COMPOSITION OPERATORS FROM WEAK TO STRONG SPACES OF VECTOR-VALUED ANALYTIC FUNCTIONS

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Abstract. Let φ be an analytic self-map of the unit disk, X a complex infinite-dimensional Banach space and $2 \le p < \infty$. It is shown that the composition operator $C_\varphi; f \mapsto f \circ \varphi$, is bounded $wH^p(X) \to H^p(X)$ if and only if C_φ is a Hilbert-Schmidt operator $H^2 \to H^2$. Here $H^p(X)$ is the X-valued Hardy space and $wH^p(X)$ is a related weak vector-valued Hardy space. A similar result is established for vector-valued Bergman spaces.

1. Introduction

Let X be a complex Banach space and $1 \le p < \infty$. The vector-valued Hardy space $H^p(X)$ consists of the analytic functions $f\colon \mathbb{D} \to X$ which satisfy

$$\|f\|_{H^p(X)} := \sup_{0<r<1} \left(\int_{\mathbb{T}} \|f(r\xi)\|_X^p \, dm(\xi) \right)^{1/p} < \infty,$$

where $\mathbb{D}$ is the unit disk in the complex plane and dm is the normalized Lebesgue measure on the unit circle $\mathbb{T} = \partial\mathbb{D}$. Analogously, the vector-valued Bergman space $B_p(X)$ consists of the analytic functions $f\colon \mathbb{D} \to X$ such that

$$\|f\|_{B_p(X)} := \left(\int_{\mathbb{D}} \|f(z)\|_X^p \, dA(z) \right)^{1/p} < \infty,$$

where dA is the normalized 2-dimensional Lebesgue measure on $\mathbb{D}$. (The customary notation $H^p(\mathbb{C}) = H^p$ and $B_p(\mathbb{C}) = B_p$ will be used in the scalar-valued case.) These classes of vector-valued spaces have been studied quite extensively, see e.g. [B2], [H] and the survey [B4]. The following weak versions of these spaces were considered by e.g. Blasco [B1] and Bonet, Domański and Lindström [BDL]: the weak spaces $wH^p(X)$ and $wB_p(X)$ consist of the analytic functions $f\colon \mathbb{D} \to X$ for which

$$\|f\|_{wH^p(X)} := \sup_{\|x^*\| \le 1} \|x^* \circ f\|_{H^p}, \quad \|f\|_{wB_p(X)} := \sup_{\|x^*\| \le 1} \|x^* \circ f\|_{B_p}$$

are finite, respectively. Such weak spaces $wE(X)$ can be introduced under fairly general conditions on the Banach space E consisting of analytic maps $\mathbb{D} \to \mathbb{C}$, see section 4.

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Let φ be an analytic self-map of $\mathbb{D}$ into itself. There is recent interest into properties of the analytic composition maps

$$C_\varphi; \quad f \mapsto f \circ \varphi,$$

in various vector-valued settings, see e.g. [LST], [BDL], [L1], [LT], [Wa] and [L2]. It is known (cf. [LST, p. 298]) that C_φ always defines a bounded linear operator $H^p(X) \to H^p(X)$ and $B_p(X) \to B_p(X)$ for any Banach space X and $1 \le p < \infty$, and it is easily checked that C_φ is also bounded on the weak spaces $wH^p(X)$ and $wB_p(X)$. Hence it is a natural problem to characterize the analytic maps $\varphi : \mathbb{D} \to \mathbb{D}$ for which C_φ is bounded from $wH^p(X)$ to $H^p(X)$, or from $wB_p(X)$ to $B_p(X)$. This problem is motivated e.g. by the fact that $H^p(X)$ and $wH^p(X)$ are completely different spaces for any infinite-dimensional Banach space X. In fact, $H^p(X) \subsetneq wH^p(X)$ and $\|\cdot\|_{wH^p(X)}$ is not equivalent to $\|\cdot\|_{H^p(X)}$ on $H^p(X)$, see [FGR, Cor. 12], or [L1, Ex. 15], [LT, sect. 6]. The properties of C_φ from $wH^p(X)$ to $H^p(X)$ further reflect these differences. Note that $wH^p(\mathbb{C}) = H^p$ and $wB_p(\mathbb{C}) = B_p$, so our question does not arise for $X = \mathbb{C}$. The theory of composition operators on various spaces of scalar-valued analytic functions is very extensive, see e.g. [CM] and [S] for comprehensive overviews.

Our main results establish that for $2 \le p < \infty$ and any complex infinite-dimensional Banach space X the operator C_φ is bounded $wH^p(X) \to H^p(X)$ if and only if

$$\int_{\mathbb{T}} \frac{1}{1 - |\varphi(\xi)|^2} dm(\xi) < \infty, \tag{1.1}$$

and C_φ is bounded $wB_p(X) \to B_p(X)$ if and only if

$$\int_{\mathbb{D}} \frac{1}{(1 - |\varphi(z)|^2)^2} dA(z) < \infty. \tag{1.2}$$

In (1.1) the a.e. radial limit function of φ on $\mathbb{T}$ is also denoted $\xi \mapsto \varphi(\xi)$. The appearence of (1.1) and (1.2) in this context is somewhat surprising. In fact, φ satisfies (1.1) if and only if C_φ is a Hilbert-Schmidt operator $H^2 \to H^2$, while analogously φ satisfies (1.2) if and only if C_φ is a Hilbert-Schmidt operator $B_2 \to B_2$ (see Remarks 4 and 8 for a more careful discussion). As a contrasting example we observe that C_φ is bounded $wBMOA(\ell^2) \to BMOA(\ell^2)$ if and only if C_φ is bounded $\mathcal{B} \to BMOA$, where $\mathcal{B}$ is the Bloch space. For completeness we also include concrete examples where the norms $\|\cdot\|_{wB_p(X)}$ and $\|\cdot\|_{B_p(X)}$ are not equivalent on $B_p(X)$ for any infinite-dimensional X and $1 \le p < \infty$.

We are indebted to Sten Kaijser for asking during a conference at Oxford, Ohio, about the boundedness of composition operators from $wH^2(\ell^2)$ to $H^2(\ell^2)$, as well as to Paweł Domański for a subsequent discussion.

2. Composition operators from weak to strong Hardy spaces

The following straightforward upper bound for the norm of C_φ between weak and strong Hardy spaces holds for any $1 \le p < \infty$.

Lemma 1. *Let X be any complex Banach space and $1 \leq p < \infty$. Then*

$$\|C_\varphi \colon wH^p(X) \to H^p(X)\| \leq \sup_{0<r<1} \left(\int_{\mathbb{T}} \frac{1}{1-|\varphi(r\zeta)|^2} dm(\zeta) \right)^{1/p}.$$

Proof. Any analytic map $f \colon \mathbb{D} \to \mathbb{C}$ satisfies $|f(z)|^p \leq (1-|z|^2)^{-1}\|f\|^p_{H^p}$ for $z \in \mathbb{D}$ (see e.g. [CM, p. 18]). Hence

$$\|f(z)\|^p_X = \sup_{\|x^*\|\leq 1} |(x^* \circ f)(z)|^p \leq \frac{1}{1-|z|^2}\|f\|^p_{wH^p(X)}$$

for $f \in wH^p(X)$. Consequently

$$\begin{aligned}\|C_\varphi f\|^p_{H^p(X)} &= \sup_{0<r<1} \int_{\mathbb{T}} \|f(\varphi(r\zeta))\|^p_X dm(\zeta)\\ &\leq \|f\|^p_{wH_p(X)} \sup_{0<r<1} \int_{\mathbb{T}} \frac{1}{1-|\varphi(r\zeta)|^2} dm(\zeta).\end{aligned}$$

□

We will require Dvoretzky's well-known theorem: *for any $n \in \mathbb{N}$ and $\varepsilon > 0$ there is $m(n,\varepsilon) \in \mathbb{N}$ so that for any Banach space X of dimension at least $m(n,\varepsilon)$ there is a linear (into) embedding $T_n : \ell^n_2 \to X$ so that*

$$(1+\varepsilon)^{-1}(\sum_{j=1}^n |a_j|^2)^{1/2} \leq \|\sum_{j=1}^n a_j T_n e_j\| \leq (\sum_{j=1}^n |a_j|^2)^{1/2} \tag{2.1}$$

for any scalars $a_1, \dots, a_n$. Here $(e_1, \dots, e_n)$ is some fixed orthonormal basis of ℓ^n_2. For proofs see e.g. [DJT, Ch. 19] or [P, Ch. 4].

The following result is the main one of this section. Here "$\approx$" means equivalence up to constants only depending on p.

Theorem 2. *Let X be any complex infinite-dimensional Banach space. Then*

$$\|C_\varphi \colon wH^p(X) \to H^p(X)\| \approx \left(\int_{\mathbb{T}} \frac{1}{1-|\varphi(\zeta)|^2} dm(\zeta) \right)^{1/p} \tag{2.2}$$

for $2 < p < \infty$, and

$$\|C_\varphi \colon wH^2(X) \to H^2(X)\| = \left(\int_{\mathbb{T}} \frac{1}{1-|\varphi(\zeta)|^2} dm(\zeta) \right)^{1/2}. \tag{2.3}$$

Note that it is already hard to compute the norm of $C_\varphi : H^2 \to H^2$ (cf. [BFHS] and its references), so the general identity (2.3) comes as a pleasant bonus. Before embarking on the proof of Theorem 2 we record an elementary numerical estimate that will be applied below.

Lemma 3. *There is $c > 0$ such that for any $-1 < \alpha \leq 1$ and $1/2 \leq t < 1$ one has*

$$\sum_{k=1}^\infty k^\alpha t^k \geq \frac{c}{(1-t)^{\alpha+1}}.$$

Proof. Suppose first that $-1 < \alpha \leq 0$. Then $\sum_{k=1}^\infty k^\alpha t^k \geq \int_1^\infty x^\alpha t^x dx$, since the map $x \mapsto x^\alpha t^x = x^\alpha e^{-x\log(1/t)}$ decreases on $[1,\infty)$. By changing variables

$x = y/(\log(1/t))$, and applying $0 < \log(1/t) \leq 2(1-t)$ for $1/2 \leq t < 1$, we get that

$$\sum_{k=1}^{\infty} k^\alpha t^k \geq \int_1^\infty x^\alpha e^{-x\log(1/t)}dx = \frac{1}{(\log(1/t))^{\alpha+1}} \int_{\log(1/t)}^\infty y^\alpha e^{-y} dy$$
$$\geq \frac{1}{2^{\alpha+1}(1-t)^{\alpha+1}} \int_{\log 2}^\infty y^\alpha e^{-y} dy.$$

If $0 < \alpha \leq 1$, then $x \mapsto x^\alpha e^{-x\log(1/t)}$ decreases for $x \geq \alpha/(\log(1/t))$. By arguing as before we obtain (with $a(t,\alpha) = \frac{\alpha}{\log(1/t)+1}$) that

$$\sum_{k=1}^{\infty} k^\alpha t^k \geq \int_{a(t,\alpha)}^\infty x^\alpha e^{-x\log(1/t)}dx \geq (2(1-t))^{-\alpha-1} \int_{\alpha+\log 2}^\infty y^\alpha e^{-y} dy.$$

The above calculations yield the claim with $c = 2^{-2}\int_{1+\log 2}^\infty y^{-1}e^{-y}dy$. □

Proof of Theorem 2. We first recall how the upper estimate

$$\|C_\varphi\| \leq \left(\int_{\mathbb{T}} \frac{1}{1-|\varphi(\zeta)|^2} dm(\zeta)\right)^{1/p} \tag{2.4}$$

follows from Lemma 1 for $2 \leq p < \infty$. If the right-hand side of (2.4) is finite, then $|\varphi(\zeta)| < 1$ for a.e. $\zeta \in \mathbb{T}$, so that $(1-|\varphi(\zeta)|^2)^{-1} = \sum_{k=0}^\infty |\varphi(\zeta)|^{2k}$ a.e. on $\mathbb{T}$. Monotone convergence and the subharmonicity of $|\varphi(\cdot)|^{2k}$ yield that

$$\int_{\mathbb{T}} \frac{1}{1-|\varphi(\zeta)|^2} dm(\zeta) = \sum_{k=0}^{\infty} \sup_{0<r<1} \int_{\mathbb{T}} |\varphi(r\zeta)|^{2k} dm(\zeta)$$
$$\geq \sup_{0<r<1} \int_{\mathbb{T}} \frac{1}{1-|\varphi(r\zeta)|^2} dm(\zeta).$$

We next derive the lower estimate for $\|C_\varphi\|$ in the case $2 < p < \infty$, before indicating the modifications required for (2.3). Suppose that $x \in X$ satisfies $\|x\| = 1$, and let $g : \mathbb{D} \to X$ be the constant map $g(z) = x$ for $z \in \mathbb{D}$. Clearly $\|g\|_{wH^p(X)} = 1$, so that $\|C_\varphi\| \geq \|g \circ \varphi\|_{H^p(X)} = \|x\| = 1$. Hence

$$\int_{\{\zeta\in\mathbb{T}:\ |\varphi(r\zeta)|^2<\frac{1}{2}\}} \frac{1}{1-|\varphi(r\zeta)|^2} dm(\zeta) \leq 2 \leq 2\|C_\varphi\|^p, \tag{2.5}$$

for $0 < r < 1$. Consequently it will suffice towards (2.2) to find a uniform constant $K > 0$ so that

$$\int_{\{\zeta\in\mathbb{T}:\ |\varphi(r\zeta)|^2\geq\frac{1}{2}\}} \frac{1}{1-|\varphi(r\zeta)|^2} dm(\zeta) \leq K\|C_\varphi\|^p, \tag{2.6}$$

for $0 < r < 1$.

Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Use Dvoretzky's theorem to fix a linear embedding $T_n : \ell_2^n \to X$ so that $\|T_n\| = 1$ and $\|T_n^{-1}\| \leq 1+\varepsilon$ as in (2.1). Put $x_k^{(n)} = T_n e_k$ for $k = 1, \ldots, n$, where $(e_1, \ldots, e_n)$ is some fixed ortonormal basis of ℓ_2^n. Let $\lambda_k = k^{1/p-1/2}$ for $k \in \mathbb{N}$ and consider the sequence (f_n) of analytic polynomials $\mathbb{D} \to X$ defined by

$$f_n(z) = \sum_{k=1}^{n} \lambda_k z^k x_k^{(n)} = T_n(\sum_{k=1}^{n} \lambda_k z^k e_k), \quad z \in \mathbb{D}.$$

According to Duren [D, Thm. 1] the sequence (λ_k) is a bounded coefficient multiplier from H^2 to H^p for $2 < p < \infty$. This means that there is $c_1 > 0$ so that

$$\|\sum_{k=1}^{n} \lambda_k a_k z^k\|_{H^p} \leq c_1 \|\sum_{k=1}^{n} a_k z^k\|_{H^2} = c_1 \left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2}, \tag{2.7}$$

for all $n \in \mathbb{N}$ and complex polynomials $\sum_{k=1}^{n} a_k z^k$. We get from (2.7) for any $x^* \in B_{X^*}$ that

$$\begin{aligned}\|x^* \circ f_n\|_{H^p} = \|\sum_{k=1}^{n} \lambda_k x^*(x_k^{(n)}) z^k\|_{H^p} &\leq c_1 \left(\sum_{k=1}^{n} |x^*(x_k^{(n)})|^2\right)^{1/2} \\ &= c_1 \left(\sum_{k=1}^{n} |T_n^* x^*(e_k)|^2\right)^{1/2} = c_1 \|T_n^* x^*\| \leq c_1.\end{aligned}$$

Thus $\sup_n \|f_n\|_{wH^p(X)} \leq c_1$ and $\|C_\varphi\| \geq c_1^{-1} \limsup_n \|f_n \circ \varphi\|_{H^p(X)}$. We get from Fatou's lemma that

$$\begin{aligned}\|C_\varphi\|^p &\geq \frac{1}{c_1^p} \limsup_n \int_{\mathbb{T}} \|T_n(\sum_{k=1}^{n} \lambda_k \varphi(r\zeta)^k e_k)\|_X^p dm(\zeta) \\ &\geq \frac{1}{c_1^p (1+\varepsilon)^p} \limsup_n \int_{\mathbb{T}} \|\sum_{k=1}^{n} \lambda_k \varphi(r\zeta)^k e_k\|_{\ell_2^n}^p dm(\zeta) \\ &= \frac{1}{c_1^p (1+\varepsilon)^p} \limsup_n \int_{\mathbb{T}} \left(\sum_{k=1}^{n} k^{2/p-1} |\varphi(r\zeta)|^{2k}\right)^{p/2} dm(\zeta) \\ &\geq \frac{1}{c_1^p (1+\varepsilon)^p} \int_{\mathbb{T}} \left(\sum_{k=1}^{\infty} k^{2/p-1} |\varphi(r\zeta)|^{2k}\right)^{p/2} dm(\zeta)\end{aligned}$$

for any $0 < r < 1$. Lemma 3, applied with $\alpha = 2/p - 1$ and $t = |\varphi(r\zeta)|^2$, yields that

$$\sum_{k=1}^{\infty} k^{2/p-1} |\varphi(r\zeta)|^{2k} \geq \frac{c_2}{(1 - |\varphi(r\zeta)|^2)^{2/p}}$$

for those $\zeta \in \mathbb{T}$ that satisfy $|\varphi(r\zeta)|^2 \geq 1/2$. Consequently

$$\|C_\varphi\|^p \geq \frac{c_2^{p/2}}{c_1^p (1+\varepsilon)^p} \int_{\{\zeta \in \mathbb{T}\colon |\varphi(r\zeta)|^2 \geq 1/2\}} \frac{1}{1 - |\varphi(r\zeta)|^2} dm(\zeta),$$

for $0 < r < 1$. This proves (2.6) with $K = c_1^p 2^p c_2^{-p/2}$ (and $\varepsilon = 1$). Hence, from Fatou once more, (2.5) and (2.6), there is $c_p > 0$ with

$$\begin{aligned}\|C_\varphi\colon wH^p(X) \to H^p(X)\| &\geq c_p \cdot \limsup_{r \to 1} \left(\int_{\mathbb{T}} \frac{1}{1 - |\varphi(r\zeta)|^2} dm(\zeta)\right)^{1/p} \\ &\geq c_p \cdot \left(\int_{\mathbb{T}} \frac{1}{1 - |\varphi(\zeta)|^2} dm(\zeta)\right)^{1/p},\end{aligned}$$

so that (2.2) holds.

For (2.3) it is convenient to use the X-valued polynomials

$$g_n(z) = \sum_{k=1}^{n} z^{k-1} x_k^{(n)} = T_n(\sum_{k=1}^{n} z^{k-1} e_k), \quad z \in \mathbb{D},$$

for $n \in \mathbb{N}$. Since (z^k) is orthonormal in H^2 it follows that $\|x^* \circ g_n\|_{H^2}^2 = \sum_{k=1}^n |T_n^* x^*(e_k)|^2 \leq 1$ for $x^* \in B_{X^*}$, so that $\|g_n\|_{wH^2(X)} \leq 1$ for each n. We obtain as above that

$$\begin{aligned}\|C_\varphi\|^2 &\geq \frac{1}{(1+\varepsilon)^2} \limsup_n \int_{\mathbb{T}} \|\sum_{k=1}^{n} \varphi(r\xi)^{k-1} e_k\|_{\ell_2^n}^2 dm(\xi) \\ &\geq \frac{1}{(1+\varepsilon)^2} \int_{\mathbb{T}} \sum_{k=1}^{\infty} |\varphi(r\xi)|^{2k-2} dm(\xi)\end{aligned}$$

for any $0 < r < 1$. Thus

$$\begin{aligned}\|C_\varphi\|^2 &\geq \frac{1}{(1+\varepsilon)^2} \limsup_{r\to 1} \int_{\mathbb{T}} \frac{1}{1-|\varphi(r\xi)|^2} dm(\xi) \\ &\geq \frac{1}{(1+\varepsilon)^2} \int_{\mathbb{T}} \frac{1}{1-|\varphi(\xi)|^2} dm(\xi),\end{aligned}$$

so that (2.3) holds as $\varepsilon > 0$ was arbitrary. □

Remarks 4. (i) The preceding argument was suggested by the case $X = \ell^2$ and $p = 2$. Let $f(z) = \sum_{k=0}^\infty z^k e_{k+1}$, where (e_k) is the unit vector basis of ℓ^2. Then $\|f(\varphi(z))\|_{\ell^2}^2 = \frac{1}{1-|\varphi(z)|^2}$ for $z \in \mathbb{D}$ and $f \in B_{wH^2(\ell^2)}$, so that as above

$$\|C_\varphi\|^2 \geq \|f \circ \varphi\|_{H^2(\ell^2)}^2 = \lim_{r\to 1} \int_{\mathbb{T}} \frac{1}{1-|\varphi(r\xi)|^2} dm(\xi) \geq \int_{\mathbb{T}} \frac{1}{1-|\varphi(\xi)|^2} dm(\xi).$$

(ii) The boundedness of $C_\varphi : wH^p(X) \to H^p(X)$ forces φ to belong to a restricted class of symbols, but (1.1) is unexpected here. Recall that C_φ is a Hilbert-Schmidt operator on H^2 if and only if (1.1) is satisfied, see [ST, Thm. 3.1] or [CM, p. 146], where the right-hand side of (2.3) equals the Hilbert-Schmidt norm. Thus (1.1) is much stricter than the compactness condition for $C_\varphi : H^2 \to H^2$ due to J.H. Shapiro, see e.g. [CM, Thm. 3.20] or [S, p. 26]. Moreover, if φ maps $\mathbb{D}$ into a polygon inscribed in the unit circle, then (1.1) holds (cf. [ST, Cor. 3.2] or [CM, Prop. 3.25]) so that C_φ is bounded $wH^2(X) \to H^2(X)$. In particular, there are φ so that $\|\varphi\|_\infty = 1$ and C_φ maps $wH^2(X)$ boundedly into $H^2(X)$ for any X.

(iii) (Suggested by Eero Saksman.) Let U be a bounded operator $H^2 \to H^2$. Suppose that

$$(U \otimes I_{\ell^2})(gx) = (Ug)x, \quad g \in H^2,\ x \in \ell^2, \tag{2.8}$$

extends to a well-defined bounded operator $U \otimes I_{\ell^2} : wH^2(\ell^2) \to H^2(\ell^2)$, where gx denotes the analytic map $z \mapsto g(z)x$ for $g \in H^2, x \in \ell^2$ and $z \in \mathbb{D}$. Then U is a Hilbert-Schmidt operator $H^2 \to H^2$, that is, $\sum_{n=0}^\infty \|Ug_n\|_{H^2}^2$ is finite, where $g_n(z) = z^n$ for $n = 0, 1, \ldots$ and $z \in \mathbb{D}$.

To see this fact note first that $\sum_{n=0}^{\infty} g_n e_{n+1} \in B_{wH^2(\ell^2)}$ by orthonormality. Hence one gets from (2.8) that

$$\begin{aligned}\sum_{n=0}^{\infty} \|Ug_n\|_2^2 &= \int_{\mathbb{T}} \big(\sum_{n=0}^{\infty} |(Ug_n)(\zeta)|^2\big) dm(\zeta) = \int_{\mathbb{T}} \|\sum_{n=0}^{\infty} (Ug_n)(\zeta) e_{n+1}\|_{\ell^2}^2 dm(\zeta)\\ &= \|\sum_{n=0}^{\infty} (Ug_n) e_{n+1}\|_{H^2(\ell^2)}^2 = \|(U \otimes I_{\ell^2})(\sum_{n=0}^{\infty} g_n e_{n+1})\|_{H^2(\ell^2)}^2\\ &\le \|U \otimes I_{\ell^2} : wH^2(\ell^2) \to H^2(\ell^2)\|^2.\end{aligned}$$

An analogous comment also applies to the Bergman case in section 3.

It remains unclear whether (2.2) holds for $1 \le p < 2$. In this case the bounded coefficient multipliers $H^2 \to H^p$ correspond precisely to $(\lambda_k) \in \ell^\infty$, see [JJ, Thm. 2]. By applying the ideas of Theorem 2 to $(\lambda_k) = (1, 1, 1, \dots)$ one only obtains the weaker lower bound

$$\|C_\varphi \colon wH^p(X) \to H^p(X)\| \ge c_p \cdot \left(\int_{\mathbb{T}} \left(\frac{1}{1 - |\varphi(\zeta)|^2} \right)^{p/2} dm(\zeta) \right)^{1/p},$$

where $c_p > 0$ is independent of φ. We leave the details to the reader.

3. Composition operators from weak to strong Bergman spaces

Let X be an arbitrary infinite-dimensional complex Banach space and $2 \le p < \infty$. In this section we relate the norm of $C_\varphi : wB_p(X) \to B_p(X)$ to the known condition for C_φ to be a Hilbert-Schmidt operator $B_2 \to B_2$.

We include concrete examples demonstrating that $wB_p(X)$ and $B_p(X)$ differ for any $p \in [1, \infty)$ and infinite-dimensional X, since this fact does not seem to have been made explicit in the literature. (Theorem 7 below also implies this for $2 \le p < \infty$, but only indirectly.) The argument will use the following fact about lacunary series in $B_p(X)$: *let X be any complex Banach space and $p \in [1, \infty)$. Then there are $a_p, b_p > 0$ so that*

$$a_p \left(\sum_{n=0}^{\infty} \|x_n\|^p 2^{-n} \right)^{1/p} \le \|\sum_{n=0}^{\infty} z^{2^n} x_n\|_{B_p(X)} \le b_p \left(\sum_{n=0}^{\infty} \|x_n\|^p 2^{-n} \right)^{1/p} \tag{3.1}$$

for any sequence $(x_n) \subset X$. (See the survey [B4, Prop. 4.4 and Cor. 4.5] for a proof.)

Proposition 5. *Let X be any complex infinite-dimensional Banach space and $p \in [1, \infty)$. Then $B_p(X) \subsetneq wB_p(X)$ and $\|\cdot\|_{wB_p(X)}$ is not equivalent to $\|\cdot\|_{B_p(X)}$ on $B_p(X)$.*

Proof. Fix for $n \in \mathbb{N}$ a linear embedding $T_n : \ell_2^n \to X$ so that $\|T_n\| = 1$ and $\|T_n^{-1}\| \le 2$ as in (2.1). Put $x_k^{(n)} = T_n e_k$ for $k = 1, \dots, n$, where $(e_1, \dots, e_n)$ is some fixed ortonormal basis of ℓ_2^n. Consider the sequence of X-valued lacunary polynomials

$$f_n(z) = \sum_{k=1}^{n} 2^{k/p} z^{2^k} x_k^{(n)} = T_n(\sum_{k=1}^{n} 2^{k/p} z^{2^k} e_k), \quad z \in \mathbb{D},$$

for $n \in \mathbb{N}$. Observe that

$$\|f_n\|_{B_p(X)} \approx n^{1/p} \qquad \text{and} \qquad \|f_n\|_{wB_p(X)} \leq c_p, \tag{3.2}$$

where the constants are independent of n. In fact, by applying (3.1) for $X = \ell^2$ we get that

$$\|f_n\|_{B_p(X)} \approx \|\sum_{k=1}^{n} 2^{k/p} z^{2^k} e_k\|_{B_p(\ell_2^n)} \approx \left(\sum_{k=1}^{n} \|2^{k/p} e_k\|_{\ell_2^n}^p 2^{-k}\right)^{1/p} = n^{1/p}$$

uniformly in n for any fixed $p \in [1, \infty)$.

Let $2 \leq p < \infty$ and $x^* \in B_{X^*}$. The scalar version of (3.1) yields that

$$\begin{aligned}\|x^* \circ f_n\|_{B_p} = \|\sum_{k=1}^{n} 2^{k/p} z^{2^k} T_n^* x^*(e_k)\|_{B_p} &\leq b_p \left(\sum_{k=1}^{n} |T_n^* x^*(e_k)|^p\right)^{1/p} \\ &\leq b_p \left(\sum_{k=1}^{n} |T_n^* x^*(e_k)|^2\right)^{1/2} \leq b_p.\end{aligned}$$

For $p \in [1, 2)$ Hölder's inequality and the above estimate imply that

$$\|f_n\|_{wB_p(X)} \leq \|f_n\|_{wB_2(X)} \leq b_2.$$

Concrete functions $f \in wB_p(X) \setminus B_p(X)$ can be produced e.g. by mimicking the argument for the vector-valued Hardy spaces in [LT, Ex. 6.2]. Consecutive applications of Dvoretzky's theorem as above yield embeddings $T_n \colon \ell_2^{2^n} \to X_n$ for each n, where $X_n = [y_{m_n+1}, \dots, y_{m_{n+1}}]$ are suitable block subspaces of some fixed Schauder basic sequence $(y_k) \subset X$. Here $(m_n) \subset \mathbb{N}$ is some rapidly enough increasing sequence. The desired analytic function $f \colon \mathbb{D} \to X$ can be chosen as

$$f(z) = \sum_{n=1}^{\infty} 2^{-\alpha n/p} T_n\left(\sum_{k=1}^{2^n} 2^{k/p} z^{2^k} e_k\right), \quad z \in \mathbb{D},$$

where $0 < \alpha < 1/2$. In fact, the series converges geometrically in $wB_p(X)$ by (3.2). Since (X_n) is a finite-dimensional Schauder decomposition in X there is $c > 0$ so that $\|\sum_{n=1}^{\infty} x_n\| \geq c \cdot \sup_n \|x_n\|$ whenever $\sum_{n=1}^{\infty} x_n$ converges in X and $x_n \in X_n$ for each n, see [LTz, p. 47]. By combining these estimates

$$\begin{aligned}\|\sum_{n=1}^{N} 2^{-\alpha n/p} T_n\left(\sum_{k=1}^{2^n} 2^{k/p} z^{2^k} e_k\right)\|_{B_p(X)} &\geq c \cdot 2^{-\alpha N/p} \|T_N\left(\sum_{k=1}^{2^N} 2^{k/p} z^{2^k} e_k\right)\|_{B_p(X)} \\ &\geq c \cdot d_p \cdot 2^{(N/p)(1-\alpha)} \to \infty\end{aligned}$$

as $N \to \infty$. Above $d_p > 0$ is independent of N. □

We next give a general upper bound for the norm of the composition operators $C_\varphi \colon wB_p(X) \to B_p(X)$.

Lemma 6. *Let X be any complex Banach space and $1 \leq p < \infty$. Then*

$$\|C_\varphi \colon wB_p(X) \to B_p(X)\| \leq \left(\int_{\mathbb{D}} \frac{1}{(1 - |\varphi(z)|^2)^2} dA(z)\right)^{1/p}.$$

Proof. Any analytic map $f \in B_p$ satisfies $|f(z)| \leq (1-|z|^2)^{-2/p}\|f\|_{B_p}$ for $z \in \mathbb{D}$, see [V1]. Thus

$$\|f(z)\|_X = \sup_{\|x^*\|\leq 1} |(x^* \circ f)(z)| \leq (1-|z|^2)^{-2/p}\|f\|_{wB_p(X)},$$

for $f \in wB_p(X)$ and $z \in \mathbb{D}$. It follows that

$$\begin{aligned}\|C_\varphi f\|_{B_p(X)} &= \left(\int_{\mathbb{D}} \|f(\varphi(w))\|_X^p dA(w)\right)^{1/p} \\ &\leq \|f\|_{wB_p(X)} \left(\int_{\mathbb{D}} \frac{1}{(1-|\varphi(w)|^2)^2} dA(w)\right)^{1/p}.\end{aligned}$$

□

The following result is the analogue of Theorem 2 in the Bergman case.

Theorem 7. *Let X be any complex infinite-dimensional Banach space. Then*

$$\|C_\varphi \colon wB_p(X) \to B_p(X)\| \approx \left(\int_{\mathbb{D}} \frac{1}{(1-|\varphi(z)|^2)^2} dA(z)\right)^{1/p} \tag{3.3}$$

for $2 < p < \infty$ and

$$\|C_\varphi \colon wB_2(X) \to B_2(X)\| = \left(\int_{\mathbb{D}} \frac{1}{(1-|\varphi(z)|^2)^2} dA(z)\right)^{1/2}. \tag{3.4}$$

Proof. The upper estimate $\|C_\varphi\| \leq \left(\int_{\mathbb{D}} \frac{1}{(1-|\varphi(z)|^2)^2} dA(z)\right)^{1/p}$ holds by the preceding lemma for $2 \leq p < \infty$. The strategy of the rest of the proof will be similar to that of Theorem 2, but involving different functions.

It will again suffice as in the Hardy case to verify for $2 < p < \infty$ that

$$\int_{\{z\in\mathbb{D}\colon |\varphi(z)|^2 \geq \frac{1}{2}\}} \frac{1}{(1-|\varphi(z)|^2)^2} dA(z) \leq K\|C_\varphi\|^p,$$

where $K > 0$ is a suitable constant. Fix for any given $n \in \mathbb{N}$ and $\varepsilon > 0$ a linear embedding $T_n : \ell_2^n \to X$ so that $\|T_n\| = 1$ and $\|T_n^{-1}\| \leq 1 + \varepsilon$ as in (2.1). Let $x_k^{(n)} = T_n e_k$ for $k = 1, \ldots, n$, where $(e_1, \ldots, e_n)$ is an ortonormal basis of ℓ_2^n. Consider the X-valued polynomials

$$f_n(z) = \sum_{k=1}^n \lambda_k z^k x_k^{(n)}, \quad z \in \mathbb{D},$$

where $\lambda_k = k^{2/p-1/2}$ for $k \in \mathbb{N}$. By a result of Vukotić [V2, Thm. 2] the sequence $(k^{2/p-1})$ is a coefficient multiplier $B_2 \to B_p$ for $2 < p < \infty$. Hence there is $c_1 > 0$ so that

$$\|\sum_{k=1}^n \lambda_k a_k z^k\|_{B_p} \leq c_1 \|\sum_{k=1}^\infty k^{1/2} a_k z^k\|_{B_2} \leq c_1 \left(\sum_{k=1}^n |a_k|^2\right)^{1/2},$$

for all $n \in \mathbb{N}$ and complex polynomials $\sum_{k=1}^n a_k z^k$, since $(\sqrt{n+1}z^n)$ is an orthonormal sequence in B_2. If $x^* \in B_{X^*}$ then we get that

$$\|x^* \circ f_n\|_{B^p} = \|\sum_{k=1}^n \lambda_k x^*(x_k^{(n)}) z^k\|_{B^p} \leq c_1 \left(\sum_{k=1}^n |x^*(x_k^{(n)})|^2\right)^{1/2} \leq c_1,$$

so that $\|f_n\|_{wB^p(X)} \le c_1$ for all n.

It follows that $\|C_\varphi\| \ge c_1^{-1} \limsup_n \|f_n \circ \varphi\|_{B_p(X)}$. By applying Lemma 3, with $\alpha = 4/p - 1 \in (-1, 1]$ and $t = |\varphi(z)|^2$, for those $z \in \mathbb{D}$ which satisfy $|\varphi(z)|^2 \ge 1/2$ we get from Fatou's lemma that

$$\begin{aligned}\|C_\varphi\|^p &\ge \frac{1}{c_1^p(1+\varepsilon)^p} \limsup_n \int_{\mathbb{D}} \| \sum_{k=1}^n \lambda_k \varphi(z)^k e_k\|_{\ell_2^n}^p dA(z)\\ &\ge \frac{1}{c_1^p(1+\varepsilon)^p} \int_{\mathbb{D}} \left(\sum_{k=1}^\infty k^{4/p-1} |\varphi(z)|^{2k} \right)^{p/2} dA(z)\\ &\ge \frac{c_2^{p/2}}{c_1^p(1+\varepsilon)^p} \int_{\{z\in\mathbb{D}\colon |\varphi(z)|^2 \ge 1/2\}} \frac{1}{(1-|\varphi(z)|^2)^2} dA(z).\end{aligned}$$

This proves the claim with $K = c_1^p 2^p c_2{}^{-p/2}$, so that (3.3) holds.

Towards (3.4) consider instead

$$g_n(z) = \sum_{k=0}^{n-1} \sqrt{k+1}\ z^k x_k^{(n)} = T_n(\sum_{k=0}^{n-1} \sqrt{k+1}\ z^k e_k), \quad z \in \mathbb{D},$$

for $n \in \mathbb{N}$. It follows that $\|g_n\|_{wB_2(X)} \le 1$ for any n, since $\|x^* \circ g_n\|_{B_2}^2 = \sum_{k=1}^n |T_n^* x^*(e_k)|^2 \le 1$ by orthonormality for any $x^* \in B_{X^*}$. We obtain as above, using some elementary calculus, that

$$\begin{aligned}\|C_\varphi\|^2 &\ge \int_{\mathbb{D}} \|T_n(\sum_{k=0}^\infty \sqrt{k+1}\ \varphi(z)^k e_k)\|_X^2 dA(z)\\ &\ge \frac{1}{(1+\varepsilon)^2} \int_{\mathbb{D}} \left(\sum_{k=0}^\infty (k+1) \cdot |\varphi(z)|^{2k} \right) dA(z)\\ &\ge \frac{1}{(1+\varepsilon)^2} \int_{\mathbb{D}} \frac{1}{(1-|\varphi(z)|^2)^2} dA(z).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we get the desired lower bound in (3.4). □

Remarks 8. (i) Define $f\colon \mathbb{D} \to \ell_2$ by $f(z) = \sum_{k=0}^\infty \sqrt{k+1}\ z^k e_{k+1}$ for $z \in \mathbb{D}$, where (e_k) is the standard unit basis of ℓ^2. One verifies as above that $f \in B_{wB_2(\ell_2)}$, while $\|f(z)\|_{\ell_2}^2 = \frac{1}{(1-|z|^2)^2}$ for $z \in \mathbb{D}$. Hence the lower bound

$$\|C_\varphi\colon wB_2(\ell_2) \to B_2(\ell_2)\|^2 \ge \int_{\mathbb{D}} \frac{1}{(1-|\varphi(w)|^2)^2} dA(w)$$

is immediate in this special case.

(ii) Boyd [Bo, Thm. 4.1] showed that C_φ is a Hilbert-Schmidt operator on B_2 if and only if (1.2) holds. Moreover, if φ maps $\mathbb{D}$ into a polygon inscribed in the unit circle, then C_φ is Hilbert-Schmidt on B_2, see [Bo, Thm. 4.3]. Thus the class of self-maps φ for which $C_\varphi\colon wH^p(X) \to H^p(X)$ is bounded for $2 \le p < \infty$ lies strictly between those where $\|\varphi\|_\infty < 1$ and where C_φ is compact on B_2. Compactness was characterized by MacCluer and Shapiro in terms of the angular derivatives of φ, see [CM, Thm. 3.22]).

For $1 \le p < 2$ the preceding ideas only yield a weaker lower bound, and this case remains unresolved. In fact, here (k^α) is a bounded coefficient multiplier $B_2 \to B_p$ if and only if $\alpha < 1/p - 1/2$, see [W, Prop. 4]. The computations of Theorem 7 applied to these sequences yield that

$$\|C_\varphi\colon wB_p(X) \to B_p(X)\| \geq c_{p,\beta} \cdot \left(\int_{\mathbb{D}} \frac{1}{(1-|\varphi(z)|^2)^\beta} dA(z) \right)^{1/p}$$

for $1 < \beta < 1 + p/2$. The details are left to the reader.

4. Other weak and strong spaces

Suppose that $(E, \|\cdot\|_E)$ is a Banach space consisting of analytic functions $\mathbb{D} \to \mathbb{C}$ such that

(i) E contains the constant functions,

(ii) the unit ball B_E is compact in the topology of uniform convergence on compact subsets of $\mathbb{D}$.

For any complex Banach space X the analytic function $f\colon \mathbb{D} \to X$ belongs to the weak vector-valued space $wE(X)$ if

$$\|f\|_{wE(X)} = \sup_{x^* \in B_{X^*}} \|x^* \circ f\|_E < \infty.$$

Then $wE(X)$ is a Banach space which is isometric to the space $L(V_*, X)$ of bounded operators, where V_* is a certain predual of E, see [BDL, p. 244]. Here $wE(X) = E(X)$ may occur. This is so e.g. if $E = H^\infty$ or $E = \mathcal{B}$, the Bloch space, but recall that $wH^p(X)$ and $wB_p(X)$ always differ from the respective strong spaces.

It is easy to check that C_φ is bounded $wE(X) \to wE(X)$ if and only if C_φ is bounded $E \to E$, and some results for composition operators on weak spaces of analytic (or even harmonic) functions are found in [BDL], [L1] and [LT]. We point out here as an example that the condition for C_φ to be bounded $wBMOA(\ell^2) \to BMOA(\ell^2)$ is unrelated to the Hilbert-Schmidt conditions (2.3) and (3.4). Recall that $BMOA(X)$ consists of the analytic functions $f\colon \mathbb{D} \to X$ for which

$$\|f\|_{BMOA(X)} = \|f(0)\|_X + \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^2(X)} < \infty,$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ for $a \in \mathbb{D}$. The weak space $wBMOA(X)$ differs from $BMOA(X)$ for any infinite-dimensional X, see [L1, Ex. 15].

Example 9. C_φ is bounded $wBMOA(\ell^2) \to BMOA(\ell^2)$ if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2 (1 - |\sigma_a(z)|^2)}{(1-|\varphi(z)|^2)^2} dA(z) < \infty. \tag{4.1}$$

Proof. The known estimates for the point evaluations on $BMOA$ (see e.g. [G, p. 95]) imply that

$$\|f(z)\|_{\ell^2} \leq M(z) \|f\|_{wBMOA(\ell^2)}, \quad \|f'(z)\|_{\ell^2} \leq \frac{1}{1-|z|^2} \|f\|_{wBMOA(\ell^2)}$$

for $z \in \mathbb{D}$, where $M(z) = 1 + \frac{1}{2}\log\frac{1+|z|}{1-|z|}$. If $f \in B_{wBMOA(\ell^2)}$, then [B3, Cor. 1.1] yields that

$$\begin{aligned}\|C_\varphi f\|_{BMOA(\ell^2)} &\leq C \cdot \big(\|f(\varphi(0))\|_{\ell^2} + \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \|f'(\varphi(z))\|_{\ell^2}^2 |\varphi'(z)|^2 d\mu_a(z)\big)\\ &\leq C \cdot \big(M(\varphi(0)) + \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} d\mu_a(z)\big),\end{aligned}$$

where $C > 0$ is a uniform constant and $d\mu_a(z) = (1 - |\sigma_a(z)|^2)dA(z)$.

Conversely, define $g : \mathbb{D} \to \ell^2$ by $g(z) = \sum_{k=0}^\infty \frac{z^{k+1}}{\sqrt{k+1}} e_{k+1}$ for $z \in \mathbb{D}$. It follows that $g \in wBMOA(\ell^2)$ (e.g. use Hardy's inequality, see [L1, Ex. 15]) and $\|g'(z)\|_{\ell^2}^2 = \frac{1}{(1-|z|^2)^2}$ as above. Thus

$$\begin{aligned}\|C_\varphi\| &\geq c \cdot \|C_\varphi g\|_{BMOA(\ell^2)} \geq c \cdot \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \|g'(\varphi(z))\|_{\ell^2}^2 |\varphi'(z)|^2 d\mu_a(z)\\ &= c \cdot \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} d\mu_a(z).\end{aligned}$$

□

Remark 10. C_φ is bounded from the Bloch space $\mathcal{B}$ to $BMOA$ if and only if (4.1) holds, see e.g. [T, Prop. 3.8] or [MT, Prop. 3.1].

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