WEAK CONVERGENCE OF SINGULAR INTEGRALS

VASILIS CHOUSIONIS

Abstract. We show that the truncated singular integral operators

$$T^{\varepsilon}_{\mu,K}(f)(x)=\int_{\mathbb{R}^n\setminus B(x,\varepsilon)}K(x-y)f(y)d\mu y.$$

converge weakly in some dense subspaces of $L^2(\mu)$, under mild assumptions for the measures and the kernels.

1. Introduction

Let $K:\mathbb{R}^n\setminus\{0\}\rightarrow\mathbb{R}$ be some continuously differentiable function and μ some Radon measure in $\mathbb{R}^n$. The truncated singular integral operators associated with μ and K are given by

$$T^{\varepsilon}_{\mu,K}(f)(x)=\int_{\mathbb{R}^n\setminus B(x,\varepsilon)}K(x-y)f(y)d\mu y.$$

Here $B(x,\varepsilon)$ is the closed ball centered at x with radius ε. Since the kernels we are interested in will remain fixed in the proofs, although the measures might vary, we will use the notation T^{ε}_{μ} instead of $T^{\varepsilon}_{\mu,K}$. Following this convention, the maximal singular integral operator is defined as

$$T^{*}_{\mu}(f)(x)=\sup_{\varepsilon>0}\left|T^{\varepsilon}_{\mu}(f)(x)\right|.$$

We are interested in limit properties of the operators T^{ε}_{μ}. First consider the direct question as to whether the limit, the so called principal value of T,

$$\lim_{\varepsilon\rightarrow 0}T^{\varepsilon}_{\mu}(f)(x),$$

exists μ almost everywhere. When $\mu=\mathcal{L}^n$, the Lebesgue measure in $\mathbb{R}^n$, and K is a standard Calderón-Zygmund kernel, due to cancelations and the denseness of smooth functions in L^1, the principal values exist almost everywhere for L^1-functions. For more general measures, the question is more complicated. Let m be an integer, $0<m<n$, and consider the coordinate Riesz kernels

$$R^{m}_{i}(x)=\frac{x_i}{|x|^{m+1}}\text{ for }i=1,...,n.$$

2000 *Mathematics Subject Classification.* Primary 42B20.

Key words and phrases. Singular Integrals, Lipschitz graphs.

The author is supported by the Finnish Graduate School in Mathematical Analysis.

Let C some compact set with finite m-dimensional Hausdorff measure $\mathcal{H}^m$, and denote by $\mu = \mathcal{H}^m\lfloor C$ the restriction of $\mathcal{H}^m$ on C. By the works of Mattila and Preiss [MP], Mattila and Melnikov [MM], Verdera [V] and Tolsa [T2] the principal values

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B(x,\varepsilon)}\frac{x_i-y_i}{|x-y|^{m+1}}d\mu y$$

exist μ almost everywhere if and only if the set C is m-rectifiable i.e. if there exist m-dimensional Lipschitz surfaces M_i, $i\in\mathbb{N}$, such that

$$\mathcal{H}^m(C\setminus\cup_{i=1}^{\infty}M_i)=0.$$

For $m=1$, Tolsa in [T1] showed, among other things, that if the operators

$$C_\mu^\varepsilon(f)(x)=\int_{\mathbb{R}^n\setminus B(x,\varepsilon)}R_i^2(x-y)d\mu y, \text{ for i=1,2},$$

are uniformly bounded in L^2, which means that there exists some constant C such that

$$\int |C_\mu^\varepsilon(f)|^2 d\mu \leq C\int |f|^2 d\mu \text{ for every } f\in L^2 \text{ and every } \varepsilon>0,$$

then the principal values exist μ almost everywhere. For $m>1$ the question remains open.

In different settings L^2-boundedness does not always imply the almost everywhere existence of principal values. Let C be the 1-dimensional four corners Cantor set and μ its natural (1-dimensional Hausdorff) measure. David in [D3], constructed Calderón-Zygmund standard kernels that define operators bounded in $L^2(\mu)$ whose principal values fail to exist μ almost everywhere. Although David's kernels can be chosen odd or even, they are not homogeneous of degree -1. In [C] families of Calderón-Zygmund standard, smooth, odd and homogeneous kernels were constructed on Sieprinski gaskets E_d of Hausdorff dimension d, $0<d<1$. These kernels give rise to singular integral operators bounded in $L^2(\mu_d)$ with principal values diverging μ_d almost everywhere. Here $\mu_d=\mathcal{H}^d\lfloor E_d$.

Recently, in [MV], Mattila and Verdera showed for general measures and kernels that the $L^2(\mu)$-boundedness of the operators $T_{\mu,K}^\varepsilon$ forces them to converge weakly in $L^2(\mu)$. This means that there exists a bounded linear operator $T_{\mu,K}:L^2(\mu)\to L^2(\mu)$ such that for all $f,g\in L^2(\mu)$,

$$\lim_{\varepsilon\to 0}\int T_{\mu,K}^\varepsilon(f)(x)g(x)d\mu x=\int T(f)(x)g(x)d\mu x.$$

Furthermore it was remarked in [MV], that by the Banach-Steinhaus theorem the converse also holds often. Motivated by this recent development it is natural to ask if limits of this type might exist if we remove the very strong L^2-boundedness assumption.

We prove that the operators $T^{\varepsilon}_{\mu,K}$ converge weakly in the sense of Theorem 1.4 under some mild assumptions for the measures and the kernels. It is of interest that weak convergence of this type holds for many $(n-1)$-purely unrectifiable measures μ, that is when $\mu(E) = 0$ for all $(n-1)$-rectifiable sets E. Recall that for 1-purely unrectifiable measures and 1-dimensional Riesz kernels the principal values diverge almost everywhere and the weak convergence in L^2 fails.

Our setting is determined by the following definitions.

Definition 1.1. The class Δ will contain all finite Radon measures μ on $\mathbb{R}^n$ such that

$$\mu(B(x,r)) \leq C_\mu r^{n-1} \text{ for } r > 0, \tag{1.1}$$

where C_μ is some constant depending on μ. The subclass $\Sigma \subset \Delta$ will contain all the measures $\mu \in \Delta$ such that for $0 < r < \operatorname{diam}(\operatorname{spt}\mu)$ and $x \in \operatorname{spt}\mu$,

$$C_\mu^{-1} r^{n-1} \leq \mu(B(x,r)) \leq C_\mu r^{n-1}, \tag{1.2}$$

where C_μ depends on μ. Radon measures, not necessarily finite, satisfying (1.2) are also referred as $(n-1)$ Ahlfors-David regular.

Definition 1.2. The class $\mathcal{K}$ will contain all continuously differentiable kernels $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ satisfying for all $x \in \mathbb{R}^n \setminus \{0\}$,

(i) $K(-x) = -K(x)$ (Antisymmetry)
(ii) $|K(x)| \leq C_0 |x|^{-(n-1)}$
(iii) $|\nabla K(x)| \leq C_1 |x|^{-n}$

where the constants C_0 and C_1 depend on K.

The classes $\mathcal{K}$ and Δ have been studied widely, see e.g. [D2] and the references therein. Notice also that both $\mathcal{K}$ and Δ are quite broad. For example the class Δ contains measures supported on balls intersected with $(n-1$-dimensional planes and Lipschitz graphs but it also contains measures whose support is some fractal set like the 1-dimensional four corners Cantor set in $\mathbb{R}^2$. Moreover Riesz kernels for $m = n-1$ belong to $\mathcal{K}$, as well as stranger kernels like the ones appearing in [D3].

We continue with some basic notation. For $x \in \mathbb{R}^n$ and $m = 1, .., n$ let $x\lfloor_m = (x_1, .., x_m)$. Denote the graph of a given function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$C_f = \{x \in \mathbb{R}^n : x_n = f(x\lfloor_{n\text{-}1})\}$$

and the corresponding half spaces by

$$H_f^+ = \{x \in \mathbb{R}^n : x_n > f(x\lfloor_{n\text{-}1})\} \text{ and } H_f^- = \{x \in \mathbb{R}^n : x_n < f(x\lfloor_{n\text{-}1})\}.$$

The following theorem is the main tool used to establish weak convergence.

Theorem 1.3. Let $\mu \in \Delta$ and $K \in \mathcal{K}$. Then for any Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ the limit

$$\lim_{\varepsilon \to 0} \int_{\substack{\mathbb{R}^n \setminus H_f^- \\ |x-y|>\varepsilon}} \int_{H_f^-} K(x-y) d\mu y d\mu x \tag{1.3}$$

exists.

Consider the following function spaces, which are dense subsets of $L^2(\mu)$ for $\mu \in \Delta$,

$$\mathcal{X}_Q(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R}, f \text{ is a finite linear combination of characteristic functions of rectangles in } \mathbb{R}^n\}$$

and

$$\mathcal{X}_B(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R}, f \text{ is a finite linear combination of characteristic functions of balls in } \mathbb{R}^n\}.$$

Rectangles in $\mathcal{X}_Q$ need not have their sides parallel to the axis. Our principal result reads as follows.

Theorem 1.4. If $\mu \in \Delta$ and $K \in \mathcal{K}$ the limit

$$\lim_{\varepsilon \to 0} \int T_\mu^\varepsilon(f)(x) g(x) d\mu x$$

exists for $f, g \in \mathcal{X}_B(\mathbb{R}^n)$ and $f, g \in \mathcal{X}_Q(\mathbb{R}^n)$.

As noted earlier, by [MV], the weak convergence in $L^2(\mu)$ implies that the operators T_μ^ε are uniformly bounded in $L^2(\mu)$. Therefore, since the singular integral operators associated with 1-dimensional Riesz kernels and 1-purely unrectifiable measures are not bounded in $L^2(\mu)$, one cannot hope of replacing the function spaces $\mathcal{X}_B(\mathbb{R}^n)$ and $\mathcal{X}_Q(\mathbb{R}^n)$ with $L^2(\mu)$ in theorem 1.4.

Before starting proving Theorems 1.3 and 1.4, we are going to state, applied to our setting, some known results that are going to be used in the proofs. The first one was proved by David in [D1].

Theorem 1.5. Let $K \in \mathcal{K}$, $\mu \in \Delta$ and $\sigma \in \Sigma$ such that

$$T_\sigma^* : L^2(\sigma) \to L^2(\sigma)$$

is bounded. Then

$$T_\sigma^* : L^2(\sigma) \to L^2(\mu) \text{ and } T_\mu^* : L^2(\mu) \to L^2(\sigma)$$

are also bounded.

Coifman, David and Meyer proved the following theorem in [CDM] based on earlier results by Coifman, McIntosh and Meyer,see [CMM].

Theorem 1.6. Let $S \subset \mathbb{R}^n$ be some $(n-1)$-dimensional Lipschitz graph and let $\sigma = \mathcal{H}^{n-1}\lfloor S$. Then if $K \in \mathcal{K}$ the corresponding maximal operator,

$$T^*_\sigma : L^2(\sigma) \to L^2(\sigma)$$

is bounded.

The following theorem was first proved by Mattila and Melnikov in [MM] for the Cauchy transform in the plane, and in the general form stated below by Verdera in [V].

Theorem 1.7. Let $S \subset \mathbb{R}^n$ be some $\mathcal{H}^{n-1}$ measurable, $(n-1)$-rectifiable set of finite $\mathcal{H}^{n-1}$ measure. Then if $K \in \mathcal{K}$ and ν is any finite Radon measure in $\mathbb{R}^n$ the principal values

$$\lim_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} K(x-y) d\nu y$$

exist for $\mathcal{H}^{n-1}$ almost all $x \in S$.

Throughout this paper $A \lesssim B$ means $A \lesssim CB$ for some absolute constant C.

2. Proof of Theorem 1.3

Let $L > \max\{1, \mathrm{Lip}(f)\}$, $V = \{x \in \mathbb{R}^n : x_n = 0\}$ and without loss of generality assume that $\mathrm{spt}\mu \subset B(0,1)$. We can also assume that $C_f \cap \mathrm{spt}\mu \neq \emptyset$, since otherwise the quantity

$$\int_{\substack{\mathbb{R}^n \setminus H_f^- \\ |x-y|>\varepsilon}} \int_{H_f^-} K(x-y) d\mu y d\mu x$$

is constant for ε small enough. In that case $pr_V^{-1}(pr_V(\mathrm{spt}\mu)) \cap C_f \subset B(0,5L)$. Let $S = B(0,5L) \cap C_f$ and denote

$$\sigma = \mathcal{H}^{n-1}\lfloor S \text{ and } \nu = \mu\lfloor H_f^-.$$

It follows easily that $\sigma \in \Sigma$.

By Theorem 1.6

$$T^*_\sigma : L^2(\sigma) \to L^2(\sigma)$$

is bounded. Furthermore since $\nu \in \Delta, \sigma \in \Sigma$ and T^*_σ is bounded Theorem 1.5 implies that

$$T^*_\nu : L^2(\nu) \to L^2(\sigma)$$

is bounded. Therefore by Hölder's inequality and the L^2 boundedness of T^*_ν,

$$\begin{aligned} \int T^*_\nu(1)(x) d\sigma x &\leq \|T^*_\nu(1)\|_{L^2(\sigma)} \|1\|_{L^2(\sigma)} \\ &\lesssim \|T^*_\nu(1)\|_{L^2(\sigma)} \\ &\lesssim \|1\|_{L^2(\nu)}. \end{aligned}$$

Hence
$$\int T_\nu^*(1)(x)d\sigma x < \infty. \tag{2.1}$$
Since $\sigma \in \Sigma$ there exists some constant C_S such that for $x \in S$ and $0 < r < 5L$,
$$\frac{r^{n-1}}{C_S} \leq \sigma(B(x,r)) \leq C_S r^{n-1}.$$
Therefore
$$\begin{aligned} \mu(S \cap B(x,r)) &\leq C_\mu r^{n-1} \\ &\leq C_\mu C_S \sigma(B(x,r)) \end{aligned}$$
for $x \in S$ and $0 < r < 5L$. Using Vitalli's covering theorem for μ we deduce that $\mu\lfloor S \leq C_\mu C_S \sigma$, which combined with (2.1) gives
$$\int_S T_\nu^*(1)(x)d\mu x < \infty. \tag{2.2}$$

The following Lemma, roughly speaking, allows us to compare the values of $T_\nu^*(1)$ on Whitney cubelike sets in H_f^+ and on their projections on C_f.

Lemma 2.1. Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be some Lipschitz function with $L > \max\{1, \mathrm{Lip}(f)\}$. If
$$\begin{aligned} Q &= \prod_{i=1}^{n-1}[a,b) \subset V \text{ with } b-a=r, \\ A &= \{x \in \mathbb{R}^n : x\lfloor_{n\text{-}1} \in Q \text{ and } f(x\lfloor_{n\text{-}1}) + 2Lr \leq x_n < f(x\lfloor_{n\text{-}1}) + 4Lr\}, \\ A' &= \{x \in C_f : x\lfloor_{n\text{-}1} \in Q\}, \end{aligned}$$
then for $z \in A$ and $z' \in A'$,
$$T_\nu^*(1)(z) \leq 3T_\nu^*(1)(z') + D$$
where $D = D(\mu, K, f, n)$.

Proof. Let $z \in A$, $z' \in A'$ and $\varepsilon > 0$. Since
$$|\, z\lfloor_{n\text{-}1} - z'\lfloor_{n\text{-}1} \,| \leq \sqrt{n-1}r$$
and
$$\begin{aligned} |z_n - z_n'| &\leq |z_n - f(z\lfloor_{n\text{-}1})| + |f(z\lfloor_{n\text{-}1}) - f(z'\lfloor_{n\text{-}1})| \\ &\leq Lr(4 + \sqrt{n-1}), \end{aligned}$$
we get
$$\begin{aligned} |z - z'| &= \sqrt{|\, z\lfloor_{n\text{-}1} - z'\lfloor_{n\text{-}1} \,|^2 + |z_n - z_n'|^2} \\ &\leq C(n)r \end{aligned}$$

where $C(n) = \sqrt{\left(4L+\sqrt{n-1}\right)^2 + n - 1}$. We have to consider two cases for $\varepsilon > 0$.

For $\varepsilon \leq |z - z'|$,

$$\begin{aligned}
|T^{\varepsilon}_{\nu}(1)(z) - T^{\varepsilon}_{\nu}(1)(z')| &= \left| \int_{B(z,\varepsilon)^c} K(z-y) d\nu y - \int_{B(z',\varepsilon)^c} K(z'-y) d\nu y \right| \\
&\leq \left| \int_{B(z',4C(n)r)\setminus B(z,\varepsilon)} K(z-y) d\nu y \right| \\
&\quad + \left| \int_{B(z',4C(n)r)\setminus B(z',\varepsilon)} K(z'-y) d\nu y \right| \\
&\quad + \int_{B(z',4C(n)r)^c} |K(z-y) - K(z'-y)|\, d\nu y
\end{aligned}$$

At this point notice that

$$d(C_f, C_f + 2Lr) \geq r \text{ where } C_f + 2Lr = \{x \in \mathbb{R}^n : x_n = f(x\lfloor_{n\text{-}1}) + 2Lr\}.$$

To see this, by way of contradiction, suppose that there exist $x \in C_f + 2Lr$ and $x' \in C_f$ such that $|x - x'| < r$. Then

$$\begin{aligned}
|f(x\lfloor_{n\text{-}1}) - f(x'\lfloor_{n\text{-}1})| &\geq |f(x\lfloor_{n\text{-}1}) - x_n| - |x_n - x'_n| \\
&> L|\, x\lfloor_{n\text{-}1} - x'\lfloor_{n\text{-}1}\,|.
\end{aligned}$$

Therefore for $z \in A$ and $y \in \operatorname{spt}\nu \subset H^-_f \cup C_f$ we get that $|z - y| \geq r$. Hence

$$\begin{aligned}
\int_{B(z',4C(n)r)\setminus B(z,\varepsilon)} |K(z-y)|\, d\nu y &\leq C_0 \int_{B(z',4C(n)r)\setminus B(z,\varepsilon)} \frac{1}{|z-y|^{n-1}} d\nu y \\
&\leq C_0 r^{-(n-1)} \nu(B(z', 4C(n)r)) \\
&\leq (4C(n))^{n-1} C_0 C_\mu.
\end{aligned}$$

Furthermore,

$$\left| \int_{B(z',4C(n)r)\setminus B(z',\varepsilon)^c} K(z'-y) d\nu y \right| \leq 2T^*_\nu(1)(z').$$

By the Mean Value Theorem we also derive that

$$\begin{aligned}
|K(z-y) - K(z'-y)| &\leq |\nabla K(\xi(y))|\, |z - z'| \\
&\leq \frac{C_1\, |z-z'|}{|\xi(y)|^n}
\end{aligned}$$

where $\xi(y)$ lies in the line segment joining $y-z$ to $y-z'$. Hence

$$\int_{B(z',4C(n)r)^c} |K(z-y)-K(z'-y)|\,d\nu y \leq \sum_{j=1}^{\infty}\int_{B(z',2^j4C(n)r)\setminus B(z',2^{j-1}4C(n)r)} \frac{C_1\,|z-z'|}{|\xi(y)|^n}d\nu y$$

For $j\in\mathbb{N}$ and $y\in B(z',2^j4C(n)r)\setminus B(z',2^{j-1}4C(n)r)$,

$$\begin{aligned} |\xi(y)| &\geq |y-z'|-|\xi(y)-(y-z')| \\ &\geq |y-z'|-|y-z-(y-z')| \\ &\geq C(n)2^jr. \end{aligned}$$

Consequently,

$$\sum_{j=1}^{\infty}\int_{B(z',2^j4C(n)r)\setminus B(z',2^{j-1}4C(n)r)} \frac{C_1\,|z-z'|}{|\xi(y)|^n}d\nu y \;\leq 4^{n-1}C_\mu C_1.$$

Combining all the above we conclude that for $z\in A$, $z'\in A'$ and $0<\varepsilon\leq|z-z'|$,

$$|T^\varepsilon_\nu(1)(z)|\leq 3T^*_\nu(1)(z')+D_1 \tag{2.3}$$

where $D_1=4^{n-1}C_\mu(C_1+C_0C(n)^{n-1})$.

Now we consider the case where $\varepsilon>|z-z'|$. Then

$$\begin{aligned} |T^\varepsilon_\nu(1)(z)-T^\varepsilon_\nu(1)(z')| &= \left|\int_{B(z,\varepsilon)^c} K(z-y)d\nu y-\int_{B(z',\varepsilon)^c} K(z'-y)d\nu y\right| \\ &\leq \left|\int_{B(z',2\varepsilon)\setminus B(z,\varepsilon)} K(z-y)d\nu y\right| \\ &\quad+\left|\int_{B(z',2\varepsilon)\setminus B(z',\varepsilon)} K(z'-y)d\nu y\right| \\ &\quad+\int_{B(z',2\varepsilon)^c} |K(z-y)-K(z'-y)|\,d\nu y \end{aligned}$$

Exactly as before

$$\left|\int_{B(z',2\varepsilon)\setminus B(z,\varepsilon)} K(z-y)d\nu y\right|\leq 2^{n-1}C_\mu C_0$$

$$\left|\int_{B(z',2\varepsilon)\setminus B(z',\varepsilon)} K(z'-y)d\nu y\right|\leq 2T^*_\nu(1)(z')$$

and

$$\int_{B(z',2\varepsilon)^c} |K(z-y)-K(z'-y)|\,d\nu y\leq 2^{2n-1}C_\mu C_1.$$

Thus for $z \in A$, $z' \in A'$ and $\varepsilon > |z - z'|$,

$$T_\nu^\varepsilon(1)(z) \le 3T_\nu^*(1)(z') + D_2 \tag{2.4}$$

where $D_2 = 2^{2n-1}C_\mu(2C_1 + C_0)$. Finally combining (2.3) and (2.4) we conclude that for $z \in A$, $z' \in A'$,

$$T_\nu^*(1)(z) \le 3T_\nu^*(1)(z') + D$$

where $D = \max\{D_1, D_2\}$. □

Therefore if A, A' as in Lemma 2.1 and $E \subset A'$ such that $\mathcal{H}^{n-1}(E) = c\mu(A)$ for some $c > 0$, we get

$$\begin{aligned}\int_A T_\nu^*(1)(z)d\mu z &\le \inf_{z'\in E}\left(3T_\nu^*(1)(z') + D\right)\mu(A)\\ &= c^{-1}\inf_{z'\in E}\left(3T_\nu^*(1)(z') + D\right)\mathcal{H}^{n-1}(E)\\ &\le 3c^{-1}\int_E T_\nu^*(1)(z')d\mathcal{H}^{n-1}z' + Dc^{-1}\mathcal{H}^{n-1}(E).\end{aligned} \tag{2.5}$$

In the following our purpose is to show that

$$\int_{H_f^+} T_\nu^*(1)(z)d\mu z < \infty, \tag{2.6}$$

which combined with (2.2) implies that

$$\int_{H_f^+\cup C_f} T_\nu^*(1)(z)d\mu z < \infty. \tag{2.7}$$

For $k \in \mathbb{N}$ let

$$S_k = \{x \in H_f^+ : x_n \ge f(x\lfloor_{n\text{-}1}) + L2^{1-k}\}.$$

In order to prove (2.6) it is enough to establish that

$$\int_{S_k} T_\nu^*(1)(z)d\mu z \le C, \tag{2.8}$$

where C is some constant not depending on k. The idea is to use some appropriate Whitney type decomposition on H_f^+. For $m \in \mathbb{N}$ and $j = (j_1, .., j_{n-1}) \in \mathbb{Z}^{n-1}$ denote

$$\begin{aligned}I_j^m &= \prod_{i=1}^{n-1}[(j_i - 1)2^{-m}, j_i 2^{-m})\\ Q_j^m &= \left\{x \in \mathbb{R}^n : x\lfloor_{n\text{-}1} \in I_{m,j} \text{ and } L2^{1-m} + f(x\lfloor_{n\text{-}1}) \le x_n < L2^{2-m} + f(x\lfloor_{n\text{-}1})\right\}\\ \mathcal{D}_m &= \{Q_j^m\}_{j\in\mathbb{Z}^{n-1}}\\ Q^m &= \cup_{j\in\mathbb{Z}^{n-1}} Q_j^m\\ F_j^m &= \{x \in C_f : x\lfloor_{n\text{-}1} \in I_j^m\}\end{aligned}$$

A rough illustration of the decomposition is shown in Figure A.

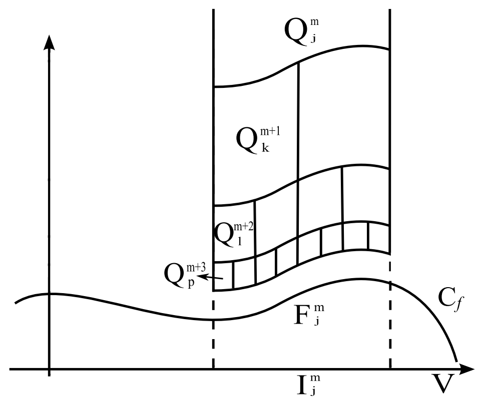

Figure A

Since $\mu \in \Delta$ and $\mathcal{H}^{n-1}\lfloor C_f$ is $(n-1)$ AD-regular, the following estimates are rather straightforward,

$$C_2^{-1}2^{-m(n-1)} \leq \mathcal{H}^{n-1}(F^m_j) \leq C_2 2^{-m(n-1)} \text{ for all } m \in \mathbb{N}, j \in \mathbb{Z}^{n-1}, \tag{2.9}$$

$$\mu(w+Q^m_j) \leq C_3 2^{-m(n-1)} \text{ for all } w \in \mathbb{R}^n, m \in \mathbb{N}, j \in \mathbb{Z}^{n-1}, \tag{2.10}$$

where C_2 depends on L and C_3 depends on μ and L.

Fix some $k \in \mathbb{N}$. For all $m \in \mathbb{N}, 1 \leq m \leq k$, our aim is to assign to each $Q^m_j \in \mathcal{D}_m$ some Borel set $E^m_j \subset C_f$ with the following properties,

(i) $E^m_j \subset F^m_j$

(ii) $\mathcal{H}^{n-1}(E^m_j) = \dfrac{\mu(Q^m_j)}{10C_2C_3}$

(iii) $E^m_j \cap (E^{m+1} \cup ... \cup E^k) = \emptyset$ where $E^l = \cup_{j\in\mathbb{Z}^{n-1}} E^l_j$ for $m+1 \leq l \leq k$.

Condition (iii) makes it clear that we need to start from $m = k$. In this case it is easy to find Borel sets E^k_j satisfying

$$E^k_j \subset F^k_j \text{ and } \mathcal{H}^{n-1}(E^k_j) = \frac{\mu(Q^k_j)}{10C_2C_3}$$

Notice that the sets E^k_j are disjoint for $j \in \mathbb{Z}^{n-1}$. In the following we can proceed inductively. Let some $m \in \mathbb{N}, 1 \leq m \leq k-1$, and suppose that for all $l \in \mathbb{N}, m < l \leq k$ there exist families $\{E^p_j : j \in \mathbb{Z}^{n-1}\}$ satisfying properties (i),(ii) and (iii). In order to demonstrate that the desired family of sets $\{E^m_j : j \in \mathbb{Z}^{n-1}\}$ exists, it is enough to show that

$$\mathcal{H}^{n-1}(F^m_j \backslash (E^{m+1} \cup ... \cup E^k)) > \frac{\mu(Q^m_j)}{10C_2C_3}$$

for all $Q^m_j \in \mathcal{D}_m$. Notice that

$$pr_V^{-1}Q_j^m \cap (Q^{m+1} \cup ... \cup Q^k) = \bigcup_{l=m+1}^{k} \bigcup_{p \in I_{l,j}} Q_p^l \tag{2.11}$$

where $I_{l,j} \subset \mathbb{Z}^{n-1}$ and $\# I_{l,j} = 2^{(l-m)(n-1)}$. This implies

$$F_j^m \cap (E^{m+1} \cup ... \cup E^k) = \bigcup_{l=m+1}^{k} \bigcup_{p \in I_{l,j}} E_p^l. \tag{2.12}$$

Furthermore,

$$\bigcup_{l=m+1}^{k} \bigcup_{p \in I_{l,j}} Q_p^l \subset Q_j^m + x_j^m \tag{2.13}$$

where $x_j^m = (0, .., 0, -L2^{1-m})$. Therefore by (2.10) and (2.13),

$$\begin{aligned}
\mathcal{H}^{n-1}\left(\bigcup_{l=m+1}^{k} \bigcup_{p \in I_{l.j}} E_p^l\right) &= \sum_{l=m+1}^{k} \sum_{p \in I_{l.j}} \mathcal{H}^{n-1}(E_p^l) \\
&= 10^{-1}C_2^{-1}C_3^{-1} \sum_{l=m+1}^{k} \sum_{p \in I_{l,j}} \mu(Q_p^l) \\
&= 10^{-1}C_2^{-1}C_3^{-1} \mu\left(\bigcup_{l=m+1}^{k} \bigcup_{p \in I_{l.j}} Q_p^l\right) \\
&\leq 10^{-1}C_2^{-1}C_3^{-1}\mu(Q_j^m + x_j^m) \\
&\leq 10^{-1}C_2^{-1}2^{-m(n-1)}.
\end{aligned}$$

Consequently by (2.12), (2.9) and (2.10)

$$\begin{aligned}
\mathcal{H}^{n-1}(F_j^m \setminus (E^{m+1} \cup ... \cup E^k)) &\geq C_2{}^{-1}2^{-m(n-1)} - \mathcal{H}^{n-1}\left(\bigcup_{l=m+1}^{k} \bigcup_{p \in I_{l.j}} E_p^l\right) \\
&> \frac{\mu(Q_j^m)}{10C_2C_3}.
\end{aligned}$$

This completes the induction.

Finally using properties of the decomposition, (2.1), (2.5) and the fact that

$$\cup\{E_j^m : \mathcal{H}^{n-1}(E_j^m) > 0\} \subset pr_V^{-1}(pr_V(\mathrm{spt}\mu)) \cap C_f \subset S,$$

we derive

$$\begin{aligned}\int_{S_k} T^*_\nu(1)(z)d\mu z &= \sum_{m=1}^{k}\sum_{j\in\mathbb{Z}^{n-1}}\int_{Q^m_j} T^*_\nu(1)(z)d\mu z\\ &\leq 30C_2C_3\sum_{m=1}^{k}\sum_{j\in\mathbb{Z}^{n-1}}\int_{E^m_j} T^*_\nu(1)(z)d\mathcal{H}^{n-1}z\\ &\quad +10C_2C_3D\sum_{m=1}^{k}\sum_{j\in\mathbb{Z}^{n-1}}\mathcal{H}^{n-1}(E^m_j)\\ &\leq 30C_2C_3\int_S T^*_\nu(1)(z)d\mathcal{H}^{n-1}z+10C_2C_3D\mathcal{H}^{n-1}(S)\\ &= 30C_2C_3\int T^*_\nu(1)(z)d\sigma z+10C_2C_3D\mathcal{H}^{n-1}(S),\end{aligned}$$

finishing the proof of (2.8).

For $z\in H^+_f$ the limit

$$\lim_{\varepsilon\to 0} T^\varepsilon_\nu(1)(z)$$

exists since $H^+_f\cap \mathrm{spt}\nu=\emptyset$. Furthermore by Theorem 1.7 the above limit also exists for μ almost every $z\in S$. Thus by (2.7) and the Lebesgue dominated convergence theorem we derive that the limit

$$\lim_{\varepsilon\to 0}\int_{H^+_f\cup C_f} T^\varepsilon_\nu(1)(z)d\mu z=\lim_{\varepsilon\to 0}\int_{H^+_f\cup C_f}\int_{\substack{H^-_f\\ |x-y|>\varepsilon}} K(z-y)d\mu y d\mu z$$

exists, completing the proof of Theorem 1.3.

Remark. As a corollary of Theorem 1.3 and Fubini's theorem we derive that the limits

$$\lim_{\varepsilon\to 0}\int_{H^+_f}\int_{\substack{\mathbb{R}^n\setminus H^+_f\\ |x-y|>\varepsilon}} K(x-y)d\mu y d\mu x$$

exist under the same assumptions with Theorem 1.3.

3. Weak Convergence in $\mathcal{X}_B(\mathbb{R}^n)$ and $\mathcal{X}_Q(\mathbb{R}^n)$

To prove Theorem 1.4 assume without loss of generality that $\mathrm{spt}\mu\subset B(0,1)$ and let $f,g\in\mathcal{X}_Q(\mathbb{R}^n)$ or $f,g\in\mathcal{X}_B(\mathbb{R}^n)$ be such that

$$f=\sum_{i=1}^{l}a_i\chi_{Q_i}\text{ and } g=\sum_{j=1}^{m}b_j\chi_{P_j},$$

where $a_i,b_j\in\mathbb{R}$ and Q_i,P_j are balls or Q_i,P_j are rectangles. Then for $\varepsilon>0$,

$$\int T_\mu^\varepsilon(f)(x)g(x)d\mu x = \sum_{j=1}^{m}\sum_{i=1}^{l} b_j a_i \int_{\substack{P_j \\ |x-y|>\varepsilon}} \int_{Q_i} K(x-y)d\mu y d\mu x.$$

Therefore it is enough to show that for balls P, Q or P, Q rectangles the limit

$$\lim_{\varepsilon\to 0} \int_{\substack{P \\ |x-y|>\varepsilon}} \int_{Q} K(x-y)d\mu y d\mu x$$

exists. But,

$$\int_{\substack{P \\ |x-y|>\varepsilon}} \int_{Q} K(x-y)d\mu y d\mu x = I_1 + I_2 + I_3 + I_4,$$

where,

$$\begin{aligned}
I_1 &= \int_{\substack{P\cap Q \\ |x-y|>\varepsilon}} \int_{P\cap Q} K(x-y)d\mu y d\mu x \\
I_2 &= \int_{\substack{P\setminus Q \\ |x-y|>\varepsilon}} \int_{P\cap Q} K(x-y)d\mu y d\mu x \\
I_3 &= \int_{\substack{P\cap Q \\ |x-y|>\varepsilon}} \int_{Q\setminus P} K(x-y)d\mu y d\mu x \\
I_4 &= \int_{\substack{P\setminus Q \\ |x-y|>\varepsilon}} \int_{Q\setminus P} K(x-y)d\mu y d\mu x.
\end{aligned}$$

By the antisymmetry of K, for every $\varepsilon > 0$,

$$I_1 = 0.$$

Furthermore by Fubini's theorem I_3 is essentially the same with I_2, allowing us to treat only I_2 and I_4. In that direction notice that for every rectangle, or ball, say P there exist some collection of rotations of Lipschitz graphs $\{F_i(P)\}_{i=1}^{2n}$, and disjoint sets $\{A_i(P)\}_{i=1}^{2n}$, such that

$$\begin{aligned}
\mathbb{R}^n \setminus P &= \cup_{i=1}^{2n} A_i(P), \\
P &\subset H^-_{F_i(P)} \cup F_i(P), \\
A_i(P) &\subset H^+_{F_i(P)}.
\end{aligned}$$

See Figure B for an illustration in the case when P is a subset of the plane. Using

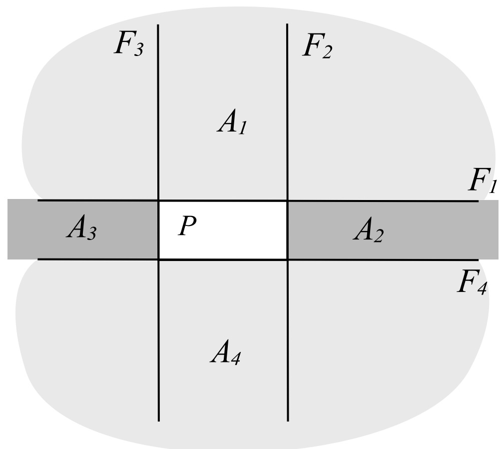

Figure B

the above geometric property I_2 and I_4 can be decomposed in the following way,

$$I_2 = \sum_{i=1}^{2n} \int_{\substack{A_i(Q)\cap P \\ |x-y|>\varepsilon}} \int_{P\cap Q} K(x-y) d\mu y d\mu x$$

and

$$I_3 = \sum_{i=1}^{2n} \int_{\substack{A_i(Q)\cap P \\ |x-y|>\varepsilon}} \int_{Q\setminus P} K(x-y) d\mu y d\mu x.$$

Therefore since limits like

$$\lim_{\varepsilon\to 0} \int_{\substack{A_i(Q)\cap P \\ |x-y|>\varepsilon}} \int_{P\cap Q} K(x-y) d\mu y d\mu x$$

and

$$\lim_{\varepsilon\to 0} \int_{\substack{A_i(Q)\cap P \\ |x-y|>\varepsilon}} \int_{Q\setminus P} K(x-y) d\mu y d\mu x$$

exist by Theorem 1.3 we finally obtain Theorem 1.4.

Acknowledgement. I would like to thank my advisor, Professor Pertti Mattila, for many ideas, discussions, suggestions and for repeated proof reading.

References

[C] V. Chousionis, *Singular integrals on Sierpinski gaskets*, preprint.

[CDM] R. R. Coifman, G. David, and Y. Meyer *La solution des conjectures de Calderón*, Adv. Math., 48, 144-148 (1983).

[CMM] R. R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Ann. of Math. (2) 116 (1982), no. 2, 361387.

[D1] G. David, *Opérateurs intégraux singuliers sur certaines courbes du plan complexe*, Ann. Sci. cole Norm. Sup. (4) 17 (1984), no. 1, 157–189.

[D2] G.David, *Wavelets and singular integrals on curves and surfaces*, Lecture Notes in Mathematics, 1465. Springer-Verlag, Berlin, 1991.

[D3] G.David, *Des intégrales singulières bornées sur un ensemble de Cantor*, C. R. Acad. Sci. Paris Sr. I Math. 332 (2001), no. 5, 391–396.

[MM] P. Mattila, M. S. Melnikov, *Existence and weak-type inequalities for Cauchy integrals of general measures on rectifiable curves and sets* , Proc. Amer.Math.Soc. 120 (1994), no. 1, 143–149.

[MMV] P. Mattila, M. S. Melnikov, J. Verdera, *The Cauchy integral, analytic capacity, and uniform rectifiability*, Ann. of Math. (2) 144 (1996), no. 1, 127–136.

[MP] P. Mattila, D. Preiss, *Rectifiable measures in $\mathbb{R}^n$ and existence of principal values for singular integrals*, J. London Math. Soc., 52 (1995), 482-496.

[MV] P. Mattila, J. Verdera, *Convergence of singular integrals with general measures*, to appear in J. Eur. Math. Soc.

[T1] X.Tolsa, *Cotlar's inequality without the doubling condition and existence of principal values for the Cauchy integral of measures*, J. Reine Angew. Math. 502 (1998), 199–235.

[T2] X.Tolsa, *Principal values for Riesz transforms and rectifiability*, Preprint.

[V] J. Verdera, *A weak type inequality for Cauchy transforms of finite measures*, Publ. Mat. 36 (1992), no. 2B, 10291034.

Department of Mathematics and Statistics , University of Helsinki, P.O. Box 68, FI-00014, Finland,

E-mail address: vasileios.chousionis@helsinki.fi