthermore for $i \in \mathbb{N}, i \geq 2$, and $z \in B(x, 2^i r) \backslash B(x, 2^{i-1} r)$,

$$\begin{aligned} |\xi(z)| &\geq |x-z| - |\xi(z) - (x-z)| \\ &\geq |x-z| - |(y-z) - (x-z)| \\ &\geq 2^{i-2} r. \end{aligned}$$

Hence

$$\begin{aligned} &\int_{\mathbb{R}^n \backslash B(x,2r)} |K(x-z) - K(y-z)||g(z)| d\nu z \\ &\qquad \leq \sum_{i=2}^{\infty} \int_{B(x,2^i r)\backslash B(x,2^{i-1}r)} \frac{C_1^K |x-y|}{|\xi(z)|^n} |g(z)| d\nu z \\ &\qquad \leq 4^n C_1^K \sum_{i=2}^{\infty} 2^{-i} \frac{1}{(2^i r)^{n-1}} \int_{B(x,2^i r)} |g(z)| d\nu z \\ &\qquad \leq 4^n C_1^K M_\nu g(x). \end{aligned}$$

For the second integral, using (2.1) we estimate,

$$\begin{aligned} \int_{H_f^- \cap B(x,2r)} |K(y-z)||g(z)| d\nu z &\leq C_0^K \int_{H_f^- \cap B(x,2r)} |y-z|^{1-n} |g(z)| d\nu z \\ &\leq (16L)^{n-1} C_0^K (2r)^{1-n} \int_{B(x,2r)} |g(z)| d\nu z \\ &\leq (16L)^{n-1} C_0^K M_\nu g(x) \end{aligned}$$

Obviously the third integral is bounded by $2T_\nu^* g(x)$. Therefore,

$$|T_\nu^\varepsilon g(y)| \leq 3 |T_\nu^* g(x)| + D_1 M_\nu g(x) \tag{2.2}$$

where $D_1 = 4^n C_1^K + (16L)^{n-1} C_0^K$.

Secondly, suppose that $\varepsilon \geq r$. Then

$$\begin{aligned} |T_\nu^\varepsilon g(x) - T_\nu^\varepsilon g(y)| &= \left| \int_{\mathbb{R}^n \backslash B(x,\varepsilon)} K(x-z) g(z) d\nu z - \int_{\mathbb{R}^n \backslash B(y,\varepsilon)} K(y-z) g(z) d\nu z \right| \\ &\leq \int_{\mathbb{R}^n \backslash B(x,2\varepsilon)} |K(x-z) - K(y-z)| \, |g(z)| d\nu z \\ &\quad + \int_{B(x,2\varepsilon)\backslash B(x,\varepsilon)} |K(y-z)||g(z)| d\nu z \\ &\quad + \left| \int_{B(x,2\varepsilon)\backslash B(x,\varepsilon)} K(x-z) g(z) d\nu z \right| \end{aligned}$$

Exactly as before

$$\int_{\mathbb{R}^n\setminus B(x,2\varepsilon)} |K(x-z)-K(y-z)|\,|g(z)|d\nu z \leq 4^n C_1^K M_\nu g(x),$$

$$\int_{B(x,2\varepsilon)\setminus B(x,\varepsilon)} |K(y-z)||g(z)|d\nu z \leq 2^{n-1} C_0^K M_\nu g(x)$$

and

$$\left|\int_{B(x,2\varepsilon)\setminus B(x,\varepsilon)} K(x-z)g(z)d\nu z\right| \leq 2T_\nu^* g(x).$$

Therefore,

$$|T_\nu^\varepsilon g(y)| \leq 3\,|T_\nu^* g(x)| + D_2 M_\nu g(x) \tag{2.3}$$

where $D_2 = 4^n C_1^K + 2^{n-1} C_0^K$. Choosing $C_N = D_1$ and combining (2.2) and (2.3) we complete the proof the Lemma 2.2. □

We can now proceed and finish the proof of Theorem 1.3. By Lemmas 2.1 and 2.2, Theorems 1.4 and 1.5, the L^p-boundedness of $M_{\nu+\sigma}$ (see, e.g., [M], Theorem 2.19) and the fact that $g(x)=0$ for $x \in C_f$,

$$\begin{aligned}
\int (T_\nu^* g)^p d\mu &\lesssim \int N(T_\nu^* g)^p d\sigma \\
&\lesssim \int (T_\nu^* g)^p d\sigma + \int (M_\nu g)^p d\sigma \\
&\lesssim \int |g|^p d\nu + \int (M_{\nu+\sigma} g)^p d(\nu+\sigma) \\
&\lesssim \int |g|^p d\nu + \int |g|^p d(\nu+\sigma) \\
&= 2\int |g|^p d\nu.
\end{aligned}$$

The proof is finished. □

Proof of Theorem 1.7. Denote $\nu = \mu\lfloor H_f^-$ and $\lambda = \mu\lfloor(H_f^+ \cup C_f)$. By Theorem 1.3

$$T_\nu^* : L^2(\nu) \to L^2(\lambda)$$

is bounded. Therefore by Hölder's inequality

$$\int T_\nu^*(1)d\lambda \leq \|T_\nu^*(1)\|_{L^2(\lambda)}\|1\|_{L^2(\lambda)} \lesssim \|T_\nu^*(1)\|_{L^2(\lambda)} \lesssim \|1\|_{L^2(\nu)} < \infty.$$

For $z \in H_f^+$ the limit

$$\lim_{\varepsilon\to 0} T_\nu^\varepsilon(1)(z)$$

exists since $H_f^+ \cap \mathrm{spt}\nu = \emptyset$. Furthermore by Theorem 1.6 the above limit also exists for μ almost every $z \in C_f$. Thus by the Lebesgue dominated convergence theorem we derive that the limit

$$\lim_{\varepsilon\to 0}\int_{H_f^+\cup C_f} T_\nu^\varepsilon(1)(z)d\mu z = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus H_f^-}\int_{\substack{H_f^- \\ |x-y|>\varepsilon}} K(x-y)d\mu y d\mu x$$

exists and is finite, completing the proof of Theorem 1.7. □

Remark. As a corollary of Theorem 1.7 and Fubini's theorem we derive that the limit

$$\lim_{\varepsilon\to 0}\int_{H_f^+}\int_{\substack{\mathbb{R}^n\setminus H_f^+ \\ |x-y|>\varepsilon}} K(x-y)d\mu y d\mu x$$

exists under the same assumptions as in Theorem 1.7.

3. Weak Convergence in $\mathcal{X}_B(\mathbb{R}^n)$ and $\mathcal{X}_Q(\mathbb{R}^n)$

To prove Theorem 1.8 let $f, g \in \mathcal{X}_Q(\mathbb{R}^n)$ or $f, g \in \mathcal{X}_B(\mathbb{R}^n)$ be such that

$$f = \sum_{i=1}^{l} a_i \chi_{Q_i} \text{ and } g = \sum_{j=1}^{m} b_j \chi_{P_j},$$

where $a_i, b_j \in \mathbb{R}$ and Q_i, P_j are closed balls or Q_i, P_j are closed rectangles. Then for $\varepsilon > 0$,

$$\int T_\mu^\varepsilon f(x) g(x) d\mu x = \sum_{j=1}^{m}\sum_{i=1}^{l} b_j a_i \int_{P_j}\int_{\substack{Q_i \\ |x-y|>\varepsilon}} K(x-y)d\mu y d\mu x.$$

Therefore it is enough to show that for balls P, Q or rectangles P, Q the limit

$$\lim_{\varepsilon\to 0}\int_P\int_{\substack{Q \\ |x-y|>\varepsilon}} K(x-y)d\mu y d\mu x$$

exists. But,

$$\int_P\int_{\substack{Q \\ |x-y|>\varepsilon}} K(x-y)d\mu y d\mu x = I_1 + I_2 + I_3 + I_4,$$

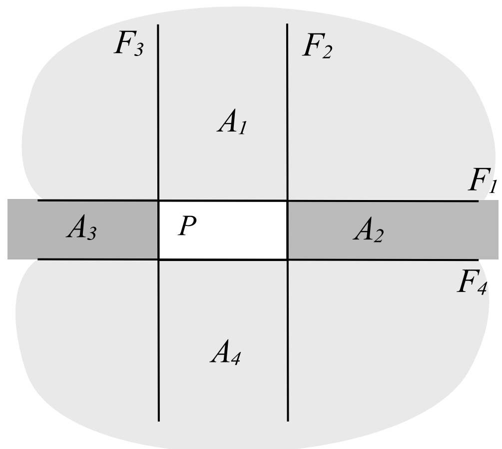

FIGURE A

where,

$$
\begin{aligned}
I_1 &= \int_{\substack{P\cap Q\\ |x-y|>\varepsilon}} \int_{P\cap Q} K(x-y)d\mu y d\mu x,\\
I_2 &= \int_{\substack{P\setminus Q\\ |x-y|>\varepsilon}} \int_{P\cap Q} K(x-y)d\mu y d\mu x,\\
I_3 &= \int_{\substack{P\cap Q\\ |x-y|>\varepsilon}} \int_{Q\setminus P} K(x-y)d\mu y d\mu x,\\
I_4 &= \int_{\substack{P\setminus Q\\ |x-y|>\varepsilon}} \int_{Q\setminus P} K(x-y)d\mu y d\mu x.
\end{aligned}
$$

By the antisymmetry of K, for every $\varepsilon > 0$,

$$I_1 = 0.$$

Furthermore by Fubini's theorem I_3 is essentially the same with I_2, allowing us to treat only I_2 and I_4. In that direction notice that for every rectangle, or ball, say P, there exist some collection of rotations of Lipschitz graphs $\{F_i(P)\}_{i=1}^{2n}$, and disjoint Borel sets $\{A_i(P)\}_{i=1}^{2n}$, such that

$$
\begin{aligned}
\mathbb{R}^n \setminus P &= \cup_{i=1}^{2n} A_i(P),\\
P &\subset H^-_{F_i(P)} \cup F_i(P),\\
A_i(P) &\subset H^+_{F_i(P)}.
\end{aligned}
$$

See Figure A for an illustration in the case when P is a subset of the plane.

Using the above geometric property I_2 and I_4 can be decomposed in the following way,

$$I_2 = \sum_{i=1}^{2n} \int_{\substack{A_i(Q)\cap P \\ |x-y|>\varepsilon}} \int_{P\cap Q} K(x-y) d\mu y d\mu x$$

and

$$I_4 = \sum_{i=1}^{2n} \int_{\substack{A_i(Q)\cap P \\ |x-y|>\varepsilon}} \int_{Q\setminus P} K(x-y) d\mu y d\mu x.$$

Therefore since limits like

$$\lim_{\varepsilon\to 0} \int_{\substack{A_i(Q)\cap P \\ |x-y|>\varepsilon}} \int_{P\cap Q} K(x-y) d\mu y d\mu x$$

and

$$\lim_{\varepsilon\to 0} \int_{\substack{A_i(Q)\cap P \\ |x-y|>\varepsilon}} \int_{Q\setminus P} K(x-y) d\mu y d\mu x$$

exist by Theorem 1.7 we finally obtain Theorem 1.8.

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