On the mapping theory in metric spaces

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Abstract

It is investigated the problem on extending to the boundary of the so–called $Q-$homeomorphisms between domains in metric spaces with measures. It is formulated conditions on the function $Q(x)$ and boundaries of the domains under which every $Q-$homeomorphism admits a continuous or homeomorphic extension to the boundary. The results can be applied, in particular, to Riemannian manifolds, the Loewner spaces, the groups by Carnot and Heisenberg.

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1 Introduction

It is studied properties of weakly flat spaces which are a far-reaching generalization of recently introduced spaces by Loewner, see e.g. [BK], [BY], [He_1], [HK] and [Ty], including, in particular, the well–known groups by Carnot and Heisenberg, see e.g. [He_2], [HH], [KRe_1], [KRe_2], [MaM], [MarV], [Mit], [Pa] and [Vo]. On this base, it is created the theory of the boundary behavior and removable singularities for quasiconformal mappings and their generalizations that can be applied to any of the counted classes of spaces. In particular, it is proved a generalization and strengthening of the known theorem by Gehring-Martio on homeomorphic extension to the boundary of quasiconformal mappings between domains of quasiextremal distance in $\mathbb{R}^n$, $n \geq 2$, see [GM].

Various modulus inequalities play a great role in the theory of quasiconformal mappings and their generalizations. In this connection, the following conception has been introduced by Professor Olli Martio, see e.g. [MRSY_1]-[MRSY_3]. Let G and G' be domains in $\mathbb{R}^n$, $n \geq 2$, and let $Q : G \to [1, \infty]$ be a measurable function. A homeomorphism $f : G \to G'$ is called a **Q−homeomorphism** if

$$M(f\Gamma) \leq \int\limits_G Q(x) \cdot \varrho^n(x)\ dm(x) \tag{1.1}$$

for every family Γ of paths in G and every admissible function ϱ for Γ. Here the notation m refers to the Lebesgue measure in $\mathbb{R}^n$. This conception is a natural generalization of the geometric definition of a quasiconformal mapping, see 13.1 and 34.6 in [Va] and it is closely related to the theory of moduli with weights, see e.g. [Ca_1], [Ca_2], [Oh_1], [Oh_2] and [Ta].

Recall that, given a family of paths Γ in $\mathbb{R}^n$, a Borel function $\varrho : \mathbb{R}^n \to [0,\infty]$ is called **admissible** for Γ, abbr. $\varrho \in adm\,\Gamma$, if

$$\int\limits_{\gamma} \varrho\, ds \ \geq \ 1 \tag{1.2}$$

for all $\gamma \in \Gamma$. The (conformal) **modulus** of Γ is the quantity

$$M(\Gamma) \ = \inf_{\varrho \in adm\,\Gamma} \int\limits_{G} \varrho^n(x) \ dm(x) \ . \tag{1.3}$$

The main goal of the theory of $Q-$homeomorphisms is to clear up various interconnections between properties of the majorant $Q(x)$ and the corresponding properties of the mappings themselves. The problem of the local and boundary behavior of $Q-$homeomorphisms is studied in $\mathbb{R}^n$ first in the case Q∈BMO (bounded mean oscillation) in the papers [MRSY_1]-[MRSY_3] and [RSY_1]-[RSY_2], and then in the case of Q∈FMO (finite mean oscillation) and other cases in the papers [IR_1]-[IR_2] and [RSY_3]-[RSY_6]. The modulus techniques for metric spaces are developed, for instance, in the papers [Fu], [He_1], [HK] and [Ma].

In what follows, (X, d, μ) denotes a space X with a metric d and a locally finite Borel measure μ. An open set in X all whose points can pairwise be connected by continuous curves is called a **domain** in X.

Now, let G and G' be domains with finite Hausdorff dimensions α and $\alpha' \geq 1$ in spaces (X, d, μ) and (X', d', μ'), and let $Q : G \to [0,\infty]$ be a measurable function. We say that a homeomorphism $f : G \to G'$ is a **Q−homeomorphism** if

$$M(f\Gamma) \leq \int\limits_{G} Q(x) \cdot \varrho^{\alpha}(x) \ d\mu(x) \tag{1.4}$$

for every family Γ of paths in G and every admissible function ϱ for Γ.

The **modulus** of Γ in the space (X, d, μ) is given by the equality

$$M(\Gamma) \ = \inf_{\varrho \in adm\,\Gamma} \int\limits_{G} \varrho^{\alpha}(x) \ d\mu(x) \tag{1.5}$$

where the admissible functions for Γ, as before, are defined by the condition of the type (1.2). Moreover, in the case of the space (X', d', μ'), we take the Hausdorff dimension α' of the domain G' in (1.5).

Recall, given a continuous way $\gamma : [a, b] \to X$ in a metric space (X, d), its length is the supremum of the sums

$$\sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i-1}))$$

over all partitions $a = t_0 \le t_1 \le ... \le t_k = b$ of the interval $[a, b]$. The curve γ is called rectifiable if its length is finite.

A space (X, d, μ) is called $\alpha-$**regular by Ahlfors** if there is a constant $C \ge 1$ such that

$$C^{-1}r^{\alpha} \le \mu(B_r) \le Cr^{\alpha} \tag{1.6}$$

for all balls B_r in X with the radius $r < diam\, X$. As known, $\alpha-$regular spaces have the Hausdorff dimension α, see e.g. [He$_1$], p. 61. We say that a space (X, d, μ) is **regular by Ahlfors** if it is $\alpha-$regular by Ahlfors for some $\alpha \in (1, \infty)$.

We will say that a space (X, d, μ) is **upper** $\alpha-$**regular at a point** $x_0 \in X$ if there is a constant $C > 0$ such that

$$\mu(B(x_0, r)) \le Cr^{\alpha} \tag{1.7}$$

for all balls $B(x_0, r)$ centered at $x_0 \in X$ with the radius $r < r_0$. We will also say that a space (X, d, μ) is **upper** $\alpha-$**regular** if the condition (1.7) holds at every point of X.

2 Connectedness in topological spaces

Let us give definitions of some topological notions and related remarks of a general character which will be useful in what follows. Let T be an arbitrary topological space. **A curve (or path) in** T is a continuous mapping $\gamma : [a, b] \to T$. Later on, $|\gamma|$ denotes the locus $\gamma([a, b])$. If A, B and C are sets in T, then $\Delta(A, B, C)$ denotes a collection of all curves γ joining A and B in C, i.e. $\gamma(a) \in A$, $\gamma(b) \in B$ and $\gamma(t) \in C$, $t \in (a, b)$.

Recall that a topological space is connected if it is impossible to split it into two non-empty open sets. Compact connected spaces are called **continua**. A topological space T is said to be **arc connected** if any two points x_1 and x_2 in T can be joined by a path $\gamma : [0, 1] \to T$, $\gamma(0) = x_1$ and $\gamma(1) = x_2$. A **domain** in T is an open arc connected set in T. We say that a metric space T is **rectifiable** if any two points x_1 and x_2 in T can be joined by a rectifiable path. In particular, we say that a domain G in T is **rectifiable** if it is a rectifiable space. A domain G in a topological space T is called **locally connected at a point** $x_0 \in \partial G$ if, for every neighborhood U of the point x_0, there is its neighborhood $V \subseteq U$ such that $V \cap G$ is connected, [Ku], c. 232. Similarly, we say that a domain G is **locally arc connected (rectifiable) at a point** $x_0 \in \partial G$ if, for every neighborhood U of the point x_0, there is its neighborhood $V \subseteq U$ such that $V \cap G$ is arc connected (rectifiable).

2.1. Proposition. *Let T be a topological (metric) space with a base of topology $\mathcal{B}$ consisting of arc connected (rectifiable) sets. Then an arbitrary open set Ω in T is connected if and only if Ω is arc connected (rectifiable).*

2.2. Corollary. *An open set Ω in $\mathbb{R}^n$, $n \ge 2$, or in any manifolds is connected if and only if Ω is arc connected (rectifiable).*

2.3. Remark. Thus, if a domain G in $\mathbb{R}^n$, $n \geq 2$, is locally connected at a point $x_0 \in \partial G$, then it is also arc connected at x_0. The same is true for manifolds. As we will show later on, the connectedness and the arc connectedness are equivalent for open sets in a wide class of the so–called weakly flat spaces which include the known spaces by Loewner and, in particular, the well–known groups by Carnot and Heisenberg.

Proof of Proposition 2.1. Let first Ω be arc connected. If Ω is simultaneously not connected, then $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are open non–empty and disjoint sets in T. Take $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$ and connect them with a curve $\gamma : [0,1] \to \Omega\,, \gamma(0) = x_1$ and $\gamma(1) = x_2$. Then the sets $\omega_1 = \gamma^{-1}\Omega_1$ and $\omega_2 = \gamma^{-1}\Omega_2$ are disjoint non–empty and open in $[0,1]$ by continuity of γ. However, the last contradicts to the connectedness of the segment $[0,1]$.

Now, let Ω is connected. Take an arbitrary point $x_0 \in \Omega$ and denote by Ω_0 the set of all points x_* in Ω which can be connected with x_0 through a finite chain of sets $B_k \subset \Omega$ in the base $\mathcal{B}$, $k = 1, ..., m$, such that $x_0 \in B_1\,, x_* \in B_m$, and $B_k \cap B_{k+1} \neq \emptyset\,, k = 1, ..., m-1$.

Note, firstly, that the set Ω_0 is open. Indeed, if a point $y_0 \in \Omega_0$, then there is its a neighborhood $B_0 \subseteq \Omega$ in the base $\mathcal{B}$ and all points of this neighborhood belongs to Ω_0. Secondly, the set Ω_0 is closed in Ω.

Really, assume that $\partial\Omega_0 \cap \Omega \neq \emptyset$. Then for every point $z_0 \in \partial\Omega_0 \cap \Omega$ there is its neighborhood $B_0 \subseteq \Omega$ in the base $\mathcal{B}$, and in this neighborhood there is a point $x_* \in \Omega_0$ because $z_0 \in \partial\Omega_0$. Thus, $z_0 \in \Omega_0$ by the definition of the set Ω_0. However, Ω_0 is open and hence $\Omega_0 \cap \partial\Omega_0 = \emptyset$. The obtained contradiction disproves the above assumption.

Thus, Ω_0 is simultaneously open and closed in Ω and, consequently, being non-empty it coincides with the set Ω in view of its connectivity. But Ω_0 by the construction is obviously arc connected.

Finally, if the space T has a base of topology $\mathcal{B}$ consisting of rectifiable domains, then, covering any path γ in T by the elements of this base, we are able to choose its finite subcovering leading to the construction of the corresponding rectifiable path.

2.4. Proposition. *If a domain G in a metric space (X, d) is locally arc connected (rectifiable) at a point $x_0 \in \partial G$, then x_0 is accessible from G through a (locally rectifiable) path $\gamma : [0,1] \to X$, $\gamma([0,1)) \subset G$, $\gamma(1) = x_0$.*

Proof. Choose a sequence of neighborhoods V_m of the point x_0 where $W_m = V_m \cap G$ are arc connected (rectifiable) and $W_m \subset B(x_0, 2^{-m})$ and also a sequence of the points $x_m \in W_m$, $m = 1, 2, \ldots$, and connect the points x_m and x_{m+1} pairwise with (rectifiable) curves γ_m in W_m. Uniting the curves γ_m, $m = 1, 2, \ldots$, and joining x_0 in the end, we obtain the desired (locally rectifiable) path to the point x_0 from G.

Recall it is said that a family of curves Γ_1 in T is **minorized** by a family of curves Γ_2 in T, abbr. $\Gamma_1 > \Gamma_2$, if, for every curve $\gamma_1 \in \Gamma_1$, there is a curve $\gamma_2 \in \Gamma_2$ such that γ_2 is a restriction of γ_1.

2.5. Proposition. *Let Ω be an open set in an arbitrary topological space T. Then*

$$\Delta(\Omega, T\setminus\Omega, T) > \Delta(\Omega, \partial\Omega, \Omega)\,.$$

Proof. Indeed, for an arbitrary curve $\gamma : [a,b] \to T$ with $\gamma(a) \in \Omega$ and $\gamma(b) \in T\setminus\Omega$, by continuity of γ the preimage $\omega = \gamma^{-1}(\Omega)$ is an open set in $[a,b]$ including the point a. Similarly, the preimage $\omega = \gamma^{-1}(T\setminus\overline{\Omega})$ is also open in $[a,b]$. Thus, in view of connectivity of the segment $[a,b]$, there is $c \in \gamma^{-1}(\partial\Omega)$ such that $\gamma([a,c)) \subset \Omega$.

2.6. Proposition. *Let γ be a rectifiable curve in a metric space (X,d) connecting points $x_1 \in \overline{B(x_0,r_1)}$ and $x_2 \in X\setminus B(x_0,r_2)$ where $0 < r_1 < r_2 < \infty$ and let $\rho : [0,\infty] \to [0,\infty]$ be a Borel function. Then*

$$\int\limits_{\gamma} \rho(d(x,x_0))\,ds \ge \int\limits_{r_1}^{r_2} \rho(r)\,dr.$$

Proof. Indeed, by the definition for the length of a curve in a metric space $\gamma : [a,b] \to X$, the length of a segment of the curve

$$s(t_1,t_2) \ge d(\gamma(t_1),\gamma(t_2))\,.$$

Moreover, by the triangle inequality

$$d(x_0,\gamma(t_2)) \le d(x_0,\gamma(t_1)) + d(\gamma(t_1),\gamma(t_2))$$

and

$$d(x_0,\gamma(t_1)) \le d(x_0,\gamma(t_2)) + d(\gamma(t_1),\gamma(t_2))\,,$$

thus,

$$d(\gamma(t_1),\gamma(t_2)) \ge |d(x_0,\gamma(t_2)) - d(x_0,\gamma(t_1))|\,.$$

Consequently,

$$ds \ge |dr|$$

where $r = d(x,x_0)\,,\quad x = x(s)$. Finally, by the Darboux property of connected sets, the continuous function $d(x,x_0)$ takes all intermediate values on γ, see e.g. [Ku]. Hence the multiplicity of any value r in the interval (r_1,r_2) of the curve is not less than 1 and the desired inequality follows.

2.7. Proposition. *If Ω and Ω' are open sets in metric spaces (X,d) and (X',d'), correspondingly, and $f : \Omega \to \Omega'$ is a homeomorphism, then the cluster set of f at every point $x_0 \in \partial\Omega$,*

$$C(x_0,f) := \{\ x' \in X' : \ x' = \lim_{n\to\infty} f(x_n),\ x_n \to x_0,\ x_n \in \Omega\ \}\,,$$

belongs to the boundary of the set Ω'.

Proof. Indeed, assume that some point $y_0 \in C(x_0,f)$ is inside of the domain Ω'. Then by the definition of the cluster set, there is a sequence $x_n \to x_0$ as $n \to \infty$ such that $y_n = f(x_n) \to y_0$. In view of continuity of the inverse mapping $g = f^{-1}$, we have that $x_n = g(y_n) \to g(y_0) = x_* \in \Omega$. However, the convergent sequence x_n cannot have two limits $x_0 \in \partial\Omega$ and $x_* \in \Omega$ in view of the triangle inequality $d(x_*,x_0) \le d(x_*,x_n) + d(x_n,x_0)$.

3 On weakly flat and strictly accessible boundaries

In this section, G is a domain of a finite Hausdorff dimension $\alpha \geq 1$ in a space (X, d, μ) with a metric d and a locally finite Borel measure μ.

We will say that the boundary of G is **weakly flat at a point** $x_0 \in \partial G$ if, for every number $P > 0$ and every neighborhood U of the point x_0 there is its neighborhood $V \subset U$ such that

$$M(\Delta(E, F; G)) \geq P \tag{3.1}$$

for all continua E and F in G intersecting ∂U and ∂V.

We will also say that the boundary of the domain G is **strictly accessible at a point** $x_0 \in \partial G$, if, for every neighborhood U of the point x_0, there exist a compact set $E \subset G$, a neighborhood $V \subset U$ of the point x_0 and a number $\delta > 0$ such that

$$M(\Delta(E, F; G)) \geq \delta$$

for every continuum F in G intersecting ∂U and ∂V.

Finally, we say that the boundary ∂G is **weakly flat** and **strictly accessible** if the corresponding properties hold at every point of the boundary.

3.2. Remark. In the definitions of the weakly flat and strictly accessible boundaries, one can restrict itself by a base of neighborhoods of a point x_0 and, in particular, one can take as the neighborhoods U and V of the point x_0 only small enough balls (open or closed) centered at the point x_0. Moreover, here one may restrict itself only by continua E and F in $\overline{U}$.

3.3. Proposition. *If the boundary ∂G is weakly flat at a point $x_0 \in \partial G$, then ∂G is strictly accessible at the point x_0.*

Proof. Let $P \in (0, \infty)$ and $U = B(x_0, r_0)$ where $0 < r_0 < d_0 = \sup\limits_{x \in G} d(x, x_0)$. Then by the condition there is $r \in (0, r_0)$ such that the inequality (3.1) holds for all continua E and F intersecting $\partial B(x_0, r_0)$ and $\partial B(x_0, r)$. By arc connectedness of G there exist points $y_1 \in G \cap \partial B(x_0, r_0)$ and $y_2 \in G \cap \partial B(x_0, r)$. Choose as a compactum E an arbitrary curve connecting the points y_1 and y_2 in G.

Then, for every continuum F in G intersecting ∂U and ∂V where $V = B(x_0, r)$, the inequality (3.1) holds.

3.4. Lemma. *Let G be a (rectifiable) domain in (X, d, μ). If ∂G is weakly flat at a point $x_0 \in \partial G$, then G is locally arc connected (rectifiable) at x_0.*

Proof. Let us assume that the domain G is not locally arc connected (rectifiable) at the point x_0. Then there is $r_0 \in (0, d_0)$, $d_0 = \sup\limits_{x \in G} d(x, x_0)$ such that $\mu_0 := \mu(G \cap B(x_0, r_0)) < \infty$ and, for every neighborhood $V \subseteq U := B(x_0, r_0)$ of the point x_0, at least one of the following conditions holds:

1) $V \cap G$ has at least two arc connected (rectifiable) components K_1 and K_2 such that $x_0 \in \overline{K_1} \cap \overline{K_2}$;

2) $V \cap G$ has infinitely many arc connected (rectifiable) components K_1, ..., K_m, ... such that $x_0 = \lim\limits_{m\to\infty} x_m$ for some $x_m \in K_m$. Note that $\overline{K_m} \cap \partial U \neq \emptyset$ for all $m = 1, 2, ...$ in view of the arc connectedness of G.

In particular, either 1) or 2) holds for the neighborhood $V = U = B(x_0, r_0)$. Let $r_* \in (0, r_0)$. Then

$$M(\Delta(K_i^*, K_j^*; G)) \leq M_0 := \frac{\mu_0}{[2(r_0 - r_*)]^\alpha} < \infty$$

where $K_i^* = K_i \cap \overline{B(x_0, r_*)}$ and $K_j^* = K_j \cap \overline{B(x_0, r_*)}$ for all $i \neq j$. Indeed, one of the admissible functions for the family Γ_{ij} of all rectifiable curves in $\Delta(K_i^*, K_j^*; G)$ is

$$\rho(x) = \begin{cases} \frac{1}{2(r_0 - r_*)}, & x \in B_0 \setminus \overline{B_*}, \\ 0, & x \in X \setminus (B_0 \setminus \overline{B_*}), \end{cases}$$

where $B_0 = B(x_0, r_0)$ and $B_* = B(x_0, r_*)$ because the components K_i and K_j cannot be connected by a (rectifiable) path in $V = B(x_0, r_0)$ and every (rectifiable) path connecting K_i^* and K_j^* at least twice intersect the ring $B_0 \setminus \overline{B_*}$.

In view of 1) - 2), the above modulus estimate contradicts to the condition of the weak flatness at the point x_0. Really, by the condition, for instance, there is $r \in (0, r_*)$ such that

$$M(\Delta(K_{i_0}^*, K_{j_0}^*; G)) \geq M_0 + 1$$

for some pair i_0 and j_0, $i_0 \neq j_0$, because in the corresponding $K_{i_0}^*$ and $K_{j_0}^*$ there is at least one curve intersecting $\partial B(x_0, r_*)$ and $\partial B(x_0, r)$.

Thus, the above assumption on the absence of the arc connectedness (rectifiability) of G at the point x_0 was not true.

3.5. Corollary. *A (rectifiable) domain with a weakly flat boundary is locally arc connected (rectifiable) at every point of its boundary.*

4 On finite mean oscillation with respect to a measure

Let G be a domain in a space (X, d, μ). Similarly to [IR$_1$], cf. also [HKM], we say that a function $\varphi : G \to \mathbb{R}$ has **finite mean oscillation at a point** $x_0 \in \overline{G}$, abbr. $\varphi \in FMO(x_0)$, if

$$\overline{\lim_{\varepsilon\to 0}} \ -\!\!\!\!\!\!\int_{G(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon| \, d\mu(x) < \infty \tag{4.1}$$

where

$$\overline{\varphi}_\varepsilon = -\!\!\!\!\!\!\int_{G(x_0,\varepsilon)} \varphi(x) \, d\mu(x) = \frac{1}{\mu(G(x_0,\varepsilon))} \int\limits_{G(x_0,\varepsilon)} \varphi(x) \, d\mu(x)$$

is the mean value of the function $\varphi(x)$ over the set

$$G(x_0, \varepsilon) = \{x \in G : d(x, x_0) < \varepsilon\}$$

with respect to the measure μ. Here the condition (4.1) includes the assumption that φ is integrable with respect to the measure μ over a set $G(x_0,\varepsilon)$ for some $\varepsilon>0$.

4.2. Proposition. *If for some collection of numbers $\varphi_\varepsilon\in\mathbb{R}$, $\varepsilon\in(0,\varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to0}} \;\; -\!\!\!\!\!\!\int_{G(x_0,\varepsilon)} |\varphi(x)-\varphi_\varepsilon| \ \ d\mu(x)<\infty\,, \tag{4.3}$$

then $\varphi\in FMO(x_0)$.

Proof. Indeed, by the triangle inequality

$$-\!\!\!\!\!\!\int_{G(x_0,\varepsilon)} |\varphi(x)-\overline{\varphi}_\varepsilon| \ \ d\mu(x) \ \ \le \ \ -\!\!\!\!\!\!\int_{G(x_0,\varepsilon)} |\varphi(x)-\varphi_\varepsilon| \ \ d\mu(x)+|\varphi_\varepsilon-\overline{\varphi}_\varepsilon| \ \ \le$$

$$\le \ \ 2\cdot -\!\!\!\!\!\!\int_{G(x_0,\varepsilon)} |\varphi(x)-\varphi_\varepsilon| \ \ d\mu(x)\,.$$

4.4. Corollary. *In particular, if*

$$\overline{\lim_{\varepsilon\to0}} \;\; -\!\!\!\!\!\!\int_{G(x_0,\varepsilon)} |\varphi(x)| \ \ d\mu(x)<\infty\,, \tag{4.5}$$

then $\varphi\in FMO(x_0)$.

Variants of the following lemma have been first proved for the BMO functions and inner points of a domain G in $\mathbb{R}^n$ under $n=2$ and $n\ge3$, correspondingly, in [RSY$_1$]-[RSY$_2$] and [MRSY$_2$]-[MRSY$_3$], and then for boundary points of G in $\mathbb{R}^n$, $n\ge2$, with the condition on doubling of a measure and for the FMO functions in [IR$_1$].

4.6. Lemma. *Let G be a domain in a space (X,d,μ) which is upper $\alpha-$regular with $\alpha\ge2$ at a point $x_0\in\overline{G}$ and*

$$\mu(G\cap B(x_0,2r))\le\gamma\cdot\log^{\alpha-2}\frac{1}{r}\cdot\mu(G\cap B(x_0,r)) \qquad \forall\, r\in(0,r_0)\,. \tag{4.7}$$

Then, for every non–negative function $\varphi:G\to\mathbb{R}$ of the class $FMO(x_0)$,

$$\int\limits_{G\cap A(\varepsilon,\varepsilon_0)} \frac{\varphi(x)\,d\mu(x)}{\left(d(x,x_0)\log\frac{1}{d(x,x_0)}\right)^{\alpha}} = O\left(\log\log\frac{1}{\varepsilon}\right) \tag{4.8}$$

as $\varepsilon\to0$ and some $\varepsilon_0\in(0,\delta_0)$ where $\delta_0=\min\,(e^{-e},d_0)$, $d_0=\sup\limits_{x\in G} d(x,x_0)$,

$$A(\varepsilon,\varepsilon_0) \ = \ \{x\in X: \varepsilon<d(x,x_0)<\varepsilon_0\}.$$

Proof. Choose $\varepsilon_0\in(0,\delta_0)$ such that the function φ is integrable in $G_0=G\cap B_0$ with respect the measure μ where $B_0=B(x_0,\varepsilon_0)$,

$$\delta=\sup_{r\in(0,\varepsilon_0)} \ -\!\!\!\!\!\!\int_{G(r)} |\varphi(x)-\overline{\varphi}_r|\,d\mu(x) \ <\infty\,,$$

$G(r) = G \cap B(r)$, $B(r) = B(x_0, r) = \{x \in X : d(x, x_0) < r\}$. Further, let $\varepsilon < 2^{-1}\varepsilon_0$, $\varepsilon_k = 2^{-k}\varepsilon_0$, $A_k = \{x \in X : \varepsilon_{k+1} \le d(x, x_0) < \varepsilon_k\}$, $B_k = B(\varepsilon_k)$ and let φ_k be the mean value of the function $\varphi(x)$ in $G_k = G \cap B_k, k = 0, 1, 2\ldots$ with respect to the measure μ. Choose a natural number N such that $\varepsilon \in [\varepsilon_{N+1}, \varepsilon_N)$ and denote $\varkappa(t) = (t \log_2 1/t)^{-\alpha}$. Then $G \cap A(\varepsilon, \varepsilon_0) \subset \Delta(\varepsilon) := \bigcup\limits_{k=0}^{N} \Delta_k$ where $\Delta_k = G \cap A_k$ and

$$\eta(\varepsilon) = \int\limits_{\Delta(\varepsilon)} \varphi(x)\ \varkappa(d(x, x_0))\ d\mu(x)\ \le\ |S_1|\ +\ S_2,$$

$$S_1(\varepsilon)\ =\ \sum_{k=1}^{N} \int\limits_{\Delta_k} (\varphi(x) - \varphi_k)\ \varkappa(d(x, x_0))\ d\mu(x)\ ,$$

$$S_2(\varepsilon)\ =\ \sum_{k=1}^{N} \varphi_k \int\limits_{\Delta_k} \varkappa(d(x, x_0))\ d\mu(x)\ .$$

Since $G_k \subset G(2d(x, x_0))$ for $x \in \Delta_k$, then by the condition (1.7) $\mu(G_k) \le \mu(G(2d(x, x_0))) \le C \cdot 2^\alpha \cdot d(x, x_0)^\alpha$, .. $\frac{1}{d(x,x_0)^\alpha} \le C \cdot 2^\alpha \frac{1}{\mu(G_k)}$.

Moreover, $\frac{1}{(\log_2 \frac{1}{d(x,x_0)})^\alpha} \le \frac{1}{k^\alpha}$ for $x \in \Delta_k$ and, thus,

$$|S_1|\ \le\ \delta C \cdot 2^\alpha \sum_{k=1}^{N} \frac{1}{k^\alpha} \le\ 2\delta C \cdot 2^\alpha$$

because under $\alpha \ge 2$

$$\sum_{k=2}^{\infty} \frac{1}{k^\alpha} < \int_1^\infty \frac{dt}{t^\alpha} = \frac{1}{\alpha - 1} \le 1.$$

Further,

$$\int\limits_{\Delta_k} \varkappa(d(x, x_0))\ d\mu(x)\ \le\ \frac{1}{k^\alpha} \int\limits_{A_k} \frac{d\mu(x)}{d(x, x_0)^\alpha}\ \le$$

$$\le\ \frac{C \cdot 2^\alpha}{k^\alpha} \cdot \frac{\mu(G_k) - \mu(G_{k+1})}{\mu(G_k)} \le \frac{C 2^\alpha}{k^\alpha}\,.$$

Moreover, by the condition (4.7)

$$\mu(G_{k-1}) = \mu(B(2\varepsilon_k) \cap G)) \le \gamma \cdot \log_2^{\alpha-2} \frac{1}{\varepsilon_k} \cdot \mu(G_k)$$

and hence

$$|\varphi_k - \varphi_{k-1}|\ =\ \frac{1}{\mu(G_k)} \left| \int\limits_{G_k} (\varphi(x) - \varphi_{k-1})\ d\mu(x) \right|\ \le$$

$$\le\ \frac{\gamma \cdot \log_2^{\alpha-2} \frac{1}{\varepsilon_k}}{\mu(G_{k-1})} \int\limits_{G_{k-1}} |(\varphi(x) - \varphi_{k-1})|\ d\mu(x)\ \le\ \delta \cdot \gamma \cdot \log_2^{\alpha-2} \frac{1}{\varepsilon_k}$$

and, by decreasing ε_k,

$$\varphi_k \;=\; |\varphi_k| \;\le\; \varphi_1 \;+\; \sum_{l=1}^{k} |\varphi_l - \varphi_{l-1}| \;\le\; \varphi_1 \;+ k\delta\gamma \cdot \log_2^{\alpha-2} \frac{1}{\varepsilon_k}.$$

Consequently, because under $\alpha \ge 2$

$$\sum_{k=1}^{\infty} \frac{1}{k^\alpha} \le 1 + \int_1^\infty \frac{dt}{t^\alpha} \;=\; 1 + \frac{1}{\alpha - 1} \;\le\; 2\,,$$

we have the following estimates

$$S_2 \;=\; |S_2| \;\le\; C2^\alpha \sum_{k=1}^{N} \frac{\varphi_k}{k^\alpha} \le\; C2^\alpha \sum_{k=1}^{N} \frac{\varphi_1 \;+ k\delta\gamma \cdot \log_2^{\alpha-2} \frac{1}{\varepsilon_k}}{k^\alpha} \le$$

$$\le C2^\alpha \left(2\varphi_1 + \delta\gamma \sum_{k=1}^{N} \frac{(k + \log_2 \varepsilon_0^{-1})^{\alpha-2}}{k^{\alpha-1}}\right) =$$

$$= C2^\alpha \left(2\varphi_1 + \delta\gamma \sum_{k=1}^{N} \frac{1}{k} \frac{(k + \log_2 \varepsilon_0^{-1})^{\alpha-2}}{k^{\alpha-2}}\right) \le$$

$$\le C2^\alpha \left(2\varphi_1 + \delta\gamma (1 + \log_2 \varepsilon_0^{-1})^{\alpha-2} \sum_{k=1}^{N} \frac{1}{k}\right)$$

and

$$\eta(\varepsilon) \le 2^{\alpha+1} C(\delta + \varphi_1) + 2^\alpha C\delta\gamma (1 + \log_2 \varepsilon_0^{-1})^{\alpha-2} \sum_{k=1}^{N} \frac{1}{k}\,.$$

Since

$$\sum_{k=2}^{N} \frac{1}{k} < \int_1^N \frac{dt}{t} = \log N < \log_2 N$$

and, for $\varepsilon_0 \in (0, 2^{-1})$ and $\varepsilon < \varepsilon_N$,

$$N \;<\; N \;+\; \log_2 \left(\frac{1}{\varepsilon_0}\right) \;=\; \log_2 \frac{1}{\varepsilon_N} \;<\; \log_2 \frac{1}{\varepsilon}\,,$$

then under $\varepsilon_0 \in (0, \delta_0)$, $\delta_0 = min(e^{-e}, d_0)$ and $\varepsilon \to 0$

$$\eta(\varepsilon) \le 2^{\alpha+1} C(\delta + \varphi_1) + 2^\alpha C\delta\gamma (1 + \log_2 \varepsilon_0^{-1})^{\alpha-2} \left(1 + \log_2 \log_2 \frac{1}{\varepsilon}\right) =$$

$$= O\left(\log\log \frac{1}{\varepsilon}\right).$$

4.9. Remark. Note that the condition (4.7) is weaker than the condition on doubling of a measure,

$$\mu(G \cap B(x_0, 2r)) \le \gamma \cdot \mu(G \cap B(x_0, r)) \qquad \forall\, r \in (0, r_0) \tag{4.10}$$

applied before it in the context of $\mathbb{R}^n, n \ge 2$, in the paper [IR$_1$]. Note also that the condition (4.10) automatically holds in the inner points of the domain G if X is regular by Ahlfors.

5 On a continuous extension to the boundary

In what follows, (X, d, μ) and (X', d', μ') are spaces with metrics d and d' and locally finite Borel measures μ and μ', and G and G' domains with finite Hausdorff dimensions α and $\alpha' \geq 1$ in (X, d) and (X', d'), correspondingly.

5.1. Lemma. *Let a domain G be locally arc connected at a point $x_0 \in \partial G$, $\overline{G'}$ be compact and let $f : G \to G'$ be a $Q-$homeomorphism such that $\partial G'$ is strictly accessible at least at one point of the cluster set*

$$C(x_0, f) = \{y \in X' : \ y = \lim_{k\to\infty} f(x_k), \ x_k \to x_0, \ x_k \in G\}\,, \tag{5.2}$$

$Q : G \to [0, \infty]$ is a measurable function satisfying the condition

$$\int\limits_{G(x_0,\varepsilon)} Q(x) \cdot \psi^{\alpha}_{x_0,\varepsilon}(d(x, x_0))\, d\mu(x) \ = \ o(I^{\alpha}_{x_0}(\varepsilon)) \tag{5.3}$$

as $\varepsilon \to 0$ where

$$G(x_0, \varepsilon) = \{x \in G : \varepsilon < d(x, x_0) < \varepsilon(x_0)\},\ \varepsilon(x_0) \in (0, d(x_0)),\ d(x_0) = \sup_{x\in G} d(x, x_0),$$

and $\psi_{x_0,\varepsilon}(t)$ is a family of non-negative measurable (by Lebesgue) functions on $(0, \infty)$ such that

$$0 < I_{x_0}(\varepsilon) \ = \ \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{x_0,\varepsilon}(t)\, dt \ < \ \infty \qquad \forall\, \varepsilon \in (0, \varepsilon_0)\,. \tag{5.4}$$

Then f can be extended to the point x_0 by continuity in (X', d').

Proof. Let us show that the cluster set $E = C(x_0, f)$ is a singleton. Note that $E \neq \emptyset$ in view of the compactness of $\overline{G'}$, see e.g. Remark 3 of the section 41 in [Ku]. By the condition of the lemma, $\partial G'$ is strictly accessible at a point $y_0 \in E$. Assume that there is one more point $y^* \in E$. Let $U = B(y_0, r_0)$ where $0 < r_0 < d(y_0, y^*)$.

In view of the local arc connectedness of the domain G at the point x_0, there is a sequence of neighborhoods V_m of the point x_0 such that $G_m = G \cap V_m$ are domains and $d(V_m) \to 0$ as $m \to \infty$. Then there exist points y_m and $y^*_m \in F_m$ which a close enough to y_0 and y^*, correspondingly, for which $d'(y_0, y_m) < r_0$ and $d'(y_0, y^*_m) > r_0$ and which can be joined by curves C_m in the domains $F_m = fG_m$. By the construction

$$C_m \cap \partial B(x_0, r_0) \neq \emptyset$$

in view of the connectedness of C_m.

By the condition of the strict accessibility there is a compact set $C \subset G'$ and a number $\delta > 0$ such that

$$M(\Delta(C, C_m, G')) \geq \delta$$

for large m because $dist(y_0, C_m) \to 0$ as $m \to \infty$. Note that $K = f^{-1}(C)$ is compact as a continuous image of a compact set. Thus, $\varepsilon_0 = dist(x_0, K) > 0$.

Let Γ_ε be the family of all paths in G connecting the ball $B_\varepsilon = \{x \in X : d(x,x_0) < \varepsilon\}$, $\varepsilon \in (0,\varepsilon_0)$, with the compactum K. Let $\psi^*_{x_0,\varepsilon}$ be a Borel function such that $\psi^*_{x_0,\varepsilon}(t) = \psi_{x_0,\varepsilon}(t)$ for a.e. $t \in (0,\infty)$ which exists by the Lusin theorem, see e.g. 2.3.5 in [Fe].

Then the function

$$\rho_\varepsilon(x) \;=\; \begin{cases} \psi^*_{x_0,\varepsilon}(d(x,x_0))/I_{x_0}(\varepsilon), & x \in G(x_0,\varepsilon), \\ 0, & x \in X\setminus G(x_0,\varepsilon), \end{cases}$$

is admissible for Γ_ε by Proposition 2.6 and, consequently,

$$M(f\Gamma_\varepsilon) \;\le\; \int\limits_G Q(x)\cdot\rho^\alpha_\varepsilon(x)\; d\mu(x)\;.$$

Hence $M(f\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$ in view of (5.3).

On the other hand, for any $\varepsilon \in (0,\varepsilon_0)$, $G_m \subset B_\varepsilon$ for large m, hence $C_m \subset fB_\varepsilon$ for such m and, thus,

$$M(f\Gamma_\varepsilon) \ge M(\Delta(C,C_m;G'))\,.$$

The obtained contradiction disproves the above assumption that the cluster set E is not degenerated to a point.

5.5. Corollary. *In particular, if*

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0} Q(x)\cdot\psi^\alpha(d(x,x_0))\,d\mu(x) \;<\; \infty \tag{5.6}$$

where $\psi(t)$ is a measurable function on $(0,\infty)$ such that

$$0 < I(\varepsilon,\varepsilon_0) \;:=\; \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\,dt \;<\; \infty \qquad\quad \forall\; \varepsilon \in (0,\,\varepsilon_0)$$

and $I(\varepsilon,\varepsilon_0) \to \infty$ as $\varepsilon \to 0$, then any $Q-$homeomorphism $f : G \to G'$ can be extended to the point x_0 by continuity in (X',d').

Here we assume that the function Q is extended by zero outside of G.

5.7. Remark. In the other words, it is sufficient for the singular integral (5.6) to be convergent in the sense of the principal value at the point x_0 at least for one kernel ψ with a non-integrable singularity at zero. Furthermore, as the lemma shows it is even sufficient for the given integral to be divergent but with the controlled speed

$$\int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0} Q(x)\cdot\psi^\alpha(d(x,x_0))\,d\mu(x) \;=\; o(I^\alpha(\varepsilon,\varepsilon_0)) \tag{5.8}$$

Choosing in Lemma 5.1 $\psi(t) \equiv 1/t$, we obtain the following theorem.

5.9. Theorem. *Let G be locally arc connected at a point $x_0 \in \partial G$, $\overline{G'}$ compact and $\partial G'$ strictly accessible. If a measurable function $Q : G \to [0, \infty]$ satisfies the condition*

$$\int\limits_{G(x_0,\varepsilon,\varepsilon_0)} \frac{Q(x)d\mu(x)}{d(x,x_0)^\alpha} = o\left(\left[\log\frac{1}{\varepsilon}\right]^\alpha\right) \tag{5.10}$$

as $\varepsilon \to 0$ where $G(x_0,\varepsilon,\varepsilon_0) = \{x \in G : \varepsilon < d(x,x_0) < \varepsilon_0\}$ for $\varepsilon_0 < d(x_0) = \sup\limits_{x\in G} d(x,x_0)$, then any $Q-$homeomorphism $f : G \to G'$ can be extended to x_0 by continuity in (X', d').

5.11. Corollary. *In particular, the conclusion of Theorem 5.9 is valid if the singular integral*

$$\int \frac{Q(x)d\mu(x)}{d(x,x_0)^\alpha} \tag{5.12}$$

is convergent at the point x_0 in the sense of the principal value.

Here as in Corollary 5.5 we assume that Q is extended by zero outside of G.

Combining Lemmas 4.6 and 5.1, choosing $\psi_\varepsilon(t) \equiv t\log\frac{1}{t}$, $t \in (0,\delta_0)$, we obtain the following theorem.

5.13. Theorem. *Let X be upper $\alpha-$regular at a point $x_0 \in \partial G$, $\alpha \geq 2$, where G is locally arc connected and satisfies the condition (4.7), and let $\overline{G'}$ be compact and $\partial G'$ strictly accessible. If $Q \in FMO(x_0)$, then any $Q-$homeomorphism $f : G \to G'$ can be extended to the point x_0 by continuity in (X', d').*

Finally, combining Theorem 5.13 and Corollary 4.4, we obtain the following statement.

5.14. Corollary. *In particular, if*

$$\overline{\lim_{\varepsilon\to 0}} \quad -\!\!\!\!\!\!\int_{G(x_0,\varepsilon)} Q(x)\ d\mu(x) < \infty \tag{5.15}$$

where $G(x_0,\varepsilon) = \{x \in G : d(x,x_0) < \varepsilon\}$, then any $Q-$homeomorphism $f : G \to G'$ can be extended to the point x_0 by continuity in (X', d').

6 On extending the inverse mappings to the boundary

Here, as it was before, see (5.2), $C(x_0, f)$ denotes the cluster set of the mapping f at a point $x_0 \in \partial G$.

6.1. Lemma. *Let $f : G \to G'$ be a $Q-$homeomorphism with $Q \in L^1_\mu(G)$. If the domain G is locally arc connected at points x_1 and $x_2 \in \partial G$, $x_1 \neq x_2$, and G' has a weakly flat boundary, then $C(x_1, f) \cap C(x_2, f) = \emptyset$.*

Proof. Set $E_i = C(x_i, f)$, $i = 1, 2$, and $\delta = d(x_1, x_2)$. Let us assume that $E_1 \cap E_2 \neq \emptyset$.

Since the domain G is locally arc connected at the points x_1 and x_2, there exist neighborhoods U_1 and U_2 of the points x_1 and x_2, correspondingly, such that $W_1 = G \cap U_1$ and $W_2 = G \cap U_2$ are domains and $U_1 \subset B_1 = B(x_1, \delta/3)$ and $U_2 \subset B_2 = B(x_2, \delta/3)$. Then by the triangle inequality $dist(W_1, W_2) \geq \frac{\delta}{3}$ and the function

$$\rho(x) = \begin{cases} \frac{3}{\delta}, & x \in G, \\ 0, & x \in X \setminus G \end{cases}$$

is admissible for the path family $\Gamma = \Delta(W_1, W_2; G)$. Thus,

$$M(f\Gamma) \ \leq \ \int\limits_X Q(x)\, \rho^\alpha(x)\, d\mu(x) \ \leq \ \frac{3^\alpha}{\delta^\alpha} \int\limits_G Q(x)\, d\mu(x) \ < \ \infty$$

because $Q \in L^1_\mu(G)$.

The last estimate contradicts, however, to the condition of the weak flatness (3.1) if there is a point $y_0 \in E_1 \cap E_2$. Indeed, then $y_0 \in \overline{fW_1} \cap \overline{fW_2}$ and in the domains $W_1^* = fW_1$ and $W_2^* = fW_2$ there exist paths intersecting any prescribed spheres $\partial B(y_0, r_0)$ and $\partial B(y_0, r_*)$ with small enough radii r_0 and r_*, see Proposition 2.5 and Lemma 3.4. Hence the assumption that $E_1 \cap E_2 \neq \emptyset$ was not true.

By Lemma 6.1 we obtain, in particular, the following conclusion.

6.2. Theorem. *Let G be locally arc connected at all its boundary points and $\overline{G}$ compact, G' with a weakly flat boundary, and let $f : G \to G'$ be a $Q-$homeomorphism with $Q \in L^1_\mu(G)$. Then the inverse homeomorphism $g = f^{-1} : G' \to G$ admits a continuous extension $\overline{g} : \overline{G'} \to \overline{G}$.*

6.3. Remark. In fact, as it is clear from the above proof, see also Proposition 2.7, it is sufficient in Lemma 6.1 and Theorem 6.2 as well as in all successive theorems to request instead of the condition $Q \in L^1_\mu(G)$ the integrability of Q in a neighborhood of ∂G assuming Q to be extended by zero outside of G.

7 On a homeomorphic extension to the boundary

Combining the results of the previous sections, we obtain the following theorems.

7.1. Lemma. *Let G be locally arc connected at its boundary, G' have a weakly flat boundary and $\overline{G}$, $\overline{G'}$ be compact. If a function $Q : G \to [0, \infty]$ of the class $L^1_\mu(G)$ satisfies the condition (5.3) at every point $x_0 \in \partial G$, then any $Q-$homeomorphism $f : G \to G'$ is extended to a homeomorphism $\overline{f} : \overline{G} \to \overline{G'}$.*

7.2. Theorem. *Let G and G' have weakly flat boundaries and $\overline{G}$ and $\overline{G'}$ be compact and let $Q : G \to [0, \infty]$ be a function of the class $L^1_\mu(G)$ with*

$$\int\limits_{G(x_0,\varepsilon,\varepsilon_0)} \frac{Q(x)d\mu(x)}{d(x,x_0)^\alpha} \ = \ o\left(\left[\log\frac{1}{\varepsilon}\right]^\alpha\right) \qquad (7.3)$$

at every point $x_0 \in \partial G$ *where* $G(x_0, \varepsilon, \varepsilon_0) = \{x \in G : \varepsilon < d(x, x_0) < \varepsilon_0\}$, $\varepsilon_0 = \varepsilon(x_0) < d(x_0) = \sup\limits_{x \in G} d(x, x_0)$. *Then any* $Q-$*homeomorphism* $f : G \to G'$ *admits an extension to a homeomorphism* $\overline{f} : \overline{G} \to \overline{G'}$.

7.4. Corollary. *In particular, the conclusion of Theorem 7.2 holds if the singular integral*

$$\int \frac{Q(x)d\mu(x)}{d(x,x_0)^{\alpha}} \tag{7.5}$$

is convergent in the sense of the principal value at all boundary points.

As before, here it is assumed that Q has been extended by zero outside of G.

7.6. Theorem. *Let* G *be a domain in an upper* $\alpha-$*regular space* (X, d, μ), $\alpha \geq 2$, *which is locally arc connected and satisfies the condition (4.7) at all boundary points,* G' *be a domain with a weakly flat boundary in a space* (X', d', μ') *and* $\overline{G}$ *and* $\overline{G'}$ *be compact. If a function* $Q : G \to [0, \infty]$ *has finite mean oscillation at all boundary points, then any* $Q-$*homeomorphism* $f : G \to G'$ *can be extended to a homeomorphism* $\overline{f} : \overline{G} \to \overline{G'}$.

7.7. Corollary. *In particular, the conclusion of Theorem 7.6 holds if*

$$\overline{\lim_{\varepsilon \to 0}} \, ⨍_{G(x_0,\varepsilon)} Q(x) \; d\mu(x) < \infty \tag{7.8}$$

at all points $x_0 \in \partial G$ *where* $G(x_0, \varepsilon) = \{x \in G : \; d(x, x_0) < \varepsilon\}$.

7.9. Remark. If the conditions of the type (5.3), (7.3), (7.5), (7.8) or finiteness of the mean oscillation hold only on a closed set $E \subset \partial G$, Q, extended by zero outside of the set E, is integrable in a neighborhood of E, $\overline{G}$ and $\overline{G'}$ are compact, G is locally connected at every point of E, and $\partial G'$ is weakly flat at all points of the cluster set

$$E' = C(E, f) = \{x' \in X' : \; x' = \lim_{k \to \infty} f(x_k), \; x_k \in G, \; x_k \to x_0 \in E\}\,, \tag{7.10}$$

then the Q–homeomorphism $f : G \to G'$ admits a homeomorphic extension $\overline{f} : G \cup E \to G' \cup E'$.

8 On moduli of families of paths going through a point

In this section we establish conditions on a measure μ under which the modulus of a family of all paths in a space (X, d, μ) going through a fixed point is equal zero.

8.1. Lemma. *Let the condition*

$$\int\limits_{A(x_0,r,R_0)} \psi^{\alpha}(d(x,x_0)) \; d\mu(x) = o\left(\left[\int\limits_{r}^{R_0} \psi(t) \; dt\right]^{\alpha}\right) \tag{8.2}$$

holds as $r \to 0$ where

$$A(x_0, r, R_0) = \{x \in X : r < d(x, x_0) < R_0\}, \qquad R_0 \in (0, \infty)\ ,$$

and let $\psi(t)$ be a non–negative function on $(0, \infty)$ such that

$$0 < \int\limits_r^{R_0} \psi(t)\ dt < \infty \qquad \forall\, r \in (0, R_0)$$

Then the family of all paths in X going through the point x_0 has the modulus zero.

8.3. Remark. The condition (8.2) implies that under $r \to 0$

$$\int\limits_{A(x_0,r,r_0)} \psi^\alpha(d(x, x_0))\ d\mu(x) = o\left(\left[\int\limits_r^{r_0} \psi(t)\ dt\right]^\alpha\right) \qquad \forall\ r_0 \in (0, R_0)\ . \tag{8.4}$$

Proof of Lemma 8.1. Let Γ be the family of all paths in X going through the point x_0. Then $\Gamma = \bigcup\limits_{k=1}^{\infty} \Gamma_k$ where Γ_k are the families of all paths in X going through x_0 and intersecting the spheres $S_k = S(x_0, r_k)$ for some sequence such that $r_k \in (0, R_0)\, , r_k \to 0$ as $k \to \infty$.

However, $M(\Gamma_k) = 0$. Indeed, the function

$$\rho(x) = \begin{cases} \psi(d(x, x_0)) \left(\int\limits_r^{r_k} \psi(t)\ dt\right)^{-1}, & x \in A_k(r), \\ 0, & x \in X \backslash A_k(r), \end{cases}$$

where $A_k(r) = A(x_0, r, r_k)$, is admissible for the family $\Gamma_k(r)$ of all paths intersecting the spheres S_k and $S(x_0, r)$, $r \in (0, r_k)$. Since $\Gamma_k > \Gamma_k(r)$, then

$$M(\Gamma_k) \le M(\Gamma_k(r)) \le \left(\int\limits_r^{r_k} \psi(t)\ dt\right)^{-\alpha} \int\limits_{A_k(r)} \psi^\alpha(d(x, x_0))\ d\mu(x)$$

and by the condition (8.2), cf. also (8.4), it follows that $M(\Gamma_k) = 0$ because $r \in (0, r_k)$ is arbitrary.

Finally, from the subadditivity of the modulus, it follows that

$$M(\Gamma) \le \sum_{k=1}^{\infty} M(\Gamma_k) = 0.$$

8.5. Theorem. *Let, for some $R_0 \in (0, \infty)$, under $r \to 0$*

$$\int\limits_{A(x_0,r,R_0)} \frac{d\mu(x)}{d^\alpha(x, x_0)} = o\left(\left[\log \frac{R_0}{r}\right]^\alpha\right)\ . \tag{8.6}$$

Then the family of all paths in X going through the point x_0 has the modulus zero.

8.7. Remark. For $X=\mathbb{R}^n$, $n\geq 2$, and $R_0\in(0,\infty)$,

$$\int\limits_{A(x_0,r,R_0)}\frac{dm(x)}{|x-x_0|^n}=\omega_{n-1}\log\left(\frac{R_0}{r}\right)=o\left(\left[\log\frac{R_0}{r}\right]^n\right) \tag{8.8}$$

where m denotes the Lebesgue measure and ω_{n-1} the area of the unit sphere in $\mathbb{R}^n$.

For spaces (X,d,μ) which are upper α-regular at the point x_0 with $\alpha>1$,

$$\int\limits_{r<d(x_0,x)<R_0}\frac{d\mu(x)}{d(x,x_0)^{\alpha}}=O\left(\log\frac{R_0}{r}\right), \tag{8.9}$$

see [He$_1$], cf. 54, and, thus, the condition (8.6) is also automatically holds in such spaces.

9 On weakly flat and strictly connected spaces

Recall that a topological space T is said to be **locally (arc) connected at a point** $x_0\in T$ if, for every neighborhood U of the point x_0, there is a neighborhood $V\subseteq U$ of the point x_0 which is (arc) connected, see [Ku], p. 232. We will say that a space T is **(arc) connected at a point** x_0 if, for every neighborhood U of the point x_0, there is a neighborhood $V\subseteq U$ of the point x_0 such that $V\setminus\{x_0\}$ is (arc) connected. Note that (arc) connectedness of a space T at a point x_0 implies its local (arc) connectedness at the point x_0. The inverse conclusion is, generally speaking, not true.

Here (X,d,μ) is a space with a metric d, a locally finite Borel measure μ and a finite Hausdorff dimension $\alpha\geq 1$.

We say that an arc connected space (X,d,μ) is **weakly flat at a point** $x_0\in X$ if, for every neighborhood U of the point x_0 and every number $P>0$, there is a neighborhood $V\subseteq U$ of x_0 such that

$$M(\Delta(E,F;X))\geq P$$

for any continua E and F in X intersecting ∂V and ∂U.

We also say that an arc connected space (X,d,μ) is **strictly connected at a point** $x_0\in X$ if, for every neighborhood U of the point x_0, there is a neighborhood $V\subseteq U$ of x_0, a compact set E in X and a number $\delta>0$ such that

$$M(\Delta(E,F;X))\geq\delta$$

for any continua F in X intersecting ∂V and ∂U.

Finally, we say that a space (X,d,μ) is **weakly flat (strictly connected)** if it is weakly flat (strictly connected) at every point.

9.1. Remark. In the definitions of weakly flat and strictly connected spaces, we may restrict ourselves by a base of neighborhoods of a point x_0 and, in particular, one may take as U and V (open or closed) only small enough balls centered at the point x_0. Moreover, here one may restrict itself only by continua E and F in $\overline{U}$. It is also obvious that every domain in a weakly flat space is a weakly flat space.

The following statement is not so important and proved similarly to Proposition 3.3 and hence we omit its proof here.

9.2. Proposition. *If a space (X, d, μ) is weakly flat at a point $x_0 \in X$, then X is strictly connected at the point x_0.*

In what follows, the following statement is much more important.

9.3. Lemma. *If a space (X, d, μ) is weakly flat at a point $x_0 \in X$, then (X, d, μ) is locally arc connected at the point x_0.*

Proof. Let us assume that the space X is not locally arc connected at the point x_0. Then there is $r_0 \in (0, d_0)$, $d_0 = \sup\limits_{x \in G} d(x, x_0)$, such that $\mu_0 := \mu(\overline{B(x_0, r_0)}) < \infty$ and every neighborhood $V \subseteq U := \overline{B(x_0, r_0)}$ of the point x_0 has an arc connected component K_0 including x_0 and infinitely many arc connected components K_1, ..., K_m, ... such that $x_0 = \lim\limits_{m \to \infty} x_m$ for some $x_m \in K_m$. Note that $K_m \cap \partial U \neq \emptyset$ for all $m = 1, 2, ...$ in view of the arc connectedness of X, see Proposition 2.5.

In particular, this is true for the neighborhood $V = U = \overline{B(x_0, r_0)}$. Let $r_* \in (0, r_0)$. Then for all $i = 1, 2, ...$

$$M(\Delta(K_i^*, K_0^*; G)) \leq M_0 := \frac{\mu_0}{[2(r_0 - r_*)]^\alpha} < \infty$$

where $K_i^* = K_i \cap \overline{B(x_0, r_*)}$ and $K_0^* = K_0 \cap \overline{B(x_0, r_*)}$. Indeed, one of the admissible functions for the family Γ_i of all rectifiable curves in $\Delta(K_i^*, K_0^*; G)$ is

$$\rho(x) = \begin{cases} \frac{1}{2(r_0 - r_*)}, & x \in B_0 \setminus \overline{B_*}, \\ 0, & x \in X \setminus (B_0 \setminus \overline{B_*}), \end{cases}$$

where $B_0 = B(x_0, r_0)$ and $B_* = B(x_0, r_*)$ because the components K_i and K_0 cannot be connected by a path in $V = \overline{B(x_0, r_0)}$ and every path connecting K_i^* and K_0^* at least twice intersects the ring $B_0 \setminus \overline{B_*}$, see Proposition 2.6.

However, the above modulus estimate contradicts to the condition of the weak flatness at the point x_0. Really, by this condition, for instance, there is $r \in (0, r_*)$ such that

$$M(\Delta(K_{i_0}^*, K_0^*; G)) \geq M_0 + 1$$

for some $i_0 = 1, 2, ...$ because in the corresponding $K_{i_0}^*$ and K_0^* there exist paths intersecting $\partial B(x_0, r_*)$ and $\partial B(x_0, r)$, see Proposition 2.5.

Thus, the above assumption on the absence of the arc connectedness of the space X at the point x_0 was not true.

Combining Lemma 9.3 with Proposition 2.1, we obtain the following conclusion.

9.4. Corollary. *An open set Ω in a weakly flat space (X,d,μ) is arc connected if and only if it is connected.*

9.5. Corollary. *A domain G in a weakly flat space (X,d,μ) is locally arc connected at a point $x_0 \in \partial G$ if and only if G is locally connected at the point x_0.*

Combining Lemmas 8.1 and 9.3, we obtain the following result.

9.6. Theorem. *If a space (X,d,μ) is weakly flat at a point $x_0 \in X$ and the condition (8.2), in particular, (8.6) holds, then (X,d,μ) is arc connected at the point x_0.*

By Remark 8.7 we come to the following conclusion.

9.7. Corollary. *If a space X is weakly flat and upper α–regular at a point $x_0 \in X$ with $\alpha > 1$, then X is arc connected at the point x_0.*

9.8. Remark. $\mathbb{R}^n$, $n \geq 2$, is a weakly flat space because

$$M(\Delta(E,F;\mathbb{R}^n)) \geq c_n \log \frac{R}{r} \tag{9.9}$$

for all continua E and F intersecting the boundaries of the balls $\mathbb{B}^n(R)$ and $\mathbb{B}^n(r)$, see e.g. 10.12 in [Va].

10 On quasiextremal distance domains

Similarly to [GM], we say that a domain G in (X,d,μ) is a **quasiextremal distance domain**, abbr., a **QED domain**, if

$$M(\Delta(E,F;X)) \leq K\, M(\Delta(E,F;G)) \tag{10.1}$$

for a finite number $K \geq 1$ and all continua E and F in G.

As it is easy to see from the definitions, a QED domain G in a weakly flat space has a weakly flat boundary and, as a consequence, ∂G is strictly accessible and, moreover, G is locally arc connected at all points of the boundary. Thus, all the above results on the extension of $Q-$homeomorphisms to the boundary hold for QED domains in weakly flat spaces. Let us give a resume of these results.

10.2. Lemma. *Let f be a $Q-$homeomorphism between QED domains G and G' in weakly flat spaces X and X', correspondingly, $\overline{G'}$ compact and let at a point $x_0 \in \partial G$*

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)} Q(x)\, \psi^{\alpha}(d(x,x_0))\, d\mu(x) \;=\; o\left(\left[\int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt\right]^{\alpha}\right) \tag{10.3}$$

as $\varepsilon \to 0$ where

$$A(x_0,\varepsilon,\varepsilon_0) = \{x \in G : \varepsilon < d(x,x_0) < \varepsilon_0\}$$

and $\psi(t)$ is a non–negative function on $(0,\infty)$ such that

$$0<\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\ dt<\infty \qquad\qquad \forall\,\varepsilon\in(0,\varepsilon_0)$$

Then there is a limit of $f(x)$ as $x\to x_0$.

10.4. Corollary. *In particular, the limit of $f(x)$ as $x\to x_0$ exists if*

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)} Q(x)\,\psi^{\alpha}(d(x,x_0))\ d\mu(x)\ <\ \infty \tag{10.5}$$

and

$$\lim_{\varepsilon\to 0}\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\ dt\ =\ \infty\,. \tag{10.6}$$

10.7. Theorem. *Let f be a $Q-$homeomorphism between QED domains G and G' in weakly flat spaces X and X', correspondingly, and let $\overline{G'}$ be compact. If at a point $x_0\in\partial G$*

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)}\frac{Q(x)\,d\mu(x)}{d(x,x_0)^{\alpha}}\ =\ o\left(\left[\log\frac{\varepsilon_0}{\varepsilon}\right]^{\alpha}\right)\,, \tag{10.8}$$

then f admits a continuous extension to the point x_0.

10.9. Corollary. *In particular, the conclusion of Theorem 10.7 holds if the singular integral*

$$\int\frac{Q(x)\,d\mu(x)}{d(x,x_0)} \tag{10.10}$$

is convergent at x_0 in the sense of the principal value.

Here we assume that Q is extended by zero outside of the domain G.

10.11. Lemma. *Let f be a $Q-$homeomorphism between QED domains G and G' in weakly flat spaces X and X', correspondingly, and let $\overline{G}$ be compact. If $Q\in L^1_{\mu}(G)$, then the inverse homeomorphism $g=f^{-1}$ admits a continuous extension $\overline{g}:\overline{G'}\to\overline{G}$.*

10.12. Theorem. *Let f be a $Q-$homeomorphism between $QED-$domains G and G' in weakly flat spaces X and X' and let $\overline{G}$ and $\overline{G'}$ be compact. If $Q\in L^1_{\mu}(G)$ satisfies one of the conditions (10.8) or (10.10) at every point $x_0\in\partial G$, then f admits a homeomorphic extension $\overline{f}:\overline{G}\to\overline{G'}$.*

10.13. Theorem. *Let f be a $Q-$homeomorphism between QED domains G and G' in weakly flat spaces X and X', correspondingly, and let $\overline{G}$ and $\overline{G'}$ be* *compact. If at a point $x_0 \in \partial G$ the function $Q : X \to [0,\infty]$ has finite mean oscillation and*

$$\text{(10.14)} \qquad \mu(B(x_0,2r)) \le \gamma \cdot \log^{\alpha-2}\frac{1}{r} \cdot \mu(B(x_0,r)) \qquad \forall\, r \in (0,r_0)$$

and (X,d,μ) is upper $\alpha-$regular with $\alpha \ge 2$ at x_0, then f admits a continuous extension to the point x_0. If the last two conditions hold at every point of ∂G, then f admits a homeomorphic extension to the boundary.

10.15. Remark. In the case of regular by Ahlfors spaces, even the condition on doubling measure holds which stronger than the condition (10.14), see Remark 4.9. In view of the compactness of $\overline{G}$, Q is integrable in a neighborhood of ∂G that follows from the condition of finite mean oscillation at all points of ∂G, see Remark 6.3. If Q is given only in a domain G, then it can be extended by zero outside of G. In particular, to have $Q \in FMO(x_0)$ for $x_0 \in \partial G$ it is sufficient to have the condition

$$\text{(10.16)} \qquad \overline{\lim_{\varepsilon\to 0}} \; ⨍_{B(x_0,\varepsilon)} Q(x)\; d\mu(x) < \infty\,.$$

By [GM], the QED domains coincide in the class of finitely connected plane domains with the so–called uniform domains introduced in the work [MS]. The following example shows that, even among simply connected plane domains, the class of domains with weakly flat boundaries is more wide than the class of QED domains. The example is based on the fact that QED domains satisfy the condition on doubling measure (4.10) at every boundary point, see Lemma 2.13 in [GM]. The below example just shows that the property on doubling measure is, generally speaking, not valid for domains with weakly flat boundaries.

Example. Consider a simply connected plane domain D of the form

$$D \;=\; \bigcup_{k=1}^{\infty} R_k$$

where

$$R_k \;=\; \{\ (x,y) \in \mathbb{R}^2 :\ 0<x<w_k,\ 0<y<h_k\ \}$$

is a sequence of rectangles with quickly decreasing widths $w_k = 2^{-\alpha 2^k} \to 0$ as $k \to \infty$ where $\alpha > \frac{1}{\log 2} > 1$ and monotonically increasing heights $h_k = 2^{-1} + \ldots + 2^{-k} \to 1$ as $k \to \infty$.

It is easy to see that D has a weakly flat boundary. This fact is not obvious only for its boundary point $z_0 = (0,1)$. Take according to Remark 9.1 as a base of neighborhoods of the point z_0 the rectangles centered at z_0

$$P_k \;=\; \{\ (x,y) \in \mathbb{R}^2 :\ |x| < w_k,\ |y-1| \le 1 - h_{k-1} = 2^{-(k-1)}\ \}\ ,$$

$k = 1,2,\ldots$. Note that

$$P_k \cap D \;=\; \bigcup_{l=k}^{\infty} S_l$$

for all $k > 1$ where

$$S_l \;=\; \{\ (x,y) \in \mathbb{R}^2 :\ 0 < x < w_l,\ h_{l-1} \le y < h_l\ \}$$

Let E and F be an arbitrary pair of continua in D intersecting ∂S_l, i.e. intersecting the horizontal lines $y = h_{l-1}$ and $y = h_l$. Denote by S_l^0 the interiority of S_l. Then $\Delta(E, F, S_l^0) \subset \Delta(E, F, D)$ and $\Delta(E, F, S_l^0)$ minorizes the family Γ_l of all paths joining the vertical sides of S_l^0 in S_l^0. Hence, see e.g. 7.2 in [Va],

$$M(\Delta(E,F,D)) \;\ge\; 2^{-l}/w_l \;\ge\; 2^{(\alpha-1)l} \;\to \infty\ .$$

Thus, the domain D really has a weakly flat boundary.

Now, set $r_k = 1 - h_{k-1} = 2^{-k}(1 + 2^{-1} + \ldots) = 2^{-(k-1)}$ and $B_k = B(z_0, r_k)$. Then

$$\lim_{k\to\infty} \frac{|D \cap P_k|}{|D \cap B_k|} \;=\; 1$$

because $w_k/r_k \le 2^{-(\alpha-1)k} \to 0$. However,

$$|D \cap P_k| \;=\; \sum_{l=k}^{\infty} |S_l| \;=\; \sum_{l=k}^{\infty} w_l \cdot (h_l - h_{l-1}) \;=\; \sum_{l=k}^{\infty} w_l\, 2^{-l}$$

and hence

$$\frac{|D \cap P_k|}{|D \cap P_{k+1}|} \;=\; \frac{\sum\limits_{l=k}^{\infty} w_l 2^{-l}}{\sum\limits_{l=k+1}^{\infty} w_l 2^{-l}} \;=\; \frac{w_k 2^{-k} + \sum\limits_{l=k+1}^{\infty} w_l 2^{-l}}{\sum\limits_{l=k+1}^{\infty} w_l 2^{-l}} \;=$$

$$=\; 1 \;+\; \frac{1}{\sum\limits_{m=1}^{\infty} \frac{w_{k+m}}{w_k}\, 2^{-m}} \;\ge\; 1 \;+\; \frac{1}{\frac{w_{k+1}}{w_k}} \;=\; 1 \;+\; \frac{w_k}{w_{k+1}} \;=\; 1 \;+\; 2^{\alpha 2^k} \;\to \infty\ .$$

Consequently,

$$\lim_{k\to\infty} \frac{|D \cap B_k|}{|D \cap B_{k+1}|} \;=\; \infty\ .$$

Thus, the domain D has not the property on doubling measure at the point $z_0 \in \partial D$ and then D is not a QED domain.

11 On nullsets for extremal distance

We say that a closed set A in a space (X, d, μ) is a **nullset for extremal distance**, abbr., **NED set**, if

$$M(\Delta(E, F; D)) = M(\Delta(E, F; D \setminus A)) \tag{11.1}$$

for any domain D in X and any continua E and F in D.

As in $\mathbb{R}^n, n \ge 2$, NED sets A in a weakly flat space X cannot have inner points and, moreover, they do not split the space X even locally, i.e., $G \setminus A$ has only one component of the arc connectedness for any domain G in X. Thus, the complement of a NED set A in such X is a very partial case of QED domains. Hence NED sets in weakly flat spaces play the same role in the problems of removability of singular sets under quasiconformal mappings and their generalizations as in $\mathbb{R}^n$, $n \geq 2$.

11.2. Proposition. *Let A be a NED set in a weakly flat space (X, d, μ) that is not a singleton. Then*

1) A has no inner point;

2) $G \setminus A$ is a domain for every domain G in X.

Proof. 1) Let us assume that there is a point $x_0 \in A$ such that $B(x_0, r_0) \subseteq A$ for some $r_0 > 0$. Let $x_* \in X$, $x_* \neq x_0$, and γ be a curve joining x_0 and x_* in X, $\gamma : [0,1] \to X$, $\gamma(0) = x_0$ and $\gamma(1) = x_*$. For small enough t, the continuum $C_t = \gamma([0,t])$ is in the ball $B(x_0, r_0)$ and, consequently, $\gamma([0,t]) \cap \overline{(X \setminus A)} = \emptyset$. Moreover, by Proposition 2.5 one can choose $t = t_0$ such that $C_{t_0} \setminus \{x_0\} \neq \emptyset$. Hence, setting $E = F = C_{t_0}$, we have $M(\Delta(E, F, X)) = \infty$ because the space X is weakly flat and, on the other hand, $M(\Delta(E, F; X \setminus A)) = 0$. The obtained contradiction disproves the above assumption.

2) Denote by Ω_* one of the connected components of the open set $G \setminus A$. Let us assume that there is one more connected component of $G \setminus A$. Then $\Omega = G \setminus \overline{\Omega_*} \neq \emptyset$ and, considering G as a topological space T, and Ω as its (open) set, by Proposition 2.5 we have that there is a path $\gamma_0 : [0,1] \to G$ such that $\gamma_0([0,1)) \subseteq \Omega$ and $x_0 := \gamma_0(1) \in \partial\Omega \cap \partial\Omega_* \cap G$. Note that the mutually complement sets Ω and $\overline{\Omega_*}$ in the space G have a common boundary and $\partial\overline{\Omega_*} \subset \partial\Omega_*$. Let $x_* \in \Omega_*$ and $x_n \in \Omega_*$, $n = 1, 2, ..., x_n \to x_0$ and γ_n be paths joining x_* and x_n in Ω_*. Then $M(\Delta(|\gamma_0|, |\gamma_n|, G)) \to \infty$ as $n \to \infty$, but $M(\Delta(|\gamma_0|, |\gamma_n|, G \setminus A)) = 0$.

The obtained contradiction disproves the above assumption that $G \setminus A$ has more than one connected components.

11.3. Lemma. *Let X and X' be compact weakly flat spaces, G be a domain in X, $A \subset G$ be a NED set in G and let f be a homeomorphism of $D = G \setminus A$ into X'. If the cluster set*

$$A' := C(A, f) = \{x' \in X' : x' = \lim_{k\to\infty} f(x_k), x_k \in D, \lim_{k\to\infty} x_k \in A\} \tag{11.4}$$

is a NED set in X' and $D' = f(D)$, then $G' = D' \cup A'$ is a domain in X'. Moreover, there exist domains G^* in X with the property $A \subset G^* \Subset G$ and $A' \cap A^* = \varnothing$ where $A^* := C(\partial G^*, f)$.

Proof. First note that the NED set A is compact as a closed set in a compact space X and hence $\varepsilon_0 = dist(A, \partial G) > 0$. Thus, A belongs to the open set

$$\Omega = \{x \in X : dist\,(x, A) < \varepsilon\}$$

for any (fixed) $\varepsilon \in (0, \varepsilon_0)$ which is itself in G. Since A is compact, A is contained in a finite number of the connected components $\Omega_1, \ldots, \Omega_m$ of Ω. Let x_0 be an arbitrary point of the domain G and let $x_k \in \Omega_k$, $k = 1, \ldots, m$. Then there exist paths $\gamma_k : [0, 1] \to G$ with $\gamma_k(0) = x_0$ and $\gamma_k(1) = x_k$, $k = 1, \ldots, m$. Note that the set $C = \bigcup\limits_{k=1}^{m} |\gamma_k|$ is compact and hence $\delta_0 = dist(C, \partial G) > 0$.

Consider the open sets

$$G_\delta = \{x \in G : dist(x, \partial G) > \delta\}.$$

By the triangle inequality the set

$$C_0 = C \bigcup \left(\bigcup_{k=1}^{m} \Omega_k \right)$$

is contained in G_δ for any $\delta \in (0, d_0)$ where $d_0 = \min(\varepsilon_0 - \varepsilon, \delta_0)$. Furthermore, C_0 is contained in only one of the connected components G^*_δ of the set G_δ because the set C_0 is connected.

By the construction $\overline{G^*_\delta} \subset G$, G^*_δ are domains in X and, consequently, they are weakly flat spaces and by Proposition 11.2 the sets $D_\delta = G^*_\delta \setminus A$ are domains with weakly flat boundaries A in the spaces G^*_δ, $\delta \in (0, d_0)$. All the more, A is a weakly flat boundary of the domains $D^*_\delta = G^*_\delta \setminus A$ in the compact spaces $\overline{G^*_\delta}$, $\delta \in (0, d_0)$.

Let $f^*_\delta = f|_{D^*_\delta}$ and $g^*_\delta = (f^*_\delta)^{-1} : D'_\delta \to D^*_\delta$ where $D'_\delta = f_\delta(D^*_\delta)$. Then as it follows by Proposition 2.7 we have the symmetry

$$A = C(A', g^*_\delta), \quad A' := C(A, f^*_\delta), \qquad \forall\, \delta \in (0, d_0).$$

Note that ∂G^*_δ are compact subsets of the domain D and, consequently, $f\partial G^*_\delta$ is a compact subsets of the domain $D' = f(D)$ which by Proposition 2.7 do not intersect A'. Thus, $d_\delta = dist(A', f\partial G^*_\delta) > 0$ for all $\delta \in (0, d_0)$. By Lemma 9.3 the space X' is locally arc connected and hence, for every point $x_0 \in A'$, there is a domain $U \subset B(x_0, d_\delta)$ which is a neighborhood of x_0 and by Proposition 11.2 $V = U \setminus A'$ is also a domain which is a subdomain of D' by the construction. Thus, $G' = D' \cup A'$ is a domain in X'.

Finally, by Proposition 11.2 and Lemma 11.3 we obtain the following consequences for NED sets, see also Remarks 6.3 and 7.9.

11.5. Lemma. *Let X and X' be compact weakly flat spaces, G a domain in X, A a NED set in X and let f be a $Q-$homeomorphism of $D = G \setminus A$ into X' such that the cluster set $C(A, f)$ is a NED set in X'. If at a point $x_0 \in A$ the condition (10.3) holds, then f admits a continuous extension to the point x_0.*

11.6. Remark. In particular, f admits an extension to $x_0 \in A$ by continuity if at least one of the conditions (10.5)–(10.6), (10.8), (10.10) or (10.14) with $Q \in FMO(x_0)$, (10.16) holds at the point.

11.7. Lemma. *Let X and X' be compact weakly flat spaces, G a domain in X, A a NED set in G and let f be a $Q-$homeomorphism of $D = G \setminus A$ into X' such that the cluster set $A' = C(A, f)$ is a NED set in X'. If $Q \in L^1_\mu(G)$, then* *the inverse homeomorphism $g = f^{-1} : D' \to D$, $D' = f(D)$, admits a continuous extension $\overline{g} : G' \to G$ where $G' = D' \cup A'$.*

11.8. Remark. Thus, if $Q \in L^1_\mu(D)$ satisfies at least one of the conditions (10.5)–(10.6), (10.8), (10.10) or (10.14) with $Q \in FMO(x_0)$, (10.16) at every point $x_0 \in A$, then any $Q-$homeomorphism f of the domain $D = G \setminus A$ into X' with NED sets A and $A' = C(A, f)$ admits a homeomorphic extension $\overline{f} : G \to G'$ where $G' = D' \cup A'$, $D' = f(D)$.

11.9. Theorem. *Let X and X' be compact weakly flat spaces, G be a domain in X, $A \subset G$ be a NED set in G and let f be a $Q-$homeomorphism of $D = G \setminus A$ into X' with a NED set $A' := C(A, f)$. If Q has finite mean oscillation and X is upper α–regular by Ahlfors with $\alpha \geq 2$ at every point $x_0 \in A$, then f admits a homeomorphic extension $\overline{f}G \to G'$ where $G' = D' \cup A'$ and $D' = f(D)$.*

12 On a continuous extension to an isolated singular point

As before, here (X, d, μ) and (X', d', μ') are spaces with metrics d and d' and locally finite Borel measures μ and μ', G and G' are domains in X and X' with finite Hausdorff dimensions α and $\alpha' \geq 1$, correspondingly.

12.1. Lemma. *Let a space X be arc connected at a point $x_0 \in G$ which has a compact neighborhood, X' be a compact weakly flat space and let $f : G \setminus \{x_0\} \to G'$ be a $Q-$homeomorphism where $Q : G \to [0, \infty]$ is a measurable function satisfying the condition*

$$\text{(12.2)} \qquad \int\limits_{\varepsilon<d(x_0,x)<\varepsilon_0} Q(x) \cdot \psi^\alpha_{x_0,\varepsilon}(d(x, x_0))\, d\mu(x) \;=\; o(I^\alpha_{x_0}(\varepsilon))$$

as $\varepsilon \to 0$ where $\varepsilon_0 < dist(x_0, \partial G)$ and $\psi_{x_0,\varepsilon}(t)$ is a family of non–negative (Lebesgue) measurable functions on $(0, \infty)$ such that

$$\text{(12.3)} \qquad 0 < I_{x_0}(\varepsilon) \;=\; \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{x_0,\varepsilon}(t)\, dt \;<\; \infty\,, \qquad \varepsilon \in (0, \varepsilon_0)\,.$$

Then f can be extended to the point x_0 by continuity in X'.

Proof. Let us show that the cluster set $E = C(x_0, f)$ is a singleton. The set E is contained in $\partial G'$ by Proposition 2.7. Moreover, E is a continuum because the domain G is connected at the point x_0. Indeed,

$$E = \limsup_{m\to\infty} f(G_m) \;=\; \bigcap_{m=1}^{\infty} \overline{f(G_m)}$$

where $G_m \;=\; G \cap U_m$ is a decreasing sequence of domains with neighborhoods U_m of the point x_0 and $d(G_m) \to 0$ as $m \to \infty$. Note that $\liminf\limits_{m\to\infty} \overline{f(G_m)} = \liminf\limits_{m\to\infty} f(G_m) \neq \emptyset$ in view of the compactness of X', see e.g. Remark 3, Section

41 in [Ku]. Consequently, $E \neq \emptyset$ is connected, see e.g. I(9.12) in [Wh], p. 15. Moreover, E is closed by the construction and hence it is compact as a closed subspace of the compact space X', see e.g. Theorem 2, IV, Section 41 in [Bou].

In view of the connectedness of G at the point x_0, there is a connected component G_* of the set $G \setminus \{x_0\} \cap B(x_0, r_0)$, $0 < r_0 < dist(x_0, \partial G)$, containing $G \setminus \{x_0\} \cap B(x_0, r_*)$ for some $r_* \in (0, r_0)$. If $\partial G = \emptyset$, then we set here $dist(x_0, \partial G) = \infty$. Since x_0 has a compact neighborhood one may suppose that $\overline{B(x_0, r_0)}$ is compact.

Consider $G'_* = fG_*$. Let us show that the cluster set $E = C(x_0, f)$ is an isolated connected component of $\partial G'_*$. Indeed, $K = \partial G_* \setminus \{x_0\}$ is a compact set as a closed subset of the compact set $\overline{B(x_0, r_0)}$ and, consequently, $K_* = fK \subset G'$ is compact. On the other hand, the compact set E is contained in $\partial G'$, i.e., $E \cap K_* = \emptyset$. Thus, $dist(E, K_*) > 0$. Finally, if $y_0 \in \partial G'_*$, then by Proposition 2.7 $C(y_0, g) \subset \partial G_* = K \cup \{x_0\}$ where $g = f^{-1}|_{G'_*}$ and, consequently, either $y_0 \in E$ or $y_0 \in K_*$.

Let $z_0 \in G'_*$. Then by Proposition 2.4 there is a path $\gamma_0 : [a, b) \to G_*$ from $\gamma_0(a) = f^{-1}(z_0)$ to $x_0 = \lim\limits_{t \to b} \gamma_0(t)$ in G_*. Setting $\gamma'_0 = f\gamma_0 : [a, b) \to G'_*$, we have that $dist(\gamma'_0(t), E) \to 0$ as $t \to b$ by definition of $E = C(x_0, f)$ in view of the compactness of the space X'. Set $C_* = \gamma'_0([a, b))$ and

$$\Gamma = \Delta(C_*, E, X')\,.$$

Consider also the families of paths

$$\Gamma_0 = \Delta(C_*, E, G'_*)$$

and

$$\Gamma_* = \{\gamma \in \Gamma : \ |\gamma| \cap R \neq \emptyset\}$$

where

$$R = X' \setminus \{G'_* \cup E\}\,.$$

Note, firstly, that $M(\Gamma_0) = M(\widetilde{\Gamma})$ where $\widetilde{\Gamma} = \Gamma \setminus \Gamma_*$. Indeed, on one hand, $\Gamma_0 \subset \widetilde{\Gamma}$ and hence $M(\Gamma_0) \le M(\widetilde{\Gamma})$. On the other hand, $\Gamma_0 < \widetilde{\Gamma}$ by Proposition 2.5 and hence $M(\Gamma_0) \ge M(\widetilde{\Gamma})$, see e.g. Theorem 1 in [Fu]. Note, secondly, that

$$M(\Gamma_*) \le M_* := \frac{\mu(X')}{(2\, dist(C_* \cup E,\, \partial G'_* \setminus E))^{\alpha'}} < \infty$$

because $C_* \cup E$ and $\partial G'_* \setminus E$ are non–intersecting compact sets and $\mu'(X') < \infty$ in view of the compactness of X' and the local finiteness of the measure μ'.

Let us assume that the continuum E is not degenerate. Let $y_0 \in E$ is a limit point of $\gamma'_0(t)$ as $t \to b$ and $y_* \in E\,, y_* \neq y_0$. By the Darboux property of connected sets $\partial B(y_0, r)$ is intersecting C_* and E for all $r \in (0, r_0)$ where $r_0 = min\{d'(y_0, \gamma_0(a)), d'(y_0, y_*)\}$. Consider continua $C(t) = \gamma_0([a, t])$, $t \in [a, b)$. Note that $dist\,(C(t), E) \to 0$ as $t \to b$ by the construction. Thus,

$$M(\Delta(C(t), E, X')) \to \infty$$

as $t\to b$ because the space X' is weakly flat. Consequently, there is $t_0\in[a,b)$ such that
$$M_0:=M(\Delta(C(t_0),E,X'))>M_*\,.$$

Recall that $\Gamma=\widetilde{\Gamma}\cup\Gamma_*$ and we obtain by monotonicity and subadditivity of the modulus

$$M_*<M_0\le M(\Gamma)\le M(\widetilde{\Gamma})+M(\Gamma_*)=M(\Gamma_0)+M(\Gamma_*)\le M(\Gamma_0)+M_*\,.$$

Consequently,
$$M(\Gamma_0)>0\,.$$

However,
$$\Gamma_0=\bigcup_{n=1}^{\infty}\Gamma_n$$

where $\Gamma_n=\Delta(C(t_n),E,G_*')$, $t_n\to b$ as $n\to\infty$, and by subadditivity of the modulus
$$M(\Gamma_0)\ \le\ \sum_{n=1}^{\infty}M(\Gamma_n)\,.$$

Thus, there is a continuum $C=C(t_n)$ such that

$$M(\Delta(C,E,G_*'))>0\,.$$

Note that $C_0=f^{-1}(C)$ is a compact set as a continuous image of a compact set. Thus, $\varepsilon_0=dist(x_0,C_0)>0$. Let

$$\Gamma_\varepsilon=\Delta(C_0,B(x_0,\varepsilon),G_*),\qquad\qquad \varepsilon\in(0,\varepsilon_0)\,,$$

and let $\psi^*_{x_0,\varepsilon}$ is a Borel function such that $\psi^*_{x_0,\varepsilon}(t)=\psi_{x_0,\varepsilon}(t)$ for a.e. $t\in(0,\infty)$ which there is in view of the Lusin theorem, see e.g. 2.3.5 in [Fe].

Then by Proposition 2.6 the function

$$\rho_\varepsilon(x)\ =\left\{\begin{array}{rr}\psi^*_{x_0,\varepsilon}(d(x,x_0))/I(\varepsilon,\varepsilon_0), & x\in A(x_0,\varepsilon,\varepsilon_0),\\ 0, & x\in X\backslash A(x_0,\varepsilon,\varepsilon_0),\end{array}\right.$$

where
$$A(x_0,\varepsilon,\varepsilon_0)=\{x\in X:\varepsilon<d(x,x_0)<\varepsilon_0\}\,,$$

is admissible for Γ_ε and, consequently,

$$M(f\Gamma_\varepsilon)\ \le\ \int\limits_G Q(x)\cdot\rho_\varepsilon^\alpha(x)\ d\mu(x)\ ,$$

i.e. $M(f\Gamma_\varepsilon)\to 0$ as $\varepsilon\to 0$ in view of (12.2).

On the other hand, $M(f\Gamma_\varepsilon)\ge M(\Delta(C,E,G_*'))>0$ because $\Delta(C_0,\{x_0\},G_*)>\Gamma_\varepsilon$ and
$$f^{-1}\Delta(C,E,G_*')\subseteq\Delta(C_0,\{x_0\},G_*)$$

for any $\varepsilon \in (0, \varepsilon_0)$ by Proposition 2.7 applied to the homeomorphism f^{-1} and $g = f^{-1}|_{G'_*}$, and $x'_0 \in E$, $x'_0 = \gamma(b)$, $\gamma \in \Delta(C, E, G'_*)$. The obtained contradiction disproves the assumption that E is not degenerate.

12.4. Corollary. *In particular, if*

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0} Q(x)\cdot\psi^{\alpha}(d(x,x_0))\, d\mu(x) \;<\; \infty \tag{12.5}$$

where $\psi(t)$ is a non–negative measurable function on $(0,\infty)$ such that

$$0 < I(\varepsilon, \varepsilon_0) \;:= \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt \;<\; \infty\;, \qquad \forall\, \varepsilon \in (0,\, \varepsilon_0)\;,$$

and $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$, then any $Q-$homeomorphism $f : G\backslash\{x_0\} \to G' \subset X'$ is extended to the point x_0 by continuity in X'.

12.6. Remark. In the other words, it is sufficient for the singular integral (12.5) to be convergent in the sense of the principal value at the point x_0 at least for one kernel ψ with a non–integrable singularity at zero. Furthermore, as Lemma 12.1 shows it is sufficient for the given integral even to be divergent but with the controlled speed:

$$\int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0} Q(x)\cdot\psi^{\alpha}(d(x,x_0))\, d\mu(x) \;=\; o(I^{\alpha}(\varepsilon,\varepsilon_0)) \tag{12.7}$$

Choosing in Lemma 12.1 $\psi(t) \equiv 1/t$, we obtain the following theorem.

12.8. Theorem. *Let X and X' be compact spaces, X be arc connected at a point $x_0 \in G$, X' be weakly flat. If a measurable function $Q : G \to [0,\infty]$ satisfies the condition*

$$\int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0} \frac{Q(x)\, d\mu(x)}{d(x,x_0)^{\alpha}} \;=\; o\left(\left[\log\frac{1}{\varepsilon}\right]^{\alpha}\right) \tag{12.9}$$

as $\varepsilon \to 0$ where $\varepsilon_0 < dist(x_0, \partial G)$, then any $Q-$homeomorphism $f : G\backslash\{x_0\} \to G'$ is extended by continuity to the point x_0.

12.10. Corollary. *In particular, the conclusion of Theorem 12.8 holds if the singular integral*

$$\int \frac{Q(x)\, d\mu(x)}{d(x,x_0)^{\alpha}} \tag{12.11}$$

is convergent in a neighborhood of the point in the sense of the principal value.

Combining Lemmas 4.6 and 12.1, choosing $\psi_{\varepsilon}(t) \equiv t\log\frac{1}{t}$, $t \in (0, \delta_0)$, in the latter, we obtain the following theorem.

12.12. Theorem. *Let X and X' be compact weakly flat spaces, G be a domain in X which is upper $\alpha-$regular with $\alpha \geq 2$ and arc connected at a point $x_0 \in G$ and*

$$\mu(B(x_0, 2r)) \leq \gamma \cdot \log^{\alpha-2} \frac{1}{r} \cdot \mu(B(x_0, r)) \qquad \forall\, r \in (0, r_0)\,. \tag{12.13}$$

If $Q \in FMO(x_0)$, then any $Q-$homeomorphism f of the domain $G \setminus \{x_0\}$ into X' is extended by continuity to the point x_0.

Combining Corollary 4.4 and Theorem 12.8, we obtain the following statement.

12.14. Corollary. *In particular, if*

$$\overline{\lim_{\varepsilon \to 0}} \; ⨍_{B(x_0,\varepsilon)} Q(x)\; d\mu(x) < \infty\,, \tag{12.15}$$

then any $Q-$homeomorphism $f : G \setminus \{x_0\} \to G' \subset X'$ is extended by continuity to the point x_0.

The following simple example shows that the above extension $\overline{f}$ of f to x_0 can be not a homeomorphism.

Example. Let $G = X$ where X is a space which coincides with a closed equilateral triangle T on one of the coordinate planes in $\mathbb{R}^3$ minus one of its vertices v. It is clear that X is not compact although it is locally compact. Let us roll up the triangle T without any distortion in such a way that the vertex v will be touched to the center c of its opposite side. The obtained space X' is compact. Let $x_0 = c$. The above (rolling up) mapping $f : X \setminus \{x_0\} \to X' \setminus \{x_0\}$ is conformal if we take in X the usual Euclidean distance as the metric d and the usual area as the Borel measure μ and in X' set d' to be geodesic (thus, the arc length is invariant under f) and $\mu'(B' \setminus \{x_0\}) = \mu(f^{-1}(B' \setminus \{x_0\}))$ for every Borel set in X' and $\mu'(\{x_0\}) = \mu(\{x_0\}) = 0$. By the construction, the mapping f can continuously be extended to x_0 and the extension $\overline{f}$ is injective, of course, but not a homeomorphism (the inverse mapping of $\overline{f}$ is not continuous).

12.16. Remark. By Proposition 2.7 the extension of f at the point x_0 is an injective mapping and, thus, a homeomorphism on any subdomain $G_* \subset\subset G$, i.e., if $\overline{G_*}$ is compact in G. The latter is, generally speaking, not true for the domain G itself as it was shown by the above example. However, this is true if, for instance, $G = X$ is compact, see e.g. [Ku].

Moreover, if the family of all paths in X' (or only in G_*) going through the point $y_0 = \overline{f}(x_0)$ has the modulus zero, see Section 8, then the restriction of the mapping $g = \overline{f}|_{G_*}$ will be a $Q-$homeomorphism. For the regular by Ahlfors spaces this always holds, see Lemma 7.18 in the book [He$_1$]. Thus, an isolated singular point of $Q-$homeomorphism in regular weakly flat spaces is locally removable under the conditions on Q enumerated above.

13 On conformal and quasiconformal mappings

Finally, let us give a resume of results for conformal and quasiconformal mappings which are direct consequences of the theory of $Q-$homeomorphisms in metric spaces with measures developed above. Namely, let as before (X, d, μ) and (X', d', μ') be spaces with metrics d and d' and locally finite Borel measures μ and μ' , and with finite Hausdorff dimensions α and $\alpha' \geq 1$, correspondingly.

Similarly to the geometric definition by Vaisala in $\mathbb{R}^n$, $n \geq 2$, see 13.1 in [Va], we say that a homeomorphism $f : G \to G'$ is called $K-$**quasiconformal**, $K \in [1, \infty]$, if

$$K^{-1}M(\Gamma) \leq M(f\Gamma) \leq KM(\Gamma) \tag{13.1}$$

for every family Γ of paths in G. We say also that a homeomorphism $f : G \to G'$ is **quasiconformal** if f is $K-$quasiconformal for some $K \in [1, \infty)$, i.e., if the distortion of moduli of path families under the mapping f is bounded. In particular, we say that a homeomorphism $f : G \to G'$ is **conformal** if

$$M(f\Gamma) = M(\Gamma) \tag{13.2}$$

for any paths families in G.

By Theorem 6.2 we obtain the following important conclusion.

13.3. Theorem. *Let G have a weakly flat boundary, G' be locally arc connected at all its boundary points and let $\overline{G'}$ be compact. Then any quasiconformal mapping $f : G \to G'$ admits a continuous extension to the boundary $\overline{f} : \overline{G} \to \overline{G'}$.*

Combining Theorem 13.3 with Lemma 3.4, we come to the following statement.

13.4. Corollary. *If G and G' are domains with weakly flat boundaries and compact closures $\overline{G}$ and $\overline{G'}$, then any quasiconformal mapping $f : G \to G'$ admits a homeomorphic extension $\overline{f} : \overline{G} \to \overline{G'}$.*

13.5. Remark. In particular, the last conclusion holds for quasiconformal mappings between QED domains with compact closures in weakly flat spaces. Note that the closures of the domains are always compact in compact spaces. Recall also that locally compact spaces always admits the so–called one–point compactification, see e.g. I.9.8. [Bou].

On the base of Lemmas 3.4 and 11.3 and Theorem 13.3, we obtain the following theorem.

13.6. Theorem. *Let X and X' be compact weakly flat spaces, G a domain in X, $A \subset G$ a NED set and f a quasiconformal mapping of the domain $D = G \setminus A$ into X'. If the cluster set $A' = C(A, f)$ is also a NED set, then f admits a quasiconformal extension to G.*

By results of the previous section, single out also the following consequences on removability of isolated singularities.

13.7. Lemma. *Let X be arc connected at a point $x_0 \in G$ with a c compact neighborhood, X' a compact weakly flat space and $f : G \setminus \{x_0\} \to G'$ a quasiconformal mapping. If μ satisfies the condition*

$$\int\limits_{\varepsilon<d(x_0,x)<\varepsilon_0} \psi^{\alpha}(d(x,x_0))\, d\mu(x) \;=\; o(I^{\alpha}(\varepsilon,\varepsilon_0)) \tag{13.8}$$

as $\varepsilon \to 0$ where $\varepsilon_0 < dist(x_0, \partial G)$ and $\psi(t)$ is a non-negative (Lebesgue) measurable function on $(0,\infty)$ such that

$$0 < I(\varepsilon,\varepsilon_0) \;=\; \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt \;<\; \infty \qquad \forall\, \varepsilon \in (0,\varepsilon_0)\,,$$

then the mapping f is extended by continuity to the point x_0.

13.9. Theorem. *Let X be arc connected at a point $x_0 \in G$ with a c compact neighborhood, X' a compact weakly flat space and $f : G \setminus \{x_0\} \to G'$ a quasiconformal mapping. If μ satisfies the condition*

$$\int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0} \frac{d\mu(x)}{d(x,x_0)^{\alpha}} \;=\; o\left(\left[\log\frac{1}{\varepsilon}\right]^{\alpha}\right) \tag{13.10}$$

as $\varepsilon \to 0$ where $\varepsilon_0 < dist(x_0, \partial G)$, then the mapping f is extended by continuity to the point x_0.

Finally, in view of Remarks 4.9 and 8.7, we have the following important conclusion from Theorem 13.9.

13.11. Corollary. *Let X and X' be regular by Ahlfors compact weakly flat spaces. Then any quasiconformal mapping $X \setminus \{x_0\}$ into X' is extended to quasiconformal mapping of X into X'.*

13.12. Corollary. *Isolated singularities of quasiconformal mappings are locally removable in regular by Ahlfors weakly flat spaces X and X' if in addtion X is locally compact and X' is compact.*

Thus, the results of the paper extend (and strengthen) the well–known theorems by J. Vaisala, M. Vuorinen, F. Gehring, O. Martio, P. Nakki and others on quasiconformal mappings in $\mathbb{R}^n$, $n \geq 2$, to $Q-$homeomorphisms in metric spaces, see e.g. [GM], [MV], [Na], [Va] [Vu], cf. also [IR_1]–[IR_2], [KR] and [MRSY_1]–[MRSY_3].

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