CONVEX-TRANSITIVITY AND FUNCTION SPACES

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Abstract. It is shown that the Bochner space $L^p([0,1],\mathrm{X})$ is convex-transitive for any convex-transitive Banach space X and $1\leq p\leq\infty$. If H is an infinite-dimensional Hilbert space and $C_0(L)$ is convex-transitive, then $C_0(L,\mathrm{H})$ is convex-transitive. Some new fairly concrete examples of convex-transitive spaces are provided.

1. Introduction

This paper is concerned with several aspects of symmetries of real Banach spaces. We denote the closed unit ball of a real Banach space X by $\mathbf{B}_{\mathrm{X}}$ and the unit sphere of X by $\mathbf{S}_{\mathrm{X}}$. A Banach space X is called *transitive* if for each $x\in\mathbf{S}_{\mathrm{X}}$ the orbit $\mathcal{G}_{\mathrm{X}}(x)\doteq\{T(x)|\ T\colon \mathrm{X}\to\mathrm{X}$ is an isometric automorphism$\}=\mathbf{S}_{\mathrm{X}}$. If $\overline{\mathcal{G}_{\mathrm{X}}(x)}=\mathbf{S}_{\mathrm{X}}$ (resp. $\overline{\mathrm{conv}}(\mathcal{G}_{\mathrm{X}}(x))=\mathbf{B}_{\mathrm{X}}$) for all $x\in\mathbf{S}_{\mathrm{X}}$, then X is called *almost transitive* (resp. *convex-transitive*).

These concepts are motivated by the *Banach-Mazur rotation problem* appearing in [2, p.242] which remains unsolved to this day and asks whether each separable transitive space is isometrically a Hilbert space. A branch of the isometric theory of Banach spaces has developed around this problem, and we refer to [3] for an extensive survey of this field. In this article we will investigate the abovementioned transitivity conditions, mainly convex-transitivity, in different settings. For some important results related to this investigation, see [10], [11], [16] and [23].

The main line of study in this paper involves the rotations of certain Banach-valued function spaces. It is already a classical result that the space L^p is almost transitive for $p\in[1,\infty)$ and convex-transitive for $p=\infty$ (see [24]). In [10] the almost transitivity of the Bochner space $L^p([0,1],\mathrm{X})$ was established in the case where X is almost transitive and $1\leq p<\infty$. See [1] for the analogous study of the spaces $C_0(L,\mathrm{X})$. We will extend these investigations into the convex-transitive setting. We will show that if X is convex-transitive, then $L^\infty([0,1],\mathrm{X})$ is convex-transitive. Similarly, it turns out that if L is a locally compact space such that $C_0(L)$ is convex-transitive and H is an infinite-dimensional Hilbert space, then $C_0(L,\mathrm{H})$ is convex-transitive. The arguments are different compared to [10] and [1], since, for example, the rotation group $\mathcal{G}_{L^\infty}$ differs considerably from $\mathcal{G}_{L^p}$ for $p<\infty$. Some implications of the transitivity conditions of $C_0(L,\mathrm{X})$ for the topology of L are also discussed.

We will also give some new, fairly concrete examples of almost transitive and convex-transitive spaces. These examples are built starting from classical function spaces. For example, it turns out that the subspace of L^1 consisting of all the functions whose average is 0, is almost transitive in the usual norm.

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Finally, we will study non-separable transitive spaces. Recall that any Banach space X of density character κ can be isometrically embedded as a subspace of an almost transitive space of the same density character (see [21] and [3, p.8-11]) or of a transitive space (see [3, Cor 2.21]). Starting from a transitive space one can also pass to a separable almost transitive subspace (see [5] and also [3, Thm. 2.24]). We will prove here that each transitive space contains a transitive subspace, the density character of which does not exceed $2^{\aleph_0}$.

Preliminaries. Throughout this article we will consider *real* Banach spaces denoted by X, Y and Z unless otherwise stated. For non-empty subsets $A, B \subset \mathrm{X}$ we put $\mathrm{dist}(A, B) = \inf_{a\in A, b\in B} ||a - b||$. The group of rotations $\mathcal{G}_{\mathrm{X}}$ of X consists of isometric automorphisms $T\colon \mathrm{X} \to \mathrm{X}$, the group operation being the composition of the maps and the neutral element is the identity map $\mathbf{I}\colon \mathrm{X} \to \mathrm{X}$. If $f \in \mathrm{X}^*$ and $x \in \mathrm{X}$, then we denote by $f \otimes x$ the map $f(\cdot)x\colon \mathrm{X} \to [x]$.

We refer to [18] for background on measure algebras and isometries of L^p-spaces. In what follows Σ is the completed sigma algebra of Lebesgue measurable sets on $[0, 1]$. We denote by $m\colon \Sigma \to \mathbb{R}$ the Lebesgue measure. Let us define an equivalence relation $\stackrel{m}{\sim}$ on Σ by setting $A \stackrel{m}{\sim} B$ if $m((A \cup B) \setminus (A \cap B)) = 0$.

For an introduction to ordinal numbers and such matters we refer to [7]. We denote the cardinality of a set A by $|A|$. Here we will apply cardinal arithmetic notations, so that $2^{\aleph_0} = |\mathbb{R}|$. Recall that the density character of X, dens(X) for short, is the smallest cardinal κ such that X contains a dense subset of cardinality κ. Given a limit ordinal κ, its cofinality is $\mathrm{cf}(\kappa) \doteq \min\{|A| :\ A \subset \kappa,\ \sup A = \kappa\}$.

2. Convex-transitivity of Banach-valued function spaces

For notational simplicity we abbreviate the Lebesgue-Bochner space $L^p([0, 1], \mathrm{X})$ by $L^p(\mathrm{X})$ for $1 \leq p \leq \infty$. Recall that this space consists of strongly measurable maps $f\colon [0, 1] \to \mathrm{X}$ endowed with the norm

$$||f||^p_{L^p(\mathrm{X})} = \int_0^1 ||f(t)||^p_{\mathrm{X}}\ \mathrm{d}t, \quad \text{for } p \in [1, \infty)$$

and $||f||_{L^\infty(\mathrm{X})} = \operatorname*{ess\,sup}_{t\in[0,1]} ||f(t)||_X$. We refer to [6] for precise definitions and background information regarding the Banach-valued function spaces appearing in this section.

2.1. **Convex-transitivity of $L^p(\mathrm{X})$.** Recall that L^∞ is convex-transitive (see [22] and [24]). Greim, Jamison and Kaminska proved that $L^p(\mathrm{X})$ is almost transitive if X is almost transitive and $1 \leq p < \infty$, see [10, Thm. 2.1]. We will prove the analogous result for convex-transitive spaces, that is, if X is convex-transitive, then $L^\infty(\mathrm{X})$ is also convex-transitive.

Recall also that $L^p(\ell^p) = L^p$ isometrically and that ℓ^p is not convex-transitive for $p \neq 2$, so that the above implications can not be reversed. We do not know an example of a transitive space X such that $L^2(\mathrm{X})$ is not transitive.

Theorem 2.1. *Let* X *be a convex-transitive space and* $1 \leq p \leq \infty$*. Then* $L^p(\mathrm{X})$ *is convex-transitive.*

We will make some preparations before giving the proof. The following two facts are obtained from the definition of the strong measurability and the triangle inequality, respectively.

Fact 2.2. *Let $F = \chi_{A_1} x_1 + \chi_{A_2} x_2 + \ldots + \chi_{A_n} x_n$, where $n \in \mathbb{N}$, $\{A_k | 1 \leq k \leqq n\}$ is a measurable partition of $[0,1]$ and $x_1, \ldots, x_n \in \mathbf{B}_{\mathrm{X}}$. Such simple functions F are dense in $\mathbf{B}_{L^\infty(\mathrm{X})}$.*

Fact 2.3. *Let X be a Banach space and $T_1, ..., T_n \in \mathcal{G}_{\mathrm{X}}$, $n \in \mathbb{N}$. Assume that $x, y, z \in \mathrm{X}$ satisfy $\mathrm{dist}(y, \mathrm{conv}(\{T_j(x) | j = 1, ..., n\})) = \epsilon > 0$ and $||x - z|| = \delta > 0$. Then*

$$\mathrm{dist}(y, \mathrm{conv}(\{T_j(z) | j = 1, ..., n\})) \leq \epsilon + \delta.$$

Proof of Theorem 2.1. Our strategy for the case $p = \infty$, which we mainly concentrate on, is to show that for a given $x \in \mathbf{S}_{\mathrm{X}}$ the constant function $\chi_{[0,1]} x \in L^\infty(\mathrm{X})$ is a big point and that $\chi_{[0,1]} x \in \overline{\mathrm{conv}}(\mathcal{G}_{L^\infty(\mathrm{X})}(F))$ for each $F \in \mathbf{S}_{L^\infty(\mathrm{X})}$. This suffices for the claim according to Fact 2.3.

First we will show that for any $x \in \mathbf{S}_{\mathrm{X}}$ the function $\chi_{[0,1]} x$ is a big point. By using Fact 2.2 it suffices to show that any simple function $F \in \mathbf{S}_{L^\infty(\mathrm{X})}$ (as in Fact 2.2) is contained in $\overline{\mathrm{conv}}(\mathcal{G}_{L^\infty(\mathrm{X})}(\chi_{[0,1]} x))$. Let $F = \chi_{A_1} y_1 + \ldots + \chi_{A_n} y_n$, where $\{A_k | 1 \leq k \leq n\}$ is a measurable partition of $[0,1]$ and $y_1, \ldots, y_n \in \mathbf{B}_{\mathrm{X}}$. Let $\epsilon > 0$.

Since X is convex-transitive, there are for each $j \in \{1, \ldots, n\}$ finite sets $M_j \subset \mathbb{N}$, families of isometries $(T_k^{(j)})_{k \in M_j} \subset \mathcal{G}_{\mathrm{X}}$ and convex weights $(a_k^{(j)})_{k \in M_j} \in \mathbf{S}_{\ell^1_+} \cap c_{00}$ such that

$$\left|\left| \sum_{k \in M_j} a_k^{(j)} T_k^{(j)}(x) - y_j \right|\right| < \epsilon \quad \text{for } j \in \{1, \ldots, n\}. \tag{2.1}$$

Observe that, given $(i_j)_j \in \prod_{j=1}^n M_j$, one obtains rotations $\boldsymbol{R} \in \mathcal{G}_{L^\infty(\mathrm{X})}$ by setting

$$\boldsymbol{R}(g)(t) = \sum_{1 \leq j \leq n} \chi_{A_j}(t) T_{i_j}^{(j)}(g(t)) \text{ for a.e. } t \in [0,1]. \tag{2.2}$$

Next we will take a convex combination of this type of rotations. By induction on the number of terms in the product one can check that

$$\sum_{(i_1, i_2, \ldots, i_n)} \prod_{j=1}^n a_{i_j}^{(j)} = 1, \tag{2.3}$$

where the summation is taken over all the combinations $(i_1, i_2, \ldots, i_n) \in \prod_{j=1}^n M_j$. One can also verify by induction on the number of terms in the product that

$$\sum_{(i_1, i_2, \ldots, i_n)} \left(\prod_{l=1}^n a_{i_l}^{(l)} \right) \sum_{l=1}^n \chi_{A_l}(t) T_{i_l}^{(l)} = \sum_{l=1}^n \sum_{i_l \in M_l} a_{i_l}^{(l)} \chi_{A_l}(t) T_{i_l}^{(l)} \tag{2.4}$$

where the leftmost sum is taken over all combinations $(i_1, \ldots, i_n) \in \prod_{j=1}^n M_j$. Indeed, the case $n = 1$ is clear. To verify the induction step, observe that

$$\begin{aligned} & \sum_{(i_1, \ldots, i_{m+1})} \left(\prod_{l=1}^{m+1} a_{i_l}^{(l)} \right) \sum_{l=1}^{m+1} \chi_{A_l}(t) T_{i_l}^{(l)} \\ = & \sum_{r \in M_{m+1}} a_r^{(m+1)} \sum_{(i_1, \ldots i_m)} \left(\prod_{l=1}^m a_{i_l}^{(l)} \right) \left(\left(\sum_{l=1}^m \chi_{A_l}(t) T_{i_l}^{(l)} \right) + \chi_{A_{m+1}}(t) T_r^{(m+1)} \right) \\ = & \sum_{(i_1, \ldots, i_m)} \left(\prod_{l=1}^m a_{i_l}^{(l)} \right) \sum_{l=1}^m \chi_{A_l}(t) T_{i_l}^{(l)} + \sum_{r \in M_{m+1}} a_r^{(m+1)} \chi_{A_{m+1}}(t) T_r^{(m+1)} \\ = & \sum_{l=1}^m \sum_{i_l \in M_l} a_{i_l}^{(l)} \chi_{A_l}(t) T_{i_l}^{(l)} + \sum_{r \in M_{m+1}} a_r^{m+1} \chi_{A_{m+1}}(t) T_r^{(r)} \end{aligned}$$

for $m \leq n - 1$. Above we used that $\sum_{r \in M_{m+1}} a_r^{(m+1)} = 1 = \sum_{(i_1, \ldots, i_m)} \prod_{l=1}^m a_{i_l}^{(l)}$ and the second equality is provided by the induction hypothesis.

Hence we obtain by (2.4), (2.1) and the definition of F that

$$\begin{aligned} & \left|\left| \sum_{(i_1, \ldots, i_n)} \left(\prod_{l=1}^n a_{i_l}^{(l)} \right) \sum_{l=1}^n \chi_{A_l}(\cdot) T_{i_l}^{(l)}(x) - F(\cdot) \right|\right|_{L^\infty(\mathrm{X})} \\ = & \left|\left| \sum_{l=1}^n \sum_{i_l \in M_l} a_{i_l}^{(l)} \chi_{A_l}(\cdot) T_{i_l}^{(l)}(x) - \sum_{l=1}^n \sum_{i_l \in M_l} a_{i_l}^{(l)} \chi_{A_l}(\cdot) F(\cdot) \right|\right|_{L^\infty(\mathrm{X})} \leq \epsilon. \end{aligned}$$

Note that according to (2.2) and (2.3) the map

$$g \mapsto \sum_{(i_1,\ldots,i_n)} \left(\prod_{l=1}^n a_{i_l}^{(l)}\right) \sum_{l=1}^n \chi_{A_l} T_{i_l}^{(l)} \circ g$$

above belongs to $\mathrm{conv}(\mathcal{G}_{L^\infty(\mathrm{X})})$. We conclude that $F \in \overline{\mathrm{conv}}(\mathcal{G}_{L^\infty(\mathrm{X})}(\chi_{[0,1]}x))$, since $\epsilon > 0$ was arbitrary.

Let us check next that for each $F \in \mathbf{S}_{L^\infty(\mathrm{X})}$ and $x \in \mathbf{S}_\mathrm{X}$ it holds that $\chi_{[0,1]}x \in \overline{\mathrm{conv}}(\mathcal{G}_{L^\infty(\mathrm{X})}(F))$. To achieve this we employ a kind of sliding hump argument. Fix $F \in \mathbf{S}_{L^\infty(\mathrm{X})}$. Since $||F|| = 1$, one can find by Fact 2.2 a sequence $(x_n)_n \subset \mathbf{S}_\mathrm{X}$ and a sequence $(B_n)_n$ of measurable (but not necessarily pairwise disjoint) subsets of $[0,1]$ such that

$$\sup_{t \in B_n} ||F(t) - x_n||_\mathrm{X} < 2^{-(n+1)} \tag{2.5}$$

and $0 < m(B_n) < \frac{1}{2}$ for all n. Put $\Delta_n = [1-2^{-n}, 1-2^{-(n+1)}]$ for $n \in \mathbb{N}$. We claim that there is a sequence $g_n \colon [0,1] \to [0,1]$ of measurable mappings such that

$$g_n(B_n) \overset{m}{\sim} [0,1] \setminus \Delta_n \text{ and } g_n(B_n^c) \overset{m}{\sim} \Delta_n \tag{2.6}$$

and

$$\text{the measure } \mu_n(\cdot) \doteq m(g_n(\cdot)) \colon \Sigma \to \mathbb{R} \text{ is equivalent to } m. \tag{2.7}$$

for each $n \in \mathbb{N}$. Indeed, let us define two sequences of auxiliary mappings. Define $e_n \colon [0,1] \to [0,1]$ by

$$e_n(t) = \begin{cases} (1-2^{-(n+1)}) \frac{m(B_n \cap [0,t])}{m(B_n)} & \text{if } t \in B_n \\ (1-2^{-(n+1)}) + 2^{-(n+1)} \frac{m([0,t]\setminus B_n)}{m([0,1]\setminus B_n)} & \text{if } t \in [0,1] \setminus B_n \end{cases}$$

and $h_n \colon [0,1] \to [0,1]$ by

$$h_n(t) = \begin{cases} t & \text{if } t \in [0, 1-2^{-n}] \\ (1-2^{-(n+1)}) - (t - (1-2^{-(n+1)})) & \text{if } t \in (1-2^{-n}, 1] \end{cases}$$

for $n \in \mathbb{N}$. Then $g_n = h_n \circ e_n$ is the desired mapping for $n \in \mathbb{N}$. Above $g_n(B_n) \overset{m}{\sim} [0,1] \setminus \Delta_n$, whose complement Δ_n is the support of the sliding hump.

Next we will apply some observations which appear e.g. in [9] and [8]. Denote by $\Sigma\backslash_m$ the quotient sigma algebra of Lebesgue measurable sets on $[0,1]$ formed by identifying the m-null sets with $\emptyset$. Note that (2.7) gives in particular that the map $\phi_n \colon \Sigma\backslash_m \to \Sigma\backslash_m$ determined by $\phi_n(A) \overset{m}{\sim} g_n(A)$ for $A \in \Sigma$ is a Boolean isomorphism for each $n \in \mathbb{N}$. According to the convex-transitivity of X, there are for each $n \in \mathbb{N}$ a finite set $K_n \subset \mathbb{N}$, a set of rotations $(T_i^{(n)})_{i \in K_n} \subset \mathcal{G}_\mathrm{X}$ and convex weights $(d_i^{(n)})_{i \in K_n} \in \mathbf{S}_{\ell^1_+} \cap c_{00}$ such that

$$\left|\left| x - \sum_{i \in K_n} d_i^{(n)} T_i^{(n)}(x_n) \right|\right|_\mathrm{X} < \frac{1}{2^{n+1}} \quad \text{for all } n \in \mathbb{N}. \tag{2.8}$$

For each $n \in \mathbb{N}$ denote by $t \mapsto g_n^{-1}(t)$ a measurable mapping $[0,1] \to [0,1]$ defined m-a.e., which is induced by the Boolean isomorphism $g_n^{-1} \colon \Sigma\backslash_m \to \Sigma\backslash_m$ (see e.g. [9, p.209] or [19, p.463]). Define mappings $S_i^{(n)} \colon L^\infty(\mathrm{X}) \to L^\infty(\mathrm{X})$ for $n \in \mathbb{N}$ and $i \in K_n$ by

$$S_i^{(n)}(F)(t) = T_i^{(n)}(F(g_n^{-1}(t))) \quad \text{for a.e. } t \in [0,1],\ F \in L^\infty(\mathrm{X}).$$

By (2.7) we get that $S_i^{(n)} \in \mathcal{G}_{L^\infty(\mathrm{X})}$ (see also [8, p.467-468]).

The function $\chi_{[0,1]}x$ can be approximated by convex combinations coming from $\mathrm{conv}(\mathcal{G}_{L^\infty(\mathrm{X})}(F))$ as follows:

$$\left|\left|\chi_{[0,1]}x - \tfrac{1}{k}\sum_{n=1}^{k}\sum_{i\in K_n} d_i^{(n)} S_i^{(n)}(F)\right|\right|_{L^\infty(\mathrm{X})} \leq \tfrac{2+\sum_{i=1}^{k}2^{-i}}{k} \overset{k\to\infty}{\longrightarrow} 0. \tag{2.9}$$

Indeed, for $n\in\mathbb{N}$ and a.e. $t\in[0,1]\setminus\Delta_n$ it holds by (2.5) and (2.8) that

$$\left|\left|x-\sum_{i\in K_n} d_i^{(n)} S_i^{(n)}(F)(t)\right|\right|_{\mathrm{X}} \leq \left|\left|x-\sum_{i\in K_n} d_i^{(n)} T_i^{(n)}(x_n)\right|\right|_{\mathrm{X}} + \tfrac{1}{2^{n+1}} \leq \tfrac{1}{2^n}.$$

On the other hand, $||x-\sum_{i\in K_n} d_n^{(n)} S_i^{(n)}(F)(t)||_{\mathrm{X}} \leq 2$ for a.e. $t\in\Delta_n$.

Consequently, $\chi_{[0,1]}x \in \overline{\mathrm{conv}}(\mathcal{G}_{L^\infty(\mathrm{X})}(F))$ for all $F\in\mathbf{S}_{L^\infty(\mathrm{X})}$ by (2.9). Thus $L^\infty(\mathrm{X})$ is convex-transitive.

The case $1\leq p<\infty$ is a straightforward modification of the proof of [10, Thm. 2.1], where one replaces $U_i x_i$ by suitable elements of $\mathrm{conv}(\mathcal{G}_{\mathrm{X}}(x_i))$ for each i. We will not reproduce the details here. □

Remark 2.4. The Bochner space $L^\infty(\mathrm{X})$ is not almost transitive for any non-trivial separable X. Indeed, take any isometric automorphism $T\colon L^\infty(\mathrm{X})\to L^\infty(\mathrm{X})$. Then it follows that $\mathrm{ess\,inf}_{t\in[0,1]}\ ||T(F)(t)||_{\mathrm{X}} = \mathrm{ess\,inf}_{t\in[0,1]}\ ||F(t)||_{\mathrm{X}}$ for any $F\in L^\infty(\mathrm{X})$ (see [8, Thm. 2]). We obtain that for $G=\chi_{[0,\frac{1}{2}]}x\in\mathbf{S}_{L^\infty(\mathrm{X})}$, where $x\in\mathbf{S}_{\mathrm{X}}$, one has

$$||\chi_{[0,1]}x - T(G)||_{L^\infty(\mathrm{X})} \geq 1$$

for any isometric automorphism T, which yields the claim.

2.2. **Convex-transitivity of $C_0(L,\mathrm{X})$.** If L is a locally compact space and X is a Banach space, then $C_0(L,\mathrm{X})$ consists of the X-valued continuous functions on L tending to 0 at infinity. This is a Banach space endowed with the norm $||f||=\sup_{t\in L}||f(t)||_{\mathrm{X}}$.

Let us recall that for the Cantor set K the space $C(K)$ is convex-transitive but not almost transitive (see [24]). There exist plenty of convex-transitive spaces of the type $C_0(L)$ (see e.g. [27]).

The following problem about the almost transitivity of $C_0(L)$ spaces has attracted attention: Wood [27] conjectured that if L is a locally compact Hausdorff space and $C_0(L)$ is an almost transitive space over the scalars $\mathbb{K}$, then L is a singleton. This problem was answered by Greim and Rajagopalan in [11] in the case $\mathbb{K}=\mathbb{R}$ in the *positive* and was recently *refuted* independently by Kawamura in [16] and by Rambla in [23] in the case $\mathbb{K}=\mathbb{C}$. Theorem 2.5 below extends the work in [11] (and here $\mathbb{K}=\mathbb{R}$).

We will require the following generalization of Stone's theorem due to Jerison (see e.g. [4, p.145]):

Let (L,τ) be a locally compact Hausdorff space and let Y *be a strictly convex Banach space. Then a map $T\colon C_0(L,\mathrm{Y})\to C_0(L,\mathrm{Y})$ is a rotation if and only if T has the form*

$$T(F)(t)=\sigma(t)(F(h(t))) \quad \text{for } F\in C_0(L,\mathrm{Y});\ t\in L, \tag{2.10}$$

where $\sigma\colon L\to\mathcal{G}_{\mathrm{Y}}$ is (τ,SOT)-continuous and $h\colon L\to L$ is a homeomorphism.

Theorem 2.5. *Let (L,τ) be locally compact Hausdorff space that is locally connected at some point $x\in L$. Suppose that* X *is a non-trivial strictly convex Banach space.*

(i) *If $C_0(L,\mathrm{X})$ is almost transitive, then L is a singleton.*

(ii) *If $C_0(L, \mathrm{X})$ is convex-transitive, then L is connected.*

Proof. Let us first make some preparations towards both these claims. Let $x, y \in L$ be such that $x \neq y$ and L is locally connected at x. Since L is completely regular and Hausdorff, there are open neighbourhoods V_x and V_y of x and y, respectively, such that $\overline{V}_x^{\tau} \cap \overline{V}_y^{\tau} = \emptyset$. According to the complete regularity of L, there is a continuous map $f_0 \colon L \to [0,1]$ such that $f_0(L \setminus V_x) = \{0\}$ and $f_0(x) = 1$. Since L is locally connected at x and $f_0^{-1}((\frac{1}{2}, 1])$ is an open neighbourhood of x, there is an open connected neighbourhood $U_x \subset f_0^{-1}((\frac{1}{2}, 1])$ of x. Again, by the complete regularity there is a positive continuous map $e \colon L \to [0,1]$ such that $e(L \setminus U_x) = \{0\}$ and $e(x) = 1$. Define $f = \min(f_0, \frac{1}{2}) + \max(e \cdot (f_0 - \frac{1}{2}), 0)$. Note that $f \in C_0(L)$, its range is in $[0,1]$, $f(L \setminus U_x) \subset [0, \frac{1}{2}]$ and $f(x) = 1$. Let $g \colon L \to [0,1]$ be a positive map in $C_0(L)$ such that $g(L \setminus V_y) = \{0\}$ and $g(y) = 1$.

Fix $w \in \mathbf{S}_{\mathrm{X}}$. Note that $f \otimes w, (f+g) \otimes w \in \mathbf{S}_{C_0(L,\mathrm{X})}$ by the construction of f and g.

Claim (i). Suppose that $x, y \in L$ are as above. Then $C_0(L, \mathrm{X})$ is not almost transitive.

Indeed, assume to the contrary that $C_0(L, \mathrm{X})$ is almost transitive. Hence there must be a rotation $T \colon C_0(L, \mathrm{X}) \to C_0(L, \mathrm{X})$ such that

$$| \, ||T(f \otimes w)|| - ||(f+g) \otimes w|| \, | \leq ||T(f \otimes w) - (f+g) \otimes w|| < \frac{1}{10}. \tag{2.11}$$

By (2.10) we know that $T(Q \otimes w)(t) = \sigma(t)(Q(h(t))w)$ for any $Q \in C_0(L)$, where $h \colon L \to L$ is a homeomorphism and $\sigma \colon L \to \mathcal{G}_{\mathrm{X}}$. In particular

$$||T(Q \otimes w)(t)||_{\mathrm{X}} = ||Q(h(t))w||_{\mathrm{X}} = |Q(h(t))| \text{ for each } t \in L, \tag{2.12}$$

since $||w||_{\mathrm{X}} = 1$. Thus by (2.11), (2.12) and the positivity of f and g we obtain that

$$f(h(t)) - \frac{1}{10} < (f+g)(t) < f(h(t)) + \frac{1}{10} \quad \text{for } t \in L. \tag{2.13}$$

Note that since $f(x) = g(y) = 1$, and $f(y) = g(x) = 0$, it holds that $f(h(x)), f(h(y)) \geq \frac{9}{10}$. Since $f(L \setminus U_x) \subset [0, \frac{1}{2}]$, we obtain that $h(x), h(y) \in U_x$ and hence $x, y \in h^{-1}(U_x)$. By (2.13) and the selection of U_x we have that

$$(f+g)(h^{-1}(t)) > f(t) - \frac{1}{10} > \frac{1}{2} - \frac{1}{10} = \frac{2}{5}$$

for all $t \in U_x$. Since f and g are disjointly supported, we get that

$$x, y \in h^{-1}(U_x) \subset f^{-1}((2/5, 1]) \ \cup \ g^{-1}((2/5, 1]).$$

Moreover, there exists a separation between the open sets $f^{-1}((2/5, 1]) \ni x$ and $g^{-1}((2/5, 1]) \ni y$. Thus $h^{-1}(U_x)$ is not connected and hence U_x is not connected; a contradiction.

Claim (ii). Suppose that $x, y \in L$ are as in the beginning of the proof. Since y was arbitrary, it suffices to show that x and y belong to the same connected component.

Let $w^* \in S_{\mathrm{X}^*}$ be a support functional for $w \in \mathbf{S}_{\mathrm{X}}$. Define $\boldsymbol{F} \in \mathbf{S}_{C_0(L,\mathrm{X})^*}$ by $\boldsymbol{F}(u) = \frac{w^*(u(x) - u(y))}{2}$ for each $u \in C_0(L, \mathrm{X})$. Note that $\boldsymbol{F}$ is a support functional for $(f - g) \otimes w \in \mathbf{S}_{C_0(L,\mathrm{X})}$ as

$$\boldsymbol{F}((f-g) \otimes w) = \frac{f(x) - g(x) - (f(y) - g(y))}{2} w^*(w) = \frac{f(x) + g(y)}{2} = 1.$$

Since $C_0(L, \mathrm{X})$ was assumed to be convex-transitive we obtain that

$$\sup \boldsymbol{F}(\overline{\mathrm{conv}}(\mathcal{G}_{C_0(L,\mathrm{X})}(f \otimes w))) = \boldsymbol{F}((f - g) \otimes w) = 1.$$

In particular we may select by the linearity of $\boldsymbol{F}$ an element $G \in \mathcal{G}_{C_0(L,\mathrm{X})}(f \otimes w)$ such that $\boldsymbol{F}(G) > \frac{9}{10}$. This means that $w^*(G(x)), -w^*(G(y)) \in (\frac{8}{10}, 1]$. Hence

$$||G(x)||_{\mathrm{X}}, ||G(y)||_{\mathrm{X}} \in (4/5, 1], \tag{2.14}$$

since $||w^*||_{\mathrm{X}^*} = 1$.

Again, by (2.10) we may write $G(t) = \sigma(t)(f(h(t))w)$. Note that $||G(t)||_{\mathrm{X}} = |f(h(t))|$ for all $t \in L$. Thus $||G(t)||_{\mathrm{X}} \leq \frac{1}{2}$ for all $t \in L \setminus h^{-1}(U_x)$ and hence $x, y \in h^{-1}(U_x)$ by (2.14). Clearly x and y belong to the same connected component as U_x is connected. □

Theorem 2.6. *Let (L, τ) be a locally compact Hausdorff space.*

(i) *If* X *is a strictly convex Banach space and* $C_0(L, \mathrm{X})$ *is convex-transitive, then* X *is convex-transitive.*

(ii) *If* H *is a Hilbert space,* $\dim(\mathrm{H}) = \infty$ *and* $C_0(L)$ *is convex-transitive, then* $C_0(L, \mathrm{H})$ *is convex-transitive.*

Our argument below relies strongly on the assumption that $\dim(\mathrm{H}) = \infty$. We do not know if this assumption can be removed.

Proof. (i). Fix $x, y \in \mathbf{S}_{\mathrm{X}}$. Let $F, G \in C_0(L, \mathrm{X})$ be given by $F = e(\cdot)x$, $G = e(\cdot)y$, where $e \in \mathbf{S}_{C_0(L)}$. By the local compactness of L and the continuity of e there is $t_0 \in L$ such that $|e(t_0)| = 1$. Since $C_0(L, \mathrm{X})$ is convex-transitive, there exists a sequence $(T_n)_{n\in\mathbb{N}} \subset \mathcal{G}_{C_0(L,\mathrm{X})}$ such that $G \in \overline{\mathrm{conv}}((T_n(F))_{n\in\mathbb{N}})$. Since X is strictly convex, representation (2.10) gives that $T_n(F)(t_0) = \sigma_n(t_0)(e(h_n(t_0))x)$ for suitable sequences (σ_n) and (h_n). Recall that here $\sigma_n \colon L \to \mathcal{G}_{\mathrm{X}}$ are continuous and $h_n \colon L \to L$ are homeomorphisms for $n \in \mathbb{N}$. Since $G(t_0) \in \{\pm y\}$ we obtain for $a_n \doteq e(h_n(t_0)) \in [-1, 1]$ that

$$\begin{aligned} y \in\ & \overline{\mathrm{conv}}(\{\sigma_n(t_0)(a_n x) | n \in \mathbb{N}\}) \subset \overline{\mathrm{conv}}(\mathrm{conv}(\{\sigma_n(t_0)(\pm x) | n \in \mathbb{N}\})) \\ =\ & \overline{\mathrm{conv}}(\{\pm\sigma_n(t_0)(x) | n \in \mathbb{N}\}) \subset \overline{\mathrm{conv}}(\mathcal{G}_{\mathrm{X}}(x)). \end{aligned} \tag{2.15}$$

Hence X is convex-transitive.

(ii). Let H be an infinite-dimensional Hilbert space and let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an orthonormal basis of H. Fix $\gamma_0 \in \Gamma$, $0 < \epsilon < 1$ and $F_0, G_0 \in \mathbf{S}_{C_0(L,\mathrm{H})}$. Define $\mathrm{r} \colon \mathrm{H} \to \mathbf{S}_{\mathrm{H}} \cup \{0\}$ by $\mathrm{r}(z) = \frac{z}{||z||}$ for $z \neq 0$ and $\mathrm{r}(0) = 0$. Recall that r is continuous away from 0.

We will make crucial use of the fact that there exists a $(||\cdot||, \mathrm{SOT})$ -continuous map $A \colon \mathbf{S}_{\mathrm{H}} \setminus \{e_{\gamma_0}\} \to \mathcal{G}_{\mathrm{H}}$ such that

$$A(x)(e_{\gamma_0}) = x \text{ for all } x \in \mathbf{S}_{\mathrm{H}} \setminus \{e_{\gamma_0}\}, \tag{2.16}$$

see [1, Prop 5.4]. We will advance in two steps to complete the proof. The first, rather technical step is needed in order to apply the above fact (2.16). This step uses the assumption that $\dim(\mathrm{H}) = \infty$. The second step involves applying the fact (2.10).

Step 1. Note that the subset $\{t \in L : \ ||F_0(t)|| \geq \frac{\epsilon}{8}\} \subset L$ is compact, and its image under the continuous mapping F_0 is also compact. Since $\dim(\mathrm{H}) = \infty$ one can find by approximation an index $\mu \in \Gamma$, $\mu \neq \gamma_0$, such that $||P_\mu \circ F_0(t)|| < \frac{\epsilon}{8}$ for $t \in L$ such that $||F_0(t)|| \geq \frac{\epsilon}{8}$. Above $P_\mu \colon \mathrm{H} \to [e_\mu]$ is the orthogonal projection. It follows that $||P_\mu \circ F_0(t)|| < \frac{\epsilon}{8}$ for all $t \in L$.

Towards an application of (2.16) we define an auxiliary mapping $F \in C_0(L, \mathrm{H})$ by perturbing F_0 as follows:

$$F(t) = \begin{cases} ||F_0(t)||\mathrm{r}\big(F_0(t) + \frac{\epsilon}{8}e_\mu\big) & \text{if } ||F_0(t)|| \geq \frac{\epsilon}{2} \\ ||F_0(t)||\mathrm{r}\big(||F_0(t)||\mathrm{r}\big(F_0(t) + \frac{\epsilon}{8}e_\mu\big) + (\frac{\epsilon}{8} - \frac{1}{4}||F_0(t)||)e_\mu\big) & \text{if } ||F_0(t)|| < \frac{\epsilon}{2} \end{cases}$$

Put $\mathrm{r}_F(t) = \mathrm{r}(F(t))$ for $t \in L$ such that $F(t) \neq 0$ and $\mathrm{r}_F(t) = e_\mu$ otherwise. Then F and r_F satisfy the following conditions:

(a) $||F(t)|| = ||F_0(t)||$ for $t \in L$.
(b) $||F_0 - F|| < \epsilon$.
(c) $F(L) \cap [e_{\gamma_0}] \subset \{0\}$.
(d) $\lim_{||F_0(t)|| \to 0} \mathrm{r}_F(t) = e_\mu$.

Indeed, conditions (a) and (d) are immediate. Condition (b) follows by analyzing the upper and lower cases separately. In the upper case one applies geometric estimates in the euclidean plane spanned by $F_0(t)$ and e_μ in H. If one writes $||F_0(t)||\mathrm{r}(F_0(t) + \frac{\epsilon}{8}e_\mu) = bF_0(t) + ce$ for suitable $b, c \in \mathbb{R}$ and $e \perp F_0(t)$, where $||e|| = 1$ and $||F_0(t)|| \geq \frac{\epsilon}{2}$, then $|(||F_0(t)|| - b)| \leq \frac{\epsilon}{8}$ and $|c| \leq \frac{\epsilon}{4}$. The lower case follows immediately by applying the triangle inequality. Towards condition (c), given $t \in L$, write $P_\mu(F_0(t) + \frac{\epsilon}{2}e_\mu) = ae_\mu$ for suitable $a \in \mathbb{R}$. The fact that $||P_\mu \circ F_0(t)|| < \frac{\epsilon}{8}$ for $t \in L$ yields that $a > 0$ above. Thus $P_\mu \mathrm{r}\big(F_0(t) + \frac{\epsilon}{8}e_\mu\big) \neq 0$ for all $t \in L$ and $P_\mu \mathrm{r}\big(||F_0(t)||\mathrm{r}\big(F_0(t) + \frac{\epsilon}{8}e_\mu\big) + (\frac{\epsilon}{8} - \frac{1}{4}||F_0(t)||)e_\mu\big) \neq 0$ for t such that $||F_0(t)|| < \frac{\epsilon}{2}$. Hence (c) holds, since $e_\mu \perp e_{\gamma_0}$.

Similarly, by using a suitable $\nu \in \Gamma \setminus \{e_{\gamma_0}\}$, one can define $G \in C_0(L, \mathrm{H})$ and r_G, which correspond to G_0 and have analogous properties. Observe that r_F and r_G are continuous mappings $L \to \mathbf{S}_\mathrm{H}$ by condition (d) and the continuity of F and G.

Step 2. We claim that the map

$$E(t) \mapsto A(\mathrm{r}_G(t)) \circ A(\mathrm{r}_F(t)))^{-1}(E(t)), \quad t \in L,\ E \in C_0(L, \mathrm{H})$$

defines an element in $\mathcal{G}_{C_0(L,\mathrm{H})}$. Indeed, recall that A is $(||\cdot||, \mathrm{SOT})$-continuous and $(\mathcal{G}_\mathrm{H}, \mathrm{SOT})$ is a topological group, which means that the group operation and the taking of the inverse are continuous. Thus the composition $L \to \mathbf{S}_\mathrm{H} \times \mathbf{S}_\mathrm{H} \to \mathcal{G}_\mathrm{H}$ given by

$$t \mapsto (\mathrm{r}_G(t), \mathrm{r}_F(t)) \mapsto A(\mathrm{r}_G(t)) \circ (A(\mathrm{r}_F(t)))^{-1}$$

is (τ, SOT)-continuous. Clearly $A(\mathrm{r}_G(t)) \circ (A(\mathrm{r}_F(t)))^{-1} \in \mathcal{G}_\mathrm{H}$ for $t \in L$. By (2.16) we obtain that

$$\mathrm{r}_G(t) = A(\mathrm{r}_G(t))(e_{\gamma_0}) = A(\mathrm{r}_G(t)) \circ (A(\mathrm{r}_F(t)))^{-1}(\mathrm{r}_F(t)) \quad \text{for } t \in L. \tag{2.17}$$

Consider $||G(\cdot)||, ||F(\cdot)|| \in C_0(L)$. One can find by the convex-transitivity of $C_0(L)$ and (2.10) homeomorphisms $h_n \colon L \to L$ and continuous functions $\theta_n \colon L \to \{-1, 1\}$ for $n \in \mathbb{N}$ such that $||G(\cdot)|| \in \overline{\mathrm{conv}}(\{\theta_n(\cdot)||F(h_n(\cdot))|| :\ n \in \mathbb{N}\})$.

Put $T_n(E)(t) = A(\mathrm{r}_G(t)) \circ (A(\mathrm{r}_F(h_n(t))))^{-1}(E(t))$ for $n \in \mathbb{N},\ E \in C_0(L, \mathrm{H})$ and $t \in L$. By (2.17) we obtain that $T_n(F \circ h_n)(t) = ||F(h_n(t))||\mathrm{r}_G(t)$ for all $t \in L$.

Since $||G(\cdot)|| \in \overline{\mathrm{conv}}(\{\theta_n(\cdot)||F(h_n(\cdot))|| :\ n \in \mathbb{N}\}) \subset C_0(L)$, we obtain that

$$\begin{aligned} G(\cdot) &= ||G(\cdot)||\mathrm{r}_G(\cdot) \in \overline{\mathrm{conv}}(\{\theta_n(\cdot)||F(h_n(\cdot))||\mathrm{r}_G(\cdot) :\ n \in \mathbb{N}\}) \\ &= \overline{\mathrm{conv}}(\{(\theta_n(\cdot)\mathbf{I}) \circ T_n(F(h_n(\cdot))) :\ n \in \mathbb{N}\}) \subset C_0(L, \mathrm{H}). \end{aligned}$$

Note that according to (2.10) the map $E(\cdot) \mapsto (\theta_n(\cdot)\mathbf{I}) \circ T_n(E(h_n(\cdot)))$ defines an element in $\mathcal{G}_{C_0(L,\mathrm{H})}$ for $n \in \mathbb{N}$. Hence $G \in \overline{\mathrm{conv}}(\mathcal{G}_{C_0(L,\mathrm{H})}(F))$. Since F_0, G_0 and ϵ

were arbitrary, Fact 2.3 and condition (b) yield that $C_0(L, \mathrm{H})$ is convex-transitive. □

3. Some new examples

Even though there exist plenty of transitive spaces, very few *concrete* examples of almost transitive and convex-transitive (scalar-valued) function spaces are known (see [3]). Next we will demonstrate transitivity conditions for a few fairly concrete spaces by direct proofs. These new examples are obtained by performing natural operations on classical function spaces.

In this section we denote

$$\ell^\infty_\lambda(\kappa) = \{x \in \ell^\infty(\kappa) :\ |\mathrm{supp}(x)| \leq \lambda\} \subset \ell^\infty(\kappa),$$

where $\kappa,\ \lambda$ are infinite cardinals and $\lambda \leq \kappa$. This clearly defines a closed subspace of $\ell^\infty(\kappa)$. Recall that κ^+ stands for the successor cardinal of κ, i.e. the least cardinal strictly larger than κ, and $\kappa^{++} = (\kappa^+)^+$.

In [25] the special role of 1-codimensional subspaces of L^p in connection with the rotations is discussed. There it is also verified that the subspace

$$M^1 \doteq \left\{ x \in L^1 \Big| \int_0^1 x(t)\ \mathrm{d}t = 0 \right\} \subset L^1$$

appearing in the following result is neither 1-complemented, nor isometric to L^1.

Theorem 3.1. *The subspace $M^1 \subset L^1$ is almost transitive.*

Proof. For each $f \in L^1$ we denote $f_+(t) = \max(f(t), 0)$ and $f_-(t) = \min(f(t), 0)$, so that $f = f_+ + f_-$. Fix $x, y \in \mathbf{S}_{M^1}$ and $\epsilon > 0$. Let $A \cup B = [0,1]$ be a measurable partition such that $A = \mathrm{supp}(x_+)$. Similarly, let $C \cup D = [0,1]$ be a measurable partition such that $C = \mathrm{supp}(y_+)$. Observe that $||x_+|| = ||x_-|| = ||y_+|| = ||y_-|| = \frac{1}{2}$.

Note that if $E, F \subset [0,1]$ are measurable sets with strictly positive measure and $R\colon L^1(E) \to L^1(F)$ is a positive linear isometry onto, then $\int_E f(t)\ \mathrm{d}t = \int_F R(f)(t)\ \mathrm{d}t$ for $f \in L^1$. Indeed,

$$\begin{aligned} & \textstyle\int_E f(t)\ \mathrm{d}t = \int_E f_+(t)\ \mathrm{d}t + \int_E f_-(t)\ \mathrm{d}t = ||f_+||_{L^1(E)} - ||f_-||_{L^1(E)} \\ = \ & ||R(f_+)||_{L^1(F)} - ||R(f_-)||_{L^1(F)} = \textstyle\int_F R(f_+)(t)\ \mathrm{d}t - \int_F -R(f_-)(t)\ \mathrm{d}t \\ = \ & \textstyle\int_F R(f)(t)\ \mathrm{d}t, \end{aligned}$$

since $f = f_+ + f_-$, $R(f_+) \geq 0$ and $R(f_-) \leq 0$.

Let $R_1\colon L^1(A) \to L^1(C)$ and $R_2\colon L^1(B) \to L^1(D)$ be positive linear isometries onto. Then $\int_A x_+(t)\ \mathrm{d}t = \int_C R_1(x_+|_A)(t)\ \mathrm{d}t$ and $\int_B x_-(t)\ \mathrm{d}t = \int_D R_2(x_-|_B)(t)\ \mathrm{d}t$ by the above observation. Since both $R_1(x_+|_A), y_+|_C \in L^1(C)$ are positive and have norm $\frac{1}{2}$, it follows by the standard argument, which is used to prove that L^1 is almost transitive (see e.g. [19, Thm. 3.1]), that there is a positive rotation $T_1\colon L^1(C) \to L^1(C)$ such that $||T_1 R_1(x_+|_A) - y_+|_C|| < \frac{\epsilon}{2}$. Similarly there is a positive rotation $T_2\colon L^1(D) \to L^1(D)$ such that $||T_2 R_2(x_-|_B) - y_-|_D|| < \frac{\epsilon}{2}$. Now

$$T_1 R_1 \oplus T_2 R_2\colon L^1 = L^1(A) \oplus_1 L^1(B) \to L^1(C) \oplus_1 L^1(D) = L^1$$

is a positive linear isometry onto. Hence $\int_{[0,1]} f(t)\ \mathrm{d}t = \int_{[0,1]} R(f)(t)\ \mathrm{d}t$ for each $f \in L^1$, so that $(T_1 R_1 \oplus T_2 R_2)|_{M^1}\colon M^1 \to M^1$ is a linear isometry onto. Note that $||(T_1 R_1 \oplus T_2 R_2)x - y|| < \epsilon$. Since x, y and ϵ were arbitrary, we conclude that M^1 is almost transitive. □

We refer to [13] for background in ultraproducts in Banach space theory. Recall that an ultrafilter $\mathcal{U}$ is non-principal if it does not contain any singletons. The writer thanks Å. Hirvonen and T. Hyttinen for providing help in proving the following fact.

Lemma 3.2. *(Homogeneity) Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and $A, B \subset \mathbb{N}$ be infinite sets such that $A, B \notin \mathcal{U}$. Then there is a permutation $h\colon \mathbb{N} \to \mathbb{N}$ such that*

(i) $h(A) = B$.
(ii) $h(U) \in \mathcal{U}$ *if and only if* $U \in \mathcal{U}$.

Proof. Observe that $\mathbb{N} \setminus (A \cup B) \in \mathcal{U}$. Let $L, M \subset \mathbb{N} \setminus (A \cup B)$ be infinite disjoint subsets such that $L \cup M = \mathbb{N} \setminus (A \cup B)$. Without loss of generality assume that $M \in \mathcal{U}$ and $L \notin \mathcal{U}$.

Since A and L are infinite and disjoint, there is a bijection $g\colon A \cup L \to A \cup L$ such that $g(A) = L$ and $g(L) = A$. Define $\tilde{g}\colon \mathbb{N} \to \mathbb{N}$ by $\tilde{g}|_{A\cup L} = g$ and $\tilde{g}|_{\mathbb{N}\setminus(A\cup L)} = \mathrm{id}|_{\mathbb{N}\setminus(A\cup L)}$. Note that $V \in \mathcal{U}$ if and only if $V \setminus (A \cup L) \in \mathcal{U}$. Hence $\tilde{g}(U) \in \mathcal{U}$ if and only if $U \in \mathcal{U}$.

Similarly, one can construct a bijection $\tilde{f}\colon \mathbb{N} \to \mathbb{N}$ such that $\tilde{f}(L) = B$, $\tilde{f}(B) = L$ and $\tilde{f}(U) \in \mathcal{U}$ if and only if $U \in \mathcal{U}$. Observe that $h = \tilde{f} \circ \tilde{g}\colon \mathbb{N} \to \mathbb{N}$ is the claimed permutation. $\square$

Fact 3.3. *Let Γ be a non-empty set. Then $\mathbf{B}_{\ell^\infty(\Gamma)} = \overline{\mathrm{conv}}(\{-1, 1\}^\Gamma) \subset \ell^\infty(\Gamma)$.*

The proof is left as an exercise (see [12, p.99]). Note that each $x \in \mathbf{B}_{\ell^\infty(\Gamma)}$ can be approximated by elements of $\{-1, -\frac{n-1}{n}, \ldots, \frac{n-1}{n}, 1\}^\Gamma$, where $n \in \mathbb{N}$.

Theorem 3.4. *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and*

$$\mathrm{Y} = \{(x_n) \in \ell^\infty |\ \lim_{\mathcal{U}} x_n = 0\}.$$

Then Y/c_0 is convex-transitive.

Let us make a few comments before the proof. Recall that ℓ^∞ itself is far away from being convex-transitive in view of the characterization of the rotations of ℓ^∞ (see [20, 2.f.14]). Note that *exactly* one of the sets $\{2n|\ n \in \mathbb{N}\}$ and $\{2n+1|\ n \in \mathbb{N}\}$ is a member of $\mathcal{U}$, say $\{2n+1|n \in \mathbb{N}\} \in \mathcal{U}$. Thus $(x_n) \mapsto (0, x_1, 0, x_2, 0, x_3, \ldots)$ defines an isometric embedding $\ell^\infty \to \mathrm{Y}$. In particular, Y/c_0 is a non-separable space containing an isometric copy of ℓ^∞/c_0.

Proof. First note that if $x = (x_n) \in \ell^\infty$ is such that $\lim_{\mathcal{U}} |x_n| = c > 0$, then $\mathrm{dist}(x, \mathrm{Y}) \geq c > 0$ because $||x - y|| \geq c$ for each $y \in \mathrm{Y}$. Thus $\mathrm{Y} \subset \ell^\infty$ is a closed subspace.

Recall (see [20, 2.f.14]) that $T \in \mathcal{G}_{\ell^\infty}$ if and only if

$$T((x_n)_n) = (\theta(n)x_{\pi(n)})_n, \tag{3.1}$$

where $\pi\colon \mathbb{N} \to \mathbb{N}$ is a bijection and $\theta\colon \mathbb{N} \to \{-1, 1\}$. Also note that such an isometry T restricted to c_0 is a member of $\mathcal{G}_{c_0}$.

Next we will check that if $T \in \mathcal{G}_{\ell^\infty}$, then $\widehat{T}\colon x + c_0 \mapsto T(x) + c_0$, for $x \in \ell^\infty$, defines a rotation $\ell^\infty/c_0 \to \ell^\infty/c_0$. Indeed, it is well-known that $\widehat{T}\colon \ell^\infty/c_0 \to \ell^\infty/c_0$ is a linear bijection. Moreover, $\inf_{z\in c_0} ||x - z|| = \inf_{z\in c_0} ||T(x) - T(z)|| = \inf_{z\in c_0} ||T(x) - z||$, so that $\widehat{T}\colon \ell^\infty/c_0 \to \ell^\infty/c_0$ is an isometry.

Note that if the permutation π above is such that $\pi^{-1}(U) \in \mathcal{U}$ exactly when $U \in \mathcal{U}$, then $T|_{\mathrm{Y}}\colon \mathrm{Y} \to \mathrm{Y}$ is a rotation. To summarize, $\widehat{T}|_{\mathrm{Y}/c_0} \in \mathcal{G}_{\mathrm{Y}/c_0}$ for such π.

Fix $u, v \in \mathbf{S}_{\mathrm{Y}/c_0}$. If $x, y \in \mathrm{Y}$ are such that $u = x + c_0$ and $v = y + c_0$, then

$$\operatorname{dist}(x, c_0) = \limsup_{n\to\infty} |x_n| = 1 = \operatorname{dist}(y, c_0) = \limsup_{n\to\infty} |y_n|, \tag{3.2}$$

since $u, v \in \mathbf{S}_{\mathrm{Y}/c_0}$. Hence we may pick $x, y \in \mathbf{S}_{\mathrm{Y}}$ such that $u = x+c_0$ and $v = y+c_0$.

Fix $k \in \mathbb{N}$. Let $A = \{n \in \mathbb{N} : \ |x_n| \geq 1 - \frac{1}{k}\}$ and $B = \{n \in \mathbb{N} : \ |y_n| \geq \frac{1}{k}\}$. Observe that these are *infinite* sets by (3.2) and that $\mathbb{N} \setminus A,\ \mathbb{N} \setminus B \in \mathcal{U}$. Thus, by Lemma 3.2 there is a bijection $\pi \colon \mathbb{N} \to \mathbb{N}$ such that $\pi(B) = A$ and $\pi(U) \in \mathcal{U}$ if and only if $U \in \mathcal{U}$. Observe that $T \colon \mathrm{Y} \to \mathrm{Y};\ (x_n)_{n\in\mathbb{N}} \mapsto (x_{\pi(n)})_{n\in\mathbb{N}}$ is a rotation. Put $w = (w_n)_{n\in\mathbb{N}} = (x_{\pi(n)})_{n\in\mathbb{N}} \in \mathcal{G}_{\mathrm{Y}}(x)$. By the definition of the sets A and B we obtain that $|w_n| \geq 1 - \frac{1}{k}$ if and only if $n \in B$.

Fix an auxiliary point $z \in \mathrm{Y}$, which satisfies

$$||y - z||_{\mathrm{Y}} \leq \frac{1}{k},\ |z_n| \leq 1 - \frac{1}{k} \text{ for } n \in B \text{ and } z_n = 0 \text{ for } n \in \mathbb{N} \setminus B.$$

Write $\ell^\infty = \ell^\infty(B) \oplus_\infty \ell^\infty(\mathbb{N} \setminus B)$. It follows from Fact 3.3 that

$$z|_B \in \overline{\operatorname{conv}}(\{(\theta(n)|\ \theta(n) = \pm 1 \text{ for } n \in B\}).$$

Since

$$||(w_n)_{n\in B} - (\operatorname{sign}(w_n))_{n\in B}||_{\ell^\infty(B)} \leq \frac{1}{k} \tag{3.3}$$

we obtain by Fact 2.3 that

$$\operatorname{dist}(z|_B, \overline{\operatorname{conv}}(\{(\theta(n)w_n)_{n\in B}|\ \theta(n) = \pm 1 \text{ for } n \in B\}) \leq \frac{1}{k}. \tag{3.4}$$

It follows that

$$\operatorname{dist}(z, \overline{\operatorname{conv}}(\{(\theta(n)w_n)_{n\in\mathbb{N}}|\ \theta(n) = \pm 1 \text{ for } n \in \mathbb{N}\})) \leq \frac{1}{k}. \tag{3.5}$$

Indeed, one can use convex combinations

$$\frac{1}{2}\left(\sum_i a_i(f_i + g) + \sum_i a_i(f_i - g)\right) \in \ell^\infty, \tag{3.6}$$

where $f_i \in \{(\theta(n)w_n)_{n\in B}|\ \theta(n) = \pm 1 \text{ for } n \in B\}$ are obtained from (3.4) such that $||\sum_i a_i f_i - z|_B||_{\ell^\infty(B)} \leq \frac{1}{k}$ and $g = (w_n)_{n\in\mathbb{N}\setminus B} \in \ell^\infty(\mathbb{N} \setminus B)$. Above we used that $z(n) = 0$ for $n \in \mathbb{N} \setminus B$.

Since $||y - z|| \leq \frac{1}{k}$ by the selection of z and $(w_n) = (x_{\pi(n)}) \in \mathcal{G}_{\mathrm{Y}}(x)$, Fact 2.3 yields that

$$\operatorname{dist}(y, \overline{\operatorname{conv}}(\mathcal{G}_{\mathrm{Y}}(x))) \leq \operatorname{dist}(y, \overline{\operatorname{conv}}(\{(\theta(n)x_{\pi(n)})_{n\in\mathbb{N}}|\ \theta \colon \mathbb{N} \to \{\pm 1\}\ \})) \leq \frac{2}{k}.$$

Since k was arbitrary we obtain that $y \in \overline{\operatorname{conv}}(\mathcal{G}_{\mathrm{Y}}(x))$. Thus $v = y + c_0 \in \overline{\operatorname{conv}}(\mathcal{G}_{\mathrm{Y}/c_0}(u))$, so that Y/c_0 is convex-transitive. □

We note that the technique applied above does not give that Y or ℓ^∞/c_0 should be convex-transitive. However, by applying similar ideas we obtain another fairly concrete example of a convex-transitive space:

Theorem 3.5. *The space $\ell^\infty_{\kappa^+}(\kappa^{++})/\ell^\infty_{\kappa}(\kappa^{++})$ is convex-transitive for any infinite cardinal κ.*

Proof. Let κ be an infinite cardinal and denote $\mathrm{X} = \ell^\infty_{\kappa^+}(\kappa^{++})/\ell^\infty_\kappa(\kappa^{++})$. Fix $x, y \in \mathbf{S}_\mathrm{X}$. Pick $u, v \in \ell^\infty_{\kappa^+}(\kappa^{++})$ such that $x = u + \ell^\infty_\kappa(\kappa^{++})$ and $y = v + \ell^\infty_\kappa(\kappa^{++})$. Thus $||u|| \geq 1$ and $||v|| \geq 1$. Observe that

$$|\{\alpha \in \kappa^{++} : \ |u(\alpha)| \geq 1 + \frac{1}{k}\}| \leq \kappa \text{ and } |\{\alpha \in \kappa^{++} : \ |v(\alpha)| \geq 1 + \frac{1}{k}\}| \leq \kappa$$

for each $k \in \mathbb{N}$ since $x, y \in \mathbf{S}_\mathrm{X}$. Thus, by using the fact that $|\omega_0 \times \kappa| = \kappa$ we obtain that $|\{\alpha \in \kappa^{++} : \ |v(\alpha)| > 1\}| \leq \kappa$ and $|\{\alpha \in \kappa^{++} : \ |u(\alpha)| > 1\}| \leq \kappa$. Hence u and v above can be chosen such that $u, v \in \mathbf{S}_{\ell^\infty_{\kappa^+}(\kappa^{++})}$.

Put $A_i = \{\alpha \in \kappa^{++} : \ |u(\alpha)| \geq 1 - \frac{1}{i}\}$ for $i \in \mathbb{N}$. Observe that $|A_i| = \kappa^+$ for $i \in \mathbb{N}$ since $x \in \mathbf{S}_\mathrm{X}$. Also note that $|\mathrm{supp}(u)| = |\mathrm{supp}(v)| = \kappa^+$. Fix $B \subset \kappa^{++} \setminus (\mathrm{supp}(u) \cup \mathrm{supp}(v))$ such that $|B| = \kappa^+$. Write $\Gamma = \mathrm{supp}(u) \cup \mathrm{supp}(v) \cup B$.

This finishes the combinatorial part of the argument at hand. The essential point above was to find suitably normalized u, v and Γ. Next we aim to show that when regarding $u, v \in \mathbf{S}_{\ell^\infty(\Gamma)}$ it holds that

$$v \in \overline{\mathrm{conv}}(\mathcal{G}_{\ell^\infty(\Gamma)}(u)) \subset \ell^\infty(\Gamma). \tag{3.7}$$

In such a case one can see similarly as in the proof of Theorem 3.4 that $y \in \overline{\mathrm{conv}}(\mathcal{G}_\mathrm{X}(x))$, which gives the claim.

To prove (3.7) observe that for each $i \in \mathbb{N}$ there is a bijection $\pi_i \colon \Gamma \to \Gamma$ such that $\pi_i(A_i) = \mathrm{supp}(v)$. For each $i \in \mathbb{N}$ define $w^{(i)} \colon \Gamma \to \mathbb{R}$ by

$$w^{(i)}(\alpha) = \chi_{\mathrm{supp}(v)}(\alpha)\mathrm{sign}(u(\pi_i^{-1}(\alpha))) + \chi_{\Gamma \setminus \mathrm{supp}(v)}(\alpha) u(\pi_i^{-1}(\alpha)).$$

One can check that $v \in \overline{\mathrm{conv}}(\{(\theta(\alpha) w^{(i)}(\alpha))_{\alpha \in \Gamma} |\ \theta \colon \Gamma \to \{\pm 1\}\ \})$ by applying Fact 3.3 on $\mathrm{supp}(v)$ and the (trivial) equality $\frac{w^{(i)}(\alpha) - w^{(i)}(\alpha)}{2} = 0$ on $\Gamma \setminus \mathrm{supp}(v)$ similarly as in (3.6) and (3.5). Finally, Fact 2.3 and the observation that $||u(\pi_i^{-1}) - w^{(i)}||_{\ell^\infty(\Gamma)} \leq \frac{1}{i}$ for $i \in \mathbb{N}$ yield (3.7). □

4. Final remarks: Constructions for transitive spaces

Recall that each transitive space contains a separable almost transitive subspace (see [5, Cor.1.3], also [3, Thm. 2.24]). In fact, it turns out below that one can always pass to a suitable transitive subspace of density character at most $2^{\aleph_0}$.

Theorem 4.1. *Let* X *be a transitive space and let* $\mathrm{Y} \subset \mathrm{X}$ *be a subspace with* $\mathrm{dens}(\mathrm{Y}) \leq 2^{\aleph_0}$. *Then there exists a closed subspace* $\mathrm{Z} \subset \mathrm{X}$ *such that*

(1) $\mathrm{Y} \subset \mathrm{Z}$
(2) $\mathrm{dens}(\mathrm{Z}) \leq 2^{\aleph_0}$
(3) Z *is transitive with respect to subgroup* $\mathcal{T} = \{T \in \mathcal{G}_\mathrm{X} |\ T(\mathrm{Z}) = \mathrm{Z}\}$ *of* $\mathcal{G}_\mathrm{X}$.

Let us recall the following folklore fact which is proved here for convenience.

Fact 4.2. *Let* (Y, d) *be a metric space,* κ *a cardinal with* $\mathrm{cf}(\kappa) > \omega_0$ *and* $\{Y_\alpha\}_{\alpha < \kappa}$ *an increasing family of closed subsets. Then* $\bigcup_{\alpha < \kappa} Y_\alpha$ *is closed.*

Proof. Assume to the contrary that there is $x \in \overline{\bigcup_{\alpha < \kappa} Y_\alpha}^d \setminus \bigcup_{\alpha < \kappa} Y_\alpha$. For each $n < \omega_0$ let $\sigma_n < \kappa$ be the least ordinal such that $\mathrm{dist}(x, Y_{\sigma_n}) < \frac{1}{n+1}$. It follows from the selection of x that $\sup_{n < \omega_0} \sigma_n = \kappa$, which contradicts $\mathrm{cf}(\kappa) > \omega_0$. □

Proof of Theorem 4.1. If $\mathrm{dens}(\mathrm{X}) \leq 2^{\aleph_0}$, then the claim holds trivially by putting $\mathrm{Z} = \mathrm{X}$, so let us assume that $\mathrm{dens}(\mathrm{X}) > 2^{\aleph_0}$. We will apply the following fact. If E is a Banach space with $\dim(E) \geq 2$ and $\mathrm{dens}(E) \leq 2^{\aleph_0}$, then $|\mathbf{S}_E| = 2^{\aleph_0}$. Indeed, clearly $|\mathbf{S}_E| \geq 2^{\aleph_0}$. If $D \subset E$ is dense and $|D| \leq 2^{\aleph_0}$, then of course $D \cap \mathbf{B}_E$ is dense in $\mathbf{B}_E$ and $|D \cap \mathbf{B}_E| \leq 2^{\aleph_0}$. Note that for each $e \in \mathbf{B}_E$ there is a sequence $(d_n) \subset D \cap \mathbf{B}_E$ such that $d_n \to e$ as $n \to \infty$. Thus, in order to estimate $|\mathbf{B}_E|$, let us estimate the number $|(D \cap \mathbf{B}_E)^{\omega_0}|$ of sequences of $D \cap \mathbf{B}_E$ as follows:

$$|(D \cap \mathbf{B}_E)^{\omega_0}| \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}.$$

Hence $|\mathbf{S}_E| \leq |\mathbf{B}_E| \leq 2^{\aleph_0}$ and consequently $|\mathbf{S}_E| = 2^{\aleph_0}$. Observe that there are

$$|\mathbf{S}_E \times \mathbf{S}_E| = |2^{\aleph_0} \times 2^{\aleph_0}| = 2^{\aleph_0} \tag{4.1}$$

many pairs $(x, y) \in \mathbf{S}_E \times \mathbf{S}_E$.

We may assume without loss of generality, possibly by passing to a larger subspace Y, that $\mathrm{dens}(\mathrm{Y}) = 2^{\aleph_0}$. Next we will apply the transitivity of X in a transfinite induction. We will construct increasing sequences $\{\mathcal{T}_\alpha\}_{\alpha<2^{\aleph_0}}$ and $\{Z_\alpha\}_{\alpha<2^{\aleph_0}}$, which consist of subgroups of $\mathcal{G}_{\mathrm{X}}$ and closed subspaces of X, respectively. Put $Z_0 = \mathrm{Y}$ and $\mathcal{T}_0 = \{\mathbf{I}_{\mathrm{X}}\}$. For each $0 < \alpha < 2^{\aleph_0}$ we will construct a subgroup $\mathcal{T}_\alpha \subset \mathcal{G}_{\mathrm{X}}$ and a subspace $Z_\alpha \subset \mathrm{X}$, which satisfy the following conditions:

(a) $|\mathcal{T}_\alpha| \leq 2^{\aleph_0}$.
(b) $\mathcal{T}_\beta \subset \mathcal{T}_\alpha$ for $\beta < \alpha$.
(c) For each $x, y \in \mathbf{S}_{\mathrm{X}} \cap \overline{\bigcup_{\beta<\alpha} Z_\beta}$ there is $T \in \mathcal{T}_\alpha$ such that $T(x) = y$.
(d) $Z_\alpha = \overline{\mathrm{span}}(\bigcup\{T(Z_\beta)|\ T \in \mathcal{T}_\alpha,\ \beta < \alpha\})$.
(e) $\mathrm{dens}(Z_\alpha) = 2^{\aleph_0}$.

In the case $\alpha = 0$ only (a) and (e) apply. Suppose that we have obtained this kind of construction for all $\alpha < \gamma$, where $0 < \gamma < 2^{\aleph_0}$. Observe that

$$\mathrm{dens}\left(\overline{\bigcup_{\alpha<\gamma} Z_\gamma}\right) \leq |\gamma \times 2^{\aleph_0}| = 2^{\aleph_0}.$$

Thus, by (4.1) and the transitivity of X there is a subset $\mathcal{F} \subset \mathcal{G}_{\mathrm{X}}$ such that for each pair $x, y \in \mathbf{S}_{\mathrm{X}} \cap \overline{\bigcup_{\alpha<\gamma} Z_\gamma}$ there is $T \in \mathcal{F}$ such that $T(x) = y$ and $|\mathcal{F}| = 2^{\aleph_0}$. Let $\mathcal{T}_\gamma \subset \mathcal{G}_{\mathrm{X}}$ be the subgroup generated by the set $\bigcup_{\alpha<\gamma} \mathcal{T}_\alpha \cup \mathcal{F}$. Then $\mathcal{T}_\gamma$ satisfies conditions (b) and (c) (for $\alpha = \gamma$). Note that $|\mathcal{T}_\gamma| \leq |(\gamma + 1) \times 2^{\aleph_0} \times \aleph_0| = 2^{\aleph_0}$, so that $\mathcal{T}_\gamma$ satisfies condition (a). Observe that $\mathrm{dens}(Z_\gamma) \leq |\mathcal{T}_\gamma \times \gamma \times 2^{\aleph_0} \times \aleph_0| = 2^{\aleph_0}$, so that condition (e) holds, as $\mathrm{Y} \subset Z_\gamma$. This completes the induction.

Put $\mathrm{Z} = \bigcup_{\alpha<2^{\aleph_0}} \mathrm{Z}_\alpha$ and observe that $\mathrm{dens}(\mathrm{Z}) = 2^{\aleph_0}$. Note that $\mathrm{Y} \subset \mathrm{Z}$ by the construction. Recall that according to König's Lemma $\mathrm{cf}(2^{\aleph_0}) > \aleph_0$ (see e.g. [7, Lm. 10.40]). Hence by Fact 4.2 we obtain that

$$\mathrm{Z} = \overline{\bigcup_{\alpha<2^{\aleph_0}} \mathrm{Z}_\alpha}. \tag{4.2}$$

Fix $x, y \in \mathbf{S}_Z$ and $T \in \bigcup_{\alpha<2^{\aleph_0}} \mathcal{T}_\alpha$. Let $\sigma < 2^{\aleph_0}$ be such that $T \in \mathcal{T}_\sigma$. According to (4.2), there is $\beta < 2^{\aleph_0}$ such that $x, y \in Z_\beta$. Then $T(x) \in Z_{\max(\sigma,\beta)+1} \subset Z$ by the construction of Z. Since T and $x \in \mathbf{S}_Z$ were arbitrary, we get that in fact both $T(Z) \subset Z$ and $T^{-1}(Z) \subset Z$ hold. This means that $T|_Z$ is a bijection $Z \to Z$.

Observe that there is $S \in \mathcal{T}_{\beta+1}$ such that $S(x) = y$ by the construction of Z. Since x and y were arbitrary and $\bigcup_{\alpha<2^{\aleph_0}} \mathcal{T}_\alpha \subset \mathcal{T}$, we conclude that Z is transitive with respect to $\mathcal{T}$. □

Remark 4.3. Observe that in Theorem 4.1 one has some control of the density character of Z, which, even though being a superspace of Y, is a still subspace of X. To put Theorem 4.1 into perspective, recall that every Banach space can be isometrically regarded as a subspace of a suitable transitive Banach space (see [3, Cor. 2.21]). Recently Hirvonen and Hyttinen observed using model theoretic tools [15, Thm.2.10] (see [14, Thm. 1.13] for details) that there exist very homogeneous Banach spaces containing all the Banach spaces up to a given density character: Let μ be an infinite cardinal. Then there exists a Banach space X such that

(1) If Y is any Banach space such that $\mathrm{dens}(\mathrm{Y}) < \mu$, then X contains an isometric copy of Y.
(2) If closed subspaces $\mathrm{Y}_1, \mathrm{Y}_2 \subset \mathrm{X}$ with $\mathrm{dens}(\mathrm{Y}_1) = \mathrm{dens}(\mathrm{Y}_2) < \mu$ are mutually isometric, then there is $T \in \mathcal{G}_\mathrm{X}$ such that $T(\mathrm{Y}_1) = \mathrm{Y}_2$.

The construction of X above is not economical in the sense that typically dens(X) is much larger than μ. Since the 1-dimensional subspaces $[x]$ and $[y]$ are isometric for all $x, y \in \mathbf{S}_\mathrm{X}$, it follows that in (2) either $T(x) = y$ or $-T(x) = y$, so that X is in particular transitive. For recent results regarding somewhat economically produced homogeneous spaces, see [17].

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