On C^1-extension and C^1-reflection of subharmonic functions from Lyapunov-Dini domains to $\mathbb{R}^N$

P.V. Paramonov*

For Lyapunov-Dini domains D in $\mathbb{R}^N$ ($N \in \{2, 3, \ldots\}$) we study the possibility of C^1-extension and C^1-reflection of subharmonic functions in D of the class $C^1(\overline{D})$ through the boundary of D to all of $\mathbb{R}^N$.

Bibliography: 14 titles.

1 Introduction

For previous results on C^m-extension of subharmonic functions we refer the reader to [1] – [5] and literature therein. In these papers one can find several different settings of the problem. Here we deal with the following particular question (and related results).

For which compact sets X in $\mathbb{R}^N$ any function $f \in C^1(X)$ subharmonic on the interior of X can be extended to a function F subharmonic and C^1 on all of $\mathbb{R}^N$ with the property $\|F\|_{C^1(\mathbb{R}^N)} \leq A_X \|f\|_{C^1(X)}$ (with $A_X \in (0, +\infty)$ depending only on X)?

The main result of this paper (Theorem 3) says that the previous property is satisfied by any C^1-smooth closed bounded domain X in $\mathbb{R}^N$ ($N \geq 3$) with connected complement and with the so-called Log-Dini-property. An analogous result for the case $N = 2$ was obtained in [3] by different methods (for balls in $\mathbb{R}^N$ it appeared earlier in [2]). We also prove several auxiliary results (having their own interests) on harmonic and subharmonic C^1-reflection (Theorems 1, 3.1 and 3.5) and give several examples (see Section 4) showing that the (sufficient) conditions of our theorems are close to be sharp. Now we go to precise definitions, notations and statements.

A function $\varepsilon(\cdot) \in C([0, +\infty))$ with the properties $\varepsilon(0) = 0$, $\varepsilon : (0, +\infty) \to (0, +\infty)$, $\varepsilon(\cdot)$ is (nonstrongly) increasing and $\varepsilon(t)/t$ is decreasing on $(0, +\infty)$,

$$\int_0^1 \frac{\varepsilon(t)}{t} \, dt < +\infty \,, \tag{1.1}$$

is called a "*Dini-type*" function.

A C^1-smooth bounded domain D in $\mathbb{R}^N$ ($N \in \{2, 3, \ldots\}$ is fixed) is called "*Lyapunov-Dini*" (L-D) domain if there exists a Dini-type function $\varepsilon(\cdot)$ (called a *Dini-function* for D) such that for each $\mathbf{x}$ and $\mathbf{y}$ on $S = \partial D$ one has

$$|\mathbf{n}^i_{\mathbf{x}} - \mathbf{n}^i_{\mathbf{y}}| \leq \varepsilon(|\mathbf{x} - \mathbf{y}|) \,, \tag{1.2}$$

where $\mathbf{n}^i_{\mathbf{x}}$ means the *inward* (with respect to D) unit normal to S at $\mathbf{x} \in S$.

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As usual, $\overline{E}$, E°, ∂E mean the closure, the interior and the boundary of the set $E \neq \emptyset$ in $\mathbb{R}^N$, $\|f\|_E = \sup_{\mathbf{x}\in E}|f(\mathbf{x})|$ is the uniform norm of the function f on E (and $\|\cdot\| = \|\cdot\|_{\mathbb{R}^N}$). For an open set Ω in $\mathbb{R}^N$ we denote by $H(\Omega)$ (respectively, $SH(\Omega)$) the class of all (real) functions harmonic (respectively, subharmonic) in Ω.

Recall, that for a closed set X in $\mathbb{R}^N$ and $m \in \{0, 1, 2, \dots\}$ one defines $C^m(X)$ as $C^m(\mathbb{R}^N)|_X$ with the norm

$$\|f\|_{m,X} = \inf \|F\|_m,$$

where the last infimum is taken over all functions $F \in C^m(\mathbb{R}^N)$ with the property $F|_X = f$ and $\|F\|_m := \max_{|\beta|\le m}\{\|\partial^\beta F\|\} < +\infty$. Notice that in the case $X = \overline{X^\circ}$, for each $f \in C^m(X)$ the derivatives

$$\partial^\beta f(\mathbf{x}) = \frac{\partial^{|\beta|} f(\mathbf{x})}{\partial x_1^{\beta_1} \dots \partial x_N^{\beta_N}}$$

with $|\beta| := \beta_1 + \dots + \beta_N \le m$ ($\beta = (\beta_1, \dots \beta_N)$, $\beta_n \in \{0, 1, 2, \dots\}$) are uniquely defined for all $\mathbf{x} \in X$, and so in this case $C^m(X)$ can be identified with the Whitney-jet space $C^m_{jet}(X)$ (see [6]). If $m = 0$, we omit the index m in notations of $C^m(X)$ and $||\cdot||_{m,X}$.

In what follows we fix $N \in \{2, 3, \dots\}$, an arbitrary Dini-function $\varepsilon(\cdot)$ and $d \in (0, +\infty)$. Let D be a (L-D) domain in $\mathbb{R}^N$ with Dini-function $\varepsilon(\cdot)$ and $\operatorname{diam} D < d$ (d should be large enough for D to exist). Set $D_o = \mathbb{R}^N \backslash \overline{D}$. The constant $A \in (0, +\infty)$ in the following Theorems 1-3 depends only on N, ε and d .

Theorem 1. *Let $u_i \in H(D) \cap C^1(\overline{D})$ and u_o be the (only) solution of the Dirichlet problem in D_o with the boundary data $u_o|_{\partial D_o} = u_i|_{\partial D_o}$ (in the unbounded component of D_o we additionally require $u_o(\infty) = 0$ for $N \ge 3$ or $|u_o(\infty)| < +\infty$ for $N = 2$, where $u_o(\infty) = \lim_{|\mathbf{x}|\to+\infty} u_o(\mathbf{x})$ must exist). Then $u_o \in C^1(\overline{D_o})$ and*

$$\|u_o\|_{1,\overline{D_o}} \le A\|u_i\|_{1,\overline{D}} \ . \tag{1.3}$$

We shall say that u_o is the C^1-*reflection* of u_i over (or through) the boundary S of the domain D. We have a useful generalization of this result in Theorem 3.1 below. From Theorem 1 we obtain the following "C^1-extension" result.

Theorem 2. *Suppose that D has connected complement and $u_i \in H(D) \cap C^1(\overline{D})$. Then one can find a function $F \in C^1(\mathbb{R}^N) \cap SH(\mathbb{R}^N)$ for $N \ge 3$ ($F \in C^1_{loc}(\mathbb{R}^N) \cap SH(\mathbb{R}^N)$ for $N = 2$) such that $F|_{\overline{D}} = u_i$ and*

$$\begin{aligned} &\|F\|_1 \le A\|u_i\|_{1,\overline{D}} \ , \quad N \ge 3 \, , \\ &\|\nabla F\| \le A\|\nabla u_i\|_{\overline{D}} \ , \quad N = 2 \, . \end{aligned} \tag{1.4}$$

It is well known that $\mathbb{R}^N \backslash D$ (and then D_o) is connected if and only if $S = \partial D$ is. The next "localization" property can be useful in applications.

Corollary 1. *Suppose that D has connected complement and $f \in SH(D) \cap C^1(\overline{D})$. If for each $\mathbf{a} \in \partial D$ there is a ball $B_\mathbf{a}$ centered at $\mathbf{a}$ and $g_\mathbf{a} \in SH(B_\mathbf{a}) \cap C^1(\overline{B_\mathbf{a}})$ with $g_\mathbf{a}|_{B_\mathbf{a}\cap D} = f$ then there exists $F \in SH(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ if $N \ge 3$ (respectively, $F \in SH(\mathbb{R}^N) \cap C^1_{loc}(\mathbb{R}^N)$ if $N = 2$) with $F|_{\overline{D}} = f$ and $||F||_1 < +\infty$ (respectively, $\|\nabla F\| < +\infty$ if $N = 2$).*

And the main goal of this paper is the following.

Theorem 3. *Suppose that D has connected complement and $\varepsilon(\cdot)$ satisfies the so-called "Log-Dini" property*

$$\int_0^1 \frac{\varepsilon(t)}{t} \log(\frac{1}{t})\, dt < +\infty\,.$$

Then for each $f \in SH(D) \cap C^1(\overline{D})$ one can find $F \in SH(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ if $N \geq 3$ (or $F \in SH(\mathbb{R}^N) \cap C^1_{loc}(\mathbb{R}^N)$ if $N = 2$) with $F|_{\overline{D}} = f$ and

$$\begin{aligned} \|F\|_1 &\leq A\|f\|_{1,\overline{D}}\,, \quad N \geq 3\,, \\ \|\nabla F\| &\leq A\|\nabla f\|_{\overline{D}}\,, \quad N = 2\,. \end{aligned} \tag{1.5}$$

The last theorem is based on a constructive, but rather technical result, Theorem 3.5, that seems to be useful in applications (the C^1-reflection property for subharmonic functions).

As far as we know, Theorems 1-3 are new for all $N \geq 3$ even for the so-called *Lyapunov* domains ((L-D) domains with $\varepsilon(t) = t^\alpha$, $\alpha \in (0,1)$).

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2 Proofs of Theorems 1 and 2

In the sequel we denote by A_0, $A_1 \ldots$ some (*fixed in this section*) positive constants, which (in the long run) *depend only on* N, $\varepsilon(\cdot)$ and d (this is important for the proofs of Theorems 1-3 and will be discussed in each nontrivial situation). The constants A_N (depending only on N) and A (depending on N, $\varepsilon(\cdot)$ and d) can be different in different accuracies. Set $B(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^N | \, |\mathbf{x} - \mathbf{a}| < r\}$ and $\overline{B}(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^N | \, |\mathbf{x} - \mathbf{a}| \leq r\}$ ($\mathbf{a} \in \mathbb{R}^N$, $r > 0$).

First we formulate several auxiliary results, which basically (but sometimes not so easy) follow from [7, Theorems 2.2 - 2.5]. We decided, for completeness and for the reader's convenience, to present the detailed proofs of these results in Section 4 below (see Theorems 4.4-4.9).

Theorem W1. *Let D be a (L-D) domain in $\mathbb{R}^N$ with Dini-function $\varepsilon(\cdot)$ and* $\operatorname{diam} D < d$, *$S = \partial D$. Let $\psi \in C(S)$ and u_i (respectively, u_o) be the solution of the Dirichlet problem in D (respectively, $D_o = \mathbb{R}^N \backslash \overline{D}$) with the boundary data ψ.*

(1) If $\psi \in C^2(S)$ then u_i and u_o are of the class $C^1(\overline{D})$ and $C^1(\overline{D_o})$ respectively, and satisfy the estimates:

$$\begin{aligned} \|u_i\|_{1,\overline{D}} &\leq A_0\|\psi\|_{2,S}\,, \\ \|u_o\|_{1,\overline{D_o}} &\leq A_0\|\psi\|_{2,S}\,. \end{aligned} \tag{2.1}$$

(2) Let $\mathbf{a} \in S$, $r \in (0, d/2)$, and suppose that $\psi = 0$ on $S \cap B(\mathbf{a}, r)$. Then u_i and u_o are of the class $C^1(\overline{D} \cap \overline{B}(\mathbf{a}, r/2))$ and $C^1(\overline{D_o} \cap \overline{B}(\mathbf{a}, r/2))$ respectively, with

$$\left|\frac{\partial u_i}{\partial \mathbf{n}^i_{\mathbf{a}}}\right| + \left|\frac{\partial u_o}{\partial \mathbf{n}^o_{\mathbf{a}}}\right| \leq \frac{A_0}{r} \log(\frac{d}{r})\, ||\psi||_S\ ,$$

where $\mathbf{n}^o_{\mathbf{a}} = -\mathbf{n}^i_{\mathbf{a}}$ is the outward normal to S at $a \in S$.

(3) Let $N \geq 3$. Let D_* be the unbounded component of D_o, $S_* = \partial D_*$, and $w \in H(D_*) \cap C(\overline{D_*})$ be such that $w|_{S_*} = 0$ and $w(\infty) = 1$. Then $w \in C^1(\overline{D_*})$, $||w||_{1,\overline{D_*}} \leq A_0$ and for each $\mathbf{a} \in S_*$ one has

$$A_0 \frac{\partial w}{\partial \mathbf{n}^o_{\mathbf{a}}} \geq 1 \ .$$

In Theorem W1 (1) we cannot, in general, put $\|\psi\|_{1,S}$ instead of $\|\psi\|_{2,S}$ (see [7, Remark 1] and Example 4.1 below). It should be said that the dependence of A_0 only on N, ε, d was not ascertained in [7].

Proof of Theorem 1. First we reduce the proof to the case when u_i can be extended as a harmonic function on some neighborhood of $\overline{D}$, so that $u_i|_{\partial D}$ belongs to the class $C^2(S)$ and so, by Theorem W1 (1), we have $u_o \in C^1(\overline{D_o})$. In fact, suppose we have proved (1.3) for all such u_i. In general case, by [8, Corollary 6.3] we can find a sequence $\{v_{is}\}_{s=1}^{+\infty}$, each v_{is} is harmonic on (it's own) neighborhood of $\overline{D}$, such that $u_i = \sum_{s=1}^{+\infty} v_{is}$ and $\|v_{is}\|_{1,\overline{D}} \leq 2^{2-s}\|u_i\|_{1,\overline{D}}$. Define v_{os} by v_{is} (like u_o by u_i), so that $v_{os} \in H(D_o) \cap C^1(\overline{D_o})$, $\|v_{os}\|_{1,\overline{D_o}} \leq A\|v_{is}\|_{1,\overline{D}}$. Then $v_o = \sum_{s=1}^{+\infty} v_{os}$ gives the result. So we can always assume that $u_o \in C^1(\overline{D_o})$.

Set $h_i(\mathbf{x}) = \partial u_i/\partial \mathbf{n}^i_{\mathbf{x}}$ (respectively, $h_o(\mathbf{x}) = \partial u_o/\partial \mathbf{n}^o_{\mathbf{x}}$), where $\mathbf{x} \in S$ and $\mathbf{n}^i_{\mathbf{x}}$ (respectively, $\mathbf{n}^o_{\mathbf{x}}$) is the unit inward normal to S at $\mathbf{x}$ with respect to the domain D (respectively, D_o).

It is well known (see [9, Ch. 2, §6.5, (27)]) that if we set $u = u_i$ in $\overline{D}$ and $u = u_o$ in D_o then

$$\Delta u = (h_i + h_o)\,\sigma \tag{2.2}$$

in the distributional sense, where σ is the $(N-1)$-dimensional (Lebesque) surface measure on S. Denote by $\Phi_N(\mathbf{x}) = \Phi(\mathbf{x})$ the standard fundamental solution for the Laplacian Δ in $\mathbb{R}^N$:

$$\Phi_2(\mathbf{x}) = (1/2\pi)\log|\mathbf{x}| \ , \ \ \Phi_N(\mathbf{x}) = -\frac{1}{\sigma_N(N-2)\,|\mathbf{x}|^{N-2}} \ , \ \ N \geq 3\,,$$

where σ_N is the (value of the) surface measure of the unit sphere in $\mathbb{R}^N$.

Put $h = h_i + h_o$ on S, so that by the classical Liouville theorem we have

$$u(\mathbf{x}) = (\Phi * (h\sigma))(\mathbf{x}) \ + \ c_2 = \int_S \Phi(\mathbf{x}-\mathbf{y})h(\mathbf{y})d\sigma_{\mathbf{y}} \ + \ c_2\,, \tag{2.3}$$

where $c_2 = 0$ for all $N > 2$ and $|c_2| \leq ||u_i||_S$ for $N = 2$. It is enough, by the maximum principle, to (properly) estimate $\|h_o\|_S$. In fact, since $u_i \in C^1(\overline{D})$ and $u_o \in C^1(\overline{D_o})$, we have

$$\nabla u_o(\mathbf{y}) \to h_o(\mathbf{x})\mathbf{n}^o_{\mathbf{x}} + \nabla u_i(\mathbf{x}) - h_i(\mathbf{x})\mathbf{n}^i_{\mathbf{x}}$$

as $\mathbf{y} \to \mathbf{x} \in S$, $\mathbf{y} \in D_o$, so that, since $\|u_o\|_{\overline{D_o}} = \|u_i\|_{\overline{D_i}}$, it would be enough to prove that $\|h_o\|_S \leq A\|u_i\|_{1,S}$. Then the jet $(u_o, \nabla u_o)_{\overline{D_o}}$ can be extended to some function of the class $C^1(\mathbb{R}^N)$ by Whitney theorem [6] which also gives the estimate (1.3).

We are first going to obtain *a priori type estimate* of h_o.

One can easily check the "doubling" property $\varepsilon(kt) \leq k\varepsilon(t)$ ($\forall t > 0$, $\forall k \geq 1$) that we shall often use below (without remarks). Fix (say, maximal) $r_0 = r_0(\varepsilon) \in (0, 1]$ with condition $\varepsilon(r_0) \leq 1/8$. Take any $\mathbf{a} \in S$. Rotating and shifting the initial coordinate system we can assume that $\mathbf{a} = \mathbf{0}$, $\mathbf{n}^i_{\mathbf{0}} = \{0, \ldots, 0, 1\}$ (so that D is "above" $\mathbf{0}$), and find $r \in (0, r_0]$ and a function $\varphi(\mathbf{x}')$ (we set $\mathbf{x} = (\mathbf{x}', x_N)$, $\mathbf{x}' = (x_1, \ldots, x_{N-1})$) such that $\varphi \in C^1(|\mathbf{x}'| \le r)$, $\varphi(\mathbf{0}') = 0$, $\nabla\varphi(\mathbf{0}') = \mathbf{0}'$, $|\nabla\varphi(\mathbf{x}')| \le 1/4$ for $|\mathbf{x}'| \le r$ and for $Q_r = \{|\mathbf{x}'| \le r,\ |x_N| \le r\}$ one has

$$Q_r \cap S = \{(\mathbf{x}, \varphi(\mathbf{x}')),\ |\mathbf{x}'| \le r\}\,. \tag{2.4}$$

Take $\mathbf{x} = (\mathbf{x}', \varphi(\mathbf{x}'))$, $|\mathbf{x}'| < r$, so that $\mathbf{n}^i_{\mathbf{x}} = \{-\nabla\varphi(\mathbf{x}'), 1\}/\sqrt{1 + |\nabla\varphi(\mathbf{x}')|^2}$. Then, using (1.2) for the taken $\mathbf{x}$ and for $\mathbf{y} = \mathbf{a} = \mathbf{0}$, we see that for all $\mathbf{x}'$, $|\mathbf{x}'| < r$,

$$\begin{aligned} &|\nabla\varphi(\mathbf{x}')| \le 2\varepsilon(|\mathbf{x}'|), \\ &|\varphi(\mathbf{x}')| \le 2\varepsilon(|\mathbf{x}'|)|\mathbf{x}'|\,, \end{aligned} \tag{2.5}$$

which show that in the last considerations we can take $r = r_0$. Put $Q_{\mathbf{0}} = Q_{r_0}$.

By (2.3) and Lebesgue's convergence theorem, since $\partial\Phi(\mathbf{x})/\partial x_n = x_n/(\sigma_N\,|\mathbf{x}|^N)$, one has

$$\begin{aligned} h_i(\mathbf{0}) - h_o(\mathbf{0}) &= \lim_{\delta\to 0} \int_S \frac{\Phi(\delta\mathbf{n}^i_{\mathbf{0}} - \mathbf{x}) - (\pm\Phi(-\mathbf{x})) - \Phi(-\delta\mathbf{n}^i_{\mathbf{0}} - \mathbf{x})}{\delta} h(\mathbf{x})d\sigma_{\mathbf{x}} = \\ &= -\int_{S\cap Q_{\mathbf{0}}} \frac{2\varphi(\mathbf{x}')}{\sigma_N\,|\mathbf{x}|^N} h(\mathbf{x})d\sigma_{\mathbf{x}} - \int_{S\setminus Q_{\mathbf{0}}} 2\frac{\partial\Phi(\mathbf{x})}{\partial x_N} h(\mathbf{x})d\sigma_{\mathbf{x}} \end{aligned}$$

so that, by (2.4) and (2.5),

$$\begin{aligned} |h_i(\mathbf{0}) - h_o(\mathbf{0})| &\le \frac{5}{\sigma_N} \int_{|\mathbf{x}'|\le r_0} \frac{\varepsilon(|\mathbf{x}'|)}{|\mathbf{x}'|^{N-1}} h(\mathbf{x}', \varphi(\mathbf{x}'))\,d\mathbf{x}' + \frac{2}{\sigma_N}\,\|h\|_{S\setminus Q_{\mathbf{0}}} \int_{S\setminus Q_{\mathbf{0}}} \frac{d\sigma(x)}{|\mathbf{x}|^{N-1}} \le \\ &\le A_1\,\|h\|_{Q_{\mathbf{0}}\cap S} \int_0^{r_0} \frac{\varepsilon(t)}{t} dt + A_2\,\|h\|_{S\setminus Q_{\mathbf{0}}}\,, \end{aligned} \tag{2.6}$$

where $A_1 = 5\sigma_{N-1}/\sigma_N$ and $A_2 \le A_N d/r_0$. The first integral in (2.6) is estimated using spherical coordinates in $\mathbb{R}^{N-1}_{\mathbf{x}'}$ (the case $N \ge 3$, or just put $\sigma_1 = 2$ if $N = 2$):

$$\int_{B(\mathbf{0}',r)} f(|\mathbf{x}'|)\,d\mathbf{x}' = \sigma_{N-1} \int_0^r f(t)\,t^{N-2}dt\,.$$

The second integral (that is, A_2) in (2.6) is estimated in (4.11) below (Section 4).

By (1.1), take maximal $r_1 = r_1(\varepsilon) \in (0, r_0]$ with the property $A_1 \int_0^{r_1} (\varepsilon(t)/t)dt \le 1/2$.

So, finally, we have for *each* $\mathbf{a} \in S$ (using (2.6) for r_1 instead of r_0 as we clearly can):

$$|h_i(\mathbf{a}) - h_o(\mathbf{a})| \le \frac{1}{2}\|h\|_{S\cap B(\mathbf{a},2r_1)} + A_3\|h\|_{S\setminus B(\mathbf{a},r_1)}\,.$$

From this one immediately obtains (for each $\mathbf{a} \in S$):

$$|h_o(\mathbf{a})| \le \frac{1}{2}\|h_o\|_{S\cap B(\mathbf{a},r_1)} + A_4(\|h_o\|_{S\setminus B(\mathbf{a},r_1)} + \|h_i\|_S)\,, \tag{2.7}$$

which is the mentioned above a priori estimate for h_o (here $A_4 \le A_N d/r_1$).

It is worth mentioning that the case $u_i \equiv 1$ (so that $h_i \equiv 0$), $u_o = 1 - w$ (see Theorem W1 (3)), shows that (2.7) does not imply the estimate $||h_o||_S \le A||h_i||_S$, and it is in fact not so simple to obtain the desired estimate $\|h_o\|_S \le A\|u_i\|_{1,S}$ from (2.7). We continue the proof of Theorem 1 by using *localization* arguments [10], [8].

For the rest of the proof we explicitly consider the case $N \geq 3$ (for $N = 2$ one should use (and estimate) the norm $||\nabla f|| + ||f||_{B(\mathbf{0},d+1)}$ instead of $||f||_1$). We can assume that $\mathbf{0} \in \overline{D}$, so that $B(\mathbf{0}, d)$ contains $\overline{D}$. By definition of $\|u_i\|_{1,\overline{D}}$ we can find $U \in C^1(\mathbb{R}^N)$, $U|_{\overline{D}} = u_i$, $\operatorname{Supp} U \subset B(0, d+1)$ and $\|U\|_1 \leq 3\|u_i\|_{1,\overline{D}}$. Let, for short, $m = \|u_i\|_{1,\overline{D}}$.

Suppose that $M = \max_{\mathbf{x}\in S} |h_o(\mathbf{x})|$ is attained at some point $\mathbf{b} = \mathbf{b}_0 \in S$. Take $B_0 = B(\mathbf{b}, r_1/8)$, $\varphi_0 \in C_0^\infty(B_0)$, $\varphi_0 = 1$ on $B(\mathbf{b}, r_1/16)$, $0 \leq \varphi_0 \leq 1$, $\|\Delta\varphi_0\| \leq A_N/r_1^2$. We also can find points $\mathbf{b}_j \in \mathbb{R}^N$, $j = 1, \ldots, J$, and $\varphi_j \in C_0^\infty(B_j)$, $B_j = B(\mathbf{b}_j, r_1/64)$, such that $0 \leq \varphi_j \leq 1$, $\|\Delta\varphi_j\| \leq A_N/r_1^2$ and $\sum_{j=0}^J \varphi_j = 1$ on $B(\mathbf{0}, d+1)$. Clearly, we also can assume that J depends only on N, r_1 and d, that $B_j \cap B(\mathbf{0}, d+1) \neq \emptyset$, but $B_j \cap B(\mathbf{b}, r_1/16) = \emptyset$ for $j \geq 1$.

We now have:

$$U = \Phi * \Delta U = \Phi * (\sum_{j=0}^J \varphi_j \Delta U) = \sum_{j=0}^J U_j,$$

where $U_j = \Phi * (\varphi_j \Delta U)$. By [8, Lemms 4.1] we have $\|\nabla U_j\| \leq A_N \|\nabla U\|$. Therefore (since $U_j(\infty) = 0$), we have for all $j = 0, \ldots, J$:

$$\|U_j\|_1 \leq A_5 m\,,\ A_5 = A_5(N)\,, \tag{2.8}$$

$U_j \in H(\mathbb{R}^N \backslash \overline{B_j}) \cap H(D)$, so that (using formula $U_j = \Phi * \Delta U_j$) and integration by parts, we get

$$\|U_j\|_{2,E_j^2} \leq A_6 m\,,\ A_6 \leq A_N/r_1\,, \tag{2.9}$$

where for $k > 0$ we set $E_j^k = \mathbb{R}^N \backslash B(\mathbf{b}_j, kr_1/64)$, $j = 1, \ldots, J$, and $E_0^k = \mathbb{R}^N \backslash B(\mathbf{b}_0, kr_1/8)$.

Set $u_{ij} = U_j|_{\overline{D}}$ and define h_{ij}, u_{oj} and h_{oj} by u_{ij} the same way as h_i, u_o and h_o by u_i. We have $u_{i(o)} = \sum_{j=0}^J u_{i(o)j}$ and $h_{i(o)} = \sum_{j=0}^J h_{i(o)j}$, as well as (by (2.8)):

$$\|h_{ij}\|_S \leq \|u_{ij}\|_{1,\overline{D}} \leq \|U_j\|_1 \leq A_5 m. \tag{2.10}$$

If $B_j \subset D$, then $U_j \equiv 0$ and so $u_{ij} \equiv 0$, and we shall assume that $B_j \not\subseteq D$ for all j. Let $2B_j \equiv B(\mathbf{b}_j, r_1/32)$, $j \neq 0$. If $2B_j \cap \overline{D} = \emptyset$ then (as $S \subset E_j^2$) by (2.9) we have $\|u_{ij}\|_{2,S} = \|U_j\|_{2,S} \leq A_6 m$, so that, by (2.2), applied to $u_{ij}|_S$, we then have

$$\|h_{oj}\|_S \leq \|u_{oj}\|_{1,\overline{D_o}} \leq A_0 \|u_{ij}\|_{2,S} \leq A_0 A_6 m = A_7 m\,. \tag{2.11}$$

Changing, if necessary, the numeration, we can suppose that the indices $j = 1, \ldots, I$ ($I \leq J$) are such that $2B_j \cap S \neq \emptyset$ (and $B_j \not\subseteq D$).

Lemma 2.1. *For each $j = 0, \ldots, I$ one has*

$$\|h_{oj}\|_{E_j^3 \cap S} \leq A_8 m\,. \tag{2.12}$$

Proof. Since we have (by (2.9)) for any j

$$\|u_{ij}\|_{2,E_j^2 \cap S} \leq \|U_j\|_{2,E_j^2} \leq A_6 m\,,$$

we can find $v_{ij} \in C^2(S)$ such that $v_{ij} = u_{ij}$ on $E_j^2 \cap S$ and

$$\|v_{ij}\|_{2,S} \leq 2\|u_{ij}\|_{2,E_j^2 \cap S} \leq 2A_6 m\,. \tag{2.13}$$

Let $w_{ij} = u_{ij} - v_{ij}$, and let v_{oj} (respectively, w_{oj}) are the solutions of the Dirichlet problem in D_o with boundary data v_{ij} (respectively, w_{ij}). By (2.13) and (2.1) we have

$$\left\| \frac{\partial v_{oj}}{\partial \mathbf{n}^o_{\mathbf{x}}} \right\|_S \leq \|v_{oj}\|_{1,\overline{D_o}} \leq 2A_0 A_6 m \,.$$

Since $w_{ij} = 0$ on $E^2_j \cap S$ and $\|w_{oj}\|_{\overline{D_o}} \leq A_6 m$, by Theorem W1 (2) (applied to $\psi = w_{ij}|_S$ and $r = r_1/16$) one has

$$\left\| \frac{\partial w_{oj}}{\partial \mathbf{n}^o_{\mathbf{x}}} \right\|_{E^3_j \cap S} \leq \frac{64 A_0}{r_1} \log(\frac{d}{r_1}) \|w_{oj}\|_{\overline{D_o}} \leq \frac{256 A_0 A_6}{r_1} \log(\frac{d}{r_1})\, m \,,$$

which gives (2.12). □

Now, by (2.11) and (2.12), since $b \in E^3_j$ for all $j \neq 0$, we get

$$\left| \sum_{j=1}^{J} h_{oj}(\mathbf{b}) \right| \leq A_9 m \,, \tag{2.14}$$

$$\|h_{o0}\|_{E^3_0 \cap S} \leq A_8 m \,. \tag{2.15}$$

We can suppose that $A_8 \leq A_9$. Also assume that $M \geq 2A_9 m$, otherwise Theorem 1 is proved. By (2.14) it follows that $|h_{o0}(\mathbf{b})| \geq M/2$, and by (2.15) we can see that $|h_{o0}(\mathbf{x})|$ attains it's maximum on S at some point $\mathbf{b}_* \in B(\mathbf{b}, 3r_1/8) \cap S$. Now, applying (2.7) for h_{o0} instead of h_o, $\mathbf{b}_*$ instead of $\mathbf{a}$ and h_{i0} instead of h_i, taking into account that $B(\mathbf{b}_*, r_1)$ contains $B(\mathbf{b}, 3r_1/8)$ (so that $\|h_{o0}\|_{S \setminus B(\mathbf{b}_*, r_1)} \leq A_8 m$) and applying (2.10), we obtain:

$$|h_{o0}(\mathbf{b}_*)| \leq \frac{1}{2} |h_{o0}(\mathbf{b}_*)| + A_4 (A_8 m + A_5 m) \,,$$

and we have finally:

$$M \leq 2|h_{o0}(\mathbf{b}_*)| \leq 4A_4(A_8 + A_5) m \,,$$

which completes the proof of Theorem 1 with $A = A(N, d, r_1)$.

□

Proof of Theorem 2.

Lemma 2.2. *Let D be a (L-D) domain with the Dini-function $\varepsilon(\cdot)$ and* $\operatorname{diam} D < d$. *Then there exist a Dini-type function ε_* with*

$$\varepsilon_*(t) \leq A_N (\, t/r_0 + \varepsilon(t)) \,,$$

a neighborhood Ω of $S = \partial D$ and a function $E \in C^1_0(\mathbb{R}^N)$ such that $||E||_1 \leq A_N$, $E \equiv 0$ on S, $|\nabla E(\mathbf{x})| \geq 1$ for all $\mathbf{x} \in \overline{\Omega}$, $E > 0$ in $\Omega \cap D$ ($E \geq 0$ in D), $E < 0$ in $\Omega \backslash \overline{D}$ ($E \leq 0$ in D_o), and

$$|\nabla E(\mathbf{x}) - \nabla E(\mathbf{y})| \leq \varepsilon_*(|\mathbf{x} - \mathbf{y}|)$$

for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^N$.

Proof. Fix any $\mathbf{a} \in S$ and consider the corresponding r_0, $Q_\mathbf{a}$ and $\varphi_\mathbf{a}$ as in the proof of Theorem 1. Recall that after rotating and shifting the initial coordinate system we obtain the new coordinate system (again denoted by $\mathbf{0_x}$) and the corresponding objects translate as follows: $\mathbf{a} \to \mathbf{0}$, $Q_\mathbf{a} \to Q_\mathbf{0} = \{|\mathbf{x}'| \le r_0, |x_N| \le r_0\}$, $\varphi_\mathbf{a} \to \varphi = \varphi_\mathbf{0}$ so that $S \cap Q_\mathbf{0} = \{(\mathbf{x}', \varphi(\mathbf{x}')), |\mathbf{x}'| \le r_0\}$, $\overline{D} \cap Q_\mathbf{0} = \{|\mathbf{x}'| \le r_0, \varphi(\mathbf{x}') \le x_N \le r_0\}$, where $\varphi \in C^1(\{|\mathbf{x}'| \le r_0\})$, $\varphi(\mathbf{0}') = 0$, $\nabla\varphi(\mathbf{0}') = \mathbf{0}'$, $\|\nabla\varphi(\mathbf{x}')\|_{\{|\mathbf{x}'| \le r_0\}} \le 1/4$.

Now, from (1.2) we need to obtain some more than (2.5). Concretely, for all $\mathbf{x}'$ and $\mathbf{y}'$ with $|\mathbf{x}'| \le r_0$, $|\mathbf{y}'| \le r_0$ we have

$$|\mathbf{n}^i_{(\mathbf{x}',\varphi(\mathbf{x}'))} - \mathbf{n}^i_{(\mathbf{y}',\varphi(\mathbf{y}'))}| = \left| \frac{(-\nabla\varphi(\mathbf{y}'),1)}{\sqrt{1+|\nabla\varphi(\mathbf{y}')|^2}} - \frac{(-\nabla\varphi(\mathbf{x}'),1)}{\sqrt{1+|\nabla\varphi(\mathbf{x}')|^2}} \right| \le$$

$$\le \varepsilon(|(\mathbf{x}'-\mathbf{y}', \varphi(\mathbf{x}')-\varphi(\mathbf{y}'))|) \le 5/4\varepsilon(|\mathbf{x}'-\mathbf{y}'|)\,,$$

so that $|1/\sqrt{1+|\nabla\varphi(\mathbf{y}')|^2} - 1/\sqrt{1+|\nabla\varphi(\mathbf{x}')|^2}| \le 5/4\varepsilon(|\mathbf{x}'-\mathbf{y}'|)$.
Therefore,

$$\frac{|\nabla\varphi(\mathbf{y}') - \nabla\varphi(\mathbf{x}')|}{\sqrt{1+|\nabla\varphi(\mathbf{y}')|^2}} \le 5/4\varepsilon(|\mathbf{x}'-\mathbf{y}'|) + |\nabla\varphi(\mathbf{x}')| 5/4\varepsilon(|\mathbf{x}'-\mathbf{y}'|) \le 25/16\varepsilon(|\mathbf{x}'-\mathbf{y}'|),$$

and hence

$$|\nabla\varphi(\mathbf{y}') - \nabla\varphi(\mathbf{x}')| \le 2\varepsilon(|\mathbf{x}'-\mathbf{y}'|),$$

$|\mathbf{x}'| \le r_0$, $|\mathbf{y}'| \le r_0$.

Fix now a function χ with the following properties: $\chi \equiv 0$ outside $Q_\mathbf{0}$, $0 < \chi \le 2$ inside $Q^\circ_\mathbf{0}$, $\chi = 2$ on $Q'_\mathbf{0} = \{|\mathbf{x}'| \le r_0/2\,, |x_N| \le r_0/2\}$, $\chi \in C^2(\mathbb{R}^N)$, and $||\chi||_m \le A_N/r_0^m$ ($m = 1$ or 2). Set

$$E_\mathbf{0}(\mathbf{x}) = \chi(\mathbf{x})(x_N - \varphi(\mathbf{x}')),$$

so that we have $E_\mathbf{0} \in C^1_0(\mathbb{R}^N)$. Since we have assumed that $\overline{D} \cap Q_\mathbf{0} = \{|\mathbf{x}'| \le r_0, \varphi(\mathbf{x}') \le x_N \le r_0\}$, we have $E_\mathbf{0}(\mathbf{x}) \ge 0$ in D and $E_\mathbf{0} > 0$ on $D \cap Q^\circ_\mathbf{0}$. Clearly, $E_\mathbf{0}(\mathbf{x}) \equiv 0$ on S. We also have

$$\nabla E_\mathbf{0}(\mathbf{x})|_S = \nabla\chi(\mathbf{x})(x_N - \varphi(\mathbf{x}'))|_S + \chi(\mathbf{x})\{(-\nabla\varphi(\mathbf{x}'),1)\}|_S = \chi(\mathbf{x})\sqrt{1+|\nabla\varphi(\mathbf{x}')|^2}\mathbf{n}^i_\mathbf{x}\,,$$

$$||E_\mathbf{0}|| \le A_N r_0\,,\ ||\nabla E_\mathbf{0}|| \le A_N\,,$$

and $|\nabla E_\mathbf{0}(\mathbf{x}) - \nabla E_\mathbf{0}(\mathbf{y})| \le A_N(|\mathbf{x}-\mathbf{y}|/r_0 + \varepsilon(|\mathbf{x}-\mathbf{y}|))$, for *all* $\mathbf{x}$ and $\mathbf{y}$.

Now, denote by $E_\mathbf{a}$ the function $E_\mathbf{0}$, rewritten in the initial coordinate system, and let $Q'_\mathbf{a}$ be the cylinder corresponding to $Q'_\mathbf{0}$.

Finally, choose some covering $\{Q'_{\mathbf{a}_s}\}$ ($s = 1, \ldots, s_0$) of S by the cylinders $Q'_{\mathbf{a}_s}$, $\mathbf{a}_s \in S$ (such that each point $\mathbf{x}$ belongs at most to A_N of $Q_{\mathbf{a}_s}$), and consider the corresponding $E_{\mathbf{a}_s}$ and $\chi_{\mathbf{a}_s}$(that is, $E_\mathbf{0}$ and χ in the initial coordinate system, denoting again $\mathbf{O_x}$). Put

$$E(\mathbf{x}) = \sum_{s=1}^{s_0} E_{\mathbf{a}_s}(\mathbf{x})\,,$$

so that $|\nabla E(\mathbf{x})| \ge \sum_{s=1}^{s_0} \chi_{\mathbf{a}_s}(\mathbf{x}) \ge 2$ on S and

$$||E||_1 \le A_N\,,\ |\nabla E(\mathbf{x}) - \nabla E(\mathbf{y})| \le A_N(|\mathbf{x}-\mathbf{y}|/r_0 + \varepsilon(|\mathbf{x}-\mathbf{y}|))\,,$$

The function E and the set $\Omega = \{\mathbf{x} \in \mathbb{R}^N \,:\, |\nabla E(\mathbf{x})| > 1\}$ give the result.

□

Corollary 2.3. *In the notations of the previous lemma, for all $\delta > 0$ small enough, let D_δ be the connected component of the (open) set $\{\mathbf{x} \in \Omega \cup D \,|\, E(\mathbf{x}) > -\delta\}$ that contains D. Then $D_\delta \to D$ as $\delta \to 0$ and (for all small enough δ) the D_δ have the same Dini-function, majorized by $\varepsilon_*(\cdot)$.*

Proof. Clearly, for δ small enough, we have $S_\delta = \partial D_\delta \subset \{\mathbf{x} \in \Omega \,|\, E(\mathbf{x}) = -\delta\}$. Take $\mathbf{x}$ and $\mathbf{y}$ on S_δ and let $\mathbf{n}_{\mathbf{x}}^{i\delta}$ be the unit inward normal to S_δ at $\mathbf{x} \in S_\delta$ with respect to D_δ. Elementary planimetric arguments and Lemma 2.2 show that

$$|\mathbf{n}_{\mathbf{x}}^{i\delta} - \mathbf{n}_{\mathbf{y}}^{i\delta}| = \left| \frac{\nabla E(\mathbf{x})}{|\nabla E(\mathbf{x})|} - \frac{\nabla E(\mathbf{y})}{|\nabla E(\mathbf{y})|} \right| \leq |\nabla E(\mathbf{x}) - \nabla E(\mathbf{y})| \leq \varepsilon_*(|\mathbf{x} - \mathbf{y}|)\,,$$

since $\inf_{S_\delta} |\nabla E| \geq \inf_\Omega |\nabla E| \geq 1$. □

Now we continue the proof of Theorem 2 for $N > 2$ (the case $N = 2$ is briefly discused later). Let $u_i = u_1 \in C^1(\overline{D}) \cap H(D)$, and put $m = \|u_1\|_{1,\overline{D}}$. Suppose that u_p, $p \in \mathbb{N}$, is defined (with $u_p \in C^1(\overline{D}) \cap H(D)$ and $\|u_p\|_{1,\overline{D}} \leq m/2^{p-1}$). Extend u_p by Whitney's theorem to a function $f_p \in C_0^1(\mathbb{R}^N)$ with $\|f_p\|_1 \leq m/2^{p-2}$. By [8, Corollary 6.3] we can find $g_p \in C^1(\mathbb{R}^N)$ harmonic on some domain D_{δ_p} ($\delta_p \in (0,1)$ is small enough, so that $S_p = S_{\delta_p}$ and $D_p = D_{\delta_p}$ satisfy Corollary 2.3 with $\delta = \delta_p$ ($\operatorname{diam} D_p < d$) and $\|f_p - g_p\|_1 \leq m/2^p$. Therefore, $\|g_p\|_1 \leq 5m/2^p$. Set $u_{p+1} = (f_p - g_p)|_{\overline{D}}$. Then $\|u_{p+1}\|_{1,\overline{D}} \leq m/2^p$, $u_{p+1} \in H(D) \cap C^1(\overline{D})$. Since $u_i = \sum_{p=1}^{+\infty} g_p|_{\overline{D}}$ and $\|g_p\|_{1,\overline{D}} \leq \|g_p\|_1 \leq 5m/2^p$ it is enough to find a function $F_p \in C^1(\mathbb{R}^N) \cap SH(\mathbb{R}^N)$ such that $F_p|_{\overline{D}} = g_p|_{\overline{D}}$ and $\|F_p\|_1 \leq A\|g_p\|_{1,\overline{D}}$. The desired function F is $\sum_{p=1}^{+\infty} F_p$.

So, we have $g_p \in C^1(\overline{D_p}) \cap H(D_p)$. Put $m_p = \|g_p\|_{1,\overline{D_p}} \leq \|g_p\|_1 \leq 5m/2^p$. Since $\overline{D}$ has connected complement, we also can assume that $\overline{D_p}$ has connected complement Ω_p. By Theorem 1 there exists a function $h_p \in C^1(\overline{\Omega_p}) \cap H(\Omega_p)$, $h_p(\infty) = 0$, such that $h_p = g_p$ on $S_p = \partial\Omega_p$ and

$$\|h_p\|_{1,\overline{\Omega_p}} \leq A_{10} m_p.$$

Here A_{10} depends only on N, ε and d (because all the domains D_p, by Corollary 2.3, have the same Dini function, majorized by $\varepsilon_*(\cdot)$, and their diameters are less than d). Applying Theorem W1 (3) for Ω_p, take the function $w_p \in H(\Omega_p) \cap C^1(\overline{\Omega_p})$, $w_p = 0$ on S_p, $w_p(\infty) = 1$, with $\|w_p\|_{1,\overline{\Omega_p}} \leq A_{11}$ (the last can be checked by Theorem 1 applied to the functions $u_i \equiv -1|_{\overline{D_p}}$ and $u_o = w_p - 1$) and

$$\left.\frac{\partial w_p}{\partial \mathbf{n}_{\mathbf{x}}^p}\right|_{S_p} \geq A_0^{-1} > 0, \quad \forall \mathbf{x} \in S_p\,.$$

Here $\mathbf{n}_{\mathbf{x}}^p$ is the inward unit normal to S_p at $\mathbf{x} \in S_p$ with respect to the domain Ω_p. For $t > 0$ consider the function $F_p^t(\mathbf{x})$, which is equal to $g_p(\mathbf{x})$ on $\overline{D_p}$ and $F_p^t(\mathbf{x}) = h_p(\mathbf{x}) + t w_p(\mathbf{x})$ in $\overline{\Omega_p}$.

By (2.2), we have

$$\Delta F_p^t = \left.\left(\frac{\partial g_p}{\partial \mathbf{n}_{\mathbf{x}}^i} + \frac{\partial h_p}{\partial \mathbf{n}_{\mathbf{x}}^p} + t \frac{\partial w_p}{\partial \mathbf{n}_{\mathbf{x}}^p} \right)\right|_{S_p} \sigma^p$$

in the distributional sense (here $\mathbf{n}_{\mathbf{x}}^i = -\mathbf{n}_{\mathbf{x}}^p$ and σ^p is the surface measure on S_p).

Therefore, for $t = t_* = (1 + A_{10}) A_0 m_p$ we have $F_p^* = F_p^{t_*} \in SH(\mathbb{R}^N) \cap Lip_1(\mathbb{R}^N)$ and $\|F_p^*\|_{Lip_1} \leq A_{12} m_p$.

Final step. Regularization. Fix $\chi_1 \in C_0^\infty(B(\mathbf{0},1))$, $\chi_1 \geq 0$, $\chi_1(\mathbf{x}) = \chi_1(|\mathbf{x}|)$, $\int_{B(\mathbf{0},1)} \chi_1(\mathbf{x})d\mathbf{x} = 1$, and let $\chi_\tau(\mathbf{x}) = \chi_1(\mathbf{x}/\tau)/\tau^N$, $\tau > 0$. Put $d_p = \operatorname{dist}(S, S_p)$, then, for any $\tau \in (0, d_p)$ one can take

$$F_p = \chi_\tau * F_p^* .$$

In fact, it is easily seen that $F_p \in C^1(\mathbb{R}^N)$ and $\|F_p\|_1 \leq A_N \|F_p^*\|_{Lip_1} \leq Am_p$. By the meanvalue theorem for harmonic functions (taking into account that χ_τ is radial and $\int \chi_\tau(\mathbf{x})d\mathbf{x} = 1$) we have $F_p = F_p^* = g_p$ on $\overline{G}$. And we get (1.4) for $N \geq 3$.

For $N = 2$ the only difference in the proof is that we take, instead of w (from Theorem W1 (3)), the function w_* with the properties $w_* \in H(D_o)$, $w_*|_S = 0$ and $w_*(\mathbf{x})/(\log|\mathbf{x}|) \to 1$ as $|\mathbf{x}| \to 0$. Use Theorem 4.9 and the reflection $z \to 1/\bar{z}$, $z \in \mathbb{C}$.

Theorem 2 is proved. □

Proof of Corollary 1. We can find open balls B_j $(j = 1, \ldots, J)$ and $\varphi_j \in C_0^\infty(B_j)$ with the following properties: $\sum_{j=1}^J \varphi_j(\mathbf{x}) = 1$ on $\overline{D}$ and for each j either $\overline{B_j} \subset D$ or there exists $\mathbf{a}_j \in \partial D$ such that $\overline{B_j} \subset B_{\mathbf{a}_j}$. If j is such that $\overline{B_j} \subset D$ we define $f_j = \Phi * (\varphi_j \Delta f)$. In case $B_j \not\subseteq D$ we choose some $\mathbf{a}_j$, the corresponding $B_{\mathbf{a}_j}$ and $g_{\mathbf{a}_j}$, and set $f_j = \Phi * (\varphi_j \Delta g_{\mathbf{a}_j})$. By [8, Lemma 4.2] we have $f_j \in C^1(\mathbb{R}^N)$ $(f_j \in C^1_{loc}(\mathbb{R}^N)$ for $N = 2)$ and $\Delta f_j = \varphi_j \Delta f \geq 0$ or $\Delta f_j = \varphi_j \Delta g_{\mathbf{a}_j} \geq 0$ (in the distributional sense), so that $f_j \in SH(\mathbb{R}^N)$. Take $F_0 = \sum_{j=1}^J f_j \in C^1_{(loc)}(\mathbb{R}^N) \cap SH(\mathbb{R}^N)$ and consider $u_i = (f - F_0)|_D$. Since $g_{\mathbf{a}_j} = f$ in $B_{\mathbf{a}_j} \cap D$ we have

$$\Delta u_i = \Delta f - \sum_{j=1}^J \varphi_j \Delta f = 0$$

in D, which gives $u_i \in H(D) \cap C^1(\overline{D})$. Extend u_i by Theorem 2 to a function $F_1 \in C^1_{(loc)}(\mathbb{R}^N) \cap SH(\mathbb{R}^N)$. The function $F = F_0 + F_1$ gives the result. □

Notice, that Corollary 1 can be reformulated for the "entire" class $SH(D) \cap C^1(\overline{D})$ in (L-D) domains D by analogy with [3, Corollary 2.6] .

3 Proof of Theorem 3

In what follows the constants $A, A_1, \ldots$ (depending only on N, ε, d) and A_N (depending only on N) can be different from the corresponding above ones and even can change from one formula to others. We need the following extension of Theorem 1.

Theorem 3.1. *Let D be a (L-D) domain in $\mathbb{R}^N$ with $\operatorname{diam} D < d$ and the Dini-function $\varepsilon(\cdot)$ satisfying the Log-Dini property*

$$\int_0^1 \frac{\varepsilon(t)}{t} \log\left(\frac{1}{t}\right) dt < +\infty. \tag{3.1}$$

Let $\psi \in C^1(S)$, $S = \partial D$, and u_i (respectively, u_o) be the solution of the Dirichlet problem in D (respectively, in $D_o = \mathbb{R}^N \backslash \overline{D}$) with the boundary data ψ. Let $\mathbf{z} \in D$ and $\mathbf{a} \in S$ be (one of) the closest points to $\mathbf{z}$ on S. Assume that $|\mathbf{z} - \mathbf{a}| = \operatorname{dist}(\mathbf{z}, S) < r_\mathbf{0}/2$, and take $\mathbf{z}_\mathbf{a}^* \in D_o$ such that $\mathbf{z} - \mathbf{a} = -(\mathbf{z}_\mathbf{a}^* - \mathbf{a})$. Then

$$|\nabla u_i(\mathbf{z}) - (\nabla u_o(\mathbf{z}_\mathbf{a}^*))_\mathbf{a}^*| \leq A\|\psi\|_{1,S} , \tag{3.2}$$

where $(\cdot)_\mathbf{a}^*$ means symmetry with respect to the hyperplane $P_\mathbf{a}$ tangent to S at the point $\mathbf{a} \in S$.

Remark 3.2. *The functions* $\varepsilon_p(t) = 1/(\log(1/t))^p$ $(0 < t < e^{-p-1})$, $\varepsilon(t) = \varepsilon(e^{-p-1})$ $(t \geq e^{-p-1})$, *do satisfy* (3.1) *if and only if* $p > 2$.

The functions $\varepsilon(t) = t^\alpha$ *satisfy* (3.1) *for each* $\alpha \in (0, 1]$.

First we prove the following Lemma. Put $m = \|\psi\|_{1,S}$.

Lemma 3.3. *In the notations of Theorem 3.1, let* $\mathbf{b} \in S$ *and* $\mathbf{x} \in D$ *(respectively,* $\mathbf{x} \in D_o$*) be such that* $|\mathbf{x} - \mathbf{b}| < r_\mathbf{0}/2$ *and the angle between* $\mathbf{x} - \mathbf{b}$ *and* $\mathbf{n}^i_\mathbf{b}$ *(respectively,* $\mathbf{n}^o_\mathbf{b}$*) be less than* $\pi/6$. *Then*

$$|u_{i(o)}(\mathbf{x}) - u_{i(o)}(\mathbf{b})| \leq A\,m|\mathbf{x} - \mathbf{b}| \log \frac{1}{|\mathbf{x} - \mathbf{b}|}. \tag{3.3}$$

The Example 4.1 (see also (4.1)) shows that the last estimate is "almost" precise.

Proof. Denote by $G(\mathbf{x}, \mathbf{y})$ the *Green function* of the domain D. Recall that (in our choice) $G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - v_\mathbf{x}(\mathbf{y})$, where Φ is mentioned above (standard) fundamental solution for the Laplacean Δ, and the function $v_\mathbf{x}(\mathbf{y})$ is harmonic with respect to $\mathbf{y}$ in D, and having the boundary data $v_\mathbf{x}(\mathbf{y}) = \Phi(\mathbf{y} - \mathbf{x})$, $\mathbf{y} \in S$. In what follows $\rho(\mathbf{x})$ means the distance from $\mathbf{x}$ to S.

Theorem W2. *In the present notations,* $G(\mathbf{x}, \mathbf{y}) \in C^1(\overline{D} \setminus \{\mathbf{x}\})$ *(*$\mathbf{x}$ *being fixed), and the Green function* G *of* D *satisfies the following properties:*

$$\left| \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial y_n} \right| \leq \frac{A\,\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{y}|^N}\ , \quad 1 \leq n \leq N, \tag{3.4}$$

$$\left| \frac{\partial}{\partial x_n} \frac{\partial}{\partial y_l} G(\mathbf{x}, \mathbf{y}) \right| \leq \frac{A}{|\mathbf{x} - \mathbf{y}|^N}\ ,\ 1 \leq n \leq N\,,\ 1 \leq l \leq N\,, \tag{3.5}$$

where $A = A(N, d, \varepsilon)$.

The same estimates hold for the Green functions of (and in) bounded components of D_o. *For the unbounded component* D_* *of* D_o, *the last estimates hold also for the Green function* G_* *of* D_* *(in place of* G*) for all* $\mathbf{y} \in B(\mathbf{0}, 2d) \cap D_*$ *(presumably,* $\mathbf{0} \in \overline{D}$*) and all* $\mathbf{x} \in D_*$.

The proof of this theorem (similar to that of [7, Theorem 2.3]) is given in Section 4 (see Theorem 4.5).

The next formula is well known [11, Theorem 12.1] (it directly follows from the Gauss-Ostrogradski formula and then clearly holds for (L-D) domains):

$$u_i(\mathbf{x}) = -\int_S \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}^i_\mathbf{y}} \psi(\mathbf{y}) d\sigma_\mathbf{y}. \tag{3.6}$$

We can suppose that $\mathbf{b} = 0$ and $\mathbf{n}^i_\mathbf{0} = (0, \ldots, 0, 1)$, so that by (3.4) and (3.6) one has (for $\mathbf{y} = (\mathbf{y}', y_N)$):

$$|u_i(\mathbf{x}) - u_i(\mathbf{0})| \leq \int_S \left| \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}^i_\mathbf{y}} \right| |\psi(\mathbf{y}) - \psi(\mathbf{0})| d\sigma_\mathbf{y} \leq A_1 \int_S \frac{\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{y}|^N} m|\mathbf{y}| d\sigma_\mathbf{y} \leq$$

$$\leq A_2 \left(\int_{S \setminus Q_\mathbf{0}} \frac{x_N}{|\mathbf{x} - \mathbf{y}|^N} m|\mathbf{y}| d\sigma_\mathbf{y} + \int_{S \cap Q_\mathbf{0}} \frac{x_N}{(|\mathbf{y}'|^2 + x_N^2)^{N/2}} m|\mathbf{y}'| d\mathbf{y}' \right) \leq$$

$$\leq A_3 m x_N + A_3 m \int_0^{r_0} \frac{x_N}{(r^2+x_N^2)^{N/2}} r^{N-1}\, dr \leq A\, m x_N \left(1 + \log \frac{r_0}{x_N}\right).$$

The penultimate integral was estimated using spherical coordinates in the hyperplane $\mathbb{R}^{N-1}_{\mathbf{y}'}$.

When $\mathbf{x}$ belongs to the unbounded component of D_o, it would be enough to additionally apply the so-called Kelvin transform (see (4.21), Section 4). Lemma 3.3 is proved. $\square$

Proof of Theorem 3.1. Clearly, we can set $\mathbf{a} = \mathbf{0}$, $\mathbf{z} = (0, \dots, 0, z_N)$, $z_N \in (0, r_0/2)$, so that $\mathbf{n}^i_{\mathbf{0}} = \{0, \dots, 0, 1\}$.

There exists a domain $D' \subseteq D \cap Q_{\mathbf{0}}$ with the following properties: D' is convex (L-D) domain with Dini-function $A_N \varepsilon(\cdot)$, D' is radially symmetric with respect to the variable $\mathbf{y}'$, and

$$D' \cap \{(\mathbf{y}', y_N) \in \mathbb{R}^N \mid |\mathbf{y}'| < \frac{r_0}{3}\} =$$

$$\{(\mathbf{y}', y_N) \in \mathbb{R}^N \mid |\mathbf{y}'| < \frac{r_0}{3}\ ,\ 4\varphi_\varepsilon(|\mathbf{y}'|) < y_N < r_0 - 4\varphi_\varepsilon(|\mathbf{y}'|)\},$$

where $\varphi_\varepsilon(r) = \int_0^r \varepsilon(t)dt$. Notice that $\varepsilon(r_0) \leq 1/8$ (see also (2.5)) and $4\varphi_\varepsilon(|\mathbf{y}'|) \geq 2|\mathbf{y}'|\varepsilon(|\mathbf{y}'|) \geq \varphi(\mathbf{y}')$ (since $\varepsilon(t)/t \geq \varepsilon(r)/r$ for $t \in (0, r]$). Let D'_* be symmetric to D' with respect to the hyperplane $P_{\mathbf{0}} = \{x_N = 0\}$, so that $D'_* \subset D_o \cap Q_{\mathbf{0}}$, and let $G'(\mathbf{x}, \mathbf{y})$ and $G'_*(\mathbf{x}, \mathbf{y})$ be the Green functions of the domains D' and D'_* respectively. Put $S' = \{(\mathbf{y}', y_N) |\, |\mathbf{y}'| \leq r_0/3,\ y_N = 4\varphi_\varepsilon(|\mathbf{y}'|)\}$, $S' \subset \partial D'$.

By (1.2) we have for $\mathbf{y} = (\mathbf{y}', \varphi(\mathbf{y}')) \in S$, $|\mathbf{y}'| \leq r_0/3$,

$$|\mathbf{n}^i_{\mathbf{0}} - \mathbf{n}^i_{\mathbf{y}}| \leq \varepsilon(|\mathbf{y}|) \leq \varepsilon\left(\sqrt{2}\frac{r_0}{3}\right) \leq \varepsilon(r_0) \leq \frac{1}{8},$$

so that the angle between $\mathbf{n}^i_{\mathbf{0}}$ and $\mathbf{n}^i_{\mathbf{y}}$ is less than $\pi/6$ and we can apply (3.3) (for $\mathbf{b} = \mathbf{y}$) to obtain

$$|u_i(\mathbf{y}', 4\varphi_\varepsilon(|\mathbf{y}'|)) - (\pm\psi(\mathbf{y})) - u_o(\mathbf{y}', -4\varphi_\varepsilon(|\mathbf{y}'|))| \leq A\, m \varphi_\varepsilon(|\mathbf{y}'|) \log \frac{1}{\varphi_\varepsilon(|\mathbf{y}'|)}. \tag{3.7}$$

Put $\tilde{\varphi}(r) = \varphi_\varepsilon(r) \log(1/\varphi_\varepsilon(r))$, $r \in (0, r_0)$. Using (3.6) for u_i and u_o in D' and D'_* respectively, the property $G'(\mathbf{x}, \mathbf{y}) = G'_*(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ ($\mathbf{x}$, $\mathbf{y} \in D$, $\overline{\mathbf{x}} = \mathbf{x}^*_{\mathbf{0}}$ and $\overline{\mathbf{y}} = \mathbf{y}^*_{\mathbf{0}}$), (3.5) and (3.7), we obtain:

$$|\nabla u_i(\mathbf{z}) - \overline{\nabla u_o(\overline{\mathbf{z}})}| =$$

$$= \left| \int_{\partial D'} \nabla_{\mathbf{x}} \frac{\partial}{\partial \mathbf{n}^i_{\mathbf{y}}} G'(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{x}=\mathbf{z}} u_i(\mathbf{y}) d\sigma_{\mathbf{y}} - \int_{\partial D'_*} \overline{\nabla_{\mathbf{x}} \frac{\partial}{\partial \mathbf{n}^i_{\mathbf{y}}} G'_*(\mathbf{x}, \mathbf{y})} \Big|_{\mathbf{x}=\overline{\mathbf{z}}} u_o(\mathbf{y}) d\sigma_{\mathbf{y}} \right| =$$

$$= \left| \int_{\partial D'} \nabla_{\mathbf{x}} \frac{\partial}{\partial \mathbf{n}^i_{\mathbf{y}}} G'(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{x}=\mathbf{z}} (u_i(\mathbf{y}) - u_o(\overline{\mathbf{y}}) d\sigma_{\mathbf{y}} \right| \leq$$

$$\leq \int_{\partial D' \setminus S'} \frac{A_1}{|\mathbf{z} - \mathbf{y}|^N} 2m\, d\sigma_{\mathbf{y}} + \int_{S'} \frac{A_1}{|\mathbf{z} - \mathbf{y}|^N} m \tilde{\varphi}(|\mathbf{y}'|) d\mathbf{y}' \leq$$

$$\leq A_2 m + A_2\, m \int_0^{r_0/3} \frac{1}{t^N} \tilde{\varphi}(t) t^{N-2}\, dt \leq A\, m \tag{3.8}$$

by (3.1) and inequalities $t\varepsilon(t)/2 \leq \varphi_\varepsilon(t) \leq t\varepsilon(t)$, $\varepsilon(t) \geq t\varepsilon(1)$.

Again, the penultimate integral in (3.8) is estimated using spherical coordinates in $P_{\mathbf{0}}$.

The following Proposition in fact will not be used in the proof of Theorem 3, but it has it's own interest, and gives a clear understanding that in (3.2) we have to bother about the "normal" derivative $\frac{\partial u_i}{\partial \mathbf{n}^i_{\mathbf{a}}}\Big|_{\mathbf{z}}$.

Proposition 3.4. *In conditions of Theorem 3.1, the case* $\mathbf{a}=0$, $\mathbf{z}=(0,\dots,z_N)$, $z_N\in(0,r_0/2)$, *we have*

$$\left|\frac{\partial u_i}{\partial x_n}\Big|_{\mathbf{x}=\mathbf{z}}\right| \le A\,\|\psi\|_{1,S}\ , \quad n\in\{1,\dots,N-1\}.$$

By (3.6) and (3.5) it is enough to consider the case when $\psi(\mathbf{x})=0$ outside $Q_{\mathbf{0}}$ and for $|\mathbf{x}'|>r_0/3$. Fix $n\in\{1,\dots,N-1\}$. For $\mathbf{y}=(\mathbf{y}',4\varphi_\varepsilon(|\mathbf{y}'|))\subset S'$ define $\tilde\psi(\mathbf{y})=\psi(\mathbf{y}',\varphi(\mathbf{y}'))$ and set $\tilde\psi(\mathbf{y})=0$ on $\partial D'\backslash S'$, so that $\tilde\psi\in C^1(\partial D')$ and $\|\tilde\psi\|_{1,\partial D'}\le A\,m$. Let $\tilde u$ be the solution of the Dirichlet problem in D' with the boundary data $\tilde\psi$. We *claim* that $|\frac{\partial\tilde u}{\partial x_n}|_{\mathbf{x}=\mathbf{z}}|\le A\,m$. Consider the function $\tilde v(\mathbf{x})=\tilde u(x_1,\dots,x_{n-1},x_n,x_{n+1},\dots,x_N)-\tilde u(x_1,\dots,x_{n-1},-x_n,x_{n+1},\dots,x_N)$, so that $\tilde v\in H(D')$, $\tilde v|_{\{x_n=0\}}=0$ and $\frac{\partial\tilde v}{\partial x_n}|_{\{x_n=0\}}=2\frac{\partial\tilde u}{\partial x_n}|_{\{x_n=0\}}$. Since $\tilde\psi\in C^1(\partial D')$ we have $|\tilde v(\mathbf{x})|\le A\,m x_n$ on $\partial D'_+$, where $D'_+=D'\cap\{x_n>0\}$, so that $|\tilde v(\mathbf{x})|\le A\,m x_n$ in $\overline{D'_+}$ and the claim follows.

Finally, take $w=u_i-\tilde u$ in $\overline{D'}$. By Lemma 3.3 (with $\mathbf{b}=(\mathbf{y}',\varphi(\mathbf{y}'))$ and $\mathbf{x}=\mathbf{y}=(\mathbf{y}',4\varphi_\varepsilon(\mathbf{y}'))\in S'$),

$$|w(\mathbf{y})|\le A_1 m 8\varphi_\varepsilon(\mathbf{y}')\log\frac{1}{8\varphi_\varepsilon(|\mathbf{y}'|)}\le A\,m\tilde\varphi(|\mathbf{y}'|)\,.$$

Clearly, also $|w(\mathbf{y})|\le A\,m$ for $\mathbf{y}\in\partial D'\backslash S'$. By (3.6), the equality

$$\frac{\partial w}{\partial x_n}\Big|_{\mathbf{x}=\mathbf{z}}=-\int_{\partial D'}\frac{\partial}{\partial x_n}\frac{\partial}{\partial\mathbf{n}^i_{\mathbf{y}}}G'(\mathbf{x},\mathbf{y})\Big|_{\mathbf{x}=\mathbf{z}}w(\mathbf{y})d\sigma_{\mathbf{y}}$$

and estimates (3.5) end the proof of Proposition 3.4 as in (3.8). □

Proof of Theorem 3. Put $M=\|f\|_{1,\overline{D}}$ and $\mu=\Delta f|_D$. We claim that for each domain Ω, $\Omega\subset D$, with piecewise smooth boundary, one has

$$\mu(\Omega)\le M\sigma(\partial\Omega), \tag{3.9}$$

where $\sigma(\cdot)$ is a surface $(N-1)$-dimensional (Lebesque) measure. To prove this it suffices (after reasonable regularization and then passing to the limit) to apply the Gauss-Ostrogradski formula:

$$\mu(\Omega)=\int_\Omega\Delta f(\mathbf{x})d\mathbf{x}=\int_{\partial\Omega}(\nabla f(\mathbf{y}),\mathbf{n}^o_{\mathbf{y}})d\sigma_{\mathbf{y}}\le M\sigma(\partial\Omega).$$

Since for each ball $B=B(\mathbf{b},\delta)\subset D$ we in fact have

$$\mu(B)=\int_{\partial B}(\nabla f(\mathbf{y})-\nabla f(\mathbf{b}),\mathbf{n}^o_{\mathbf{y}})d\sigma_{\mathbf{y}}\le\omega_{\overline{D}}(\nabla f,\delta)\sigma(\partial B)=o(\delta^{N-1})$$

($\omega_E(g,\cdot)$ being the modulus of continuity of the (scalar- or vector-) function g on the set E), one can also easily prove that $\mu(\partial\Omega\cap D)=0$ for the Ω considered.

The idea of the proof of (1.5) is the following. Take $u_i \in H(D)$, $u_i|_{\partial D} = f|_{\partial D}$. If $u_i \in C^1(\overline{D})$ and $\|u_i\|_{1,\overline{D}} \le A_1 m$, then it could be possible to prove (1.5) with $A = A(A_1, N, \varepsilon, d)$ (see the end of the proof of Theorem 3 below). But it turns out (see Example 4.1 below) that *it is not always* $u_i \in C^1(\overline{D})$.

By Theorem 3.1, the "reflected" harmonic function u_o satisfy the property (3.2), and we need also appropriately "reflect" the measure μ from D to D_o. The last means that we want to find such positive measure μ^* in D_o (for the notations G, $\mathbf{z}$, $\mathbf{a}$, $\mathbf{z}^*_{\mathbf{a}}$ and $(\cdot)^*_{\mathbf{a}}$ see Theorem 3.1, $G_o(\mathbf{x},\mathbf{y})$) is the Green function of the domain D_o) that

$$\int_{D_o} G_o(\mathbf{x},\mathbf{y})\, d\mu^*_{\mathbf{y}} \in C^1(D_o)$$

and

$$\left| \int_D \nabla_{\mathbf{x}} G(\mathbf{x},\mathbf{y})|_{\mathbf{x}=\mathbf{z}} d\mu_{\mathbf{y}} - \left(\int_{D_o} \nabla_{\mathbf{x}} G_o(\mathbf{x},\mathbf{y})|_{\mathbf{x}=\mathbf{z}^*_{\mathbf{a}}} d\mu^*_{\mathbf{y}} \right)^*_{\mathbf{a}} \right| \le A\, M \tag{3.10}$$

for all $\mathbf{z} \in D$ with $\rho(\mathbf{z}) < r_0/4$. After this we shall have $f_o(\mathbf{x}) = u_o(\mathbf{x}) + \int_{D_o} G_o(\mathbf{x},\mathbf{y}) d\mu^*_{\mathbf{y}} \in C^1(\overline{D_o}) \cap SH(D_o)$, $f_o = f$ on S, and we can terminate the proof essentially as in the proof of Theorem 2.

We pass to the details supposing that $N \ge 3$ (for $N = 2$ one can follow the previous notes on this matter).

Let $\{B_j, \varphi_j\}_{j \in J}$ be the Whitney partition of unity on D (see [12, Ch.VI, §1]). Recall that J is some countable set (for "nonoverlapping" the notations we assume that $J \cap \{0, 1, \ldots N\} = \emptyset$), $B_j = B(\mathbf{b}_j, \delta_j)$, $\mathbf{b}_j \in D$ and there is $A_1 \ge 3N$ (depending only on N) such that

$$\frac{1}{A_1}\rho(\mathbf{b}_j) \le \delta_j \le \frac{1}{6}\rho(\mathbf{b}_j)\,; \tag{3.11}$$

furthermore, for each $\mathbf{z} \in D$ the number $\#(\mathbf{z}, J)$ of balls B_j, $j \in J$, that intersect $B(\mathbf{z}, \rho(\mathbf{z})/2)$ satisfies

$$\#(\mathbf{z}, J) \le A_1\,; \tag{3.12}$$

and $\varphi_j \in C_0^\infty(B_j)$, $\varphi_j \ge 0$, have the properties

$$\|\Delta \varphi_j\| \le \frac{A_1}{\delta_j^2}, \quad \sum_{j \in J} \varphi_j(\mathbf{x}) \equiv 1 \quad (\mathbf{x} \in D). \tag{3.13}$$

Put $\mu_j = \mu\varphi_j$ and define $\hat{\mu}(\mathbf{x}) = \int_D G(\mathbf{x},\mathbf{y}) d\mu_{\mathbf{y}}$, $\hat{\mu_j}(\mathbf{x}) = \int G(\mathbf{x},\mathbf{y}) d\mu_{j\mathbf{y}}$.

Since $f = u_i + \hat{\mu}$ in D (see [13, Theorem 1.24']), we have $\hat{\mu} \in C^1(D)$. We claim that also $\hat{\mu_j} \in C^1(\overline{D})$ and

$$\|\hat{\mu_j}\|_{1,\overline{D}} \le A\, M. \tag{3.14}$$

In fact, let $f_j = \Phi * \mu_j$. By (3.13) and [8, Lemma 4.2], we have $f_j \in C^1(\mathbb{R}^N)$ and

$$\|\nabla f_j\| \le A\, M, \tag{3.15}$$

and by (3.9) and (3.4), for $\mathbf{x} \in D \backslash B(\mathbf{b}_j, (3/2)\delta_j)$,

$$|\nabla \hat{\mu_j}(\mathbf{x})| \le \left| \int \nabla_{\mathbf{x}} G(\mathbf{x},\mathbf{y}) d\mu_{j\mathbf{y}} \right| \le \int_{B_j} \frac{A_1 \rho(\mathbf{y}) \varphi_j(\mathbf{y}) d\mu_{\mathbf{y}}}{|\mathbf{x}-\mathbf{y}|^N} \le A_2 \frac{\delta_j^N}{(\delta_j/2)^N} \le A\, M\,.$$

Since $\hat{\mu_j} - f_j$ is harmonic in D, it remains to apply the maximum principle in $B(\mathbf{b}_j, \frac{3}{2}\delta_j)$ for $\frac{\partial}{\partial x_n}(\hat{\mu_j} - f_j)$, $n \in \{1, \dots, N\}$, and recall that $\hat{\mu_j}|_{\partial D} = 0$. The claim (3.14) is proved.

Notice also, that the property (3.15) of f_j allows to reduce the proof Theorem 3 to the case when

$$\operatorname{Supp} \mu \subset \{\mathbf{x} \in D |\, \rho(\mathbf{x}) < r_2\}, \tag{3.16}$$

where some fixed $r_2 = r_2(\varepsilon(\cdot)) \in (0, r_0/32)$ will be chosen later (see (3.22)). In fact, let $J_0 = \{j \in J |\, \rho(\mathbf{b}_j) \geq r_2/2\}$. Then $F_0 = \sum_{j \in J_0} f_j$ is subharmonic in $\mathbb{R}^N$, $\|F_0\|_1 \leq AM$, and it remains to extend $(f - F_0)|_{\overline{D}}$ instead of f.

So, in the sequel we shall always require (3.16). The *reflection* of μ over S (having (3.16) and $\rho(\mathbf{b}_j) < r_2/2$) consist of the following. For each $j \in J$ ($J_0 = \emptyset$), let $\mathbf{a}_j \in S$ be (some) point closest to $\mathbf{b}_j$, $|\mathbf{a}_j - \mathbf{b}_j| = \rho(\mathbf{b}_j)$. Let $P_j = P_{\mathbf{a}_j}$ be hyperplane tangent to S at $\mathbf{a}_j$. Define μ_j^* as a measure, "symmetric" to μ_j with respect to P_j (that is, $\mu_j(E) = \mu_j^*(E_j^*)$ for each Borel set E and the set E_j^* symmetric to E with respect to P_j). The measure

$$\mu^* = \sum_{j \in J} \mu_j^*$$

is the desired reflection of μ "over" S.

For checking (3.10) it remains to prove the following result (the case $\mathbf{a} = 0$ in (3.10)) and use the maximum principle. Notice also, that $\operatorname{Supp} \mu^* \subset B(\mathbf{0}, 2d)$ and we can use Theorem W2 in order to estimate $G_o(\mathbf{x}, \mathbf{y})$ for $|\mathbf{y}| < 2d$.

Theorem 3.5. *Let $\hat{\mu^*}(\mathbf{x}) = \int G_o(\mathbf{x}, \mathbf{y}) d\mu^*_{\mathbf{y}}$, $\hat{\mu_j^*}(\mathbf{x}) = \int G_o(\mathbf{x}, \mathbf{y}) d\mu^*_{j\mathbf{y}}$, $\mathbf{x} \in \overline{D_o}$. Let $\mathbf{z} \in D$ be such that $\mathbf{z} = (0, \dots, z_N)$, $0 < z_N < r_0/4$, $\mathbf{a} = \mathbf{0}$ is closest (one of) to $\mathbf{z}$ on S. Then*

$$|\nabla \hat{\mu}(\mathbf{z}) - \overline{\nabla \hat{\mu^*}(\overline{\mathbf{z}})}| \leq AM\,, \tag{3.17}$$

where the "overline" (for vectors) means symmetry with respect to the hyperplane $P_{\mathbf{0}} = \{\mathbf{x} \in \mathbb{R}^N \mid x_N = 0\}$ tangent to S at $\mathbf{a} = \mathbf{0}$.

Proof. Let $Q_{\mathbf{0}}$ and $\varphi_\varepsilon(\cdot)$ be as in the proof of Theorem 3.1. We can find $\tilde{D}$ (by analogy with D') with the properties: $\tilde{D} \cap \{\mathbf{y} = (\mathbf{y}', y_N) \in \mathbb{R}^N \,|\, |\mathbf{y}'| < r_0/16\} =$

$$= \{\mathbf{y} \in \mathbb{R}^N \,|\, |\mathbf{y}'| < r_0/16\,,\ 8\varphi_\varepsilon(|\mathbf{y}'|) < y_N < r_0/2 - 8\varphi_\varepsilon(|\mathbf{y}'|)\}\,,$$

$y_N > 8\varphi_\varepsilon(|\mathbf{y}'|)$ for all $\mathbf{y} \in \tilde{D}$, $\tilde{D}$ is convex radially symmetric with respect to $\mathbf{y}'$ domain having Dini-function bounded by $A\,\varepsilon(\cdot)$, and $\tilde{D} \subset \{\mathbf{y} \in \mathbb{R}^N \,|\, |\mathbf{y}'| < r_0/8\}$. Let $\tilde{D}_o$ be symmetric to $\tilde{D}$ with respect to $P_{\mathbf{0}}$ (recall, that $\varepsilon(r) \leq 1/8$ for $0 < r \leq r_0$).

Lemma 3.6. *Let $J_1 = \{j \in J |\, \mathbf{b}_j \in D \backslash \tilde{D}\}$, then*

$$\Sigma_1 = \sum_{j \in J_1} (|\nabla \hat{\mu_j}(z)| + |\nabla \hat{\mu_j^*}(z)|) \leq AM\,. \tag{3.18}$$

Proof. Let first $j \in J_1$ be such that $|\mathbf{b}_j'| \leq r_0/16$. Then (since $\rho(\mathbf{b}_j) < r_2/2 < r_0/2$)we have $|b_{jN}| \leq 8\varphi_\varepsilon(|\mathbf{b}_j'|) \leq 8|\mathbf{b}_j'|\varepsilon(|\mathbf{b}_j'|) \leq |\mathbf{b}_j'|$, so that $\rho(\mathbf{b}_j) \leq |\mathbf{b}_j| \leq \sqrt{2}|\mathbf{b}_j'|$. Therefore, for all $\mathbf{y} \in B_j$ we have, by (3.11),

$$|\mathbf{y} - \mathbf{b}_j| \leq \frac{1}{6}\rho(\mathbf{b}_j) \leq \frac{1}{4}|\mathbf{b}_j'|$$

and so $|\mathbf{b}'_j| \le 4|\mathbf{y}'|/3$, which gives also $|\mathbf{y}| \le 2|\mathbf{y}'|$ and

$$\rho(\mathbf{y}) \le \frac{7}{6}\rho(\mathbf{b}_j) \le 20\varphi_\varepsilon(|\mathbf{b}'_j|) \le A|\mathbf{y}'|\varepsilon(\mathbf{y}').$$

And for these j we have by (3.4):

$$|\nabla\hat{\mu}_j(\mathbf{z})| \le \int |\nabla_{\mathbf{x}} G(\mathbf{x},\mathbf{y})|_{\mathbf{x}=\mathbf{z}}|d\mu_{j\mathbf{y}} \le \int \frac{A_1\rho(\mathbf{y})}{|\mathbf{z}-\mathbf{y}|^N} d\mu_{j\mathbf{y}} \le \int \frac{A_2|\mathbf{y}'|\varepsilon(|\mathbf{y}'|)d\mu_{j\mathbf{y}}}{|\mathbf{y}|^N}\,.$$

The same estimate holds also for $|\nabla\mu_j^*(\mathbf{z})|$ (see Lemma 3.8 bellow). Since the part of the sum in (3.18) for j with the property $|\mathbf{b}'_j| > r_0/16$ can be estimated easily, we obtain

$$\Sigma_1 \le A\left(M + \int_{|\mathbf{y}|\le r_0} \frac{\varepsilon(|\mathbf{y}|)d\mu_{\mathbf{y}}}{|\mathbf{y}|^{N-1}}\right) \tag{3.19}$$

and (3.18) immediately follows from the following elementary lemma.

Lemma 3.7. *Let $h(t)$ be a nondecreasing function on $[0,+\infty)$ with the property $0 \le h(t) \le t^{N-1}$, $t \ge 0$. Then, for any $r > 0$,*

$$\int_0^r \frac{\varepsilon(t)}{t^{N-1}} dh(t) \le (N-1)\int_0^r \frac{\varepsilon(t)}{t}\, dt.$$

Proof. For $\delta \in (0,r)$ put $h_\delta(t) = \varepsilon(\delta)/\delta^{N-1}$ in $(0,\delta)$ and $h_\delta(t) = \varepsilon(t)/t^{N-1}$ in $[\delta, a]$. Since h_δ is positive and decreasing, the result follows directly applying Abel summation for the Riemann sums of the integral $\int_0^r h_\delta(t)\, dh(t)$, and then letting $\delta \to 0$. □

To finish the proof of (3.18), we calculate the integral in (3.19) using spherical coordinates in $\mathbb{R}^N_{\mathbf{y}}$. Concretely, let $h(r) = \mu(B(\mathbf{0}, r))$. By (3.9) (since $\mu = 0$ outside D) we have $h(r) \le A_1 M r^{N-1}$, and so

$$\int_{|\mathbf{y}|\le r_0} \frac{\varepsilon(|\mathbf{y}|)d\mu_{\mathbf{y}}}{|\mathbf{y}|^{N-1}} \le A_2 M \int_0^{r_0} \frac{\varepsilon(r)\, dr}{r} \le AM\,. \tag{3.20}$$

Lemma 3.6 is proved. □

Lemma 3.8. *Let $j \in J$ be such that $|\mathbf{b}_j| \le r_0/2$. Then for each $\mathbf{y} \in B_j$ we have*

$$|\overline{\mathbf{y}} - \mathbf{y}_j^*| \le A_*|\mathbf{y}|\varepsilon(|\mathbf{y}|), \tag{3.21}$$

where $\mathbf{y}_j^*$ is symmetric to $\mathbf{y}$ with respect to P_j and $A_* \le 108$.

Proof. Since $|\mathbf{b}_j| \le r_0/2$, by definition of $\mathbf{a}_j$ we have $|\mathbf{a}_j| \le 2|\mathbf{b}_j| \le r_0$, so that $\mathbf{a}_j \in S \cap Q_{\mathbf{0}}$ and $a_{jN} = \varphi(\mathbf{a}'_j)$ (recall that $|\varphi(r)| \le 2r\varepsilon(r) \le r/4$ for $r \le r_0$). Since $|\mathbf{y}| \in (\frac{5}{6}|\mathbf{b}_j|, \frac{7}{6}|\mathbf{b}_j|)$ for $\mathbf{y} \in B_j$, we have $|\mathbf{a}_j| \le 3|\mathbf{y}|$ for these $\mathbf{y}$. Elementary calculations show that

$$\overline{\mathbf{y}} - \mathbf{y}_j^* = \mathbf{y} - 2(\mathbf{y}, \mathbf{n}_{\mathbf{0}}^i)\mathbf{n}_{\mathbf{0}}^i - (\mathbf{y} - 2(\mathbf{y}-\mathbf{a}_j, \mathbf{n}_j^i)\mathbf{n}_j^i) = 2(\mathbf{y}-\mathbf{a}_j, \mathbf{n}_j^i)\mathbf{n}_j^i - 2(\mathbf{y}, \mathbf{n}_{\mathbf{0}}^i)\mathbf{n}_{\mathbf{0}}^i,$$

where $\mathbf{n}_{\mathbf{0}}^i$ and $\mathbf{n}_j^i$ are the inner unit normals to S at $\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{a}_j$ respectively. Then

$$|\overline{\mathbf{y}} - \mathbf{y}_j^*| \le 2|a_{jN}| + 6|\mathbf{y} - \mathbf{a}_j||\mathbf{n}_j^i - \mathbf{n}_{\mathbf{0}}^i|.$$

Since $|\mathbf{n}_j^i - \mathbf{n}_{\mathbf{0}}^i| \le \varepsilon(|\mathbf{a}_j|)$, we easily obtain (3.21) using the "doubling" property of $\varepsilon(\cdot)$. □

The *final restriction on* r_2 is (see (3.21))

$$A_*\varepsilon(r_2) < \frac{1}{20}\,. \tag{3.22}$$

In particular, in Lemma 3.8 we also have

$$|\overline{\mathbf{y}} - \mathbf{y}_j^*| \le \frac{1}{4}|\mathbf{y}| \tag{3.23}$$

whenever $|\mathbf{y}| < 5r_2$.

Notice, that for $|\mathbf{z}| = z_N \ge 2r_2$ the proof of (3.17) is easy, because, by (3.4) and (3.9), we even have

$$|\nabla\hat{\mu}(\mathbf{z})| \le \int |\nabla_{\mathbf{x}}G(\mathbf{x},\mathbf{y})|_{\mathbf{x}=\mathbf{z}}|d\mu_{\mathbf{y}} \le A_1\int\frac{\rho(\mathbf{y})d\mu_{\mathbf{y}}}{|\mathbf{z}-\mathbf{y}|^N} \le A_2 r_2\frac{\sigma(\partial D)M}{r_2^N} \le AM\,, \tag{3.24}$$

and the same estimate holds also for $|\nabla\hat{\mu^*}(\overline{\mathbf{z}})|$. So, from now on, we suppose that $|\mathbf{z}| \le 2r_2 \le r_0/16$. Consider the set $\Omega_{\mathbf{z}} = \{\mathbf{y} \mid |\mathbf{y}|/2 > |\mathbf{y}-\mathbf{z}|\}$ which is in fact $\Omega_{\mathbf{z}} = \{\mathbf{y}|\,|\mathbf{y} - \frac{4}{3}\mathbf{z}| < \frac{2}{3}|\mathbf{z}|\}$.

The set $J_2 = \{j \in J\backslash J_1|\, B_j \cap \Omega_{\mathbf{z}} \ne \emptyset\}$ is "small" (the number of its elements can be estimated with the help of (3.12)). By (3.14) (the analogous estimate holds also for $\hat{\mu_j^*}$) we have

$$\Sigma_2 = \sum_{j\in J_2}(|\nabla\hat{\mu_j}(\mathbf{z})| + |\nabla\hat{\mu_j^*}(\overline{\mathbf{z}})|) \le AM.$$

Let now, $J_3 = \{j \in J\backslash(J_1\cup J_2)|\,|\mathbf{b_j}| > 4r_2\}$. Then, like in (3.24),

$$\Sigma_3 = \sum_{j\in J_3}(|\nabla\hat{\mu_j}(\mathbf{z})| + |\nabla\hat{\mu_j^*}(\overline{\mathbf{z}})|) \le AM.$$

Put $J_4 = J\backslash(J_1\cup J_2\cup J_3)$ and let ν_j, $j\in J_4$, be the measure "symmetric" to μ_j with respect to $P_{\mathbf{0}}$ ($\mu_j(E) = \nu_j(E_{\mathbf{0}}^*)$ for any Borel set E).

We claim that

$$\sum_{j\in J_4}|\nabla\hat{\nu_j}(\overline{\mathbf{z}}) - \nabla\hat{\mu_j^*}(\overline{\mathbf{z}}) \le AM, \tag{3.25}$$

where $\hat{\nu_j}(\mathbf{x}) = \int G_o(\mathbf{x},\mathbf{y})d\nu_{j\mathbf{y}}$. In fact, for $j\in J_4$ one has

$$\nabla\hat{\nu_j}(\overline{\mathbf{z}}) - \nabla\hat{\mu_j^*}(\overline{\mathbf{z}}) = \int\nabla_{\mathbf{x}}G_o(\mathbf{x},\overline{\mathbf{y}})|_{\mathbf{x}=\overline{\mathbf{z}}}d\mu_{j\mathbf{y}} - \int\nabla_{\mathbf{x}}G_o(\mathbf{x},\mathbf{y}_j^*)|_{\mathbf{x}=\overline{\mathbf{z}}}d\mu_{j\mathbf{y}}\,,$$

$|\mathbf{b}_j| \le 4r_2$, and (by (3.11)) $|\mathbf{y}| \le 5r_2 \le r_0/6$ whenever $\mathbf{y}\in B_j$, so that (3.23) holds for all $\mathbf{y}\in\Omega_4 = \cup_{j\in J_4}B_j$. Since also (as $j\notin J_2$) $\Omega_4\cap\Omega_{\mathbf{z}} = \emptyset$, we have finally for all $\mathbf{y}\in\Omega_4$:

$$|\mathbf{y}| \le 5r_2\,,\ |\mathbf{y}-\mathbf{z}| \ge \frac{1}{2}|\mathbf{y}|\,,\ |\mathbf{y}-\mathbf{z}| \ge |\mathbf{z}|/3\,,\ |\overline{\mathbf{y}} - \mathbf{y}_j^*| \le \frac{1}{4}|\mathbf{y}|\,. \tag{3.26}$$

Therefore, for $\mathbf{y}\in\Omega_4$ we can write:

$$|\nabla_{\mathbf{x}}G_o(\mathbf{x},\overline{\mathbf{y}})|_{\mathbf{x}=\overline{\mathbf{z}}} - \nabla_{\mathbf{x}}G_o(\mathbf{x},\mathbf{y}_j^*)|_{\mathbf{x}=\overline{\mathbf{z}}}| \le \|\nabla_{\mathbf{x}}\nabla_{\tilde{\mathbf{y}}}G_o(\mathbf{x},\tilde{\mathbf{y}})|_{\mathbf{x}=\overline{\mathbf{z}}}\|_{\tilde{\mathbf{y}}\in[\overline{\mathbf{y}},\mathbf{y}_j^*]}|\overline{\mathbf{y}} - \mathbf{y}_j^*|\,,$$

and so, by (3.5), (3.21) and (3.26),

$$\sum_{j\in J_4} |\nabla\hat{\nu_j}(\overline{\mathbf{z}}) - \nabla\hat{\mu_j^*}(\overline{\mathbf{z}})| \le \int_{B(\mathbf{0},5r_2)} \frac{A_1}{|\mathbf{z}-\mathbf{y}|^N} A_*|\mathbf{y}|\varepsilon(|\mathbf{y}|)d\mu_{\mathbf{y}} \le A\int_{B(\mathbf{0},r_0)} \frac{\varepsilon(|\mathbf{y}|)d\mu_{\mathbf{y}}}{|\mathbf{y}|^{N-1}},$$

which gives (3.25) by (3.20).

Finally, it remains to prove that

$$\Sigma_4 = \sum_{j\in J_4} |\nabla\hat{\mu_j}(\mathbf{z}) - \overline{\nabla\hat{\nu_j}(\overline{\mathbf{z}})}| \le AM\,.$$

Using (3.12) we then have

$$\Sigma_4 \le A\int_{\Omega_4} |\nabla_{\mathbf{x}}(G(\mathbf{x},\mathbf{y}) - G_o^*(\mathbf{x},\mathbf{y}))|_{\mathbf{x}=\mathbf{z}}|d\mu_{\mathbf{y}}, \tag{3.27}$$

where $G_o^*(\mathbf{x},\mathbf{y}) = G_o(\overline{\mathbf{x}},\overline{\mathbf{y}})$ (defined on $((D_o)_{\mathbf{0}}^*)^2$).

Recall, that for $j\in J_4$ (as $j\notin J_1$) we have $\mathbf{b}_j\in\tilde{D}$, but it is not necessary that also $\mathbf{y}\in\tilde{D}$. The part of the integral in (3.27) with $\mathbf{y}\in\Omega_4\backslash\tilde{D}$ looks like

$$\int_{\Omega_4\backslash\tilde{D}} |\nabla_{\mathbf{x}}(G(\mathbf{x},\mathbf{y}) - G_o^*(\mathbf{x},\mathbf{y}))|_{\mathbf{x}=\mathbf{z}}|d\mu_{\mathbf{y}} \le AM,$$

it can be estimated the same way as in (3.19)(Lemma 3.6) or as in (3.24).

So, it remains to estimate the integral

$$I_1 = \int_{\Omega_4\cap\tilde{D}} |\nabla_{\mathbf{x}}(G(\mathbf{x},\mathbf{y}) - G_o^*(\mathbf{x},\mathbf{y}))|_{\mathbf{x}=\mathbf{z}}|d\mu_{\mathbf{y}}\,.$$

Notice, that for $\mathbf{y}\in\tilde{D}$ we have $y_N > 8\varphi_\varepsilon(|\mathbf{y}'|)$, so that (since $|\varphi(\mathbf{y}')| \le 4\varphi_\varepsilon(|\mathbf{y}'|)$ and $\varphi_\varepsilon'(r) = \varepsilon(r) \le 1/8$)

$$|\mathbf{y}| \ge \rho(\mathbf{y}) \ge \frac{2}{\sqrt{5}}(y_N - 4\varphi_\varepsilon(|\mathbf{y}'|)) \ge y_N/3, \quad \rho(\mathbf{y}) \le 2y_N\,,$$

and the same holds for the distance from $\mathbf{y}$ to $(S)_{\mathbf{0}}^*$.

Also in Ω_4 we have $|\mathbf{y}-\mathbf{z}| \ge |\mathbf{z}|/3$ (as $\Omega_4\cap\Omega_{\mathbf{z}} = \emptyset$, see above). In order to estimate I_1 consider several steps.

1^0. Set $\Omega_4' = \Omega_4\cap\tilde{D}\cap\{|\mathbf{y}| < 3|\mathbf{z}|\}$. Then, by (3.4) and (3.9),

$$\int_{\Omega_4'} |\nabla_{\mathbf{x}}G(\mathbf{x},\mathbf{y})|_{\mathbf{x}=\mathbf{z}}|d\mu_{\mathbf{y}} \le A_1\int_{\Omega_4'} \frac{\rho(\mathbf{y})}{|\mathbf{z}-\mathbf{y}|^N}d\mu_{\mathbf{y}} \le A_2\int_{\Omega_4'} \frac{|\mathbf{y}|d\mu(\mathbf{y})}{|\mathbf{z}|^N} \le AM\,,$$

and the same way one estimates $\int_{\Omega_4'} |\nabla_{\mathbf{x}}G_o^*(\mathbf{x},\mathbf{y})|_{\mathbf{x}=\mathbf{z}}|d\mu_{\mathbf{y}}$.

2^0. Set $\Omega_5 = \Omega_4\cap\tilde{D}\cap\{|\mathbf{y}| \ge 3|\mathbf{z}|\}$. Take $\Psi_{\mathbf{y}}(\mathbf{x}) = G(\mathbf{x},\mathbf{y}) - G_o^*(\mathbf{x},\mathbf{y})$ as a function of $\mathbf{x}$, $\mathbf{x}\in\overline{D'}$ ($\mathbf{y}\in\Omega_5$ is fixed). We need to estimate $|\nabla\Psi_{\mathbf{y}}(\mathbf{z})|$. Since $\Psi_{\mathbf{y}}\in H(D')\cap C^1(\overline{D})$ we can use (3.6) for $\Psi_{\mathbf{y}}$ in D'. To do this let us estimate $\Psi_{\mathbf{y}}(\mathbf{x})$ on $\partial D'$. If $\mathbf{x} = (\mathbf{x}', 4\varphi_\varepsilon(|\mathbf{x}'|))\in\partial D'$ is such that $|\mathbf{x}'| < r_0/3$, we have:

$$|G(\mathbf{x},\mathbf{y}) - G(\mathbf{x}_\varphi,\mathbf{y})| \le \|\nabla G(\cdot,\mathbf{y})\|_{[\mathbf{x}_\varphi,\mathbf{x}]}|4\varphi_\varepsilon(|\mathbf{x}'|) - \varphi(\mathbf{x}')|\,,$$

where $\mathbf{x}_\varphi = (\mathbf{x}',\varphi(\mathbf{x}'))\in S$ (so that $G(\mathbf{x}_\varphi,\mathbf{y}) = 0$). We need the following lemma.

Lemma 3.9 (Elementary). *In the notations just above we have*

$$\min_{\tilde{\mathbf{x}}\in[\mathbf{x}_\varphi,\mathbf{x}]} |\mathbf{y}-\tilde{\mathbf{x}}| \geq A_1|\mathbf{y}-\mathbf{x}'|,$$

where $A_1 \in (0,+\infty)$ *is absolute constant, and we identify* $(\mathbf{x}',0)$ *and* $\mathbf{x}'$*.*

Proof. Consider a trapezium with the vertices at $\mathbf{y} = (\mathbf{y}', y_N)$, $\mathbf{y}'$, $\mathbf{x}'$, $\mathbf{x}$ and let $\mathbf{y}_\varepsilon = (\mathbf{y}', 4\varphi_\varepsilon(|\mathbf{y}'|))$. Then $y_N > 2y_{\varepsilon N}$ and it is not hard to see that for each $\tilde{\mathbf{x}} \in [\mathbf{x}_\varphi, \mathbf{x}]$ the angle between the vectors $\mathbf{y}-\mathbf{y}_\varepsilon$ and $\tilde{\mathbf{x}}-\mathbf{y}_\varepsilon$ is greater than $\pi/2 - \arctan(1/2)$ (since $4{\varphi'}_\varepsilon(r) = 4\varepsilon(r) \leq 1/2\,,\ r \leq r_0$). Simple trigonometric calculations end the proof. □

Now, by (3.4), we have

$$|G(\mathbf{x},\mathbf{y}) - G(\mathbf{x}_\varphi,\mathbf{y})| \leq \|\nabla G(\cdot,\mathbf{y})\|_{[\mathbf{x}_\varphi,\mathbf{x}]} |2\varphi_\varepsilon(|\mathbf{x}'|) - \varphi(\mathbf{x}')| \leq$$

$$\leq A_1 \frac{\rho(\mathbf{y})}{|\mathbf{y}-\mathbf{x}'|^N}\, 8\varphi_\varepsilon(|\mathbf{x}'|) \leq A\,\frac{y_N\varphi_\varepsilon(|\mathbf{x}'|)}{|\mathbf{y}-\mathbf{x}'|^N},$$

Proceeding the same way with G_o^* (and $\mathbf{x}_{-\varphi} = (\mathbf{x}', -\varphi(\mathbf{x}'))$ instead of $\mathbf{x}_\varphi$) we finally get

$$|\Psi_\mathbf{y}(\mathbf{x})| \leq A\,\frac{y_N\varphi_\varepsilon(|\mathbf{x}'|)}{|\mathbf{y}-\mathbf{x}'|^N}$$

for $\mathbf{x} \in S' = \{\mathbf{x}', 4\varphi_\varepsilon(|\mathbf{x}'|),\ |\mathbf{x}'| < r_0/3\}$. Therefore, by (3.6) and (3.5), applied in D',

$$|\nabla\Psi_\mathbf{y}(\mathbf{z})| = \left|\int_{\partial D'} \left(\nabla_\mathbf{z} \frac{\partial}{\partial \mathbf{n}_\mathbf{x}} G'(\mathbf{z},\mathbf{x})\right) \Psi_\mathbf{y}(\mathbf{x}) d\sigma_\mathbf{x}\right| \leq \left|\int_{S'}\right| + \left|\int_{\partial D'\setminus S'}\right| \leq$$

$$\leq \int_{S'} \frac{A_1 y_N \varphi_\varepsilon(|\mathbf{x}'|) d\sigma_\mathbf{x}}{|\mathbf{z}-\mathbf{x}|^N |\mathbf{y}-\mathbf{x}'|^N} + A_2\,.$$

The penultimate integral is clearly less then A_2, because $|\mathbf{y}'| \leq r_0/8$ ($\mathbf{y} \in \tilde{D}$) and $|\mathbf{x}| \geq r_0/3$ (as $\mathbf{x} \notin S'$).

Again, using Lemma 3.9 for $\mathbf{z}$ in place of $\mathbf{y}$, we see that in order to estimate $|\nabla\Psi_\mathbf{y}(\mathbf{z})|$ it remains to estimate the integral

$$I_2 = \int_{|\mathbf{x}'|\leq r_0/3} K_{\mathbf{zy}}(\mathbf{x}')\,d\mathbf{x}'\ ,\quad \mathbf{y} \in \Omega_5\,,$$

where we set

$$K_{\mathbf{zy}}(\mathbf{x}') = \frac{y_N|\mathbf{x}'|\varepsilon(|\mathbf{x}'|)}{|\mathbf{z}-\mathbf{x}'|^N|\mathbf{y}-\mathbf{x}'|^N}\,.$$

Consider 3 cases.

Case 1. Here $|\mathbf{x}'| \leq |\mathbf{y}'|/2$, and we apply spherical coordinates in $\mathbb{R}^N_{\mathbf{x}'}$:

$$\int_{|\mathbf{x}'|\leq|\mathbf{y}|/2} K_{\mathbf{zy}}(\mathbf{x}')d\mathbf{x}' \leq \frac{A_1}{|\mathbf{y}|^{N-1}} \int_0^{|\mathbf{y}|/2} \frac{r\varepsilon(r)r^{N-2}\,dr}{r^N} \leq \frac{A_1}{|\mathbf{y}|^{N-1}}\lambda_1(|\mathbf{y}|)\,,$$

where $\lambda_1(t) = \int_0^t \frac{\varepsilon(\tau)}{\tau}\,d\tau$.

Lemma 3.10. *In the previous notations, for each $r \in (0,1]$,*

$$\int_0^r \frac{\lambda_1(t)}{t} dt \le \int_0^r \frac{\varepsilon(t)}{t} \log \frac{1}{t} dt\,.$$

Proof. Apply the following corollary of Fubini's theorem: $\int_0^r dt \int_0^t f d\tau = \int_0^r d\tau \int_\tau^r f dt\,.$ □

Case 2. Here $\mathbf{x}' \in K'$, where $K' = \{\mathbf{x}' \in \mathbb{R}^{N-1} \mid |\mathbf{y}|/2 \le |\mathbf{x}'| \le 2|\mathbf{y}|\}$. Then

$$\int_{K'} K_{\mathbf{zy}}(\mathbf{x}')d\mathbf{x}' \le A_1 \frac{\varepsilon(|\mathbf{y}|)}{|\mathbf{y}|^{N-1}} \int_{K'} \frac{y_N d\mathbf{x}'}{|\mathbf{x}'-\mathbf{y}|^N} \le A_2 \frac{\varepsilon(|\mathbf{y}|)}{|\mathbf{y}|^{N-1}} \int_0^{3|\mathbf{y}|} \frac{y_N r^{N-2}\, dr}{(r^2+y_N^2)^{N/2}} \le A \frac{\varepsilon(|\mathbf{y}|)}{|\mathbf{y}|^{N-1}}\,.$$

The penultimate integral is estimated in spherical coordinates of $\mathbb{R}^{N-1}_{\mathbf{x}'-\mathbf{y}'}$.

Case 3. Here $|\mathbf{x}'| > 2|\mathbf{y}'|$, so that

$$\int_{|\mathbf{x}'|>2|\mathbf{y}|} K_{\mathbf{zy}}(\mathbf{x}')d\mathbf{x}' \le A\,|\mathbf{y}| \int_{2|\mathbf{y}|}^{r_0/3} \frac{r\varepsilon(r)r^{N-2}\,dr}{r^{2N}} \le A \int_{|\mathbf{y}|}^{r_0} \frac{\varepsilon(r)\,dr}{r^N} = \frac{A}{|\mathbf{y}|^{N-1}}\lambda_2(|\mathbf{y}|)\,,$$

where

$$\lambda_2(t) = t^{N-1} \int_t^{r_0} \frac{\varepsilon(\tau)}{\tau^N} d\tau\,.$$

Lemma 3.11. *In the previous notation,*

$$\int_0^{r_0} \frac{\lambda_2(t)}{t} dt = \frac{1}{N-1} \int_0^{r_0} \frac{\varepsilon(t)}{t} dt\,.$$

Proof. As in the proof of Lemma 3.10 (take $r = r_0$). □

Therefore, we obtain:

$$|\nabla \Psi_{\mathbf{y}}(\mathbf{z})| \le A \frac{\varepsilon(|\mathbf{y}|) + \lambda_1(|\mathbf{y}|) + \lambda_2(|\mathbf{y}|)}{|\mathbf{y}|^{N-1}} = A \frac{\lambda(|\mathbf{y}|)}{|\mathbf{y}|^{N-1}}\,.$$

To finally estimate I_1, it remains to check the following inequality:

$$\int_{\Omega_5} |\nabla \Psi_{\mathbf{y}}(\mathbf{z})|\, d\mu_{\mathbf{y}} \le A_1 \int_{|\mathbf{y}|\le r_0} \frac{\lambda(|\mathbf{y}|)}{|\mathbf{y}|^{N-1}} d\mu_{\mathbf{y}} \le A_2 M \int_0^{r_0} \frac{\lambda(r)}{r}\, dr \le AM\,,$$

which follows from (3.9) and Lemmas 3.7, 3.10 and 3.11 (since, clearly, $\lambda(\rho)$, in place of $\varepsilon(\rho)$, also satisfies the conditions of Lemma 3.7).

Theorem 3.5 is proved. □

We terminate the proof of Theorem 3 following that one of Theorem 2. Let $f \in C^1(\overline{D}) \cap SH(D)$ and $m = \|f\|_{1,\overline{D}}$. For $p \in \{1,2,\dots\}$ we can find $g_p \in C^1(\overline{D_p}) \cap SH(D_p)$ harmonic on $D_p \setminus \overline{D}$ ($D_p = D_{\delta_p}$, $\delta_p \in (0,1)$ is small enough), $\|g_p\|_{1,\overline{D_p}} \le Am/2^p$ and $f = \sum_{p=1}^{+\infty} g_p|_{\overline{D}}$. The proof of this fact is almost the same as (for balls) in [2, Lemma 5.2] (plus iterations). It remains to appropriately extend each g_p (from $\overline{D}$). Put $\Omega_p = \mathbb{R}^N \setminus \overline{D}_p$ and $S_p = \partial D_p$. By Theorem 3.5 we find the subharmonic reflection h_p of g_p over S_p (that is, $h_p \in C^1(\overline{\Omega_p}) \cap SH(\Omega_p)$, $h_p = g_p$ on S_p and $\|h_p\|_{1,\overline{\Omega_p}} \le A\|g_p\|_{1,\overline{D_p}}$). It follows from the proof of Theorem 3.5 that $h_p \in H(D_{\delta'} \setminus \overline{D}_{\delta_p})$ for some $\delta' > \delta_p$. It suffices (taking (2.2) into account) to add appropriate tw_p and make a regularization without changing g_p on $\overline{D}$ (which can be done because g_p is harmonic in $D_p \setminus \overline{D}$).

□

4 Examples and background.

Example 4.1. *Let D be any bounded convex domain in $\mathbb{R}^N$ ($N \geq 2$) such that $S = \partial D$ contains the set $B'_\delta = \{\mathbf{x} = (\mathbf{x}', x_N) \in \mathbb{R}^N \,|\, x_N = 0, |\mathbf{x}'| < \delta\}$ for some $\delta > 0$. Then there exists $\psi \in C^1(\overline{D}) \cap SH(D)$ such that the solution Ψ of the Dirichlet problem in D with the boundary data $\psi|_S$ is not in $C^1(\overline{D})$.*

Proof. We can suppose that $\delta \in (0, 1/4)$ and

$$D \subset \Omega = \{\mathbf{x} \in \mathbb{R}^N \,|\, x_N > 0, |\mathbf{x}'| < 1/2\}.$$

Fix $p \in (0,1)$ and define

$$\psi(\mathbf{x}) = \psi(|\mathbf{x}'|) = \frac{|\mathbf{x}'|}{|\log|\mathbf{x}'||^p}, \quad |\mathbf{x}'| \in (0,1),$$

$\psi(\mathbf{0}) = 0$. It can be easily checked that $\psi \in C^1(\overline{\Omega}) \cap SH(\Omega)$. Let $\psi_0(\mathbf{x}') = \psi(\mathbf{x}')$ for $|\mathbf{x}'| < 1/2$ and $\psi_0(\mathbf{x}') = \psi(1/2)$, $|\mathbf{x}'| \geq 1/2$, and let Ψ_0 be the Dirichlet solution in $\mathbb{R}^N_+ = \{\mathbf{x} \in \mathbb{R}^N \,|\, x_N > 0\}$ with the boundary data ψ_0. By the Poisson formula in $\mathbb{R}^N_+$ one has for $x_N \in (0, 1/2)$:

$$\Psi_0(\mathbf{0}', x_N) = \frac{2}{\sigma_N} \int_{\mathbb{R}^{N-1}_{\mathbf{y}'}} \frac{x_N \psi_0(\mathbf{y}')}{|(\mathbf{y}', 0) - (\mathbf{0}', x_N)|^N} d\mathbf{y}' \geq \frac{2\sigma_{N-1}}{\sigma_N} \int_0^{1/2} \frac{x_N r r^{N-2} dr}{(r^2 + x_N^2)^{N/2} (\log \frac{1}{r})^p} \geq$$

$$\geq A(N)\, x_N \int_{x_N}^{1/2} \frac{dr}{r (\log \frac{1}{r})^p} = \frac{A(N)\, x_N}{1-p} \left(\left(\log \frac{1}{x_N} \right)^{1-p} - (\log 2)^{1-p} \right),$$

so that, clearly, $\partial \Psi_0 / \partial x_N|_{\mathbf{x}=\mathbf{0}} = +\infty$. On the other hand, we can find $\delta_0 \in (0, \delta)$ and $\lambda_0 > 0$ such that $B_0^+ = \{\mathbf{x} \in \mathbb{R}^N \,|\, x_N > 0, |\mathbf{x}'| < \delta_0\} \subset D$ and $\lambda_0 \Psi_0 \leq \Psi$ on ∂B_0^+.

Therefore,

$$\Psi(\mathbf{0}', x_N) \geq \lambda_0 A(N)\, x_n \left(\log \frac{1}{x_N} \right)^{1-p}, \quad x_N \in (0, \delta_0),$$

which ends the proof and shows (letting $p \to 0+$) that the estimate (3.3) is "almost" precise. It is also easily seen that the function $-\psi|_{\partial D} \in C^1(S)$ can not be extended to $\overline{D}$ as a function of the class $C^1(\overline{D}) \cap SH(D)$. □

In the next example we construct a C^1-smooth convex "almost" (L-D) domain D in $\mathbb{R}^2$ for which the C^1-harmonic reflection property (see Theorem 1) does not hold. This example shows that the (sufficient) (L-D) condition on D in Theorem 1 is "almost" sharp. An analogous example in $\mathbb{R}^N$, $N \geq 3$ can be then easily obtained.

Example 4.2. *Set $B_+ = \{\zeta \in \mathbb{C} | \, |\zeta| < 1/e, \operatorname{Re} \zeta > 0\}$ and $\Sigma' = \{\zeta \in \mathbb{C} | \operatorname{Re} \zeta = 0, |\zeta| < 1/e\} \subset \partial B_+$. The function $k(\zeta) = -\zeta / \log(\zeta)$ maps conformally B_+ onto some domain Ω_+ and k is homeomorphism $\overline{B_+}$ onto $\overline{\Omega_+}$ (we set $k(0) = 0$). Here $\log(\zeta)$ means the main holomorphic branch of logarithm in $\mathbb{C} \backslash (-\infty, 0]$, $\log(1) = 0$. One checks the conformality of k on B_+ applying the classical inverse principal of boundaries correspondence. It can* *be easily shown that $S' = k(\Sigma')$ is C^1-smooth curve, convex "to the right". Moreover, on S' we have (for some $A \in (0, +\infty)$)*

$$0 \le -x_1 \le A \frac{x_2}{|\log|x_2||}, \quad |x_2| \le \left(1 + \frac{\pi^2}{4}\right)^{-1} e^{-1}$$

Notice that the curve $-x_2 = |x_2|/|\log|x_2||^p$ with $p > 1$ is Lyapunov-Dini curve.

Then there exists bounded convex C^1-smooth domain $D \subset \{z = x_1 + ix_2 \in \mathbb{C}|\, x_1 < 0\}$ such that $S' \subset S = \partial D$ and $S \backslash \{0\}$ is C^∞-smooth. Consider $u_i(\mathbf{x}) = -x_1 \in H(D) \cap C^\infty(\overline{D})$. We claim that the corresponding u_o (see the notations in Theorem 1) satisfies

$$\left.\frac{\partial u_o}{\partial x_1}\right|_{\mathbf{0}} = \frac{\partial u_o}{\partial \mathbf{n}^o_{\mathbf{0}}} = +\infty. \tag{4.1}$$

In fact, take $h(\mathbf{x}) = \mathrm{Re}(k^{-1}(z))$ in $\overline{\Omega_+}$ ($\mathbf{x} = (x_1, x_2)$). One can find $\lambda \in (0, +\infty)$ such that $\lambda u_o \ge h$ on $\partial\Omega_+$, and so in Ω by the maximum principle. So that (4.1) follows from the equality $\partial h/\partial x_1|_{\mathbf{0}} = +\infty$. This example also shows that the Green function of the ("almost" (L-D)) domain Ω_+^* (obtained from Ω_+ by "smoothing" $\partial\Omega_+$ near it's "angle"-points) does not satisfy [7, Theorem 2.3] (see also Theorem 4.5 below).

Example 4.3. *For $p \in (0, +\infty)$ define a C^1-function*

$$f_p(t) = -\frac{|t|}{|\log|t||^p}, \quad t \in [-1/2, 1/2], \quad t \ne 0,$$

and $f_p(0) = 0$.

(1) For $p \in (0, 1]$ there do not exist $\delta > 0$ and a function F continuous and subharmonic on $B_\delta = \{\mathbf{x} \in \mathbb{R}^2|\, |\mathbf{x}| < \delta\}$, such that $F(x_1, 0) = f(x_1)$ for $|x_1| < \delta$.

(2) For each $p \in (1, +\infty)$ one can find $F \in C^1_{loc}(\mathbb{R}^2) \cap SH(\mathbb{R}^2)$ with $F(x_1, 0) = f(x_1)$ for all $x_1 \in [-1/2, 1/2]$ and $\|\nabla F\| < +\infty$.

Proof. Set $g_p(t) = f_p(t)$ for $|t| \le 1/2$ and let $g_p(t)$ be some negative bounded even C^2-function for $|t| > 0$. Let F_p^+ (respectively, F_p^-) be the Dirichlet solution in $\mathbb{R}^2_+$ (respectively, in $\mathbb{R}^N_- = \{\mathbf{x} \in \mathbb{R}^2 | x_2 < 0\}$) with the boundary data g_p. By the Poisson formula we have for all $\alpha \in (0, \pi)$ and $r \in (0, 1/2$:

$$F_p^+(r\cos\alpha, r\sin\alpha) = F_p^-(r\cos\alpha, -r\sin\alpha) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{r\sin\alpha\, g_p(t)dt}{(t - r\cos\alpha)^2 + r^2\sin^2\alpha} \ge$$

$$\ge \frac{r\sin\alpha}{2\pi}\int_0^{+\infty} \frac{g_p(t)dt}{t^2 + r^2} \le -\frac{r\sin\alpha}{4\pi}\int_r^{1/2} \frac{dt}{t|\log|t||^p} = -r\sin\alpha\, h_p(r),$$

where $h_p(r) \to +\infty$ as $r \to 0+$ wherever $p \in (0, 1]$.

Fix $p \in (0, 1]$ and suppose, by contradiction, that such F in (1) exists.

Put $M = \sup_{|\mathbf{x}|=\delta/2}(|F(\mathbf{x})| + |F_p(\mathbf{x})|) < +\infty$ and let u_+ (respectively, u_-) be the solution of the Dirichlet problem in $B^+_{\delta/2} = B_{\delta/2} \cap \mathbb{R}^2_+$ (respectively, $B^-_{\delta/2} = B_{\delta/2} \cap \mathbb{R}^2_-$) with the boundary data 0 on $\partial B^\pm_{\delta/2} \cap \{x_2 = 0\}$ and M on the rest of the boundary. By Theorem W1(2) there is $A \in (0, +\infty)$ such that

$$|u_+(x_1, x_2)| + |u_-(x_1, x_2)| \le Ax_2$$

for each $\mathbf{x} = (x_1, x_2)$ with $|\mathbf{x}| \leq \delta/4$ and $x_2 \geq 0$.

Therefore, for $\alpha \in (0, \pi)$ and $r \in (0, \delta/4)$ one has (as $F \leq F_p^{\pm} + u_{\pm}$ in $B_{\delta/2}^{\pm}$)

$$F(r\cos\alpha, \pm r\sin\alpha) \leq -r\sin\alpha\,(h_p(r) - A), \quad F(\mathbf{0}) = 0,$$

which clearly, contradicts to the subharmoniticity of $F(\mathbf{x})$ at $\mathbf{x} = \mathbf{0}$ by the mean value property.

Let now $p > 1$. It is not difficult to check (see also Theorem 4.9 below) that $F_p^- \in C^1(\mathbb{R}^2_-)$. It suffices to apply Theorem 2.

Recall, that in [3, Theorem 3.1] it was proved that the (L-D) condition is in some sense necessary in Theorem 3. For instance, in the scale of functions $\varepsilon_p(\cdot)$ (see Remark 3.2 above) we have the lack of the extension (in the sense of Theorem 3) for $p \in (0, 1]$. For $p \geq 1$ domains with the Dini-function $\varepsilon_p(\cdot)$ are (L-D) domains. The case $p > 2$ is covered by Theorem 3. The case $p \in (1, 2]$ is still unconsidered.

We also do not know if the (L-D) condition in Theorem 2 is precise. $\square$

In the rest of the paper we discuss the proofs of Theorems W1 and W2 following basically the ideas of the original proofs in [7]. In particular, we check that all the appearing constants depend only on N, $d = \operatorname{diam} D$ and $\varepsilon(\cdot)$. The last is important for the proofs of our main results. Also, for the interested reader (especially beginner) it would be very useful to check all the details of the proofs, which look rather useful in applications. First we present the detailed proof of the main working result of [7] – [7, Theorem 2.2].

Theorem 4.4. *Let $\varepsilon_1(t)$ be a Dini-type function with $\varepsilon_1(1) \leq 1/2$. Define $\varphi_1(r) = \int_0^r \varepsilon_1(t)dt$, so that $t\varepsilon_1(t)/2 \leq \varphi_1(t) \leq t\varepsilon_1(t)$ and $-\varphi_1$ is concave. Put*

$$T_1 = \{\mathbf{x} \in \mathbb{R}^N \,|\, |\mathbf{x}'| \leq 1\,,\ -\varphi_1(|\mathbf{x}'|) \leq x_N \leq 1\}\,,$$

$\Sigma_1 = \{\mathbf{x} \in \partial T_1 \mid x_N = -\varphi_1(|\mathbf{x}'|)\}$. Let $u_1 \in H(T_1^\circ)$ have the boundary values $u_1|_{\Sigma_1} = 0$, $u_1|_{\partial T_1 \setminus \Sigma_1} = 1$ (with $\|u_1\|_{T_1} \leq 1$). Then there exists a constant $A_1 = A_1(N, \varepsilon(\cdot)) \in (0, +\infty)$ such that

$$|u_1(\mathbf{0}', x_N)| \leq A_1 x_N\,, \quad x_N \in (0, 1].$$

Proof. Set $T_2 = \{\mathbf{z}/2 |\, \mathbf{z} \in T_1\}$, $T_2(\mathbf{x}') = \{\mathbf{z} + (\mathbf{x}', -2\varphi_1(\mathbf{x}'))|\, \mathbf{z} \in T_2\}$, $\Sigma_2(\mathbf{x}') = \{\mathbf{z}/2 + (\mathbf{x}', -2\varphi_1(\mathbf{x}'))|\mathbf{z} \in \Sigma_1\} \subset \partial T_2(\mathbf{x}')$. We *claim* that for $|\mathbf{x}'| < 1/2$ one has $\Sigma_2(\mathbf{x}') \cap \Sigma_1 = \emptyset$ (that is, Σ_2 is "below" Σ_1). In fact, it is enough to check that

$$\chi_t(s) = 2\varphi_1(t) + \frac{1}{2}\varphi_1(2s) - \varphi_1(t + s) \geq 0$$

for all $t \geq 0$ and $s \geq 0$. It is easily seen that the function $\chi_t(s)$ (t fixed) has it's minimum at $s = t$, so that it is enough to see that $\lambda(t) = \chi_t(t) \geq 0$, $t \geq 0$. But $\lambda(0) = 0$ and $\lambda'(t) = 2\varepsilon_1(t) - \varepsilon_1(2t) \geq 0$, which ends the proof of the claim.

Let $u_2' \in H((T_2(\mathbf{x}'))^\circ)$ have the boundary values $u_2' = 0$ on $\Sigma_2(\mathbf{x}')$ and $u_2' = 1$ on $\partial T_2(\mathbf{x}') \setminus \Sigma_2(\mathbf{x}')$ (with $\|u_2'\|_{T_2(\mathbf{x}')} \leq 1$). Since, clearly, $(\mathbf{x}', 0) \in T_2(\mathbf{x}')$, by the maximum principle for u_1 and u_2' in $T_1 \cap T_2(\mathbf{x}')$ we have

$$u_1(\mathbf{x}', 0) \leq u_2'(\mathbf{x}', 0) = u_1(\mathbf{0}', 4\varphi_1(|\mathbf{x}'|))$$

for all $\mathbf{x}'$ with $|\mathbf{x}'| \leq 1/2$.

Let u_ψ be the solution of the Dirichlet problem in $\mathbb{R}^N_+$ with the boundary data $\psi(\mathbf{x}') = u_1(\mathbf{0}, 4\varphi_1(|\mathbf{x}'|))$, $|\mathbf{x}'| \le 1/2$, $\psi(\mathbf{x}') = 1$ for $|\mathbf{x}'| \ge 1/2$. By the Poisson formula,

$$u_\psi(\mathbf{x}', x_N) = \frac{2}{\sigma_N} \int_{\mathbb{R}^{N-1}} \frac{x_N}{((\mathbf{y}' - \mathbf{x}')^2 + x_N^2)^{N/2}} \psi(\mathbf{y}') d\mathbf{y}'.$$

It is clear that $u_\psi(\mathbf{x}', x_N) \ge 1/2$ for $|\mathbf{x}'| \ge 1/2$. For $|\mathbf{x}'| \le 1/2$ we have

$$u_\psi(\mathbf{x}', 1) \ge \frac{2}{\sigma_N} \int_1^{+\infty} \frac{\sigma_{N-1} r^{N-2} dr}{(r^2+1)^{N/2}} \ge \frac{1}{\sigma_N} \int_0^{+\infty} \frac{\sigma_{N-1} r^{N-2} dr}{(r^2+1)^{N/2}} = \frac{1}{2}$$

(one estimates the corresponding integral $\int_0^1$ changing variables $t = 1/r$). We then have $u(\mathbf{x}', x_N) \le 2u_\psi(\mathbf{x}', x_N)$ and so, for $x_N \in (0,1)$,

$$u(\mathbf{0}', x_N) \le \frac{4\sigma_{N-1}}{\sigma_N} \left(\int_0^\delta \frac{x_N u(\mathbf{0}', 4\varphi_1(r)) r^{N-2} dr}{(r^2 + x_N^2)^{N/2}} + \int_\delta^{+\infty} \frac{x_N r^{N-2} dr}{(r^2 + x_N^2)^{N/2}} \right),$$

where $\delta \in (0, 1/2]$ will be choosen later.

Suppose that

$$A_1 = \sup_{x_N \in (0,1]} \frac{u_1(\mathbf{0}', x_N)}{x_N} < +\infty$$

and let $t \in (0,1]$ be such that $u_1(\mathbf{0}', t)/t \ge A_1/2$.

Then we have for $A_2 = 32\sigma_{N-1}/\sigma_N$:

$$A_1 t \le 2u_1(\mathbf{0}', t) \le A_2 \left(\int_0^\delta \frac{t A_1 \varphi_1(r) r^{N-2} dr}{(r^2 + t^2)^{N/2}} + \frac{1}{4} \int_\delta^{+\infty} \frac{t r^{N-2} dr}{(r^2+t^2)^{N/2}} \right).$$

Therefore,

$$A_1 \le A_2 \left(A_1 \int_0^\delta \frac{\varepsilon_1(r)}{r} dr + \frac{1}{4\delta} \right).$$

Take the maximal $\delta = \delta_1 \in (0, 1/2]$ such that

$$A_2 \int_0^{\delta_1} \frac{\varepsilon_1(r)}{r} dr \le 1/2,$$

and we find $A_1 \le A_2/(2\delta_1)$.

To finish the proof of Theorem 4.4 we need to reduce the general situation to the case when we know that $A_1 < +\infty$. To this end, for each fixed $\theta \in (0, 1/2)$ define $\varepsilon_\theta(t) = \varepsilon_1(\theta) t/\theta_1$ for $t \in (0, \theta_1)$, $\varepsilon_\theta(t) = \varepsilon_1(\theta)$ for $t \in [\theta_1, \theta]$, and $\varepsilon_\theta(t) = \varepsilon_1(t)$ for $t \ge \theta$, where $\theta_1 \in (0, \theta)$ is chosen such that

$$\int_0^\theta \varepsilon_\theta(t) dt = \int_0^\theta \varepsilon_1(t) dt,$$

which gives $\theta_1 = 2\left(\theta - \int_0^\theta \frac{\varepsilon_1(t)}{\varepsilon_1(\theta)} dt\right)$. Recall that $\varepsilon_1(kt) \ge k\varepsilon_1(t)$ for $k \in (0,1]$.

The main reason to consider the function $\varepsilon_\theta(\cdot)$ is the following. Each $\varepsilon_\theta(\cdot)$ is a Dini-type function such that $\varphi_\theta(t) = \int_0^t \varepsilon_\theta(\tau) d\tau$ is equal to $\varphi_1(t)$ for $t \ge \theta$ and $\varphi_\theta(t) \le \varphi_1(t)$ for $t \in [0, \theta]$. Moreover:

$$\int_0^r \frac{\varepsilon_\theta(t)}{t} dt \le \int_0^r \frac{\varepsilon_1(t)}{t} dt, \quad \forall r > 0.$$

In fact, integration by parts gives:

$$\int_0^r \frac{\varepsilon_\theta(t)}{t}dt = \int_0^r \frac{d\varphi_\theta(t)}{t} = \frac{\varphi_\theta(r)}{r} + \int_0^r \frac{\varphi_\theta(t)}{t^2}dt \leq$$

$$\leq \frac{\varphi_1(r)}{r} + \int_0^r \frac{\varphi_1(t)}{t^2}dt = \int_0^r \frac{\varepsilon_1(t)}{t}dt. \tag{4.2}$$

Let T_θ, u_θ, A_θ be defined for $\varepsilon_\theta(\cdot)$ as T_1, u_1, A_1 for $\varepsilon_1(\cdot)$ in Theorem 4.4 above. We *claim* that A_θ are finite for each $\theta \in (0, 1/2)$. In fact, given θ one can find $\delta_\theta \in (0, 1/4)$ such that $B((\mathbf{0}', -\delta_\theta), \delta_\theta) \subset \mathbb{R}^N \backslash T_\theta$ and so the claim follows from the maximum principle (in the domain $T_\theta \subset T_1$) for u_θ and $v_\theta(\mathbf{x}) = l_\theta(\delta_\theta^{2-N} - |\mathbf{x} + (\mathbf{0}', \delta_\theta)|^{2-N})$ with an appropriate $l_\theta > 0$. It remains to note that $u_\theta \to u_1$ as $\theta \to 0$, and apply (4.2) to see that A_θ depend only on $\varepsilon_1(\cdot)$. $\square$

Theorem 4.5. *Let D be a (L-D) domain in $\mathbb{R}^N$ with the Dini function $\varepsilon(\cdot)$ and $d = \operatorname{diam} D$. Let $G(\mathbf{x}, \mathbf{y})$ be the Green function for D. Then there is $A = A(N, d, \varepsilon) \in (0, +\infty)$ such that for each $\mathbf{x}$ and $\mathbf{y}$ in D one has*

(1) $|G(\mathbf{x}, \mathbf{y})| \leq A\,\rho(\mathbf{x})|\mathbf{x} - \mathbf{y}|^{1-N}$, *here* $N \geq 3$;

(2) $|\partial G(\mathbf{x}, \mathbf{y})/\partial x_n| \leq A\,|\mathbf{x} - \mathbf{y}|^{1-N}$,

(3) $|\partial G(\mathbf{x}, \mathbf{y})/\partial y_n| \leq A\,\rho(\mathbf{x})|\mathbf{x} - \mathbf{y}|^{-N}$,

(4) $|\partial^2 G(\mathbf{x}, \mathbf{y})/(\partial x_m \partial y_n)| \leq A\,|\mathbf{x} - \mathbf{y}|^{-N}$

for all m and n in $\{1, \ldots, N\}$.

The same estimates hold for the Green functions of (and in) bounded components of D_o. For the unbounded component D_* of D_o, the estimates (1)-(4) hold also for the Green function G_* of D_* (in place of G) for all $\mathbf{y} \in B(\mathbf{0}, 2d) \cap D_*$ (presumably, $\mathbf{0} \in \overline{D}$) and all $\mathbf{x} \in D_*$.

Proof. We consider only the case $N \geq 3$. The proof of (2)-(4) for $N = 2$ can be obtained using conformal mappings [14, Theorem 3.5].

(1). Let, as before, $r_0 \in (0, 1]$ be the maximal number with the property $\varepsilon(r_0) \leq 1/8$. Fix $\mathbf{y} \in D$. We can suppose that $\mathbf{x} = (\mathbf{0}', x_N) \in D$, $x_N > 0$, is such that $\rho(\mathbf{x}) = |\mathbf{x}|$, and $\mathbf{0} \in \partial D$ is the closest to $\mathbf{x}$ on ∂D. It is trivial that

$$0 \leq -G(\mathbf{x}, \mathbf{y}) \leq \frac{A_2}{|\mathbf{x} - \mathbf{y}|^{N-2}},$$

where $A_2 = A_2(N)$. If $\rho(\mathbf{x}) \geq r_0$ then

$$|G(\mathbf{x}, \mathbf{y})| \leq \frac{A_2|\mathbf{x} - \mathbf{y}|}{|\mathbf{x} - \mathbf{y}|^{N-1}} \leq \frac{A_2 d}{|\mathbf{x} - \mathbf{y}|^{N-1}} \leq \frac{A_3\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{y}|^{N-1}},$$

where $A_3 = A_2 d/r_0$. Also, if $|\mathbf{x} - \mathbf{y}| \leq 8\rho(\mathbf{x})$ then

$$|G(\mathbf{x}, \mathbf{y})| \leq \frac{A_2|\mathbf{x} - \mathbf{y}|}{|\mathbf{x} - \mathbf{y}|^{N-1}} \leq \frac{8A_2\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{y}|^{N-1}}.$$

It remains to consider the case $\{\rho(\mathbf{x}) < r_0, \rho(\mathbf{x}) < |\mathbf{x} - \mathbf{y}|/8\}$. Put $r = \min\{r_0, |\mathbf{x} - \mathbf{y}|/8\}$, so that $0 \leq x_N = \rho(\mathbf{x}) < r \leq r_0$. Let, as before, $Q_r = \{\mathbf{z} \in \mathbb{R}^N,\ |\mathbf{z}'| \leq r, |z_N| \leq r\}$. We claim that $\mathrm{dist}(\mathbf{y}, \partial Q_r) \geq |\mathbf{x}-\mathbf{y}|/2$, which follows easily considering the cases $r < r_0$ and $r = r_0$. Therefore,

$$|G(\mathbf{x},\mathbf{y})| \leq \frac{2^N A_2}{|\mathbf{x}-\mathbf{y}|^{N-2}} = M_G, \quad \forall \mathbf{x} \in \partial(Q_r \cap D).$$

Define $\varepsilon_1(t) = 4\varepsilon(rt)$, so that $\varepsilon_1(t)$ satisfies the conditions of Theorem 4.4, and, since $\varepsilon_1(t) \leq 4\varepsilon(t)$, we have

$$\int_0^t \frac{\varepsilon_1(\tau)}{\tau} d\tau \leq 4 \int_0^t \frac{\varepsilon(\tau)}{\tau} d\tau, \quad t > 0. \tag{4.3}$$

Moreover, if (as before) $\varphi_\varepsilon(t) = \int_0^t \varepsilon(\tau) d\tau$, we have

$$\varphi_1(t) = 4 \int_0^t \varepsilon(r\tau) d\tau = \frac{4}{r} \varphi_\varepsilon(rt),$$

so that the set T_1 (see the proof of Theorem 4.4) is similar to the set

$$T_r = \{\mathbf{z} \in Q_r | -4\varphi_\varepsilon(|\mathbf{z}'|) \leq z_N \leq r\} \supset (Q_r \cap \overline{D})$$

with coefficient $1/r$. By the maximum principle (in $Q_r \cap \overline{D}$) for the functions $-G(\mathbf{x},\mathbf{y})$ and $M_G u_1(\mathbf{x}/r)$, using Theorem 4.4 and (4.3), we get

$$|G(\mathbf{x},\mathbf{y})| \leq M_G u_1 \left(\frac{x_N}{r}\right) \leq A_4 \frac{x_N}{r} \frac{1}{|\mathbf{x}-\mathbf{y}|^{N-2}}$$

with $A_4 = A_4(N, \varepsilon(\cdot))$. Finally, if $r = |\mathbf{x}-\mathbf{y}|/8$, $(r \leq r_0)$ then

$$G(\mathbf{x},\mathbf{y}) \leq \frac{8A_4 \rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{N-1}}.$$

If $|\mathbf{x}-\mathbf{y}|/8 > r_0$ (that is, $r = r_0$), we have

$$|G(\mathbf{x},\mathbf{y})| \leq \frac{A_4 \rho(\mathbf{x})}{r_0 |\mathbf{x}-\mathbf{y}|^{N-2}} \leq \frac{A_4 \rho(\mathbf{x}) d}{r_0 |\mathbf{x}-\mathbf{y}|^{N-1}} \leq \frac{A \rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{N-1}}$$

with $A = A_4 d/r_0$. So, finally, $A = A(N, \varepsilon, d)$.

(2). Let $\rho(\mathbf{x}) \leq |\mathbf{x}-\mathbf{y}|$. Take a ball $B_{\mathbf{x}} = B(\mathbf{x}, \rho)$ with $\rho = \rho(\mathbf{x})/2$, and represent $G(\mathbf{z},\mathbf{y})$ at $B_{\mathbf{x}}$ by the Poisson integral:

$$G(\mathbf{z},\mathbf{y}) = \frac{1}{\sigma_N} \int_{\partial B_{\mathbf{x}}} \frac{\rho^2 - |\mathbf{z}-\mathbf{x}|^2}{\rho |\mathbf{z}-\boldsymbol{\zeta}|^N} G(\boldsymbol{\zeta},\mathbf{y}) d\sigma_{\boldsymbol{\zeta}}.$$

After taking $\partial/\partial z_n|_{\mathbf{z}=\mathbf{x}}$ under the integral, it suffices to use (1) to have an appropriately estimate of $G(\boldsymbol{\zeta},\mathbf{y})$ for $\boldsymbol{\zeta} \in \partial B_{\mathbf{x}}$. If $\rho(\mathbf{x}) > |\mathbf{x}-\mathbf{y}|$, take $B = B(\mathbf{x}, |\mathbf{x}-\mathbf{y}|/2)$ and do the same in B using the estimate $G(\mathbf{z},\mathbf{y}) \leq 2^N A_2 |\mathbf{z}-\mathbf{y}|^{2-N}$, $\mathbf{z} \in \partial B$.

Lemma 4.6. *Fixed* $\mathbf{y} \in D$, *one has* $\frac{\partial}{\partial y_n} G(\mathbf{x},\mathbf{y}) \to 0$ *uniformly as* $\mathbf{x} \to \partial D$.

Proof. Fix $r > 0$ small enough, so that for $|\mathbf{x}-\mathbf{y}| = r$ we have $G(\mathbf{x},\mathbf{y}) \geq |\Phi(\mathbf{x}-\mathbf{y})/2| \geq A_5|\mathbf{x}-\mathbf{y}|^{2-N}$, $A_5 = A_5(N)$. If $\mathbf{z}$ is a vector with $|\mathbf{z}| \ll r$, we have, by (2), for $|\mathbf{x}-\mathbf{y}| = r$:

$$\frac{|G(\mathbf{x},\mathbf{y}+\mathbf{z}) - G(\mathbf{x},\mathbf{y})|}{|\mathbf{z}|} \leq A|\mathbf{x}-\mathbf{y}|^{1-N} \leq A_6 r^{-1}|G(\mathbf{x},\mathbf{y})|. \tag{4.4}$$

Since $G(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}+\mathbf{z}) = 0$ for $\mathbf{x} \in \partial D$, the inequality (4.4) holds for *all* $\mathbf{x} \in D\backslash\overline{B(\mathbf{y},r)} = D_r$ (notice that $|G(\mathbf{x},\mathbf{y})| = -G(\mathbf{x},\mathbf{y})$ is harmonic in D_r). Fixing $\mathbf{x} \in D_r$ and letting $|\mathbf{z}| \to 0$ we get

$$|\nabla_{\mathbf{y}} G(\mathbf{x},\mathbf{y})| \leq A_6 r^{-1}|G(\mathbf{x},\mathbf{y})| \leq A_7 r^{-1}\frac{\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{N-1}},$$

$\mathbf{x} \in D_r$, which proves the Lemma. □

Finitely, using (2) with $\partial G/\partial y_n$ instead of $\partial G/\partial x_n$ and applying Lemma 4.6, we obtain (3) the same way as in the proof of (1). Also (4) follows from (3) as (2) follows from (1).

The proof of the last part of Theorem 4.5 (that concerns G_*) goes the same way as before (whenever $|\mathbf{x}| < 4d$ and $|\mathbf{y}| < 2d$). In case $|\mathbf{x}| \geq 4d$ and $|\mathbf{y}| < 2d$ use maximum principle. □

Theorem 4.7. *In conditions of Theorem 4.5 we have $G(\mathbf{x},\mathbf{y}) \in C^1(\overline{D}\backslash\{\mathbf{y}\})$, $\mathbf{y}$ fixed.*

Proof. Fix $\mathbf{y} \in D$. By definition, $G(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) - v_{\mathbf{y}}(\mathbf{x})$, where $v(\mathbf{x}) = v_{\mathbf{y}}(\mathbf{x}) \in H(D) \cap C(\overline{D})$ and $v|_S = \Phi(\mathbf{x}-\mathbf{y})|_S \in C^2(S)$. We have to prove that $v \in C^1(\overline{D})$. By (2) of Theorem 4.5 and the maximum principle the function $|\nabla v|$ is bounded in D. It is enough to prove that there exists a Dini-type function ε_* (independent of $\mathbf{x}$) such that for all n and m in $\{1,\ldots,N\}$

$$\left|\frac{\partial^2}{\partial x_n \partial x_m} v(\mathbf{x})\right| \leq \frac{\varepsilon_*(\rho(\mathbf{x}))}{\rho(\mathbf{x})}, \quad \mathbf{x} \in D. \tag{4.5}$$

In fact, (4.5) gives that ∇v is uniformly continuous in D and then, clearly, $v \in C^1(\overline{D})$.

To prove (4.5) we can assume that $\mathbf{x} = (\mathbf{0}', x_N)$, $x_N \in (0, r_0/2)$, $\rho(\mathbf{x}) = |\mathbf{x}|$, $\mathbf{0} \in S$ is the closest to $\mathbf{x}$ on S. We use the notation from the proof of Theorem 3.1 (r_0, $Q_{\mathbf{0}}$, D', S') and denote by $G_+(\boldsymbol{\zeta},\boldsymbol{\eta})$ the Green function of $\mathbb{R}^N_+$.

Let $v|_S = \psi|_S$, where $\psi \in C^2_0(\mathbb{R}^N)$. Take $u(\mathbf{z}) = v(\mathbf{z}) - \psi(\mathbf{0}) - (\nabla\psi(\mathbf{0})\,,\mathbf{z})$, so that $u \in C(\overline{D}) \cap H(D)$, $\|\nabla u\|_D < +\infty$, $u|_S = \psi_0$, where $\psi_0(\mathbf{z}) = \psi(\mathbf{z}) - \psi(\mathbf{0}) - (\nabla\psi(\mathbf{0})\,,\mathbf{z})$ and so

$$|\psi_0(\mathbf{z})| \leq \omega_1(|\mathbf{z}|)|\mathbf{z}| \tag{4.6}$$

with some Dini-type function $\omega_1(t)$, independent of $\mathbf{x}$. It suffices to prove (4.5) for u in place of v. Using the Gauss-Ostrogradski (or just the second Green's) formula (in $D' \subset \mathbb{R}^N_+$) we get

$$\int_{\partial D'} \left(u(\boldsymbol{\eta})\frac{\partial G_+(\mathbf{z},\boldsymbol{\eta})}{\partial \mathbf{n}^o_{\boldsymbol{\eta}}} - \frac{\partial u(\boldsymbol{\eta})}{\partial \mathbf{n}^o_{\boldsymbol{\eta}}} G_+(\mathbf{z},\boldsymbol{\eta}) \right) d\sigma_{\boldsymbol{\eta}} =$$

$$= -\int_{D'} (u(\boldsymbol{\zeta})\Delta_{\boldsymbol{\zeta}} G_+(\mathbf{z},\boldsymbol{\zeta}) - \Delta u(\boldsymbol{\zeta}) G_+(\mathbf{z},\boldsymbol{\zeta}))d\boldsymbol{\zeta} = -u(\mathbf{z}),$$

where $\mathbf{n}^o_{\boldsymbol{\eta}}$ is the inward unit normal to $\partial D'$ at $\boldsymbol{\eta}$.

Since $\|\nabla u\|_D = M_u < +\infty$, we have for $\boldsymbol{\eta} = (\boldsymbol{\eta}', 4\varphi_\varepsilon(|\boldsymbol{\eta}'|))$, $|\boldsymbol{\eta}'| < r_0/3$ (that is, $\boldsymbol{\eta} \in S' \subset \partial D'$),

$$|u(\boldsymbol{\eta})| \le |u(\boldsymbol{\eta}) - u(\boldsymbol{\eta}', \varphi(\boldsymbol{\eta}'))| + |u(\boldsymbol{\eta}', \varphi(\boldsymbol{\eta}')| \le$$
$$\le M_u 8\varphi_\varepsilon(|\boldsymbol{\eta}'|) + \omega_1(2|\boldsymbol{\eta}'|)|\boldsymbol{\eta}'| \le \omega_2(|\boldsymbol{\eta}'|)|\boldsymbol{\eta}'|\,, \tag{4.7}$$

where $\omega_2(\cdot)$ is a Dini-type function and $(\boldsymbol{\eta}', \varphi(\boldsymbol{\eta}')) \subset S$. We can write

$$\left.\frac{\partial^2 u}{\partial z_n \partial z_m}\right|_{\mathbf{z}=\mathbf{x}} = \int_{\partial D'} \left[\frac{\partial u(\boldsymbol{\eta})}{\partial \mathbf{n}^o_{\boldsymbol{\eta}}} \left.\frac{\partial^2 G_+(\mathbf{z},\boldsymbol{\eta})}{\partial z_n \partial z_m}\right|_{\mathbf{z}=\mathbf{x}} - u(\boldsymbol{\eta}) \frac{\partial^2}{\partial z_n \partial z_m} \left.\left(\nabla_{\boldsymbol{\eta}} G(\mathbf{z},\boldsymbol{\eta})\,,\, \mathbf{n}^o_{\boldsymbol{\eta}}\right)\right|_{\mathbf{z}=\mathbf{x}} \right] d\sigma_{\boldsymbol{\eta}} =$$
$$= \int_{\partial D'} (H_1(\mathbf{x},\boldsymbol{\eta}) - H_2(\mathbf{x},\boldsymbol{\eta})) d\sigma_{\boldsymbol{\eta}}\,.$$

We estimate the last integral using the following elementary inequalities for the Green function G_+:

$$\left| \left.\frac{\partial^2 G_+(\mathbf{z},\boldsymbol{\eta})}{\partial z_n \partial z_m}\right|_{\mathbf{z}=\mathbf{x}} \right| \le \frac{A\boldsymbol{\eta}_N}{|\mathbf{x}-\boldsymbol{\eta}|^{N+1}}\,,$$
$$\left| \left.\frac{\partial^3 G(\mathbf{z},\boldsymbol{\zeta})}{\partial x_n \partial x_m \partial \zeta_l}\right|_{\mathbf{z}=\mathbf{x},\boldsymbol{\zeta}=\boldsymbol{\eta}} \right| \le \frac{A}{|\mathbf{x}-\boldsymbol{\eta}|^{N+1}}\,,$$

where $A = A(N)$. Then

$$\int_{S'} |H_1(\mathbf{x},\boldsymbol{\eta})|\, d\sigma_{\boldsymbol{\eta}} \le M_u \int_0^{r_0} \frac{A\varepsilon(r) r\, r^{N-2} dr}{(r^2 + x_N^2)^{\frac{N+1}{2}}} \le$$
$$\le AM_u\varepsilon(x_N) \int_0^{x_N} \frac{r^{N-1}\,dr}{x_N^{N+1}} + AM_u \int_{x_N}^{r_0} \frac{\varepsilon(r)\,dr}{r^2} = AM_u \left(\frac{\varepsilon(x_N)}{Nx_N} + \frac{\lambda_2(x_N)}{x_N} \right),$$

where $\lambda_2(t) = t\int_t^{r_0} \frac{\varepsilon(r)}{r^2} dr$ is Dini-type function by Lemma 3.11. Also, by (4.7),

$$\int_{S'} |H_2(\mathbf{x},\boldsymbol{\eta})|\, d\sigma_{\boldsymbol{\eta}} \le \int_0^{r_0} \frac{A}{(r^2 + x_N^2)^{\frac{N+1}{2}}}\, \omega_2(r) r\, r^{N-2} dr$$

can be estimated similarly. Finally, the analogous integrals over $\partial D' \backslash S'$ are estimated trivially. □

Theorem W2 now is also completely proved.

Theorem 4.8. *In conditions of Theorem 4.5 let $\psi_0 \in C^1(S)$ be a C^1-Dini function on S, which means that there is $\psi \in C^1_0(\mathbb{R}^N)$ with $\psi|_S = \psi_0$ and*

$$|\nabla\psi(\mathbf{x}) - \nabla\psi(\mathbf{y})| \le \omega(|\mathbf{x}-\mathbf{y}|)\,, \quad \forall\,\mathbf{x}\,, \forall\mathbf{y}\, \in \mathbb{R}^N\,, \tag{4.8}$$

where function $\omega(\cdot)$ is a Dini-type function. Let u_i be the Dirichlet solution in D with the boundary data ψ_0. Then $u_i \in C^1(\overline{D})$ and

$$\|u_i\|_{1,\overline{D}} \le A \left(\|\nabla\psi\|_{\partial D} + \int_0^{r_0} \frac{\omega(r)}{r} dr + \omega(d) \right). \tag{4.9}$$

For instance, if $\psi_0 \in C^2(S)$ then

$$\|u_i\|_{1,\overline{D}} \le A\|\psi_0\|_{2,S}. \tag{4.10}$$

Proof. Let us first estimate $\|\nabla u_i\|_{\overline{D}}$. To this end it suffices to estimate $|u_i(\mathbf{0}', x_N) - u_i(\mathbf{0})|/x_N$, where $\mathbf{0} \in S$, $x_N \in (0, r_0/2)$ and $\mathbf{0}$ is the closest to $\mathbf{x} = (\mathbf{0}', x_N) \in D$ on S. In fact, if $\mathbf{z} \neq 0$ is (fixed) small enough vector, then the function $(u_i(\mathbf{x}+\mathbf{z}) - u_i(\mathbf{x}))/|\mathbf{z}|$ is harmonic in $D \cap \{\mathbf{x} - \mathbf{z} |\, \mathbf{x} \in D\}$, and so attains it's extremums when $\mathbf{x} \in S$ or $\mathbf{x} + \mathbf{z} \in S$. Therefore, it is enough to take $\mathbf{x} + \mathbf{z} = \mathbf{0} \in S$ and suppose that $\mathbf{0}$ is the closest to $\mathbf{x} = (\mathbf{0}', x_N)$ on S (the last uses also the fact that $u_i \in C^1(S)$). Now, by Theorem 4.7, $G(\mathbf{x}, \mathbf{y}) \in C^1(\overline{D}\backslash\{\mathbf{y}\})$, so we already have a right to use formula (3.6), which gives

$$u_i(\mathbf{x}) - u_i(\mathbf{0}) - (\nabla\psi(\mathbf{0}), \mathbf{x}) = -\int_S \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}^i_{\mathbf{y}}} (\psi(\mathbf{y}) - \psi(\mathbf{0}) - (\nabla\psi(\mathbf{0}), \mathbf{y}))\, d\sigma_{\mathbf{y}}.$$

By (4.8) we have $|\psi(\mathbf{y}) - \psi(\mathbf{0}) - (\nabla\psi(\mathbf{0}), \mathbf{y})| \leq \omega(|\mathbf{y}|)|\mathbf{y}|$, and so, by Theorem 4.5 (3),

$$\frac{|u_i(\mathbf{x}) - u_i(\mathbf{0})|}{x_N} \leq |\nabla\psi(\mathbf{0})| + \frac{1}{x_N} \int_S \frac{A_1 x_N \omega(|\mathbf{y}|)|\mathbf{y}| d\sigma_{\mathbf{y}}}{|\mathbf{x} - \mathbf{y}|^N} \leq$$

$$\leq |\nabla\psi(\mathbf{0})| + A_2 \int_0^{r_0} \frac{\omega(r)}{r} dr + A_2 \omega(d) \int_{S^*_{r_0}} \frac{d\sigma_{\mathbf{y}}}{|\mathbf{y}|^{N-1}}\,,$$

where $S^*_{r_0} = \{\mathbf{y} \in S | |\mathbf{y}| \geq r_0\}$.

We claim that

$$\int_{S^*_{r_0}} \frac{d\sigma(\mathbf{y})}{|\mathbf{y}|^{N-1}} \leq A_N \frac{d}{r_0}\,. \tag{4.11}$$

To check this, consider the system of equal disjoint cubes $\{K_j\}_{j\in\mathbb{Z}^N}$ (with the side length $l = r_0/\sqrt{N}$) covering $\mathbb{R}^N$, that is

$$K_j = \{\mathbf{z} \in \mathbb{R}^N |\, j_n l \leq z_n < (j_n + 1)l\ ,\ \ n \in \{1, \dots, N\}\},$$

$j = (j_1, \dots, j_N) \in \mathbb{Z}^N$. If $K_j \cap S \ni \mathbf{a} \neq \emptyset$ then $K_j \subset Q_{\mathbf{a}}$ and so $\sigma(S \cap K_j) \leq \sigma(S \cap Q_{\mathbf{a}}) \leq A_N r_0^{N-1}$. For $m = 1, 2, \dots$ let N_m be the number of cubes Q_j that intersect $B(\mathbf{0}, (m+1)r_0)\backslash B(\mathbf{0}, m r_0)$, so that, clearly, $N_m \leq A_N m^{N-1}$. Therefore,

$$\int_{S^*_{r_0}} \frac{d\sigma_{\mathbf{y}}}{|\mathbf{y}|^{N-1}} \leq \sum_{m=1}^{d/r_0} \frac{A_N r_0^{N-1} m^{N-1}}{(r_0 m)^{N-1}} \leq A_N \frac{d}{r_0}\,. \tag{4.12}$$

By this we proved that $\|\nabla u_i\|_D$ is bounded and, as soon as we prove that $u_i \in C^1(\overline{D})$, we also immediately obtain (4.9). To check that $u_i \in C^1(\overline{D})$ it suffices to repeat the second part of the proof of Theorem 4.7, where we used the property (4.6), which corresponds to (4.8).

Finally, if $\psi_0 \in C^2(S)$ we can find $\psi \in C^2_0(\mathbb{R}^N)$ with $\|\psi\|_2 \leq 2\|\psi_0\|_{2,S} = M$, so that for the corresponding $\omega(\cdot)$ we have

$$\omega(t) \leq 2Mt \quad \text{ and } \quad \omega(d) \leq 2M\,.$$

Then (4.9) gives (4.10). □

Theorem 4.9. *Let D be a (L-D) domain in $\mathbb{R}^N$ with the Dini-type function $\varepsilon(\cdot)$ and $d = \operatorname{diam} D$. Then there is a function $C(t) = C(N, d, \varepsilon(\cdot), t) > 0$ on $(0, +\infty)_t$ such that for each $\mathbf{y} \in D$ we have*

$$-\frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}^i_{\mathbf{x}}} \geq C(\rho(\mathbf{y})), \quad \forall \mathbf{x} \in \partial D\,, \tag{4.13}$$

where G is the Green function for D.

Proof. Let $\mathbf{0} \in S = \partial D$, $\{0, \ldots, 0, 1\} = \mathbf{n}^i_{\mathbf{0}}$ (for D), and r_0, D', S' are defined in the proof of Theorem 3.1. Let G_+ and G' be the Green functions for $\mathbb{R}^N_+$ and D' respectively. Take $\mathbf{y} = (\mathbf{y}', y_N)$ and $\mathbf{x} = (\mathbf{x}', x_N)$ with $0 < x_N < y_N/2 < r_0/4$ (so that $|\mathbf{y} - \mathbf{x}| > y_N/2$). By the (second) Green formula,

$$G(\mathbf{x}, \mathbf{y}) - G_+(\mathbf{x}, \mathbf{y}) = \int_{\partial D'} G_+(\mathbf{z}, \mathbf{x}) \frac{\partial}{\partial \mathbf{n}^i_{\mathbf{z}}} G(\mathbf{z}, \mathbf{y}) d\sigma_{\mathbf{z}} \,. \tag{4.14}$$

It can be easily checked that there are $A_1 = A_1(N)$ and $A_2 = A_2(N)$ in $(0, +\infty)$ such that

$$-G_+(\mathbf{x}, \mathbf{y}) \geq A_1 x_N |\mathbf{x} - \mathbf{y}|^{1-N} \,, \tag{4.15}$$

$$-G_+(\mathbf{z}, \mathbf{x}) \leq A_2 x_N z_N |\mathbf{x} - \mathbf{z}|^{-N} , \quad \forall \mathbf{z} \in \mathbb{R}^N_+ \,. \tag{4.16}$$

Fix $\delta \in (0, r_0/3]$ and let $S'_\delta = \{\mathbf{z} \in \partial D' |\, |\mathbf{z}'| \leq \delta\}$, $S^*_\delta = \partial D' \backslash S'_\delta$. By (4.16) and (2) of Theorem 4.5,

$$\left| \int_{S'_\delta} G_+(\mathbf{z}, \mathbf{x}) \frac{\partial}{\partial \mathbf{n}^i_{\mathbf{z}}} G(\mathbf{z}, \mathbf{y}) d\sigma_{\mathbf{z}} \right| \leq \int_{S'_\delta} \frac{A_2 x_N z_N}{|\mathbf{x} - \mathbf{z}|^N} \frac{A_3}{|\mathbf{z} - \mathbf{y}|^{N-1}} d\sigma_{\mathbf{z}} \leq$$

$$\leq A_4 \frac{x_N}{|\mathbf{y}|^{N-1}} \int_{|\mathbf{z}'| < \delta} \frac{\varphi_\varepsilon(|\mathbf{z}'|) d\mathbf{z}'}{|\mathbf{x} - \mathbf{z}|^N} \leq A_5 \frac{x_N}{|\mathbf{y} - \mathbf{x}|^{N-1}} \int_0^\delta \frac{\varepsilon(r)}{r} \, dr \,. \tag{4.17}$$

Also by (1) and (3) of Theorem 4.5,

$$\left| \int_{S^*_\delta} G_+(\mathbf{z}, \mathbf{x}) \frac{\partial}{\partial \mathbf{n}^i_{\mathbf{z}}} G(\mathbf{z}, \mathbf{y}) d\sigma_{\mathbf{z}} \right| \leq \int_{S^*_\delta} \frac{A x_N}{|\mathbf{x} - \mathbf{z}|^{N-1}} \frac{y_N}{|\mathbf{y} - \mathbf{z}|^N} d\sigma_{\mathbf{z}} \leq$$

$$\leq A_6 x_N y_N \int_{S^*_\delta} \frac{d\sigma_N}{|\mathbf{z}|^{2N-1}} \leq A_7 \frac{x_N y_N}{\delta^N} \,, \tag{4.18}$$

where the last inequality can be checked the same way as in (4.12):

$$\int_{S^*_\delta} \frac{d\sigma_{\mathbf{z}}}{|\mathbf{z}|^{2N-1}} \leq \sum_{m=1}^{r_0/\delta} \frac{A_N \delta^{N-1} m^{N-1}}{(\delta m)^{2N-1}} \leq \frac{A_8}{\delta^N} \,.$$

Fix (maximal) $\delta_0 \in (0, r_0/3]$ with the property (recall (4.15))

$$A_5 \int_0^{\delta_0} \frac{\varepsilon(r)}{r} dr \leq \frac{1}{3} A_1 \,, \tag{4.19}$$

and let $\mathbf{y}$ be such that

$$\frac{A_7 y_N^N}{\delta_0^N} \leq \frac{1}{3} A_1 \,,$$

that is $y_N < \sqrt[N]{A_1/3A_7}\,\delta_0 = \delta_1$, and so

$$A_7 y_N / \delta_0 \leq \frac{1}{3} A_1 |\mathbf{y} - \mathbf{x}|^{N-1}.$$

So, finally, by (4.14)–(4.19), for $y_N < \delta_1$ and all $x_N < y_N/2$ we have (using maximum principle):

$$-G(\mathbf{x},\mathbf{y}) \geq -G'(\mathbf{x},\mathbf{y}) \geq \frac{A_1 x_N}{3|\mathbf{x}-\mathbf{y}|^{N-1}}\,, \tag{4.20}$$

which gives

$$\left|\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial x_N}\bigg|_{\mathbf{x}=\mathbf{0}}\right| \geq \frac{A_1}{3|\mathbf{y}|^{N-1}} \geq \frac{A_1}{3\delta_1^N}\,.$$

Lemma 4.10. *In conditions of Theorem 4.9 let $\mathbf{a} \in D$. There is $A_9 = A_9(N) \in [1,+\infty)$ such that for each $\mathbf{b} \in B(\mathbf{a},\rho(\mathbf{a})/8)$ one has*

$$A_9^{-1}|G(\mathbf{z},\mathbf{a})| \leq |G(\mathbf{z},\mathbf{b})| \leq A_9|G(\mathbf{z},\mathbf{a})|$$

for all $\mathbf{z} \in D$ with $\rho(\mathbf{z}) \leq \rho(\mathbf{a})/2$.

Proof. Use trivial inequality (the case $N \geq 3$)

$$\frac{1}{2}|\Phi(\mathbf{z}-\mathbf{a})| \leq |G(\mathbf{z},\mathbf{a})| \leq |\Phi(\mathbf{z}-\mathbf{a})|, \quad |\mathbf{z}-\mathbf{a}| < \frac{\rho(\mathbf{a})}{2},$$

and maximum principle for $-G(\mathbf{z},\mathbf{a})$ and $-lG(\mathbf{z},\mathbf{b})$ in $D\backslash B(\mathbf{a},3\rho(\mathbf{a})/16)$ with an appropriate $l = l(N) \in (0,+\infty)$. ☐

We are ready to finish the proof of Theorem 4.9. Fix any $\boldsymbol{\zeta} \in D$ and put $\rho_1(\boldsymbol{\zeta}) = \min\{\rho(\boldsymbol{\zeta})\,,\,\delta_1\}$. In our notations it suffices to prove that for all $x_N \in (0\,,\,\rho_1(\boldsymbol{\zeta})/4)$ one has

$$-G((\mathbf{0}',x_N),\boldsymbol{\zeta}) \geq x_N C(\rho(\boldsymbol{\zeta})).$$

Take $y_N = \rho_1(\boldsymbol{\zeta})$. One can find M points $\{\boldsymbol{\zeta}_m\}_{m=1}^M$ with $\boldsymbol{\zeta}_1 = (\mathbf{0}', y_N) = \mathbf{y}$, $\boldsymbol{\zeta}_M = \boldsymbol{\zeta}$, $\rho(\boldsymbol{\zeta}_m) \geq \rho_1(\boldsymbol{\zeta})/2$, $\boldsymbol{\zeta}_{m+1} \in B(\boldsymbol{\zeta}_m,\rho(\boldsymbol{\zeta}_m)/8)$ and $M = M(N,d,\varepsilon,\rho(\boldsymbol{\zeta}))$. Applying Lemma 4.10 to $\mathbf{a} = \boldsymbol{\zeta}_m$, $\mathbf{b} = \boldsymbol{\zeta}_{m+1}$ ($m \in \{1,\ldots,M-1\}$) we can finally take (by (4.20))

$$C(\rho) = A_9^{-M}A_1/3\delta_1^N\,,$$

and (4.13) is proved. ☐

Proof of Theorem W1. The proof of (1) for u_i and u_o in bounded components of D_0 follows immediately from Theorem 4.8. To prove (2) for u_i we apply (3.6) to have

$$\frac{\partial u_i(\mathbf{x})}{\partial x_n} = -\int_{S_r^*}\frac{\partial^2 G(\mathbf{x},\mathbf{y})}{\partial x_n \partial \mathbf{n}_{\mathbf{y}}^i}\psi(\mathbf{y})d\sigma_{\mathbf{y}}\,,$$

where $\mathbf{x} \in D \cap \overline{B}(\mathbf{a},r/2)$ and $S_r^* = \{\mathbf{y} \in S|\,|\mathbf{y}-\mathbf{a}| \geq r\}$. By (4) of Theorem 4.5 then

$$|\nabla u_i(\mathbf{x})| \leq A\|\psi\|_S \int_{S_r^*}\frac{d\sigma(\mathbf{y})}{|\mathbf{y}-\mathbf{a}|^N} \leq A_{10}\|\psi\|_S\frac{1}{r}\ln\frac{d}{r}\,.$$

The last inequality can be proved as in (4.12):

$$\int_{S_r^*}\frac{d\sigma_{\mathbf{y}}}{|\mathbf{y}-\mathbf{a}|^N} \leq \sum_{m=1}^{d/r}\frac{A_N r^{N-1}m^{N-1}}{(rm)^N} \leq \frac{A_N}{r}\sum_{m=1}^{d/r}\frac{1}{m}.$$

To finish the proof of (1) and (2) for u_o in D_* and (3) for ω in D_*, it is enough to apply the so called *Kelvin transform* (see (4.21) below ; for (3), additionally, use Theorem 4.9).

We can suppose that $\mathbf{0} \in D$ and $\rho(\mathbf{0}) \geq r_0$. The *inversion* $\mathbf{x} \to \mathbf{x}^* = \mathbf{x}/|\mathbf{x}|^2$ (via the unit sphere) maps some (L-D) domain $\tilde{D}$ (with the Dini-function $A\varepsilon(\cdot)$, $A = A(N, d, \varepsilon)$) onto $D_* \cup \{\infty\}$. If $f_* \in H(D_*)$ and $f_*(\infty) = 0$ (for $N \geq 3$, which gives $f_*(\mathbf{x}) = O(|\mathbf{x}|^{2-N})$) or $|f_*(\infty)| < +\infty$ (for $N = 2$), the Kelvin transform (via the unit sphere) of the function f_* is defined as

$$\tilde{f}(\mathbf{x}) = |\mathbf{x}|^{2-N} f_*(\mathbf{x}/|\mathbf{x}|^2) \ , \ \ \mathbf{x} \in \tilde{D}. \tag{4.21}$$

Then $\tilde{f} \in H(\tilde{D})$ and

$$\Delta\tilde{f}(\mathbf{x}) = |\mathbf{x}|^{-2-N} [\Delta f_*](\mathbf{x}/|\mathbf{x}|^2) \ , \ \ \mathbf{x} \in \tilde{D}$$

(see [11, Ch.13]).

□

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P.V. Paramonov
Mechanics and Mathematics Faculty,
Moscow State University,
119991 Moscow, Russian Federation.
e-mail: petr.paramonov@list.ru