BESOV CAPACITY AND HAUSDORFF MEASURES IN METRIC MEASURE SPACES

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Abstract. This paper studies Besov p-capacities as well as their relationship to Hausdorff measures in Ahlfors regular metric spaces of dimension Q for $1 < Q < p < \infty$. Lower estimates of the Besov p-capacities are obtained in terms of the Hausdorff content associated with gauge functions h satisfying the decay condition $\int_0^1 h(t)^{1/(p-1)} \frac{dt}{t} < \infty$.

1. Introduction

In this paper (X, d, μ) is a proper (that is, closed bounded subsets of X are compact) and unbounded metric space. In addition, it is Ahlfors Q-regular for some $Q > 1$. That is, there exists a constant $C = c_\mu$ such that, for each $x \in X$ and all $r > 0$,

$$C^{-1} r^Q \le \mu(B(x,r)) \le C r^Q.$$

For $Q < p < \infty$ we define

$$B_p(X) = \{u \in L^p(X) : ||u||_{B_p(X)} < \infty\},$$

where

$$||u||_{B_p(X)} = ||u||_{L^p(X)} + [u]_{B_p(X)} \tag{1}$$

with

$$[u]_{B_p(X)} = \left(\int_X \int_X \frac{|u(x) - u(y)|^p}{d(x,y)^{2Q}} \, d\mu(x) \, d\mu(y) \right)^{1/p}. \tag{2}$$

The expressions $||u||_{B_p(X)}$ and $[u]_{B_p(X)}$ from (1) and (2) are called the *Besov norm* and the *Besov seminorm* of u respectively. We have

$$[u]_{B_p(X)} = 0 \text{ if and only if } u \text{ is constant } \mu\text{-a.e.} \tag{3}$$

Besov spaces have recently been used in the study of quasiconformal mappings in metric spaces and in geometric group theory, see [Bou05] and [BP03].

Capacities associated with Besov spaces were studied by Netrusov in [Net92] and [Net96], and by Adams and Hurri-Syrjänen in [AHS03]. Bourdon in [Bou05] studied Besov B_p-capacity in the metric setting.

We develop a theory of Besov B_p-capacity on X and prove that this capacity is a Choquet set function. We also relate Hausdorff measure and Besov capacity when X is an Ahlfors Q-regular complete metric space with $Q > 1$ admitting a weak $(1, \widetilde{p})$-Poincaré inequality, where $1 \le \widetilde{p} < Q < p < \infty$. Some of the ideas used here follow [KM96], [KM00], [BP03], and [Bou05].

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2. Preliminaries

In this section we present the standard notations to be used throughout this paper. Here and throughout this paper $B(x,r) = \{y \in X : d(x,y) < r\}$ is the open ball with center $x \in X$ and radius $r > 0$, $\overline{B}(x,r) = \{y \in X : d(x,y) \le r\}$ is the closed ball with center $x \in X$ and radius $r > 0$, while $S(x,r) = \{y \in X : d(x,y) = r\}$ is the closed sphere with center $x \in X$ and radius $r > 0$. For a positive number λ, $\lambda B(a,r) = B(a,\lambda r)$ and $\lambda \overline{B}(a,r) = \overline{B}(a,\lambda r)$.

Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line. $C(a,b,\dots)$ is a constant that depends only on the parameters $a,b,\dots$. Here Ω will denote a nonempty open subset of X. For $E \subset X$, the boundary, the closure, and the complement of E with respect to X will be denoted by ∂E, $\overline{E}$, and $X \setminus E$, respectively; diam E is the diameter of E with respect to the metric d and $E \subset\subset F$ means that $\overline{E}$ is a compact subset of F.

For two sets $A, B \subset X$, we define $\operatorname{dist}(A,B)$, the distance between A and B, by

$$\operatorname{dist}(A,B) = \inf_{a \in A, b \in B} d(a,b).$$

For $\Omega \subset X$, $C(\Omega)$ is the set of all continuous functions $u : \Omega \to \mathbf{R}$. Moreover, for a measurable $u : \Omega \to \mathbf{R}$, supp u is the smallest closed set such that u vanishes on the complement of supp u. We also use the spaces

$$\begin{aligned} C_0(\Omega) &= \{\varphi \in C(\Omega) : \operatorname{supp} \varphi \subset\subset \Omega\}, \\ Lip(\Omega) &= \{\varphi : \Omega \to \mathbf{R} : \varphi \text{ is Lipschitz}\}, \\ Lip_{loc}(\Omega) &= \{\varphi : \Omega \to \mathbf{R} : \varphi \text{ is locally Lipschitz}\}, \\ Lip_0(\Omega) &= Lip(\Omega) \cap C_0(\Omega). \end{aligned}$$

Let $f : \Omega \to \mathbf{R}$ be integrable. For $E \subset \Omega$ measurable with $0 < \mu(E) < \infty$, we define

$$f_E = \frac{1}{\mu(E)} \int_E f \, d\mu(x).$$

We say that a locally integrable function $u : X \to \mathbf{R}$ belongs to BMO(X), the space of functions of bounded mean oscillation, if

$$[u]_{\mathrm{BMO(X)}} = \sup_{a \in X} \sup_{r > 0} \frac{1}{\mu(B(a,r))} \int_{B(a,r)} |u - u_{B(a,r)}| dx < \infty.$$

3. Besov spaces

In this section we prove some basic properties of the Besov spaces $B_p(X)$ and their closed subspaces $B_p(\Omega)$ and $B_p^0(\Omega)$, where $\Omega \subset X$ is an open set. We also present standard lemmas needed for the proofs of our main results.

We know that in the Euclidean case $B_p(\mathbf{R}^n)$ is a reflexive Banach space and moreover, $\mathcal{S}$ is dense in $B_p(\mathbf{R}^n)$ where $\mathcal{S} = \mathcal{S}(\mathbf{R}^n)$ is the Schwartz class. See [AH96, Theorem 4.1.3] and [Pee76, Chapter 3]. We would like to prove similar results about reflexivity and density when (X,d,μ) is an Ahlfors Q-regular metric space with $Q > 1$. It is easy to see that every Lipschitz function with compact support belongs to $B_p(X)$ whenever X is proper and unbounded.

We have the following lemma regarding the reflexivity of $B_p(X)$ when (X,d,μ) is an Ahlfors Q-regular metric space with $Q > 1$.

Lemma 3.1. *Suppose $1 < Q < p < \infty$ and that X is an Ahlfors Q-regular metric space. Then $B_p(X)$ is a reflexive space.*

Proof. Let ν be a measure on the product space $X \times X$ given by

$$d\nu(x,y) = d(x,y)^{-2Q} d\mu(x) d\mu(y).$$

We endow the product space $L^p(X,\mu) \times L^p(X \times X, \nu)$ with the product norm. Namely, for $(u,g) \in L^p(X,\mu) \times L^p(X \times X, \nu)$ we let

$$||(u,g)||_{L^p(X,\mu)\times L^p(X\times X,\nu)} = ||u||_{L^p(X,\mu)} + ||g||_{L^p(X\times X,\nu)}.$$

Clearly this product space is reflexive because it is a product of two reflexive spaces. Since $B_p(X)$ embeds isometrically into a closed subspace of this reflexive product space, we have that $B_p(X)$ is itself a reflexive space. This finishes the proof. □

Lemma 3.2. *Suppose $1 < Q < p < \infty$ and that X is an Ahlfors Q-regular metric space. There exists a constant $C = C(Q, p, c_\mu)$ such that $[u]_{\mathrm{BMO(X)}} \le C[u]_{B_p(X)}$ whenever $u \in L^1_{loc}(X)$.*

Proof. Indeed, let $u \in L^1_{loc}(X)$ be such that $[u]_{B_p(X)} < \infty$. Suppose that $B = B(a,R)$ is a ball in X. It is easy to see that there exists a constant $C = C(Q,p,c_\mu)$ such that

$$\begin{aligned}\frac{1}{\mu(B)}\int_B |u(x)-u_B|^p d\mu(x) &\le \frac{1}{\mu(B)^2}\int_B\int_B |u(x)-u(y)|^p d\mu(x)d\mu(y) \\ &\le C\int_B\int_B \frac{|u(x)-u(y)|^p}{d(x,y)^{2Q}} d\mu(x)d\mu(y).\end{aligned} \tag{4}$$

Therefore,

$$[u]_{\mathrm{BMO(X)}} \le C(Q,p,c_\mu)[u]_{B_p(X)} \tag{5}$$

and the claim follows. □

For an open set $\Omega \subset X$ we define

$$B_p(\Omega) = \{u \in B_p(X) : u = 0 \ \mu\text{-a.e. in } X \setminus \Omega\}.$$

For a function $u \in B_p(\Omega)$ we let $||u||_{B_p(\Omega)} = ||u||_{B_p(X)}$.

We notice that $B_p(\Omega)$ is a closed subspace of $B_p(X)$ with respect to the Besov norm, hence it is itself a reflexive space.

We define $B_p^0(\Omega)$ as the closure of $Lip_0(\Omega)$ in $B_p(X)$. Since $Lip_0(\Omega) \subset B_p(\Omega)$, it follows that $B_p^0(\Omega) \subset B_p(\Omega)$, so we can say that $B_p^0(\Omega)$ is the closure of $Lip_0(\Omega)$ in $B_p(\Omega)$.

Lemma 3.3. *$B_p(\Omega)$ is closed under truncations. In particular, bounded functions in $B_p(\Omega)$ are dense in $B_p(\Omega)$.*

Proof. The proof is very similar to the proof of [Cos, Lemma 2.1] and omitted. □

For a measurable function $u : \Omega \to \mathbf{R}$, we let $u^+ = \max(u,0)$ and $u^- = \min(u,0)$.

Lemma 3.4. *If $u_j \to u$ in $B_p(\Omega)$ and $v_j \to v$ in $B_p(\Omega)$, then $\min(u_j,v_j) \to \min(u,v)$ in $B_p(\Omega)$.*

Proof. The proof is similar to the proof of [Cos, Lemma 2.2] and omitted. □

Next we show that the space $B_p^0(\Omega)$ is a lattice.

Lemma 3.5. *If $u, v \in B_p^0(\Omega)$, then $\min(u, v)$ and $\max(u, v)$ are in $B_p^0(\Omega)$. Moreover, if $u \in B_p^0(\Omega)$ is nonnegative, then there is a sequence of nonnegative functions $\varphi_j \in Lip_0(\Omega)$ converging to u in $B_p(\Omega)$.*

Proof. It is enough to show, due to Lemma 3.4, that u^+ is in $B_p^0(\Omega)$ whenever u is in $Lip_0(\Omega)$. But this is immediate, because $u^+ \in Lip_0(\Omega)$ whenever $u \in Lip_0(\Omega)$. This finishes the proof. □

Lemma 3.6. *Let φ be a Lipschitz function with compact support in X. If $u \in B_p(X)$, then $u\varphi \in B_p(X)$ with*

$$||u\varphi||_{B_p(X)} \le C\,||u||_{B_p(X)},$$

where C depends on Q, p, c_μ, the Lipschitz constant of φ, and the diameter of $\operatorname{supp} \varphi$.

Proof. Let R be the diameter of $\operatorname{supp} \varphi$. We choose $x_0 \in \operatorname{supp} \varphi$ such that $\operatorname{supp} \varphi \subset \overline{B}$, where $B = B(x_0, R)$. Let $L > 0$ be a constant such that $|\varphi(x) - \varphi(y)| \le Ld(x, y)$ for every $x, y \in X$. Note that $||\varphi||_{L^\infty(X)} \le LR$. We also notice that

$$||u\varphi||_{L^p(X)} \le ||\varphi||_{L^\infty(X)}\,||u||_{L^p(X)},$$

hence $u\varphi \in L^p(X)$. Observe that

$$\int_X \int_X \frac{|u(x)\varphi(x) - u(y)\varphi(y)|^p}{d(x, y)^{2Q}} d\mu(x)\, d\mu(y) = I_1 + 2I_2,$$

where

$$I_1 = \int_{2B} \int_{2B} \frac{|u(x)\varphi(x) - u(y)\varphi(y)|^p}{d(x, y)^{2Q}} d\mu(x)\, d\mu(y) \tag{6}$$

and

$$I_2 = \int_{2B} \int_{X\setminus 2B} \frac{|u(x)\varphi(x) - u(y)\varphi(y)|^p}{d(x, y)^{2Q}} d\mu(x)\, d\mu(y). \tag{7}$$

For every $x, y \in X$ we have

$$|u(x)\varphi(x) - u(y)\varphi(y)| \le |u(x) - u(y)|\,|\varphi(x)| + |u(y)|\,|\varphi(x) - \varphi(y)|.$$

Therefore

$$I_1 \le 2^p(||\varphi||^p_{L^\infty(X)}[u]^p_{B_p(X)} + I_{11}), \tag{8}$$

where

$$I_{11} = \int_{2B} \int_{2B} \frac{|u(y)|^p\,|\varphi(x) - \varphi(y)|^p}{d(x, y)^{2Q}} d\mu(x)\, d\mu(y).$$

From the definition of I_{11} we have, since φ is Lipschitz with constant L,

$$\begin{aligned} I_{11} \;&\le\; \int_{2B} \int_{2B} \frac{L^p\,|u(y)|^p}{d(x, y)^{2Q-p}} d\mu(x)\, d\mu(y) \\ &=\; L^p \int_{2B} |u(y)|^p \left(\int_{2B} d(x, y)^{p-2Q} d\mu(x)\right) d\mu(y). \end{aligned} \tag{9}$$

We have

$$\int_{2B} |x - y|^{p-2Q} d\mu(x) \le C(Q, p, c_\mu) R^{p-Q} \tag{10}$$

for every $y \in 2B$, where we recall that R is the radius of B. From (9) and (10) we get

$$\begin{aligned} I_{11} &\leq C(Q,p,c_\mu)\, L^p\, R^{p-Q} \int_{2B} |u(y)|^p d\mu(y) \\ &\leq C(Q,p,c_\mu)\, L^p\, R^{p-Q}\, ||u||^p_{L^p(X)}. \end{aligned} \tag{11}$$

Since φ is supported in B, it follows from the definition of I_2 that

$$I_2 = \int_B \int_{X\setminus 2B} \frac{|u(y)|^p\, |\varphi(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y).$$

Hence

$$I_2 \leq ||\varphi||^p_{L^\infty(X)} \int_B \int_{X\setminus 2B} \frac{|u(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y)$$

and since $d(x,y) \geq \frac{d(x,x_0)}{2}$ whenever $x \in X \setminus 2B$ and $y \in B$, we get

$$I_2 \leq 2^{2Q}\, ||\varphi||^p_{L^\infty(X)} \int_B |u(y)|^p\, d\mu(y) \int_{X\setminus 2B} \frac{1}{d(x,x_0)^{2Q}} d\mu(x).$$

Hence

$$\begin{aligned} I_2 &\leq C(Q,p,c_\mu)\, ||\varphi||^p_{L^\infty(X)}\, R^{-Q} \int_B |u(y)|^p d\mu(y) \\ &\leq C(Q,p,c_\mu)\, ||\varphi||^p_{L^\infty(X)}\, R^{-Q}\, ||u||^p_{L^p(X)}. \end{aligned} \tag{12}$$

From (8), (11), (12), and the fact that $I = I_1 + 2I_2$, we get that $u\varphi \in B_p(X)$ with

$$||u\varphi||_{B_p(X)} \leq C||u||_{B_p(X)}, \tag{13}$$

where the constant C is as required. This finishes the proof. □

Lemma 3.7. *Let φ be a Lipschitz function with compact support in X. Suppose u_k is a sequence in $B_p(X)$ converging to u in $B_p(X)$. Then $u_k\varphi$ converges to $u\varphi$ in $B_p(X)$.*

Proof. From Lemma 3.6, we have that $u_k\varphi \in B_p(X)$ for every $k \geq 1$ and $u\varphi \in B_p(X)$. Moreover, Lemma 3.6 implies

$$||u_k\varphi - u\varphi||_{B_p(X)} \leq C||u_k - u||_{B_p(X)} \tag{14}$$

for every $k \geq 1$, and since $u_k \to u$ in $B_p(X)$, it follows that $u_k\varphi \to u\varphi$ in $B_p(X)$. This finishes the proof. □

Remark 3.8. Let $\Omega, \widetilde{\Omega}$ be bounded and open subsets of X with $\Omega \subset\subset \widetilde{\Omega}$. Suppose that φ is a function in $Lip_0(\widetilde{\Omega})$ with Lipschitz constant $C(Q,c_\mu)/\text{dist}(\Omega, X\setminus\widetilde{\Omega})$ such that

$$0 \leq \varphi \leq 1 \text{ and } \varphi = 1 \text{ in } \Omega. \tag{15}$$

By an argument similar to the one from Lemma 3.6, one can show that $u\varphi \in B_p(\widetilde{\Omega})$ whenever $u \in B_p(X)$ and $\varphi \in Lip_0(\widetilde{\Omega})$ satisfies (15). Moreover, in this case

$$||u\varphi||_{B_p(\widetilde{\Omega})} \leq C||u||_{B_p(X)}$$

for all $u \in B_p(X)$ and the constant $C > 0$ can be chosen to depend only on Q, p, c_μ, $\text{dist}(\Omega, X\setminus\widetilde{\Omega})$, and the diameter of $\widetilde{\Omega}$.

Remark 3.9. It is easy to see that $u\varphi \in B_p(X)$ whenever u, φ are bounded functions in $B_p(X)$. Moreover,

$$||u\varphi||_{L^p(X)} \leq \min(||u||_{L^\infty(X)}||\varphi||_{L^p(X)},\ ||\varphi||_{L^\infty(X)}||u||_{L^p(X)})$$

and

$$[u\varphi]_{B_p(X)} \leq ||u||_{L^\infty(X)}[\varphi]_{B_p(X)} + ||\varphi||_{L^\infty(X)}[u]_{B_p(X)}.$$

Lemma 3.10. *Let $B = B(x_0, R) \subset X$ and η be a $C(c_\mu)/R$-Lipschitz function supported in $2B$ such that $0 \leq \eta \leq 1$. Then there exists a constant $C = C(Q, p, c_\mu)$ such that*

$$[\eta(v - v_B)]_{B_p(X)} \leq C[v]_{B_p(X)}$$

whenever $v \in L^1_{loc}(X)$ with $[v]_{B_p(X)} < \infty$.

Proof. Let $v \in L^1_{loc}(X)$ such that $[v]_{B_p(X)} < \infty$. Then $v \in L^p_{loc}(X)$ and this implies, since $\eta \in Lip_0(2B)$, that $\eta(v - v_B) \in L^p(X)$. We repeat to some extent the argument of Lemma 3.6 with $\varphi = \eta$ and $u = v - v_B$. We can choose $L = \frac{C(c_\mu)}{R}$ and we note that $||\eta||_{L^\infty(X)} \leq 1$. Hence

$$\int_X \int_X \frac{|u(x)\eta(x) - u(y)\eta(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y) = I_1 + 2\, I_2, \tag{16}$$

where

$$I_1 = \int_{4B} \int_{4B} \frac{|u(x)\eta(x) - u(y)\eta(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y)$$

and

$$I_2 = \int_{4B} \int_{X\setminus 4B} \frac{|\eta(x)u(x) - \eta(y)u(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y)$$

We notice that $I_1 \leq 2^p(I_{10} + I_{11})$, where

$$I_{10} = \int_{4B} \int_{4B} \frac{|\eta(y)(u(x) - u(y))|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y)$$

and

$$I_{11} = \int_{4B} \int_{4B} \frac{|u(x)(\eta(x) - \eta(y))|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y).$$

We have

$$I_{10} \leq \int_{4B} \int_{4B} \frac{|u(x) - u(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y) \leq [v]^p_{B_p(X)} \tag{17}$$

since $||\eta||_{L^\infty(X)} \leq 1$. As in (11) we get with $L = \frac{C(c_\mu)}{R}$

$$I_{11} \leq C(Q, p, c_\mu)\, R^{-Q} \int_{4B} |v(y) - v_B|^p d\mu(y). \tag{18}$$

Because η is supported in $2B$, it follows from the definition of I_2 that in fact

$$I_2 = \int_{2B} \int_{X\setminus 4B} \frac{|v(y) - v_B|^p\, |\eta(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y).$$

As in Lemma 3.6 we get

$$I_2 \leq C(Q, p, c_\mu)\, R^{-Q} \int_{2B} |v(y) - v_B|^p d\mu(y). \tag{19}$$

From (16), (17), (18), (19), and the fact that $I_1 \leq 2^p(I_{10} + 2I_{11})$, we have that $\eta(v - v_B) \in B_p(X)$ with

$$\begin{aligned}[\eta(v - v_B)]^p_{B_p(X)} &\leq C(Q, p, c_\mu) \int_{4B}\int_{4B} \frac{|v(x) - v(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y) \\ &\leq C(Q, p, c_\mu)\, [v]^p_{B_p(X)}.\end{aligned}$$

This finishes the proof. □

We now show that every function in $B_p(X)$ can be approximated by locally Lipschitz functions in $B_p(X)$.

Proposition 3.11. *$Lip_{loc}(X) \cap B_p(X)$ is dense in $B_p(X)$. More precisely, if u has finite Besov seminorm, then there exists a sequence $u_\varepsilon, \varepsilon > 0$, in $Lip_{loc}(X)$ such that:*
(i) $[u_\varepsilon - u]_{B_p(X)} \to 0$ *as* $\varepsilon \to 0$,
(ii) $||u_\varepsilon - u||_{L^p(X)} \to 0$ *as* $\varepsilon \to 0$.

Proof. For every $\varepsilon > 0$ we construct a family of balls $B(x_i, \varepsilon)$ that cover X, have bounded overlap, and form a c_1/ε-Lipschitz partition of unity associated with that cover as in [KL02]. Here $c_1 = c_1(c_\mu)$. More precisely, we choose a family of balls $B(x_i, \varepsilon), i = 1, 2, \dots$, such that

$$X \subset \bigcup_{i=1}^{\infty} B(x_i, \varepsilon)$$

and

$$\sum_{i=1}^{\infty} \chi_{6B(x_i,\varepsilon)} < c_0 = c_0(Q, c_\mu). \tag{20}$$

Now we choose a sequence of c_1/ε-Lipschitz functions $\varphi_i, i = 1, 2, \dots$, such that $0 \leq \varphi_i \leq 1$, $\varphi_i = 0$ on $X \setminus 6B(x_i, \varepsilon)$, $\varphi_i \geq 1/c_0$ on $3B(x_i, \varepsilon)$, where c_0 is the constant from (20) and such that

$$\sum_{i=1}^{\infty} \varphi_i = 1$$

on X. We define the approximation by setting

$$u_\varepsilon(x) = \sum_{i=1}^{\infty} \varphi_i(x) u_{3B(x_i,\varepsilon)}$$

for every $x \in X$. Then u_ε is a locally Lipschitz function.

(i) We note that

$$u_\varepsilon(x) - u(x) = \sum_{i=1}^{\infty} \varphi_i(x)(u_{3B(x_i,\varepsilon)} - u(x))$$

for every $x \in X$. From this and (20) we obtain

$$[u_\varepsilon - u]^p_{B_p(X)} \leq (2c_0)^p \sum_{i=1}^{\infty} [\varphi_i(u_{3B(x_i,\varepsilon)} - u)]^p_{B_p(X)}, \tag{21}$$

where c_0 is the bounded overlap constant appearing in (20). However, from Lemma 3.10 there exists a constant $C = C(Q, p, c_\mu)$ such that

$$[\varphi_i(u_{3B(x_i,\varepsilon)} - u)]^p_{B_p(X)} \leq C \int_{12B(x_i,\varepsilon)}\int_{12B(x_i,\varepsilon)} \frac{|u(x) - u(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y)$$

for every $i = 1, 2, \dots$, From this and (21) we obtain

$$(22) \qquad [u_\varepsilon - u]^p_{B_p(X)} \le C \sum_{i=1}^{\infty} \int_{12B(x_i,\varepsilon)} \int_{12B(x_i,\varepsilon)} \frac{|u(x) - u(y)|^p}{d(x,y)^{2Q}} d\mu(x)\, d\mu(y),$$

where $C = C(Q, p, c_\mu)$. If we denote

$$A_\varepsilon = \{(x, y) \in X \times X : d(x, y) < 24\varepsilon\},$$

we have from (20) and (22) that

$$[u_\varepsilon - u]^p_{B_p(X)} \le C(Q, p, c_\mu) \int_X \int_X \frac{|u(x) - u(y)|^p}{d(x,y)^{2Q}} \chi_{A_\varepsilon}(x, y) d\mu(x)\, d\mu(y).$$

An application of Lebesgue Dominated Convergence Theorem yields $[u_\varepsilon - u]_{B_p(X)} \to 0$ as $\varepsilon \to 0$. Moreover, we also notice that $[u_\varepsilon]_{B_p(X)} \le C(Q, p, c_\mu)[u]_{B_p(X)}$ for every $\varepsilon > 0$.

(ii) By using (20) and the fact that φ_i forms a partition of unity we obtain, via an argument similar to the one from Lemma 3.2

$$(23) \qquad \begin{aligned} ||u_\varepsilon - u||^p_{L^p(X)} &\le (c_0)^p \sum_{i=1}^{\infty} ||\varphi_i(u_{3B(x_i,\varepsilon)} - u)||^p_{L^p(X)} \\ &\le (c_0)^p \sum_{i=1}^{\infty} \int_{6B(x_i,\varepsilon)} |u(x) - u_{3B(x_i,\varepsilon)}|^p d\mu(x) \\ &\le C(Q, p, c_\mu)\, \varepsilon^Q \int_X \int_X \frac{|u(x) - u(y)|^p}{d(x,y)^{2Q}} d\mu(x) d\mu(y), \end{aligned}$$

where c_0 is the constant from (20). This implies immediately that $||u_\varepsilon - u||_{L^p(X)} \to 0$ as $\varepsilon \to 0$. This finishes the proof.

□

Proposition 3.12. *$Lip_0(X)$ is dense in $B_p(X)$.*

Proof. Let $u \in B_p(X)$. Without loss of generality we can assume that u is locally Lipschitz and in particular bounded. We fix $x_0 \in X$. For every integer $k \ge 2$, we define $\varphi_k : X \to \mathbf{R}$ by

$$\varphi_k(x) = \begin{cases} 1 & \text{if } 0 \le d(x, x_0) \le k, \\ \frac{\ln \frac{k^2}{d(x,x_0)}}{\ln k} & \text{if } k < d(x, x_0) \le k^2, \\ 0 & \text{if } d(x, x_0) > k^2. \end{cases}$$

Then $\varphi_k \in B_p(X)$ and moreover, $[\varphi_k]^p_{B_p(X)} \le C(\ln k)^{1-p}$. (See (24).)

Let $u_k = u\varphi_k$. Then $u_k \in Lip_0(X)$ and

$$||u - u_k||_{L^p(X)} \le ||u\chi_{X \setminus B(x_0,k)}||_{L^p(X)} \to 0 \text{ as } k \to \infty.$$

We also have

$$\begin{aligned} [u - u_k]_{B_p(X)} &\le \left(\int_X \int_X \frac{(1 - \varphi_k(y))^p |u(x) - u(y)|^p}{d(x,y)^{2Q}} \, d\mu(x)\, d\mu(y) \right)^{1/p} \\ &+ ||u||_{L^\infty(X)} [\varphi_k]_{B_p(X)} \to 0 \end{aligned}$$

as $k \to \infty$. This finishes the proof. □

Lemma 3.13. *Let $v \in B_p(\Omega)$.*

(i) *If* supp $v \subset\subset \Omega$, then $v \in B^0_p(\Omega)$.

(ii) If $u \in B^0_p(\Omega)$ and $0 \le v \le u$ in X, then $v \in B^0_p(\Omega)$.

Proof. The proof is similar to the proof of [Cos, Lemma 2.10] and omitted. □

Lemma 3.14. *Suppose that* $\Omega \subset\subset X$*. Let* $u \in B_p(\Omega)$ *such that* $u = 0$ *on* $X \setminus \Omega$ *and* $\lim_{\Omega \ni x \to y} u(x) = 0$ *for all* $y \in \partial\Omega$*. Then* $u \in B_p^0(\Omega)$*.*

Proof. The proof is similar to the proof of [Cos, Lemma 2.11] and omitted. □

4. Relative Besov capacity

In this section, we establish a general theory of relative Besov capacity and study how this capacity is related to Hausdorff measures.

For $E \subset \Omega$ we define

$$BA(E, \Omega) = \{u \in B_p^0(\Omega) : u \geq 1 \text{ on a neighborhood of } E\}.$$

We call $BA(E, \Omega)$ the set of *admissible functions for the condenser* (E, Ω). The *relative Besov p-capacity* of the pair (E, Ω) is denoted by

$$\operatorname{cap}_{B_p}(E, \Omega) = \inf\{[u]^p_{B_p(\Omega)} : u \in BA(E, \Omega)\}.$$

If $BA(E, \Omega) = \emptyset$, we set $\operatorname{cap}_{B_p}(E, \Omega) = \infty$.

Since $B_p^0(\Omega)$ is closed under truncations and the truncation does not increase the B_p-seminorm, we may restrict ourselves to those admissible functions u for which $0 \leq u \leq 1$.

Remark 4.1. If K is a compact subset of the bounded and open set $\Omega \subset X$, we get the same Besov B_p-capacity for (K, Ω) if we restrict ourselves to a smaller set of admissible functions, namely

$$BW(K, \Omega) = \{u \in Lip_0(\Omega) : u = 1 \text{ in a neighborhood of } K\}.$$

Indeed, let $u \in BA(K, \Omega)$; we may clearly assume that $u = 1$ in a neighborhood $U \subset\subset \Omega$ of K. Then we choose a cut-off Lipschitz function η, $0 \leq \eta \leq 1$ such that $\eta = 1$ in $X \setminus U$ and $\eta = 0$ in a neighborhood $\widetilde{U}$ of K, $\widetilde{U} \subset\subset U$. Now, if $\varphi_j \in Lip_0(\Omega)$ is a sequence converging to u in $B_p^0(\Omega)$, then $\psi_j = 1 - \eta(1 - \varphi_j)$ is a sequence belonging to $BW(K, \Omega)$ which converges to $1 - \eta(1 - u)$ in $B_p^0(\Omega)$. (See Lemma 3.7.) But $1 - \eta(1 - u) = u$. This establishes the assertion, since $BW(K, \Omega) \subset BA(K, \Omega)$. In fact, it is easy to see that if $K \subset \Omega$ is compact we get the same Besov B_p-capacity if we consider

$$B\widetilde{W}(K, \Omega) = \{u \in Lip_0(\Omega) : u = 1 \text{ on } K\}.$$

It is also useful to observe that if $\psi \in B_p^0(\Omega)$ is such that $\varphi - \psi \in B_p^0(\Omega \setminus K)$ for some $\varphi \in B\widetilde{W}(K, \Omega)$, then

$$\operatorname{cap}_{B_p}(K, \Omega) \leq [\psi]^p_{B_p(\Omega)}.$$

4.1. **Basic properties of the relative Besov capacity.** A capacity is a monotone, subadditive set function. The following theorem expresses, among other things, that this is true for the relative Besov p-capacity.

Theorem 4.2. *Suppose* (X, d, μ) *is a proper and unbounded Ahlfors Q-regular metric space with* $1 < Q < p < \infty$*. Let* $\Omega \subset X$ *be a bounded open set. The set function* $E \mapsto \operatorname{cap}_{B_p}(E, \Omega)$, $E \subset \Omega$, *enjoys the following properties:*

(i) *If* $E_1 \subset E_2$*, then* $\operatorname{cap}_{B_p}(E_1, \Omega) \leq \operatorname{cap}_{B_p}(E_2, \Omega)$.

(ii) *If* $\Omega_1 \subset \Omega_2$ *are open, bounded, and* $E \subset \Omega_1$*, then*

$$\operatorname{cap}_{B_p}(E, \Omega_2) \leq \operatorname{cap}_{B_p}(E, \Omega_1).$$

(iii) $\operatorname{cap}_{B_p}(E,\Omega) = \inf\{\operatorname{cap}_{B_p}(U,\Omega) : E \subset U \subset \Omega,\, U \text{ open}\}$.

(iv) *If K_i is a decreasing sequence of compact subsets of Ω with $K = \bigcap_{i=1}^{\infty} K_i$, then*

$$\operatorname{cap}_{B_p}(K,\Omega) = \lim_{i\to\infty} \operatorname{cap}_{B_p}(K_i,\Omega).$$

(v) *If $E_1 \subset E_2 \subset \ldots \subset E = \bigcup_{i=1}^{\infty} E_i \subset \Omega$, then*

$$\operatorname{cap}_{B_p}(E,\Omega) = \lim_{i\to\infty} \operatorname{cap}_{B_p}(E_i,\Omega).$$

(vi) *If $E = \bigcup_{i=1}^{\infty} E_i \subset \Omega$, then*

$$\operatorname{cap}_{B_p}(E,\Omega) \le \sum_{i=1}^{\infty} \operatorname{cap}_{B_p}(E_i,\Omega).$$

Proof. The proof is very similar to the proof of [Cos, Theorem 3.1] and is therefore omitted. □

A set function that satisfies properties (i), (iv), (v) and (vi) is called a *Choquet capacity* (relative to Ω). We may thus invoke an important capacitability theorem of Choquet and state the following result. See [Doo84, Appendix II].

Theorem 4.3. *Suppose (X,d,μ) is a metric measure space as in Theorem 4.2. Suppose that Ω is a bounded open set in X. The set function $E \mapsto \operatorname{cap}_{B_p}(E,\Omega)$, $E \subset \Omega$, is a Choquet capacity. In particular, all Borel subsets (in fact, all analytic) subsets E of Ω are capacitable, i.e.,*

$$\operatorname{cap}_{B_p}(E,\Omega) = \sup\{\operatorname{cap}_{B_p}(K,\Omega) : K \subset E \text{ compact}\}$$

whenever $E \subset \Omega$ is analytic.

4.2. **Upper estimates for the relative Besov capacity.** Next we derive some upper estimates for the relative Besov capacity. Similar estimates have been obtained earlier by Bourdon in [Bou05]. We follow his methods.

Theorem 4.4. *Let (X,d,μ) be a metric measure space as in Theorem 4.2. There exists a constant $C = C(Q,p,c_\mu) > 0$ depending only on Q, p and c_μ such that*

$$\operatorname{cap}_{B_p}(B(x_0,r),B(x_0,R)) \le C\left(\ln\frac{R}{r}\right)^{1-p} \tag{24}$$

for every $0 < r < \frac{R}{2}$ and every $x_0 \in X$.

Proof. We use the function $u : X \to \mathbf{R}$,

$$u(x) = \begin{cases} 1 & \text{if } 0 \le d(x,x_0) \le r, \\ \frac{\ln \frac{d(x,x_0)}{R}}{\ln \frac{r}{R}} & \text{if } r < d(x,x_0) < R, \\ 0 & \text{if } d(x,x_0) \ge R. \end{cases}$$

Then $u \in B_p(X)$ because it is Lipschitz with compact support. Since u is continuous on X and 0 outside $B(x_0,R)$, we have in fact from Lemma 3.14 that $u \in B_p^0(B(x_0,R))$. In fact $u \in BA(B(x_0,r),B(x_0,R))$ since $u = 1$ on $B(x_0,r)$. Let $v(x) = \ln\frac{R}{r}\,u(x)$. We will get an upper bound for $[v]_{B_p(B(x_0,R))}$. Let $k \ge 3$ be the smallest integer such that $2^{k-1}r \ge R$. For $i = 1,\ldots,k$ we define $B_i = B(x_0,2^i r) \setminus \overline{B}(x_0,2^{i-1}r)$. We also define $B_0 = B(x_0,r)$ and $B_{k+1} = X \setminus B(x_0,2^k r)$. We have

$$[v]^p_{B_p(B(x_0,R))} = \sum_{0\le i,j\le k+1} I_{i,j} = \sum_{0\le i,j\le k+1} \int_{B_i}\int_{B_j} \frac{|v(x)-v(y)|^p}{d(x,y)^{2Q}}\, d\mu(x)\, d\mu(y).$$

Obviously we have $I_{i,j} = I_{j,i}$. We majorize $I_{i,j}$ by distinguishing a few cases. For $j \le k$ and $0 \le i \le j-2$ we have from the definition of v that $|v(x) - v(y)| \le j - i + 1$ whenever $x \in B_i$ and $y \in B_j$, hence

$$I_{i,j} \le C_0(j-i+1)^p\,(2^j r)^{-2Q}\,(2^i r)^Q\,(2^j r)^Q,$$

that is $I_{i,j} \le C_1(j-i)^p 2^{(i-j)Q}$. For $0 \le i \le j \le k$ we notice, since v is $\frac{1}{2^{i-1}r}$-Lipschitz on $\bigcup_{j\ge i} B_j$ that

$$I_{i,j} \le (2^{i-1}r)^{-p} \int_{B_i}\int_{B_j} \frac{1}{d(x,y)^{2Q-p}}\,d\mu(x)\,d\mu(y).$$

Moreover, we have

$$\int_{B_j} \frac{1}{d(x,y)^{2Q-p}}\,d\mu(x) \le C_2(\text{diam } B_j)^{p-Q}$$

for every $y \in B(x_0, 2^i r)$, where C_2 depends only on p, Q and c_μ. Hence for $0 \le i \le j \le k$ we have

$$I_{i,j} \le C_3(2^{i-1}r)^{-p}(2^i r)^Q(2^j r)^{p-Q} \le C_4 2^{(j-i)(p-Q)}.$$

In particular, for $j-1 \le i \le j \le k$, the integral $I_{i,j}$ is bounded by a constant that depends only on p, Q and c_μ. Now we have to bound $I_{i,j}$ when $j = k+1$. Since v is constant on $B_k \cup B_{k+1}$, we have $I_{i,k+1} = 0$ for $i \in \{k, k+1\}$. For $0 \le i \le k-1$ we have

$$I_{i,k+1} \le (k-i+1)^p \int_{B_i}\int_{B_{k+1}} \frac{1}{d(x,y)^{2Q}}\,d\mu(x)\,d\mu(y).$$

But there exists $C_5 > 0$ such that

$$\int_{B_{k+1}} \frac{1}{d(x,y)^{2Q}}\,d\mu(x) \le C_5(2^{k+1}r)^{-Q}$$

for every $y \in X$ with $d(y, x_0) \le 2^{k-1}r$. Hence $I_{i,k+1} \le C_6(k-i+1)^p 2^{(i-k-1)Q}$. Finally we have

$$[v]^p_{B_p(B(x_0,R))} \le C_7 k + C_8 \sum_{0\le i\le j\le k+1} (j-i)^p 2^{(i-j)Q}.$$

The last sum is equal to

$$\sum_{l=1}^{k+1} (k+1-l)l^p 2^{-lQ}.$$

But $k+1-l \le k+1$ and there exists $a > 1$ such that $l^p 2^{-lQ} \le C_9 a^{-l}$ for $l \ge 1$. Hence

$$[v]^p_{B_p(B(x_0,R))} \le C_{10} \ln \frac{R}{r}$$

and

$$[u]^p_{B_p(B(x_0,R))} \le C_{10} \left(\ln \frac{R}{r}\right)^{1-p}.$$

The claim follows with $C = C_{10}$. □

For a fixed $r > 0$ we construct the dyadic partition of X as in [Chr90, Theorem11]. That is, a family of open sets $\mathcal{D}_r = \{K^\alpha_{m,r} : m \in \mathbf{Z}, \alpha \in I_m\}$ such that

(i) $\mu(X \setminus \bigcup_\alpha K^\alpha_{m,r}) = 0, \forall m$.

(ii) If $l \ge m$ then either $K^\beta_{l,r} \subset K^\alpha_{m,r}$ or $K^\beta_{l,r} \cap K^\alpha_{m,r} = \emptyset$.

(iii) For each (m, α) and each $l < m$ there is a unique β such that $K^\alpha_{m,r} \subset K^\beta_{l,r}$.

(iv) For every (m,α) there exists a ball $B^\alpha_{m,r} = B(x^\alpha_{m,r}, 10^{-m}r)$ such that

$$\frac{1}{10}B^\alpha_{m,r} \subset K^\alpha_{m,r} \subset 3B^\alpha_{m,r}.$$

We call these open sets "dyadic cubes".

Two distinct dyadic cubes K, K' in $\mathcal{D}_r$ are *adjacent* if there exists an integer k such that either

(i) K, K' are in generation k and $\overline{K} \cap \overline{K'} \neq \emptyset$, or

(ii) one of the cubes K, K' is in generation k, the other one is in generation $k+1$ the one in generation k contains the other one.

Similarly, if $K_0 \subset X$ is a dyadic cube in $\mathcal{D}_r$, we denote by $\mathcal{D}_r(K_0)$ the dyadic subcubes of K_0.

For two adjacent cubes $K, K' \in \mathcal{D}_r$ we have

$$\begin{aligned} |f_{\overline{K}} - f_{\overline{K'}}|^p &= \left| \frac{1}{\mu(\overline{K})} \int_{\overline{K}} f(x)\, d\mu(x) - \frac{1}{\mu(\overline{K'})} \int_{\overline{K'}} f(y)\, d\mu(y) \right|^p \\ &= \left| \frac{1}{\mu(\overline{K})} \frac{1}{\mu(\overline{K'})} \int_{\overline{K}} \int_{\overline{K'}} (f(x) - f(y))\, d\mu(x)\, d\mu(y) \right|^p \\ &\leq \frac{1}{\mu(\overline{K})} \frac{1}{\mu(\overline{K'})} \int_{\overline{K}} \int_{\overline{K'}} |f(x) - f(y)|^p\, d\mu(x)\, d\mu(y) \\ &\leq C \int_{\overline{K}} \int_{\overline{K'}} \frac{|f(x)-f(y)|^p}{d(x,y)^{2Q}}\, d\mu(x)\, d\mu(y), \end{aligned}$$

where C is a constant that depends only on the Ahlfors regularity of X.

For the following lemma see [BP03, Lemma 3.5].

Lemma 4.5. *There exists a constant C depending only on the Ahlfors regularity of X such that*

$$C^{-1}|\eta - \zeta|^{-2Q} \leq \sum_{K,K' \in \mathcal{D}_r \text{ adjacent}} \frac{\chi_{\overline{K}}(\eta)\chi_{\overline{K'}}(\zeta)}{\mu(\overline{K})\mu(\overline{K'})} \leq C|\eta - \zeta|^{-2Q}$$

for μ-a.e. $\eta, \zeta \in X$.

We also have (see [BP03, Theorem 3.4]):

Lemma 4.6. *There exists a constant C depending only on p and on the Ahlfors regularity of X such that*

$$\begin{aligned} C^{-1}[f]^p_{B_p(X)} &\leq \sum_{K,K' \in \mathcal{D}_r \text{ adjacent}} \frac{1}{\mu(\overline{K})} \frac{1}{\mu(\overline{K'})} \int_{\overline{K}} \int_{\overline{K'}} |f(x) - f(y)|^p\, d\mu(x)\, d\mu(y) \\ &\leq C[f]^p_{B_p(X)} \end{aligned}$$

for every $f \in B_p(X)$.

This implies (see [BP03, Lemma 3.5]):

Lemma 4.7. *There exists a constant C depending only on p and on the Ahlfors regularity of X such that*

$$\sum_{K,K' \in \mathcal{D}_r \text{ adjacent}} |f_{\overline{K}} - f_{\overline{K'}}|^p \leq C[f]^p_{B_p(X)} \tag{25}$$

for every $f \in B_p(X)$.

4.3. **Hausdorff measure and relative Besov capacity.** Now we examine the relationship between Hausdorff measures and the B_p-capacity. Let h be a real-valued and increasing function on $[0,\infty)$ such that $\lim_{t\to 0} h(t) = h(0) = 0$ and $\lim_{t\to\infty} h(t) = \infty$. Such a function h is called a *measure function.* Let $0 < \delta \le \infty$. Suppose $\Omega \subset X$ is open. For $E \subset \overline{\Omega}$ we define

$$\Lambda^{\delta}_{h,\overline{\Omega}}(E) = \inf \sum_i h(r_i),$$

where the infimum is taken over all coverings of E by open sets G_i in $\overline{\Omega}$ with diameter r_i not exceeding δ. The set function $\Lambda^{\infty}_{h,\overline{\Omega}}$ is called the *h-Hausdorff content relative to* $\overline{\Omega}$. Clearly $\Lambda^{\delta}_{h,\overline{\Omega}}$ is an outer measure for every $\delta \in (0,\infty]$ and every open set $\Omega \subset X$. We write $\Lambda^{\delta}_{h}(E)$ for $\Lambda^{\delta}_{h,X}(E)$.

Moreover, for every $E \subset \overline{\Omega}$, there exists a Borel set $\widetilde{E}$ such that $E \subset \widetilde{E} \subset \overline{\Omega}$ and $\Lambda^{\delta}_{h,\overline{\Omega}}(E) = \Lambda^{\delta}_{h,\overline{\Omega}}(\widetilde{E})$. Clearly $\Lambda^{\delta}_{h,\overline{\Omega}}(E)$ is a decreasing function of δ. It is easy to see that $\Lambda^{\delta}_{h,\overline{\Omega_2}}(E) \le \Lambda^{\delta}_{h,\overline{\Omega_1}}(E)$ for every $\delta \in (0,\infty]$ whenever Ω_1 and Ω_2 are open sets in X such that $E \subset \overline{\Omega_1} \subset \overline{\Omega_2}$. This allows us to define the *h-Hausdorff measure relative to* $\overline{\Omega}$ of $E \subset \overline{\Omega}$ by

$$\Lambda_{h,\overline{\Omega}}(E) = \sup_{\delta>0} \Lambda^{\delta}_{h,\overline{\Omega}}(E) = \lim_{\delta\to 0} \Lambda^{\delta}_{h,\overline{\Omega}}(E).$$

The measure $\Lambda_{h,\overline{\Omega}}$ is Borel regular; that is, it is an additive measure on Borel sets of $\overline{\Omega}$ and for each $E \subset \overline{\Omega}$ there is a Borel set G such that $E \subset G \subset \overline{\Omega}$ and $\Lambda_{h,\overline{\Omega}}(E) = \Lambda_{h,\overline{\Omega}}(G)$. (See [Fed69, p. 170] and [Mat95, Chapter 4].) If $h(t) = t^s$, we write Λ_s for $\Lambda_{t^s,X}$. It is immediate from the definition that $\Lambda_s(E) < \infty$ implies $\Lambda_u(E) = 0$ for all $u > s$. The smallest $s \ge 0$ that satisfies $\Lambda_u(E) = 0$ for all $u > s$ is called the *Hausdorff dimension of E.*

For $\Omega \subset X$ open and $\delta > 0$ the set function $\Lambda^{\delta}_{h,\overline{\Omega}}$ has the following property:

(i) If K_i is a decreasing sequence of compact sets in $\overline{\Omega}$, then

$$\Lambda^{\delta}_{h,\overline{\Omega}}\Big(\bigcap_{i=1}^{\infty} K_i\Big) = \lim_{i\to\infty} \Lambda^{\delta}_{h,\overline{\Omega}}(K_i).$$

Moreover, if $\Omega \subset\subset X$ and h is a continuous measure function, then $\Lambda^{\delta}_{h,\overline{\Omega}}$ satisfies the following additional properties:

(ii) If E_i is an increasing sequence of arbitrary sets in $\overline{\Omega}$, then

$$\Lambda^{\delta}_{h,\overline{\Omega}}\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \lim_{i\to\infty} \Lambda^{\delta}_{h,\overline{\Omega}}(E_i).$$

(iii) $\Lambda^{\delta}_{h,\overline{\Omega}}(E) = \sup\{\Lambda^{\delta}_{h,\overline{\Omega}}(K) : K \subset E \text{ compact}\}$ whenever $E \subset \overline{\Omega}$ is a Borel set. (See [Rog70, Chapter 2:6].)

We have the following proposition:

Proposition 4.8. *Suppose (X, d, μ) is an Ahlfors Q-regular metric space with $Q > 1$. Let $h : [0,\infty) \to [0,\infty)$ be a measure function.*

(a) If $\liminf_{t\to 0} h(t)t^{-Q} = 0$, then $\Lambda^{\delta}_{h}(X) = 0$.

(b) If $\liminf_{t\to 0} h(t)t^{-Q} > 0$, then there is an increasing function $h^* : [0,\infty) \to [0,\infty)$ such that $h^*(0) = 0$, h^* is continuous, $t \mapsto h(t)\,t^{-Q}$, $0 < t < \infty$ is decreasing and there

exists a constant $C = C(Q, c_\mu)$ such that for all $E \subset X$ and all $\delta > 0$

$$C^{-1}\Lambda_h^\delta(E) \le \Lambda_{h^*}^\delta(E) \le C\Lambda_h^\delta(E).$$

Proof. The proof is similar to the proof of [AH96, Proposition 5.1.8] and omitted. □

If $h : [0, \infty) \to [0, \infty)$ is a continuous increasing measure function such that $t \mapsto h(t)\, t^{-Q}$, $0 < t < \infty$ is decreasing, we know that $\Lambda_h(E) = 0$ if and only if $\Lambda_h^\infty(E) = 0$. (See [AH96, Proposition 5.1.5].) If $h(t) = t^s$, $0 < s < \infty$, we write Λ_s^∞ for $\Lambda_{t^s,X}^\infty$.

Theorem 4.9. *Suppose $1 \le \widetilde{p} < Q < p < \infty$. Let (X, d, μ) be a complete and unbounded Ahlfors Q-regular metric space that supports a weak $(1, \widetilde{p})$-Poincaré inequality. Suppose $h : [0, \infty) \to [0, \infty)$ is a continuous increasing measure function such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$ is decreasing. Let $K_{0,r} \in \mathcal{D}_r$ be a dyadic cube of generation 0 and let $x_0 \in X$ be such that $B(x_0, r/10) \subset K_{0,r}$. There exists a positive constant $C_1' = C_1'(Q, p, c_\mu)$ such that*

$$\frac{\Lambda_h^\infty(E \cap \overline{K}_{k,r})}{\left(\int_0^{10^{-k}r} h(t)^{p'-1}\frac{dt}{t}\right)^{p-1}} \le C_1' k^{p-1}\mathrm{cap}_{B_p}(E \cap \overline{K}_{k,r}, B(x_0, r/10)) \tag{26}$$

for every $E \subset X$, every $k > 1, r > 0$, and for every $K_{k,r} \in \mathcal{D}_r(K_{0,r})$ cube of generation k such that $B(x_0, 10^{-k}r) \cap \overline{K}_{k,r} \neq \emptyset$.

Proof. We fix $r > 0$ and $k > 1$. Suppose $K_{k,r} \in \mathcal{D}_r(K_{0,r})$ is a dyadic subcube of $K_{0,r}$ of generation k such that $\overline{K}_{k,r} \cap B(x_0, 10^{-k}r) \neq \emptyset$.

Let $E \subset X$. From the fact that there exists a Borel set $\widetilde{E}$ such that $E \subset \widetilde{E} \subset X$ and $\mathrm{cap}_{B_p}(E \cap \overline{K}_{k,r}, B(x_0, r/10)) = \mathrm{cap}_{B_p}(\widetilde{E} \cap \overline{K}_{k,r}, B(x_0, r/10))$, we can assume that E is a Borel set. Moreover, from the discussion before Proposition 4.8 and the fact that $\mathrm{cap}_{B_p}(\cdot, B(x_0, r/10))$ is a Choquet capacity, we can assume without loss of generality that E is compact.

There is nothing to prove if either $\Lambda_h^\infty(E \cap \overline{K}_{k,r}) = 0$ or if $\int_0^{10^{-k}r} h(t)^{p'-1}\frac{dt}{t} = \infty$. So we can assume without loss of generality that $\alpha = \Lambda_h^\infty(E \cap \overline{K}_{k,r}) > 0$ and that $\int_0^{10^{-k}r} h^{p'-1}(t)\frac{dt}{t} < \infty$.

For every $\zeta \in S(x_0, r/10)$ there exists a decreasing sequence $(K_{s,\zeta})_{s \le 0}$ of dyadic subcubes of $K_{0,r}$ such that $K_{s,\zeta}$ is a cube of generation s for every integer $s \le 0$ and

$$\bigcap_{s \le 0} \overline{K}_{s,\zeta} = \{\zeta\}.$$

We denote by s_ζ^0 the sequence $(\overline{K}_{s,\zeta})_{s \le 0}$.

Similarly, for every $\eta \in \overline{K}_{k,r}$ there exists a decreasing sequence $(K_{s+k,\eta})_{s \ge 0}$ of dyadic subcubes of $K_{k,r}$ such that $K_{s+k,\eta}$ is of generation $s + k$ for every $s \ge 0$ and

$$\bigcap_{s \ge 0} \overline{K}_{s+k,\eta} = \{\eta\}.$$

We denote by s_η^1 the sequence $(\overline{K}_{s+k,\eta})_{s \ge 0}$. Let $I = \{K_{0,r}, \dots, K_{k,r}\}$ be a shortest sequence of pairwise adjacent cubes connecting $K_{0,r}$ and $K_{k,r}$.

For $(\zeta, \eta) \in S(x_0, r/10) \times \overline{K}_{k,r}$ we define $\gamma_{\zeta,\eta} = (\overline{K}_{s,\zeta,\eta})_{s \in \mathbf{Z}}$, where

$$K_{s,\zeta,\eta} = \begin{cases} K_{s,\zeta} & \text{if } s \le 0 \\ K_{s,r} & \text{if } 0 \le s \le k \\ K_{s,\eta} & \text{if } s \ge k. \end{cases}$$

For $K, K' \in \mathcal{D}_r$ we define

$$\mathcal{C}(K, K') = \{(\zeta, \eta) \in S(x_0, \frac{r}{10}) \times \overline{K}_{k,r} : K = K_{s,\zeta,\eta},\ K' = K_{s+1,\zeta,\eta} \text{ for some } s \in \mathbf{Z}\}.$$

We notice that $\mathcal{C}(K, K') = \emptyset$ if K, K' are not adjacent or if they are adjacent but of the same generation.

Since X is an Ahlfors Q-regular complete metric space that satisfies a weak $(1, \widetilde{p})$-Poincaré inequality with $1 \le \widetilde{p} < Q$, there exists (see [Kor07, Theorem 4.2]) a constant C depending only on $\widetilde{p}$ and on the data of X such that

$$C^{-1} t^{Q-\widetilde{p}} \le \Lambda^{\infty}_{Q-\widetilde{p}}(S(x,t)) \le C t^{Q-\widetilde{p}}$$

for all closed spheres $S(x,t)$ of radius t in X. We also have $\alpha = \Lambda^{\infty}_h(E \cap \overline{K}_{k,r}) > 0$. Therefore, by applying Frostman's lemma (see [Mat95, Theorem 8.8]), there exists a constant $C > 0$ and probability measures ν_0 on $S(x_0, r/10)$ and ν_1 on $E \cap \overline{K}_{k,r}$ such that for every ball $B(x,t)$ of radius t in X we have

$$\nu_0(B(x,t)) \le C \left(\frac{t}{r}\right)^{Q-\widetilde{p}} \text{ and } \nu_1(B(x,t)) \le C \frac{h(t)}{\alpha}. \tag{27}$$

For $K, K' \in \mathcal{D}_r$ we define

$$m(\overline{K}, \overline{K'}) = \nu_0 \times \nu_1(\mathcal{C}(K, K')).$$

We notice that $m(\overline{K}, \overline{K'}) m(\overline{K'}, \overline{K}) = 0$ for every pair of cubes $K, K' \in \mathcal{D}_r$. Moreover, if $m(\overline{K}, \overline{K'}) \ne 0$, then this implies that K and K' are adjacent but of different generations.

Let f be in $BW(E, B(x_0, r/10))$. Then, since f is continuous, we have that

$$f_{\overline{K}_v} \to f(y)$$

for every $y \in X$ for every nested sequence $\overline{K}_v$ of r-dyadic cubes containing y and converging to y. It follows that

$$1 \le f(\eta) - f(\zeta) \le \sum_{s \in \mathbf{Z}} (f_{\overline{K}_{s+1,\zeta,\eta}} - f_{\overline{K}_{s,\zeta,\eta}})$$

whenever $\eta \in E \cap \overline{K}_{k,r}$ and $\zeta \in S(x_0, r/10)$.

We obtain with the definition of $m(\overline{K}, \overline{K'})$ and by Hölder's inequality, that

$$\begin{aligned}
1 &\le \int_{S(x_0,r/10)} \int_{E \cap \overline{K}_{k,r}} \sum_{s \in \mathbf{Z}} (f_{\overline{K}_{s+1,\zeta,\eta}} - f_{\overline{K}_{s,\zeta,\eta}}) d\nu_0(\zeta)\, d\nu_1(\eta) \\
&\le \int_{S(x_0,r/10)} \int_{\overline{K}_{k,r}} \sum_{s \in \mathbf{Z}} |f_{\overline{K}_{s+1,\zeta,\eta}} - f_{\overline{K}_{s,\zeta,\eta}}| d\nu_0(\zeta)\, d\nu_1(\eta) \\
&= \sum_{K,K' \in \mathcal{D}_r \text{ adjacent}} |f_{\overline{K}} - f_{\overline{K'}}| m(\overline{K}, \overline{K'}) \\
&\le \left(\sum_{K,K' \in \mathcal{D}_r \text{ adjacent}} |f_{\overline{K}} - f_{\overline{K'}}|^p \right)^{1/p} \left(\sum_{K,K' \in \mathcal{D}_r \text{ adjacent}} m(\overline{K}, \overline{K'})^{p'} \right)^{1/p'} \\
&\le C [f]_{B_p(X)} \left(\sum_{K,K' \in \mathcal{D}_r \text{ adjacent}} m(\overline{K}, \overline{K'})^{p'} \right)^{1/p'},
\end{aligned}$$

where we used (25) for the last inequality. Here the constant C depends only on p and on the Ahlfors regularity of X. For a nonnegative integer s we let

$$E_{0,s} = \{(K,K') \in \mathcal{D}_r \times \mathcal{D}_r : K = K_{-s-1,\zeta},\ K' = K_{-s,\zeta} \text{ for some } \zeta \in S(x_0, r/10)\}$$

and similarly

$$E_{1,s} = \{(K,K') \in \mathcal{D}_r \times \mathcal{D}_r : K = K_{s+k,\eta},\ K' = K_{s+k+1,\eta} \text{ for some } \eta \in \overline{K}_{k,r}\}.$$

We notice that we can break $\sum = \sum_{K,K'\in\mathcal{D}_r} m(\overline{K},\overline{K'})^{p'}$ into 3 parts, namely

$$\sum = \sum_{s=0}^{\infty} \sum_{(K,K')\in E_{0,s}} m(\overline{K},\overline{K'})^{p'} + \sum_{K,K'\in I} m(\overline{K},\overline{K'})^{p'} + \sum_{s=0}^{\infty} \sum_{(K,K')\in E_{1,s}} m(\overline{K},\overline{K'})^{p'}.$$

We recall that $I = \{K_{0,r}, \dots, K_{k,r}\}$ is a shortest sequence of pairwise adjacent cubes in $\mathcal{D}_r$ connecting $K_{0,r}$ and $K_{k,r}$. Thus, the sum in the middle is exactly k. We get upper bounds for the first and the third term in the sum. We notice that for every $s \geq 0$ we have

$$\sum_{(K,K')\in E_{0,s}} m(\overline{K},\overline{K'}) = 1$$

since $\nu_0 \times \nu_1$ is a probability measure. On the other hand, there exists a constant C' depending only on p and on the Hausdorff dimension of X such that

$$m(\overline{K},\overline{K'}) \leq C' \frac{h(10^{-s-k}r)}{\alpha} \text{ for every } (K,K') \in E_{1,s}$$

for every integer $s \geq 0$ and

$$m(\overline{K},\overline{K'}) \leq C' 10^{(\widetilde{p}-Q)s} \text{ for every } (K,K') \in E_{0,s}$$

for every integer $s \geq 0$.

Therefore

$$\begin{aligned}\sum_{s=0}^{\infty} \sum_{(K,K')\in E_{1,s}} m(\overline{K},\overline{K'})^{p'} &= \sum_{s=0}^{\infty} \sum_{(K,K')\in E_{1,s}} m(\overline{K},\overline{K'})^{p'-1} m(\overline{K},\overline{K'}) \\ &\leq C\alpha^{1-p'} \sum_{s\geq 0} h(10^{-s-k}r)^{p'-1} \left(\sum_{(K,K')\in E_{1,s}} m(\overline{K},\overline{K'}) \right).\end{aligned}$$

But there exists a constant $C_0 = C_0(Q,p) > 1$ such that

$$\frac{1}{C_0} \int_0^{10^{-k}r} h(t)^{p'-1} \frac{dt}{t} \leq \sum_{s\geq 0} h(10^{-k-s}r)^{p'-1} \leq C_0 \int_0^{10^{-k}r} h(t)^{p'-1} \frac{dt}{t}$$

for every $r > 0$, every integer $k > 1$ and every continuous increasing measure function $h : [0,\infty) \to [0,\infty)$ such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$, is decreasing. Hence

$$\sum_{s=0}^{\infty} \sum_{(K,K')\in E_{1,s}} m(\overline{K},\overline{K'})^{p'} \leq C\,\alpha^{1-p'} \int_0^{10^{-k}r} h(t)^{p'-1} \frac{dt}{t}.$$

From a similar computation we get

$$\begin{aligned}\sum_{s=0}^{\infty} \sum_{(K,K')\in E_{0,s}} m(\overline{K},\overline{K'})^{p'} &= \sum_{s=0}^{\infty} \sum_{(K,K')\in E_{0,s}} m(\overline{K},\overline{K'})^{p'-1} m(\overline{K},\overline{K'}) \\ &\leq C \sum_{s\geq 0} 10^{-(p'-1)(Q-\widetilde{p})s} \left(\sum_{(K,K')\in E_{0,s}} m(\overline{K},\overline{K'}) \right) = C.\end{aligned}$$

So we get

$$\sum \leq C \left(\alpha^{1-p'} \int_0^{10^{-k}r} h(t)^{p'-1} \frac{dt}{t} + k + 1 \right).$$

It is easy to see that there exists a constant C depending only on p and on the Hausdorff dimension of X such that

$$\frac{\Lambda_h^\infty(\overline{K}_{k,r})}{\left(\int_0^{10^{-k}r} h(t)^{p'-1}\frac{dt}{t}\right)^{p-1}} \leq C.$$

for every $r > 0$, every integer $k > 1$ and every continuous increasing measure function $h : [0,\infty) \to [0,\infty)$ such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$, is decreasing. Hence

$$\sum \leq Ck\, \alpha^{1-p'} \int_0^{10^{-k}r} h(t)^{p'-1} \frac{dt}{t}.$$

Therefore we obtain

$$1 \leq C[f]_{B_p(B(x_0,r/10))} \left(k\, \alpha^{1-p'} \int_0^{10^{-k}r} h(t)^{p'-1} \frac{dt}{t} \right)^{1/p'}$$

for every integer $k > 1$ and for every $f \in BW(E \cap \overline{K}_{k,r}, B(x_0, r/10))$. This implies that there exists a constant C_1' depending only on p and on the Hausdorff dimension of X such that

$$\frac{\Lambda_h^\infty(E \cap \overline{K}_{k,r})}{\left(\int_0^{10^{-k}r} h(t)^{p'-1}\frac{dt}{t}\right)^{p-1}} k^{1-p} \leq C_1' \mathrm{cap}_{B_p}(E \cap \overline{K}_{k,r}, B(x_0, r/10)).$$

This finishes the proof. □

As a consequence of Theorem 4.9, we obtain the following theorem.

Theorem 4.10. *Suppose $1 \leq \widetilde{p} < Q < p < \infty$. Let (X, d, μ) be a complete and unbounded Ahlfors Q-regular metric space as in Theorem 4.9. Suppose $h : [0,\infty) \to [0,\infty)$ is a continuous increasing measure function such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$ is decreasing. There exists a positive constant $C_1 = C_1(Q, p, c_\mu)$ such that*

$$\frac{\Lambda_h^\infty(E \cap B(x,r))}{\left(\int_0^r h(t)^{p'-1}\frac{dt}{t}\right)^{p-1}} \leq C_1 \left(\ln \frac{R}{r} \right)^{p-1} \mathrm{cap}_{B_p}(E \cap B(x,r), B(x,R))$$

for every $E \subset X$, every $x \in X$, and every pair of positive numbers r, R such that $r < \frac{R}{2}$.

Proof. Fix $x \in X$ and r, R such that $0 < r < \frac{R}{2}$. Without loss of generality we can assume that $B(x, 100R) \subset K_{0,1000R}$. We choose $k \geq 3$ integer such that $10^{2-k}R \leq r < 10^{3-k}R$. From the construction of the dyadic cubes and the fact that X is a Q-Ahlfors regular space with $Q > 1$, it follows that there exists a constant $C = C(Q, c_\mu)$ independent of k such that every ball of radius $10^{2-k}R$ intersects with at most C dyadic subcubes of $K_{0,1000R}$ from the kth generation. We leave the rest of the details to the reader. □

It follows easily that if X is a complete and unbounded Ahlfors Q-regular metric space as in Theorem 4.10, then there exists a constant $C = C(Q, p, \widetilde{p}, c_\mu)$ such that

$$\frac{\Lambda_1^\infty(E \cap B(a,R))}{R} \leq C \mathrm{cap}_{B_p}(E \cap B(a,R), B(a,2R)) \tag{28}$$

whenever $E \subset X$, $R > 0$, and $a \in X$.

As a corollary we have the following.

Corollary 4.11. *Suppose X is a complete and unbounded Ahlfors Q-regular metric space as in Theorem 4.10. There exists a positive constant $C_2 = C_2(Q, p, \widetilde{p}, c_\mu)$ such that*

$$C_2 \left(\ln \frac{R}{r} \right)^{1-p} \le \operatorname{cap}_{B_p}(B(x,r), B(x,R)) \tag{29}$$

for every $x \in X$ and every pair of positive numbers r, R such that $r < \frac{R}{2}$.

Proof. We apply Theorem 4.10 for $h(t) = t^{Q-\widetilde{p}}$. We notice (see [Kor07, Theorem 4.2]) that there exists a constant $C_2' = C_2'(Q, p, \widetilde{p}, c_\mu)$ such that

$$\frac{1}{C_2'} \le \frac{\Lambda^\infty_{Q-\widetilde{p}}(B(x,r))}{\left(\int_0^r t^{(p'-1)(Q-\widetilde{p})} \frac{dt}{t} \right)^{p-1}} \le C_2' \tag{30}$$

for every $x \in X$ and every $r > 0$. The rest is routine.

□

Theorem 4.4 and Corollary 4.11 easily yield the following theorem, (cf. [Bou05]).

Theorem 4.12. *Suppose X is a complete and unbounded Ahlfors Q-regular metric space as in Theorem 4.10. There exists $C_0 = C_0(Q, p, c_\mu) > 0$ such that*

$$\frac{1}{C_0} \left(\ln \frac{R}{r} \right)^{1-p} \le \operatorname{cap}_{B_p}(B(x,r), B(x,R)) \le C_0 \left(\ln \frac{R}{r} \right)^{1-p} \tag{31}$$

for every $x \in X$ and every pair of positive numbers r, R such that $r < \frac{R}{2}$.

A set $E \subset X$ is said to be *of Besov B_p-capacity zero* if $\operatorname{cap}_{B_p}(E \cap \Omega, \Omega) = 0$ for all open and bounded $\Omega \subset X$. In this case we write $\operatorname{cap}_{B_p}(E) = 0$. The following lemma is obvious.

Lemma 4.13. *A countable union of sets of Besov B_p-capacity zero has Besov B_p-capacity zero.*

The next lemma shows that, if E is bounded, one needs to test only a single bounded open set Ω containing E in showing that E has zero Besov B_p-capacity.

Lemma 4.14. *Suppose that E is bounded and that there is a bounded neighborhood Ω of E with $\operatorname{cap}_{B_p}(E, \Omega) = 0$. Then $\operatorname{cap}_{B_p}(E) = 0$.*

Proof. The proof is similar to the proof of [Cos, Lemma 3.13] and omitted. □

Corollary 4.15. *Suppose X is a complete and unbounded Ahlfors Q-regular metric space as in Theorem 4.10. Let $E \subset X$ be such that $\operatorname{cap}_{B_p}(E) = 0$. Then $\Lambda_h(E) = 0$ for every measure function $h : [0, \infty) \to [0, \infty)$ such that*

$$\int_0^1 h(t)^{p'-1} \frac{dt}{t} < \infty. \tag{32}$$

In particular, the Hausdorff dimension of E is zero and $X \setminus E$ is connected.

Note that for every $\varepsilon > 0$ we can take $h = h_\varepsilon : [0, \infty) \to [0, \infty)$ in Corollary 4.15, where $h_\varepsilon(t) = (\ln t)^{1-p-\varepsilon}$ for every $t \in (0, 1/2)$.

Proof. It is enough to assume, without loss of generality, that $h : [0, \infty) \to [0, \infty)$ is a continuous measure function such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$ is decreasing. (See Proposition 4.8.) If $\operatorname{cap}_{B_p}(E) = 0$, then there exists a Borel set $\widetilde{E}$ such that $E \subset \widetilde{E}$ and $\operatorname{cap}_{B_p}(\widetilde{E}) = 0$, hence we can assume without loss of generality that E is itself Borel. Since Λ_h is a Borel regular measure and $\Lambda_h(E) = 0$ if and only if $\Lambda_h^\infty(E) = 0$, it is enough to assume that E is in fact compact. For E compact the claim follows obviously from Theorem 4.10.

The second claim is a consequence of the first claim because for every $s \in (0, Q)$, the function $h_s : [0, \infty) \to [0, \infty)$ defined by $h_s(t) = t^s$ has the property (32). The third claim is an easy consequence of the second claim. □

We also get upper bounds of the relative Besov p-capacity in terms of a certain Hausdorff measure.

Proposition 4.16. *Let $h : [0, \infty) \to [0, \infty)$ be an increasing homeomorphism such that $h(t) = (\ln \frac{1}{t})^{1-p}$ for all $t \in (0, \frac{1}{2})$. Suppose (X, d, μ) is a proper and unbounded Ahlfors Q-regular metric space. Let E be a compact subset of X. There exists a constant C depending only on p and on the Ahlfors regularity of X such that $\operatorname{cap}_{B_p}(E, \Omega) \leq C\Lambda_h(E)$ for every bounded and open set Ω containing E.*

Proof. The proof is similar to the proof of [Cos, Proposition 3.17] and omitted. □

Proposition 4.16 gives another sufficient condition to obtain sets of Besov p-capacity zero.

Theorem 4.17. *Let $h : [0, \infty) \to [0, \infty)$ be an increasing homeomorphism such that $h(t) = (\ln \frac{1}{t})^{1-p}$ for all $t \in (0, \frac{1}{2})$. Then $\Lambda_h(E) < \infty$ implies $\operatorname{cap}_{B_p}(E) = 0$ for every $E \subset X$.*

Proof. The proof is similar to the proof of [Cos, Theorem 3.16] and omitted. □

5. Besov capacity and quasicontinuous functions

In this section we study a global Besov capacity and quasicontinuous functions in Besov spaces.

5.1. **Besov Capacity.**

Definition 5.1. For a set $E \subset X$ define

$$\operatorname{Cap}_{B_p}(E) = \inf\{||u||^p_{L^p(X)} + [u]^p_{B_p(X)} : u \in S(E)\},$$

where u runs through the set

$$S(E) = \{u \in B_p(X) : \, u = 1 \text{ in a neighborhood of } E\}.$$

Since $B_p(X)$ is closed under truncations and the norms do not increase, we may restrict ourselves to those functions $u \in S(E)$ for which $0 \leq u \leq 1$. We get the same capacity if we consider the apparently larger set of admissible functions, namely

$$\widetilde{S}(E) = \{u \in B_p(X) : \, u \geq 1 \ \mu\text{-a.e. in a neighborhood of } E\}.$$

Moreover, we have the following lemma:

Lemma 5.2. *If K is compact, then*

$$\mathrm{Cap}_{B_p}(K) = \inf\{||u||^p_{L^p(X)} + [u]^p_{B_p(X)} : u \in S_0(K)\}$$

where $S_0(K) = S(K) \cap Lip_0(X)$.

Proof. Let $u \in S(K)$. Since $B_p(X) = B^0_p(X)$, we may choose a sequence of functions $\varphi_j \in Lip_0(X)$ converging to u in $B_p(X)$. Let U be a bounded and open neighborhood of K such that $u = 1$ in U. Let $\psi \in Lip(X)$, $0 \leq \psi \leq 1$ be such that $\psi = 1$ in $X \setminus U$ and $\psi = 0$ in $\widetilde{U} \subset\subset U$, an open neighborhood of K. From Lemma 3.7 we see that the functions $\psi_j = 1 - (1 - \varphi_j)\psi$ converge to $1 - (1 - u)\psi$ in $B_p(X)$. This establishes the assertion since $1 - (1 - u)\psi = u$.

□

We have a result similar to Theorem 4.2, namely:

Theorem 5.3. *The set function $E \mapsto \mathrm{Cap}_{B_p}(E), E \subset X$ is a Choquet capacity. In particular*

(i) *If $E_1 \subset E_2$, then $\mathrm{Cap}_{B_p}(E_1) \leq \mathrm{Cap}_{B_p}(E_2)$.*

(ii) *If $E = \bigcup_i E_i$, then*

$$\mathrm{Cap}_{B_p}(E) \leq \sum_i \mathrm{Cap}_{B_p}(E_i).$$

We have introduced two different capacities, and it is next shown that they have the same zero sets.

Let $\Omega, \widetilde{\Omega}$ be bounded and open subsets of X such that $\Omega \subset\subset \widetilde{\Omega}$. Let $\eta \in Lip_0(\widetilde{\Omega})$ be a cut-off function as in Remark 3.8. Suppose K is a compact subset of Ω. Then, if $u \in S_0(K)$, we have that $u\eta$ is admissible for the condenser $(K, \widetilde{\Omega})$. Therefore

$$\mathrm{cap}_{B_p}(K, \widetilde{\Omega}) \leq [u\eta]^p_{B_p(\widetilde{\Omega})} \leq ||u\eta||^p_{B_p(\widetilde{\Omega})} \leq C\,||u||^p_{B_p(X)} \tag{33}$$

where C depends only on Q, p, c_μ, diam $\widetilde{\Omega}$ and $\mathrm{dist}(\Omega, X \setminus \widetilde{\Omega})$. (See Remark 3.8.) Since $||u||_{B_p(X)} = ||u||_{L^p(X)} + [u]_{B_p(X)}$, we have

$$||u||^p_{B_p(X)} \leq 2^p(||u||^p_{L^p(X)} + [u]^p_{B_p(X)}). \tag{34}$$

From (33) and (34) we get, by taking the infimum over all $u \in S_0(K)$, that

$$\mathrm{cap}_{B_p}(K, \widetilde{\Omega}) \leq 2^p\, C\, \mathrm{Cap}_{B_p}(K), \tag{35}$$

where C is the constant from (33).

Since both $\mathrm{cap}_{B_p}(\cdot, \widetilde{\Omega})$ and $\mathrm{Cap}_{B_p}(\cdot)$ are Choquet capacities, we obtain:

Theorem 5.4. *There exists $C > 0$ depending only on Q, p, c_μ, $\mathrm{dist}(\Omega, X \setminus \widetilde{\Omega})$ and* diam $\widetilde{\Omega}$ *such that*

$$\mathrm{cap}_{B_p}(E, \widetilde{\Omega}) \leq C\, \mathrm{Cap}_{B_p}(E) \tag{36}$$

for every $E \subset \Omega$.

Corollary 5.5. *If $\mathrm{Cap}_{B_p}(E) = 0$, then $\mathrm{cap}_{B_p}(E) = 0$.*

We also have a converse result, namely:

Theorem 5.6. *If $\mathrm{cap}_{B_p}(E) = 0$, then $\mathrm{Cap}_{B_p}(E) = 0$.*

Proof. The proof is similar to the proof of [Cos, Theorem 4.6] and omitted. □

Remark 5.7. For $E \subset X$ compact we see from the proof of Lemma 4.14 and Theorem 5.6 that it is enough to have $\mathrm{cap}_{B_p}(E, \Omega) = 0$ for one bounded open set $\Omega \subset X$ with $E \subset \Omega$ in order to have $\mathrm{Cap}_{B_p}(E) = 0$.

It is desirable to know when a set is negligible for a Besov space. If there is an isometric isomorphism between two normed spaces X and Y we write $X = Y$. In particular, if E is relatively closed subset of Ω, then by

$$B_p^0(\Omega \setminus E) = B_p^0(\Omega)$$

we mean that each function $u \in B_p^0(\Omega)$ can be approximated in B_p-norm by functions from $Lip_0(\Omega \setminus E)$.

Theorem 5.8. *Suppose that E is a relatively closed subset of Ω. Then*

$$B_p^0(\Omega \setminus E) = B_p^0(\Omega)$$

if and only $\mathrm{Cap}_{B_p}(E) = 0$.

Proof. Suppose that $\mathrm{cap}_{B_p}(E) = 0$. Let $\varphi \in Lip_0(\Omega)$ and choose a sequence u_j of functions in $B_p(X)$ such that $0 \leq u_j \leq 1$, $u_j = 1$ in a neighborhood of E and $u_j \to 0$ in $B_p(X)$. For every $j \geq 1$ we define $w_j = (1 - u_j)\varphi$. Then from Remark 3.9 and the properties of the functions φ and u_j, it follows that w_j is a bounded sequence of functions in $B_p(X)$, compactly supported in $\Omega \setminus E$. Lemma 3.13 implies that w_j is a sequence in $B_p^0(\Omega \setminus E)$. Moreover, Lemma 3.7 implies, since $\varphi - w_j = u_j\varphi$ for every $j \geq 1$ and since $||u_j||_{B_p(X)} \to 0$, that w_j converges to φ in $B_p(X)$. Since w_j is a sequence in $B_p^0(\Omega \setminus E)$, it follows that $\varphi \in B_p^0(\Omega \setminus E)$. Hence

$$B_p^0(\Omega) \subset B_p^0(\Omega \setminus E)$$

and since the reverse inclusion is trivial, the sufficiency is established.

For the only if part, let $K \subset E$ be compact. It suffices to show that $\mathrm{Cap}_{B_p}(K) = 0$. Choose $\varphi \in Lip_0(\Omega)$ with $\varphi = 1$ in a neighborhood of K. Since $B_p^0(\Omega \setminus E) = B_p^0(\Omega)$, we may choose a sequence of functions $\varphi_j \in Lip_0(\Omega \setminus K)$ such that $\varphi_j \to \varphi$ in $B_p(\Omega)$. Consequently

$$\mathrm{Cap}_{B_p}(K) \leq \left(\lim_{j \to \infty} ||\varphi_j - \varphi||^p_{L^p(X)} + [\varphi_j - \varphi]^p_{B_p(X)} \right) = 0,$$

and the theorem follows. □

5.2. **Quasicontinuous functions.** We show that for each $u \in B_p(X)$ there is a function v such that $u = v$ μ-a.e. and that v is B_p-quasicontinuous, i.e. v is continuous when restricted to a set whose complement has arbitrarily small Besov B_p-capacity. Moreover, this quasicontinuous representative is unique up to a set of Besov B_p-capacity zero.

Definition 5.9. A function $u : X \to \mathbf{R}$ is B_p-*quasicontinuous* if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $\mathrm{Cap}_{B_p}(G) < \varepsilon$ and the restriction of u to $X \setminus G$ is continuous.

A sequence of functions $\psi_j : X \to \mathbf{R}$ converges B_p-*quasiuniformly* in X to a function ψ if for every $\varepsilon > 0$ there is an open set G such that $\mathrm{Cap}_{B_p}(G) < \varepsilon$ and $\psi_j \to \psi$ uniformly in $X \setminus G$.

We say that a property holds B_p-*quasieverywhere*, or simply q.e., if it holds except on a set of Besov B_p-capacity zero.

Theorem 5.10. *Let $\varphi_j \in C(X) \cap B_p(X)$ be a Cauchy sequence in $B_p(X)$. Then there is a subsequence φ_k which converges B_p-quasiuniformly in X to a function $u \in B_p(X)$. In particular, u is B_p-quasicontinuous and $\varphi_k \to u$ B_p-quasieverywhere in X.*

Proof. The proof is similar to the proof of [HKM93, Theorem 4.3] and omitted. □

Theorem 5.10 implies the following corollary.

Corollary 5.11. *Suppose that $u \in B_p(X)$. Then there exists a B_p-quasicontinuous Borel function $v \in B_p(X)$ such that $u = v$ μ-a.e.*

Proof. Since $u \in B_p(X)$, from Theorem 3.12 there exists a sequence of functions φ_j in $Lip_0(X)$ converging to u in $B_p(X)$. Passing to subsequences if necessary, we can assume that $\varphi_j \to u$ pointwise μ-a.e. in X and that

$$2^{jp}\left(||\varphi_{j+1} - \varphi_j||^p_{L^p(X)} + [\varphi_{j+1} - \varphi_j]^p_{B_p(X)}\right) < 2^{-j}$$

for every $j = 1, 2, \ldots$ Defining $E_j = \{x \in X : |\varphi_{j+1} - \varphi_j| > 2^{-j}\}$ and letting $E = \cap_{k=1}^{\infty} \cup_{j=k} E_j$, the proof of Theorem 5.10 yields the existence of a function $v \in B_p(X)$, such that $\varphi_j \to v$ in $B_p(X)$ and pointwise in $X \setminus E$. Since E is a Borel set of Besov B_p-capacity zero and the functions φ_j are continuous, this finishes the proof. □

Theorem 5.12. *Let $u \in B_p(X)$. Then $u \in B_p^0(\Omega)$ if and only if there exists a B_p-quasicontinuous function v in X such that $u = v$ μ-a.e. in Ω and $v = 0$ q.e. in $X \setminus \Omega$.*

Proof. Fix $u \in B_p^0(\Omega)$ and let $\varphi_j \in Lip_0(\Omega)$ be a sequence converging to u in $B_p(\Omega)$. By Theorem 5.10 there is a subsequence of φ_j which converges B_p-quasieverywhere in X to a B_p-quasicontinuous function v in X such that $u = v$ μ-a.e. in Ω and $v = 0$ q.e. in $X \setminus \Omega$. Hence v is the desired function.

To prove the converse, we assume first that Ω is bounded. Because the truncations of v converge to v in $B_p(\Omega)$, we can assume that v is bounded. Without loss of generality, since v is B_p-quasicontinuous and $v = 0$ q.e. outside Ω we can assume that in fact $v = 0$ everywhere in $X \setminus \Omega$. Choose open sets G_j such that v is continuous on $X \setminus G_j$ and $\mathrm{Cap}_{B_p}(G_j) \to 0$. By passing to a subsequence, we may pick a sequence φ_j in $B_p(X)$ such that $0 \le \varphi_j \le 1$, $\varphi_j = 1$ everywhere in G_j, $\varphi_j \to 0$ μ-a.e. in X, and

$$||\varphi_j||^p_{L^p(X)} + [\varphi_j]^p_{B_p(X)} \to 0.$$

Then from Remark 3.9 we have that $w_j = (1 - \varphi_j)v$ is a bounded sequence in $B_p(\Omega)$. Moreover, for every $j \ge 1$, we have $\lim_{x \to y, x \in \Omega} w_j(x) = 0$ for all $y \in \partial\Omega$. Thus, from Lemma 3.14, we have that w_j is a sequence in $B_p^0(\Omega)$. Clearly $w_j \to v$ in $L^p(X)$ and pointwise μ-a.e. in X. This, together with the boundedness of the sequence w_j in $B_p^0(\Omega)$, implies via Mazur's lemma that $v \in B_p^0(\Omega)$. The proof is complete in case Ω is bounded.

Assume that Ω is unbounded. We can assume again, without loss of generality, that v is bounded and that $v = 0$ everywhere in $X \setminus \Omega$. We fix $x_0 \in X$. For every $k \ge 2$ let $\varphi_k \in Lip_0(B(x_0, k^2))$ be such that $0 \le \varphi_k \le 1$, $\varphi_k = 1$ on $B(x_0, k)$ and $[\varphi_k]_{B_p(X)} \le C(\ln k)^{1-p}$. (See (24).) Then $v_k = v\varphi_k \in B_p^0(\Omega \cap B(x_0, k^2)) \subset B_p^0(\Omega)$ for every $k \ge 2$ and like in Theorem 3.12, we get

$$||v - v_k||_{B_p(X)} \to 0,$$

which implies that $v \in B_p^0(\Omega)$. This finishes the proof. □

We denote by

$$Q^{B_p} = Q^{B_p}(X)$$

the set of all functions $u \in B_p(X)$ such that there exists a sequence $\varphi_j \in C(X) \cap B_p(X)$ converging to u both in $B_p(X)$ and B_p-quasiuniformly. It follows immediately from Theorem 5.10 that the functions in Q^{B_p} are B_p-quasicontinuous and for each $v \in B_p(X)$ there is $u \in Q^{B_p}$ such that $u = v$ μ-a.e. We soon show that, conversely, each B_p-quasicontinuous function v of $B_p(X)$ belongs to Q^{B_p}.

Theorem 5.13. *Let $u \in Q^{B_p}$. If $u \geq 1$ B_p-quasieverywhere on E, then*

$$\mathrm{Cap}_{B_p}(E) \leq ||u||^p_{L^p(X)} + [u]^p_{B_p(X)}.$$

Proof. The proof is similar to the proof of [HKM93, Lemma 4.7] and omitted. □

This result has the following corollary.

Corollary 5.14. *Suppose that Ω is open and bounded and let $E \subset\subset \Omega$. Let $u \in Q^{B_p}$. Suppose that $u \geq 1$ quasieverywhere on E and that u has compact support in Ω. Then*

$$\mathrm{cap}_{B_p}(E, \Omega) \leq [u]^p_{B_p(\Omega)}.$$

We know that Cap_{B_p} is an outer capacity. It satisfies the following compatibility condition (see [Kil98]):

Theorem 5.15. *Suppose that G is open and $\mu(E) = 0$. Then*

$$\mathrm{Cap}_{B_p}(G) = \mathrm{Cap}_{B_p}(G \setminus E). \tag{37}$$

Proof. The proof is very similar to the proof of [Cos, Theorem 4.15] and omitted. □

We state now the uniqueness of a B_p-quasicontinuous representative.

Theorem 5.16. *Let f and g be B_p-quasicontinuous functions on X such that*

$$\mu(\{x : f(x) \neq g(x)\}) = 0.$$

Then $f = g$ B_p-quasieverywhere on X.

Proof. The proof is verbatim the proof from [Kil98, p. 262]. □

Combining Theorem 5.13 and Theorem 5.16 we obtain the following corollary.

Corollary 5.17. *Suppose that $E \subset X$. Then*

$$\mathrm{Cap}_{B_p}(E) = \inf\{||u||^p_{L^p(X)} + [u]^p_{B_p(X)}\},$$

where the infimum is taken over all B_p-quasicontinuous $u \in B_p(X)$ such that $u = 1$ B_p-quasieverywhere on E.

Corollary 5.11 and Theorem 5.16 imply that each $u \in B_p(X)$ has a "unique" quasicontinuous version.

Corollary 5.18. *Suppose that $u \in B_p(X)$. Then there exists a B_p-quasicontinuous function v such that $u = v$ μ-a.e. Moreover, if $\widetilde{v}$ is another B_p-quasicontinuous function such that $u = \widetilde{v}$ μ-a.e., then $v = \widetilde{v}$ B_p-quasieverywhere.*

We have a result similar to Corollary 5.18 for locally integrable functions with finite B_p-seminorm.

Corollary 5.19. *Suppose that $u \in L^1_{loc}(X)$ such that $[u]_{B_p(X)} < \infty$. Then there exists a B_p-quasicontinuous Borel function v such that $u = v$ μ-a.e. Moreover, if $\widetilde{v}$ is another B_p-quasicontinuous Borel function such that $u = \widetilde{v}$ μ-a.e., then $v = \widetilde{v}$ B_p-quasieverywhere.*

Proof. We prove the "uniqueness" first. Suppose $v, \widetilde{v}$ are two B_p-quasicontinuous Borel functions such that $v = u$ μ-a.e. and $\widetilde{v} = u$ μ-a.e. Let $w = v - \widetilde{v}$. We notice that w is B_p-quasicontinuous and belongs to $B_p(X)$ because $w = 0$ μ-a.e. in X. Hence from Corollary 5.18 we have that $w = 0$ B_p-quasieverywhere. The "uniqueness" is proved.

We prove now the existence. Fix $x_0 \in X$. For every integer $k \geq 1$ we choose a 2^{1-k}-Lipschitz function η_k supported in $B(x_0, 2^{k+1})$ such that $\eta_k = 1$ on $B(x_0, 2^k)$. We have

$$\eta_{k+1}\eta_k = \eta_k \tag{38}$$

for every integer $k \geq 1$. For a fixed integer $k \geq 1$, we define $u_k = \eta_k u$. Then $u_k \in L^p(X)$ because $u \in L^p_{loc}(X)$ and $\eta_k \in Lip_0(B(x_0, 2^{k+1}))$. Moreover, from Lemma 3.10, it follows that $[\eta_k u - \eta_k u_{B(x_0,2^k)}]_{B_p(X)} < \infty$. From this and the fact that $\eta_k \in B_p(X)$, imply that $u_k \in B_p(X)$. Therefore, from Corollary 5.11 it follows that there exists $\widetilde{u}_k \in B_p(X)$ a B_p-quasicontinuous Borel function such that $\widetilde{u}_k = u_k$ μ-a.e. in X. In particular, since $\eta_k = 1$ in $B(x_0, 2^k)$, this implies that $\widetilde{u}_k = u$ μ-a.e. in $B(x_0, 2^k)$. So, for every integer $k \geq 1$ we have that $\widetilde{u}_{k+1}$ is a B_p-quasicontinous Borel representative of $\eta_{k+1}u$, hence $\eta_k\widetilde{u}_{k+1}$ is a B_p-quasicontinuous Borel representative of $\eta_k\eta_{k+1}u = u_k$, where the equality follows from the definition of u_k and (38). This implies that both $\eta_k\widetilde{u}_{k+1}$ and $\widetilde{u}_k$ are two B_p-quasicontinuous Borel representatives of $u_k \in B_p(X)$, hence from Corollary 5.18 we can assume that $\widetilde{u}_k = \eta_k\widetilde{u}_{k+1}$ in $B(x_0, 2^k)$. Since $\eta_k = 1$ on $B(x_0, 2^k)$, this means in particular that we can assume that $\widetilde{u}_k(x) = \widetilde{u}_{k+1}(x)$ for every x in $B(x_0, 2^k)$.

So, we constructed a sequence of B_p-quasicontinuous Borel functions $\widetilde{u}_k$ in $B_p(X)$ satisfying the following properties:

$$\begin{aligned} \widetilde{u}_k(x) &= u(x) && \text{for } \mu\text{-a.e. } x \text{ in } B(x_0, 2^k) \\ \widetilde{u}_l(x) &= \widetilde{u}_k(x) && \text{for every } x \text{ in } B(x_0, 2^k) \text{ and } l \geq k \geq 1. \end{aligned}$$

We define $\widetilde{u} : X \to \overline{\mathbf{R}}$ by

$$\widetilde{u}(x) = \lim_{k\to\infty} \widetilde{u}_k(x).$$

Thus, $\widetilde{u}$ is a B_p-quasicontinuous Borel function and $u = \widetilde{u}$ μ-a.e. This proves the existence of a B_p-quasicontinuous Borel representative of u. The claim follows.

□

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