Measure-preserving transformations of Volterra Gaussian processes and related bridges

Céline Jost*

Abstract

We consider Volterra Gaussian processes on $[0, T]$, where $T > 0$ is a fixed time horizon. These are processes of type $X_t = \int_0^t z_X(t, s)dW_s,\ t \in [0, T]$, where z_X is a square-integrable kernel, and W is a standard Brownian motion. An example is fractional Brownian motion. By using classical techniques from operator theory, we derive measure-preserving transformations of X, and their inherently related bridges of X. As a closely connected result, we obtain a Fourier-Laguerre series expansion for the first Wiener chaos of a Gaussian martingale over $[0, \infty)$.

MSC: 60G15; 37A05; 42C10; 60G44

Keywords: Volterra Gaussian process; Measure-preserving transformation; Bridge; Series expansion; Gaussian martingale; Fractional Brownian motion

1 Introduction

Fix $T > 0$, and let $(X_t)_{t\in[0,T]}$ be a continuous Volterra Gaussian process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Hence, there exist a kernel $z_X \in L^2\left([0, T]^2\right)$ which is *Volterra*, i.e. $z_X(t, s) = 0,\ s \geq t$, and a standard Brownian motion $(W_t)_{t\in[0,T]}$, such that

$$X_t \;=\; \int_0^t z_X(t, s)dW_s,\ a.s.,\ t \in [0, T]. \tag{1.1}$$

Clearly, $X_0 = 0$, a.s., X is centered, and

$$R^X(s, t) \;=\; \int_0^{s\wedge t} z_X(t, u)z_X(s, u)du,\ s, t \in [0, T], \tag{1.2}$$

where $R^Y(s, t) := \mathrm{Cov}_{\mathbb{P}}(Y_s, Y_t),\ s, t \in [0, T]$, for a general Gaussian process Y. We assume that z_X is *non-degenerate*, meaning that the family $\{z_X(t, \cdot)\,|\, t \in (0, T]\}$ is linearly independent and generates a dense subspace of $L^2([0, T])$. On the one hand, it follows from this, that R^X is *non-degenerate on* $(0, T]$, meaning that the matrices $\left\{R^X(t_i, t_j)\right\}_{i,j=1,\ldots,n}$, where $0 < t_1 < \ldots < t_n \leq T$ and $n \in \mathbb{N}$, are positive definite. On the other hand, it follows that

$$\Gamma_t(X) \;=\; \Gamma_t(W),\ t \in [0, T], \tag{1.3}$$

i.e.

$$\mathbb{F}_T^X \;=\; \mathbb{F}_T^W. \tag{1.4}$$

*Department of Mathematics and Statistics, P.O. Box 68 (Gustaf Hällströmin katu 2b), 00014 University of Helsinki, Finland. E-mail: celine.jost@helsinki.fi

Here, $\Gamma_T(Y) := \overline{\operatorname{span}\{Y_t \,|\, t \in [0,T]\}} \subseteq L^2(\mathbb{P})$ and $\mathbb{F}_T^Y := \left(\mathcal{F}_t^Y\right)_{t\in[0,T]}$ denote the first Wiener chaos and the completed natural filtration of the Gaussian process Y with $Y_0 = 0$, a.s., over $[0,T]$, respectively. Moreover, we assume that

$$z_X(T,\cdot) \;\neq\; 0,\; \lambda\text{-}a.e., \tag{1.5}$$

where λ denotes Lebesgue measure, and

$$z_X(\cdot, s) \text{ has bounded variation on } [u,T] \text{ for all } u \in (s,T). \tag{1.6}$$

Processes of this type are a natural generalization of the nowadays, in connection with finance and telecommunications, extensively studied *fractional Brownian motion with Hurst index* $H \in (0,1)$. Fractional Brownian motion, denoted by B^H, is the unique (in the sense of equality of finite dimensional distributions) centered, H-self-similar Gaussian process with stationary increments. For $H = \frac{1}{2}$, fractional Brownian motion is standard Brownian motion. We have that

$$z_{B^H}(t,s) = \frac{C(H)}{\Gamma\left(H+\frac{1}{2}\right)}(t-s)^{H-\frac{1}{2}}\,{}_2F_1\left(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, \frac{s-t}{s}\right) 1_{[0,t)}(s),$$

$s,t \in [0,T]$, where Γ denotes the gamma function, $C(H) := \left(\frac{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-H\right)}{\Gamma(2-2H)}\right)^{\frac{1}{2}}$ and ${}_2F_1$ is Gauss' hypergeometric function. Moreover,

$$R^{B^H}(s,t) \;=\; \frac{1}{2}\left(s^{2H}+t^{2H}-|t-s|^{2H}\right),\; s,t \in [0,T].$$

For a Gaussian process $(Y_t)_{t\in[0,T]}$ with $Y_0 = 0$, a.s., recall that the process $\left(Y_t^T\right)_{t\in[0,T]}$ is a *bridge of* Y *(from* 0 *to* 0 *on* $[0,T]$*)* if

$$\mathrm{Law}_{\mathbb{P}}\left(Y^T\right) \;=\; \mathrm{Law}_{\mathbb{P}}\left(Y \,|\, Y_T = 0\right).$$

Clearly, Y^T is a Gaussian process with

$$Y_0^T \;=\; Y_T^T \;=\; 0,\; a.s.,$$

i.e. it is tied to 0 at both ends. Therefore, the bridge is an intuitive concept for interpolation. It is well-known, that the continuous process

$$\widehat{X}_t^T \;:=\; X_t \;-\; \frac{R^X(t,T)}{R^X(T,T)}X_T,\; t \in [0,T], \tag{1.7}$$

is a bridge of X, that satisfies

$$\Gamma_T\left(\widehat{X}^T\right) \;\oplus\; \operatorname{span}\{X_T\} \;=\; \Gamma_T(X). \tag{1.8}$$

(see [4], Proposition 4). $\widehat{X}^T$ is called the *anticipative* bridge of X, since

$$\mathbb{F}_T^{\widehat{X}^T} \vee \sigma(X_T) \;=\; \mathbb{F}_T^X \vee \sigma(X_T), \tag{1.9}$$

i.e. not the natural, but the initially $\sigma(X_T)$-enlarged filtrations of X and $\widehat{X}^T$ coincide. Note that, more generally, $\widehat{Y}^T$ is a bridge of Y for every Gaussian process $(Y_t)_{t\in[0,T]}$ with $Y_0 = 0$, a.s., and $R^Y(T,T) > 0$.

We implicitly assume, that $(\Omega, \mathcal{F}, \mathbb{P}) = \left(C([0,T]), B([0,T]), \mathbb{P}^X\right)$, where $C([0,T]) := \{\omega : [0,T] \to \mathbb{R} \,|\, \omega \text{ is continuous}\}$, $\mathcal{B}([0,T])$ is the Borel-σ-algebra on $C([0,T])$

equipped with the norm of uniform convergence, and $\mathbb{P}^X$ is the probability measure, with respect to which the *coordinate process* $X_t(\omega) := \omega(t)$, $\omega \in C([0,T])$, $t \in [0,T]$, is a centered Gaussian process with covariance function R^X. Then $(\Omega, \mathcal{F}, \mathbb{P})$ is called the *coordinate space of* X. Note that $\mathcal{F} = \mathcal{F}_T^X$. The measurable map

$$\begin{array}{rcl} \mathcal{T} \;:\; (\Omega, \mathcal{F}, \mathbb{P}) & \rightarrow & (\Omega, \mathcal{F}, \mathbb{P}) \\ X(\omega) & \mapsto & \mathcal{T}(X(\omega)) \end{array}$$

is a *measure-preserving transformation on* $(\Omega, \mathcal{F}, \mathbb{P})$, if it is a morphism, i.e. if $\mathbb{P}^{\mathcal{T}} = \mathbb{P}$, or equivalently, if $\mathcal{T}(X) \stackrel{d}{=} X$, where $\stackrel{d}{=}$ denotes equality of finite dimensional distributions. The measurable map

$$\begin{array}{rcl} \mathcal{B} \;:\; (\Omega, \mathcal{F}, \mathbb{P}) & \rightarrow & (\Omega, \mathcal{F}, \mathbb{P}) \\ X(\omega) & \mapsto & \mathcal{B}(X(\omega)) \end{array}$$

is a *bridge transformation on* $(\Omega, \mathcal{F}, \mathbb{P})$, if $\mathcal{B}(X) \stackrel{d}{=} X^T$. Note that, in particular, $\mathcal{T}, \mathcal{B} : C([0,T]) \rightarrow C([0,T])$ are linear maps.

In this work, we derive measure-preserving transformations, such that

$$\Gamma_T(\mathcal{T}(X)) \;=\; \Gamma_T\left(\widehat{X}^T\right).$$

As a naturally related problem, we derive bridge transformations, that satisfy

$$\Gamma_T(\mathcal{B}(X)) \;=\; \Gamma_T(X).$$

We define $\mathcal{T}(X)$ and $\mathcal{B}(X)$ as Wiener integral processes with respect to X, where the integrand kernels are based on suitable isometric operators on the Wiener integrand space of X. This method follows ideas of Jeulin and Yor (see [8]) and Peccati (see [13]), where the case $X = W$ is considered. If X is a martingale, then due to independence of increments, the operators and the resulting transformations are simple. However, if X is not a martingale, then operators and obtained transformations are technically more involved. In fact, the operators are composed by the corresponding operators for the prediction martingale of X_T, and the isometry between the Wiener integrand spaces of X and the prediction martingale, respectively. From these results, we obtain in particular an alternative, purely operator theoretic derivation for the *dynamic* bridge of X, as introduced in [4]. Furthermore, by using the same operator theoretic methods and as a closely connected result, we obtain a Fourier-Laguerre series expansion for the first Wiener chaos of a Gaussian martingale over $[0, \infty)$.

Remark 1.1. 1. Representation (1.1) is unique in the following sense: assume that there exist another non-degenerate Volterra kernel z'_X and another standard Brownian motion W', such that $X_t = \int_0^t z'_X(t,s)dW'_s$, a.s., $t \in [0,T]$. Then, it follows from (1.3), that $\Gamma_t(W) = \Gamma_t(W')$, $t \in [0,T]$, i.e. W and W' are indistinguishable. In particular,

$$0 \;=\; \mathrm{E}_{\mathbb{P}}(X_t - X_t)^2 \;=\; \int_0^t \left(z_X(t,s) - z'_X(t,s)\right)^2 ds,\; t \in [0,T],$$

which implies that $z_X(t,\cdot) = z'_X(t,\cdot)$, λ-a.e., $t \in [0,T]$.
2. Although the covariance function R^X is continuous on $[0,T]^2$, it is generally not true, that $z_X(\cdot, s)$ is continuous for all $s \in [0,T]$. For example, for the fractional Brownian motion with $H < \frac{1}{2}$, the function $z_{B^H}(\cdot, s)$ is discontinuous in $t = s$ for every $s \in (0,T)$.
3. (1.5) is not a necessary condition for the non-degeneracy of a Volterra kernel, as the example $z_X(t,s) := 1_{[\frac{t}{2},t)}(s)$, $s,t \in [0,T]$, shows.

The article is organized as follows. Section 2 is dedicated to Wiener integration with respect to X and X^T, respectively. In Section 3, we derive the measure-preserving and bridge transformations in the case, when X is a martingale. These results are generalized to Volterra Gaussian processes in Section 4. In Section 5, we derive the series expansion.

2 Wiener integrals

First, we review the construction of abstract and time domain Wiener integrals with respect to the Volterra Gaussian process X. Second, we define abstract Wiener integrals with respect to the bridge X^T, and explain their relation to Wiener integrals with respect to X.

2.1 Wiener integrals with respect to X

Let $\mathcal{E}_T := \mathrm{span}\{1_{[0,t)}\,|\,t \in (0,T]\}$ be the space of elementary functions on $[0,T]$. The Hilbert space of *abstract Wiener integrands of X on* $[0,T]$, denoted by $\Lambda_T(X)$, is defined as the completion of $\mathcal{E}_T$ with respect to the scalar product

$$(1_{[0,s)}, 1_{[0,t)})_X \;:=\; R^X(s,t),\;\; s,t \in (0,T].$$

Hence, $f \in \Lambda_T(X)$ is an equivalence class of Cauchy sequences $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{E}_T^{\mathbb{N}}$, where $\{f_n\}_{n\in\mathbb{N}} \sim \{g_n\}_{n\in\mathbb{N}} :\Leftrightarrow (f_n - g_n, f_n - g_n)_X \to 0,\; n \to \infty$. The scalar product on $\Lambda_T(X)$ is given by

$$(f,g)_{\Lambda_T(X)} := \lim_{n\to\infty} (f_n, g_n)_X,\;\; f,g \in \Lambda_T(X),\;\; \{f_n\}_{n\in\mathbb{N}} \in f,\;\; \{g_n\}_{n\in\mathbb{N}} \in g,$$

and induces the norm $|\cdot|_{\Lambda_T(X)}$. The isometric isomorphism defined by

$$\begin{array}{rcccl} I_T^X & : & \Lambda_T(X) & \to & \Gamma_T(X) \\ & & 1_{[0,t)} & \mapsto & X_t,\;\; t \in (0,T], \end{array}$$

is called the *abstract Wiener integral with respect to X on* $[0,T]$. By definition, $\int_0^T f(s)dX_s := I_T^X(f)$ is centered, Gaussian and $\mathrm{E}_{\mathbb{P}}\left(\int_0^T f(s)dX_s\right)^2 = |f|^2_{\Lambda_T(X)}$ for all $f \in \Lambda_T(X)$.

By combining (1.1), the so-called *time domain representation of* X, with the standard Wiener integral, we obtain a subspace of $\Lambda_T(X)$, whose elements can be identified with functions. For this purpose, define a linear operator

$$\left(\mathbf{K}^X f\right)(s) \;:=\; f(s) z_X(T,s) \;+\; \int_s^T \left(f(u) - f(s)\right) z_X(du,s),\;\; s \in (0,T).$$

By (1.6), the integral is well-defined for a suitably large class of functions. Clearly, $\mathbf{K}^X$ extends the linear isometry

$$\begin{array}{rcl} (\mathcal{E}_T, (\cdot\,,\cdot)_X) & \to & L^2([0,T]) \\ 1_{[0,t)} & \mapsto & z_X(t,\cdot),\;\; t \in (0,T]. \end{array}$$

We define the space of *time domain Wiener integrands of X on* $[0,T]$ by

$$\check{\Lambda}_T(X) \;:=\; \left\{ f : [0,T] \to \mathbb{R} \;\middle|\; \mathbf{K}^X f \text{ is well-defined and } \int_0^T \left(\mathbf{K}^X f\right)^2(s)ds < \infty \right\}$$

with scalar product

$$(f,g)_{\breve{\Lambda}_T(X)} \;:=\; \left(\mathbf{K}^X f, \mathbf{K}^X g\right)_{L^2([0,T])}, \; f,g \in \breve{\Lambda}_T(X).$$

By (1.2), we have that

$$(1_{[0,s)}, 1_{[0,t)})_{\breve{\Lambda}_T(X)} \;=\; (1_{[0,s)}, 1_{[0,t)})_{\Lambda_T(X)}, \; s,t \in [0,T].$$

Let $f \in \Lambda_T(X)$. If there exists $\check{f} \in \breve{\Lambda}_T(X)$, such that

$$\left(f, 1_{[0,t)}\right)_{\Lambda_T(X)} \;=\; \left(\check{f}, 1_{[0,t)}\right)_{\breve{\Lambda}_T(X)} \text{ for all } t \in (0,T],$$

then we identify f and $\check{f}$. $\mathcal{E}_T$ is dense in $\breve{\Lambda}_T(X)$, hence $\breve{\Lambda}_T(X) \subseteq \Lambda_T(X)$. However, in general, $\breve{\Lambda}_T(X) \neq \Lambda_T(X)$. The restriction $I_T^X|_{\breve{\Lambda}_T(X)}$ is called the *time domain Wiener integral with respect to X on $[0,T]$*.

Remark 2.1. Due to the non-degeneracy of z_X, we have that $|f|_{\breve{\Lambda}_T(X)} = 0$ if and only if $f = 0$, λ-a.e.

Remark 2.2. From (1.3), it follows that (1.1) has a reciprocal in the sense that there exists a Volterra kernel z_X^*, with $z_X^*(t,\cdot) \in \Lambda_T(X)$, $t \in (0,T]$, such that

$$W_t \;=\; \int_0^t z_X^*(t,s)dX_s, \; a.s., \; t \in [0,T].$$

For $t \in (0,T]$, this abstract Wiener integral is a time domain Wiener integral if and only if there exists $f \in \breve{\Lambda}_T(X)$, such that $\mathbf{K}^X f = 1_{[0,t)}$, λ-a.e.

Remark 2.3. If R^X is of bounded variation, then it determines a finite signed measure on $[0,T]^2$. Hence, one can define an alternative space of Wiener integrands of X on $[0,T]$ by

$$|\Lambda_T(X)| \;:=\; \left\{ f:[0,T]\to\mathbb{R} \;\middle|\; \int_0^T \int_0^T |f(s)||f(t)|\left|R^X\right|(dt,ds) < \infty \right\},$$

where $\left|R^X\right|$ denotes the measure of total variation of R^X, and the integral is a Lebesgue-Stieltjes integral. The corresponding scalar product is given by

$$(f,g)_{|\Lambda_T(X)|} \;:=\; \int_0^T \int_0^T f(t)g(s)R^X(dt,ds), \; f,g \in |\Lambda_T(X)|.$$

For details, see [7].

Example 2.4. For the fractional Brownian motion holds that (see [1], p.797-800)

$$\left(\mathbf{K}^{B^H} f\right)(s) \;=\; C(H)s^{\frac{1}{2}-H}\left(\mathcal{I}_{T-}^{H-\frac{1}{2}} \cdot^{H-\frac{1}{2}} f\right)(s), \; s \in (0,T),$$

where $\mathcal{I}_{T-}^{H-\frac{1}{2}}$ denotes the right-sided Riemann-Liouville fractional integral operator of order $H-\frac{1}{2}$ over $[0,T]$. Furthermore,

$$z_{B^H}^*(t,s) \;=\; \frac{C(H)^{-1}}{\Gamma\left(\frac{3}{2}-H\right)}(t-s)^{\frac{1}{2}-H}{}_2F_1\left(\frac{1}{2}-H, \frac{1}{2}-H, \frac{3}{2}-H, \frac{s-t}{s}\right)1_{[0,t)}(s),$$

$s,t \in [0,T]$. We have that $\breve{\Lambda}_T\left(B^H\right) = \Lambda_T\left(B^H\right)$ if and only if $H \le \frac{1}{2}$. Clearly, R^{B^H} is of bounded variation if and only if $H \ge \frac{1}{2}$. Then, $z_{B^H}^*(t,\cdot) \in \left|\Lambda_T\left(B^H\right)\right| \subseteq \breve{\Lambda}_T\left(B^H\right)$, $t \in (0,T]$. See [15] for full proofs.

2.2 Wiener integrals with respect to X^T

From (1.7), it follows that X^T is centered and

$$R^{X^T}(s,t) \;=\; R^X(s,t) \;-\; \frac{R^X(s,T)R^X(t,T)}{R^X(T,T)},\;\; s,t \in [0,T]. \tag{2.1}$$

Hence, R^{X^T} is non-degenerate on $(0,T)$, and $R^{X^T}(T,T) = 0$. In order to define a scalar product based on R^{X^T}, let

$$\mathcal{C}_T \;:=\; \{f : [0,T] \to \mathbb{R} \mid f \text{ is constant } \lambda\text{-}a.e.\}$$

and consider the quotient space $\mathcal{E}_T/\mathcal{C}_T$. Clearly, $1_{[0,T)} \sim 0 \bmod \mathcal{C}_T$. The Hilbert space of *abstract Wiener integrands of* X^T *on* $[0,T]$, denoted by $\Lambda_T\big(X^T\big)$, is defined as the completion of $\mathcal{E}_T/\mathcal{C}_T$ with respect to the scalar product

$$\big(\overline{1_{[0,s)}}, \overline{1_{[0,t)}}\big)_{X^T} \;:=\; R^{X^T}(s,t),\;\; s,t \in (0,T].$$

Thus, $\overline{f} \in \Lambda_T\big(X^T\big)$ is an equivalence class of Cauchy sequences $\big\{\overline{f_n}\big\}_{n\in\mathbb{N}} \in (\mathcal{E}_T/\mathcal{C}_T)^{\mathbb{N}}$, where $\big\{\overline{f_n}\big\}_{n\in\mathbb{N}} \sim \{\overline{g_n}\}_{n\in\mathbb{N}} :\Leftrightarrow \big(\overline{f_n - g_n}, \overline{f_n - g_n}\big)_{X^T} \to 0$, $n \to \infty$. The scalar product on $\Lambda_T\big(X^T\big)$ is given by

$$\big(\overline{f}, \overline{g}\big)_{\Lambda_T(X^T)} := \lim_{n\to\infty} \big(\overline{f_n}, \overline{g_n}\big)_{X^T},\;\; \overline{f}, \overline{g} \in \Lambda_T\big(X^T\big),\;\; \big\{\overline{f_n}\big\}_{n\in\mathbb{N}} \in \overline{f},\;\; \{\overline{g_n}\}_{n\in\mathbb{N}} \in \overline{g},$$

with induced norm $|\cdot|_{\Lambda_T(X^T)}$. The isometric isomorphism defined by

$$\begin{array}{rcl} I_T^{X^T} \;:\; \Lambda_T\big(X^T\big) & \to & \Gamma_T\big(X^T\big) \\ \overline{1_{[0,t)}} & \mapsto & X_t^T,\;\; t \in (0,T], \end{array}$$

is called the *abstract Wiener integral with respect to* X^T *on* $[0,T]$. We have that $\int_0^T \overline{f}(s)dX_s^T := I_T^{X^T}\big(\overline{f}\big)$ is centered, Gaussian and $\mathrm{E}_{\mathbb{P}}\left(\int_0^T \overline{f}(s)dX_s^T\right) = \big|\overline{f}\big|^2_{\Lambda_T(X^T)}$ for all $\overline{f} \in \Lambda_T\big(X^T\big)$.

Let

$$\begin{array}{rcl} \eta^X \;:\; \Lambda_T(X) & \to & \Lambda_T(X) \\ f(\cdot) & \mapsto & f(\cdot) \;-\; \dfrac{\big(f, 1_{[0,T)}\big)_{\Lambda_T(X)}}{R^X(T,T)}. \end{array}$$

η^X is the linear orthoprojection from $\Lambda_T(X)$ onto the closed subspace

$$\Lambda_{T,0}(X) \;:=\; \left\{ f \in \Lambda_T(X) \;\middle|\; \big(f, 1_{[0,T)}\big)_{\Lambda_T(X)} = 0 \right\}$$

along $\Lambda_{T,0}(X)^\perp = \mathcal{C}_T$. In particular, $\ker\big(\eta^X\big) = \mathcal{C}_T$ and

$$\eta^X(\Lambda_T(X)) \;=\; \Lambda_{T,0}(X). \tag{2.2}$$

Thus, the map

$$\begin{array}{rcl} \overline{\eta^X} \;:\; \Lambda_T(X)/\mathcal{C}_T & \to & \Lambda_{T,0}(X) \\ \overline{f} & \mapsto & \eta^X f \end{array}$$

is an isomorphism. From (2.1), we obtain that

$$\big(\overline{1_{[0,s)}}, \overline{1_{[0,t)}}\big)_{X^T} \;=\; \big(\eta^X 1_{[0,s)}, \eta^X 1_{[0,t)}\big)_X,\;\; s,t \in (0,T].$$

Hence,

$$\Lambda_T\big(X^T\big) \;=\; \Lambda_T(X)/\mathcal{C}_T,$$

i.e.

$$\overline{f} \in \Lambda_T\left(X^T\right) \;\Leftrightarrow\; f \in \Lambda_T(X).$$

From (1.7), it follows that

$$\widehat{X}_t^T \;=\; \int_0^T \left(\eta^X 1_{[0,t)}\right)(s)dX_s,\; a.s.,\; t \in [0,T]. \tag{2.3}$$

Hence,

$$\int_0^T \overline{f}(s)d\widehat{X}_s^T \;=\; \int_0^T \left(\eta^X f\right)(s)dX_s,\; a.s.,\; f \in \Lambda_T(X). \tag{2.4}$$

From (1.8), we see that that (2.3) does not have a reciprocal, i.e. X_t can not be written as an abstract Wiener integral with respect to $\widehat{X}^T$. By using (2.4), we can write (1.8) as follows:

$$\int_0^T f(s)dX_s \;=\; \int_0^T \overline{f}(s)d\widehat{X}_s^T \;+\; \frac{\left(f, 1_{[0,T)}\right)_{\Lambda_T(X)}}{R^X(T,T)} \cdot X_T,\; f \in \Lambda_T(X). \tag{2.5}$$

By combining (2.4) and (2.2), we obtain the following:

Lemma 2.5. *We have that*

$$\Gamma_T\left(\widehat{X}^T\right) \;=\; I_T^X\left(\Lambda_{T,0}(X)\right).$$

3 Gaussian martingales

We consider the special case when X is an $\mathbb{F}_T^X$-martingale (or equivalently, an $\mathbb{F}_T^W$-martingale). This corresponds to a Volterra kernel of type $z_X(t,s) = z_M(s)1_{[0,t)}(s)$, $s,t \in [0,T]$, where $z_M \in L^2([0,T])$ and $z_M \neq 0$, λ-a.e. For convenience, we write $X = M$. Let $\langle M\rangle_\cdot := \int_0^\cdot z_M^2(s)ds$ denote the quadratic variation or variance function of M. Then $\langle M\rangle_\cdot$ is strictly increasing on $[0,T]$. We denote

$$\langle M\rangle_{T,t} \;:=\; \langle M\rangle_T \;-\; \langle M\rangle_t,\; t \in [0,T].$$

We derive two types of measure-preserving and bridge transformations, present some pathwise relations between the transformed processes, and show how the types a connected. For the case $M = W$, large parts of the results have been obtained in [8] and [13], although not in the present form and not so detailed. The generalization from W to M can be obtained by using the fact that $M \overset{d}{=} W_{\langle M\rangle_\cdot}$. However, for convenience, we provide independent and complete proofs.

3.1 Measure-preserving and bridge transformations

Clearly, we have that

$$\Lambda_T(M) \;=\; L_T^2(M) \;:=\; L^2([0,T], d\langle M\rangle_\cdot).$$

Furthermore,

$$\Lambda_{T,0}(M) \;=\; L_{T,0}^2(M) \;:=\; \left\{ f \in L_T^2(M) \;\middle|\; \int_0^T f(s)d\langle M\rangle_s = 0 \right\}$$

and

$$\left(\eta^M f\right)(\cdot) \;=\; f(\cdot) \;-\; \frac{1}{\langle M\rangle_T}\int_0^T f(s)d\langle M\rangle_s,\; f \in L_T^2(M).$$

Lemma 3.1. *1. For $f \in L^2_T(M)$, let*

$$\left(\mathcal{H}^{M,1} f\right)(s) \ := \ \frac{1}{\langle M \rangle_s} \int_0^s f(u) d\langle M \rangle_u, \ s \in (0,T],$$

and

$$\left(\mathcal{H}^{M,2} f\right)(s) \ := \ \frac{1}{\langle M \rangle_{T,s}} \int_s^T f(u) d\langle M \rangle_u, \ s \in [0,T).$$

The Hardy type operators $\mathcal{H}^{M,1}$ and $\mathcal{H}^{M,2}$ are bounded endomorphisms on $L^2_T(M)$ with adjoints

$$\left(\mathcal{H}^{M,1,*} f\right)(s) \ = \ \int_s^T \frac{f(u)}{\langle M \rangle_u} d\langle M \rangle_u, \ s \in (0,T],$$

and

$$\left(\mathcal{H}^{M,2,*} f\right)(s) \ = \ \int_0^s \frac{f(u)}{\langle M \rangle_{T,u}} d\langle M \rangle_u, \ s \in [0,T),$$

respectively.
2. For $i \in \{1,2\}$, let furthermore

$$\alpha^{M,i} f \ := \ f \ - \ \mathcal{H}^{M,i} f$$

and

$$\beta^{M,i} f \ := \ f \ - \ \mathcal{H}^{M,i,*} f.$$

Then $\alpha^{M,i} : L^2_{T,0}(M) \to L^2_T(M)$ and $\beta^{M,i} : L^2_T(M) \to L^2_{T,0}(M)$ are isometric isomorphisms with $\left(\alpha^{M,i}\right)^{-1} = \beta^{M,i}$.
3. For $i \in \{1,2\}$, we have that

$$\alpha^{M,i} \eta^M f \ = \ \alpha^{M,i} f, \ f \in \ L^2_T(M), \tag{3.1}$$

and

$$\eta^M \beta^{M,i} f \ = \ \beta^{M,i} f, \ f \in \ L^2_T(M). \tag{3.2}$$

Also,

$$\beta^{M,i} \alpha^{M,i} f \ = \ \eta^M f, \ f \in L^2_T(M). \tag{3.3}$$

Proof. 1. From Hardy's inequality (see [5], Theorem 327, p. 240), it follows that

$$\int_0^1 \frac{1}{x^2} \left(\int_0^x g(z) dz \right)^2 dx \le \ 4 \int_0^1 g^2(x) dx, \ g \in L^2([0,1]). \tag{3.4}$$

By using (3.4) with $g(z) := f\left(y^{-1}(z)\right)$ for $i = 1$, and $g(z) := f\left(y^{-1}(1-z)\right)$ for $i = 2$, where $y(u) := \frac{\langle M \rangle_u}{\langle M \rangle_T}$, we obtain that $\left|\mathcal{H}^{M,i} f\right|^2_{L^2_T(M)} \le 4 |f|^2_{L^2_T(M)}$, $f \in L^2_T(M)$. From Fubini's theorem, it follows that $\mathcal{H}^{M,i,*}$ is the adjoint of $\mathcal{H}^{M,i}$.
2. By using Fubini's theorem and splitting integrals, we have that $\alpha^{M,i} : L^2_{T,0}(M) \to L^2_T(M)$ is the inverse of $\beta^{M,i} : L^2_T(M) \to L^2_{T,0}(M)$. Moreover, from part 1, it follows that $\alpha^{M,i}$ is the adjoint of $\beta^{M,i}$. So $\alpha^{M,i}$ and $\beta^{M,i}$ are unitary and hence isometric.
3. A straightforward calculation yields (3.1). (3.2) follows from part 2. By using Fubini's theorem and splitting integrals, we obtain (3.3). □

Remark 3.2. 1. Let $f \in L^2_T(M)$. For $t \in (0,T]$ and $s \in [t,T)$, we have that $\left(\alpha^{M,2} f 1_{[0,t)}\right)(s) = 0$ and $\left(\beta^{M,2} f 1_{[0,t)}\right)(s) = c_M(f,t)$, where c_M is a function independent of s.
2. It holds that $\alpha^{M,i} c \equiv 0$, $c \in \mathcal{C}_T$, $i \in \{1,2\}$.

We can state and prove the following Theorem:

Theorem 3.3. *Let $i \in \{1,2\}$. The transformation defined by*

$$\mathcal{T}_t^{(i)}(M) \;:=\; \int_0^T \left(\beta^{M,i} 1_{[0,t)}\right)(s) dM_s,\ t \in [0,T], \tag{3.5}$$

is measure-preserving, i.e. $\mathcal{T}^{(i)}(M)$ is an $\mathbb{F}_T^{\mathcal{T}^{(i)}(M)}$-martingale with $\mathcal{T}_0^{(i)}(M) = 0$, a.s., and $\langle \mathcal{T}^{(i)}(M)\rangle_\cdot = \langle M\rangle_\cdot$. Furthermore, the process

$$\mathcal{B}_t^{(i)}(M) \;:=\; \int_0^T \left(\alpha^{M,i} 1_{[0,t)}\right)(s) dM_s,\ t \in [0,T], \tag{3.6}$$

is a bridge of M.

Proof. Clearly, $\mathcal{T}^{(i)}(M)$ and $\mathcal{B}^{(i)}(M)$ are centered Gaussian processes. From part 2 of Lemma 3.1, we obtain that

$$\mathrm{Cov}_{\mathbb{P}}\left(\mathcal{T}_s^{(i)}(M), \mathcal{T}_t^{(i)}(M)\right) \;=\; \mathrm{Cov}_{\mathbb{P}}(M_s, M_t),\ s,t \in [0,T].$$

Furthermore, by combining (3.1), part 2 of Lemma 3.1 and (2.3), we obtain that

$$\mathrm{Cov}_{\mathbb{P}}\left(\mathcal{B}_s^{(i)}(M), \mathcal{B}_t^{(i)}(M)\right) \;=\; \mathrm{Cov}_{\mathbb{P}}\left(M_s^T, M_t^T\right),\ s,t \in [0,T].$$

□

3.2 Pathwise relations for the transformed processes

By using the stochastic Fubini theorem, we have that

$$\mathcal{T}_t^{(1)}(M) \;=\; M_t \;-\; \int_0^t \frac{M_s}{\langle M\rangle_s} d\langle M\rangle_s,\ a.s.,\ t \in [0,T]. \tag{3.7}$$

Similarly, we obtain that

$$\mathcal{T}_t^{(2)}(M) \;=\; M_t \;-\; \int_0^t \frac{M_T - M_s}{\langle M\rangle_{T,s}} d\langle M\rangle_s,\ a.s.,\ t \in [0,T]. \tag{3.8}$$

On the one hand, by combining (3.5), (3.2) and (2.4), we have that

$$\mathcal{T}_t^{(i)}(M) \;=\; \int_0^T \left(\overline{\beta^{M,i} 1_{[0,t)}}\right)(s) d\widehat{M}_s^T,\ a.s.,\ t \in [0,T],\ i \in \{1,2\}. \tag{3.9}$$

On the other hand, by combining (2.3), (3.3) and (3.5), we obtain the reciprocal

$$\widehat{M}_t^T \;=\; \int_0^T \left(\alpha^{M,i} 1_{[0,t)}\right)(s) d\mathcal{T}_s^{(i)}(M),\ a.s.,\ t \in [0,T],\ i \in \{1,2\}. \tag{3.10}$$

Clearly, from (3.9) and (3.10), it follows that

$$\Gamma_T\left(\mathcal{T}^{(i)}(M)\right) \;=\; \Gamma_T\left(\widehat{M}^T\right),\ i \in \{1,2\}. \tag{3.11}$$

For $i = 2$, it follows more precisely from part 1 of Remark 3.2, that

$$\Gamma_t\left(\mathcal{T}^{(2)}(M)\right) \;=\; \Gamma_t\left(\widehat{M}^T\right),\ t \in [0,T], \tag{3.12}$$

i.e.

$$\mathbb{F}_T^{\mathcal{T}^{(2)}(M)} \;=\; \mathbb{F}_T^{\widehat{M}^T}. \tag{3.13}$$

From (3.13), we obtain that $\mathcal{T}^{(2)}(M)$ is an $\mathbb{F}_T^{\widehat{M}^T}$-martingale. Furthermore, by combining (3.11) and (1.8), we have that $\mathcal{T}^{(2)}(M)$ is orthogonal to M_T, and thus independent of M_T. Hence, $\mathcal{T}^{(2)}(M)$ is also an $\left(\mathbb{F}_T^{\widehat{M}^T} \vee \sigma(M_T)\right)$-martingale and so, by using (1.9), an $\left(\mathbb{F}_T^M \vee \sigma(M_T)\right)$-martingale. From (3.8), it is then easy to see, that $\mathcal{T}^{(2)}(M)$ is the martingale component in the $\left(\mathbb{F}_T^M \vee \sigma(M_T)\right)$-semimartingale decomposition of M.

By combining (3.8) and (1.7), we obtain that

$$\mathcal{T}_t^{(2)}(M) \;=\; \widehat{M}_t^T \;+\; \int_0^t \frac{\widehat{M}_s^T}{\langle M\rangle_{T,s}} d\langle M\rangle_s,\; a.s.,\; t \in [0,T].$$

In particular, $\widehat{M}^T$ is the unique solution of the linear stochastic differential equation

$$d\widehat{M}_t^T \;=\; d\mathcal{T}_t^{(2)}(M) \;-\; \frac{\widehat{M}_t^T}{\langle M\rangle_{T,t}} d\langle M\rangle_t, \qquad \widehat{M}_0^T \;=\; 0.$$

A straightforward calculation yields

$$\mathcal{B}_t^{(1)}(M) \;=\; -\langle M\rangle_t \int_t^T \frac{1}{\langle M\rangle_s} dM_s,\; a.s.,\; t \in (0,T]. \tag{3.14}$$

Similarly,

$$\mathcal{B}_t^{(2)}(M) \;=\; \langle M\rangle_{T,t} \int_0^t \frac{dM_s}{\langle M\rangle_{T,s}},\; a.s.,\; t \in [0,T).$$

By combining part 2 of Lemma 3.1 and (3.6), we obtain that

$$M_t \;=\; \int_0^T \left(\overline{\beta^{M,i} 1_{[0,t)}}\right)(s) d\mathcal{B}_s^{(i)}(M),\; a.s.,\; t \in [0,T],\; i \in \{1,2\}. \tag{3.15}$$

By comparing identities (3.6) and (3.15) with identities (3.10) and (3.9), we observe that the bridge $\mathcal{B}^{(i)}(M)$ is related to the martingale M in the same way, as the bridge $\widehat{M}^T$ is related to the martingale $\mathcal{T}^{(i)}(M)$. Hence, from (3.11), we obtain that

$$\Gamma_T\Big(\mathcal{B}^{(i)}(M)\Big) \;=\; \Gamma_T(M),\; i \in \{1,2\}. \tag{3.16}$$

Moreover, from (3.12), we have that

$$\Gamma_t\Big(\mathcal{B}^{(2)}(M)\Big) \;=\; \Gamma_t(M),\; t \in [0,T],$$

i.e.

$$\mathbb{F}_T^{\mathcal{B}^{(2)}(M)} \;=\; \mathbb{F}_T^M.$$

From (3.10) and (3.15) we obtain the following:

Lemma 3.4. *Let $i \in \{1,2\}$. Then*

$$\widehat{M}_t^T \;=\; \mathcal{B}_t^{(i)}\left(\mathcal{T}^{(i)}(M)\right),\; a.s.,\; t \in [0,T].$$

Furthermore,

$$M_t \;=\; \mathcal{T}_t^{(i)}\left(\mathcal{B}^{(i)}(M)\right),\; a.s.,\; t \in [0,T].$$

Remark 3.5. 1. From (3.11), we have that $\mathcal{F}_T^{\mathcal{T}^{(i)}(M)} \subsetneq \mathcal{F}_T^M$. Hence, $\mathcal{T}^{(i)}$ is injective, but not surjective.
2. From (3.16), it follows that $\mathcal{F}_T^{\mathcal{B}^{(i)}(M)} = \mathcal{F}_T^M$. Hence, $\mathcal{B}^{(i)}$ is bijective.

3.3 Connection between the cases $i = 1$ and $i = 2$

Let $(Y_t)_{t\in[0,T]}$ be a Gaussian process with $Y_0 = 0$, a.s., and let

$$\mathcal{S}_t(Y) \;:=\; Y_T - Y_{T-t},\;\; t \in [0,T], \tag{3.17}$$

be the time-reversion of the process Y. Clearly, we have that $\mathcal{S}_0(Y) = 0$, a.s., and

$$\mathcal{S}_T(Y) \;=\; Y_T,\;\; a.s. \tag{3.18}$$

Furthermore, $\Gamma_T(\mathcal{S}(Y)) = \Gamma_T(Y)$ and $\mathcal{S}_t^2(Y) = Y_t$, a.s., $t \in [0,T]$.

It is easy to show, that $\mathcal{S}(M)$ is a continuous $\mathbb{F}_T^{\mathcal{S}(M)}$-martingale with

$$\langle \mathcal{S}(M)\rangle_t \;=\; \langle M\rangle_{T,T-t},\;\; t \in [0,T]. \tag{3.19}$$

Hence $\langle \mathcal{S}(M)\rangle_\cdot$ is strictly increasing on $[0,T]$. Note that $\mathcal{S}$ is a measure-preserving transformation if and only if M has stationary increments, i.e. if and only if $M = W$. By using (3.19), we obtain that

$$\left(\beta^{M,1}1_{[T-t,T)}\right)(\cdot) \;=\; \left(\beta^{\mathcal{S}(M),2}1_{[0,t)}\right)(T-\cdot),\;\; \lambda\text{-}a.e.,\;\; t \in (0,T].$$

It follows from this, that

$$\mathcal{S}_t\left(\mathcal{T}^{(1)}(M)\right) \;=\; \mathcal{T}_t^{(2)}\left(\mathcal{S}(M)\right),\;\; a.s.,\;\; t \in [0,T]. \tag{3.20}$$

Similarly, we have that

$$\left(\alpha^{M,1}1_{[T-t,T)}\right)(\cdot) \;=\; \left(\alpha^{\mathcal{S}(M),2}1_{[0,t)}\right)(T-\cdot),\;\; \lambda\text{-}a.e.,\;\; t \in (0,T].$$

Therefore,

$$\mathcal{S}_t\left(\mathcal{B}^{(1)}(M)\right) \;=\; \mathcal{B}_t^{(2)}\left(\mathcal{S}(M)\right),\;\; a.s.,\;\; t \in [0,T]. \tag{3.21}$$

4 Volterra Gaussian processes

We generalize the results of Section 3 to the case, when X is a continuous Volterra Gaussian processes with a non-degenerate Volterra kernel satisfying (1.5), (1.6) and that

$$\mathbf{K}^X f = z_X(T,\cdot)1_{[0,t)} \text{ has a solution in } \check{\Lambda}_T(X) \text{ for all } t \in (0,T], \tag{4.1}$$

which we denote by $k^*(t,\cdot) := k^*_{X,T}(t,\cdot)$, $t \in (0,T]$. We proceed similarly as in Section 3.

4.1 Measure-preserving and bridge transformations

Let

$$M_t \;:=\; M_t(X,T) \;:=\; \mathrm{E}_{\mathbb{P}}\left(X_T \middle| \mathcal{F}_t^X\right),\;\; t \in [0,T],$$

denote the *prediction martingale of* X_T *with respect to* $\mathbb{F}_T^X$. By definition,

$$M_T \;=\; X_T,\;\; a.s. \tag{4.2}$$

Clearly, by (1.4) we have that

$$M_t \;=\; \int_0^t z_X(T,s)dW_s,\;\; a.s.,\;\; t \in [0,T]. \tag{4.3}$$

From (4.3) and (1.5), it follows that M is a continuous Gaussian martingale with $M_0 = 0$, a.s., and the quadratic variation function $\langle M\rangle_\cdot = \int_0^\cdot z_X^2(T,s)ds$ is strictly increasing on $[0,T]$. In particular, all results from Section 3 hold true for M. We have that $\Gamma_t(M) = \Gamma_t(W)$, $t \in [0,T]$, and hence, from (1.3), it follows that

$$\Gamma_t(M) \;=\; \Gamma_t(X),\ t\in[0,T]. \tag{4.4}$$

More precisely, we have that

$$X_t \;=\; \int_0^t k(t,s)dM_s,\ a.s.,\ t\in[0,T], \tag{4.5}$$

where the Volterra kernel is given by

$$k(t,s) \;:=\; k_{X,T}(t,s) \;:=\; \frac{z_X(t,s)}{z_X(T,s)},\ s,t\in[0,T]. \tag{4.6}$$

Let $\kappa := \kappa^{X,T}$ be the isometric isomorphism defined by

$$\begin{aligned} \kappa \;:\; \Lambda_T(X) \;&\to\; L_T^2(M) \\ 1_{[0,t)} \;&\mapsto\; k(t,\cdot),\ t\in(0,T]. \end{aligned}$$

Then

$$\int_0^T f(s)dX_s \;=\; \int_0^T (\kappa f)(s)dM_s,\ a.s.,\ f\in\Lambda_T(X). \tag{4.7}$$

Also, from (4.3) and (4.1), it follows that

$$M_t \;=\; \int_0^T k^*(t,s)dX_s,\ a.s.,\ t\in[0,T]. \tag{4.8}$$

Hence

$$\kappa^{-1}1_{[0,t)} \;=\; k^*(t,\cdot),\ t\in(0,T].$$

From (4.4), it follows that k^* is a Volterra kernel. Clearly, by using (4.2), we have that

$$R^X(T,T) \;=\; \langle M\rangle_T \tag{4.9}$$

and

$$k^*(T,\cdot) \;\equiv\; 1. \tag{4.10}$$

From (4.6) and (4.10), it follows that

$$\kappa(c) \;=\; \kappa^{-1}(c) \;=\; c,\ c\in\mathcal{C}_T. \tag{4.11}$$

Moreover, (4.7) and (4.2) imply that

$$\begin{aligned} \left(f,1_{[0,T)}\right)_{\Lambda_T(X)} \;&=\; \mathrm{Cov}_{\mathbb{P}}\left(\int_0^T f(s)dX_s, X_T\right) \\ &=\; \mathrm{Cov}_{\mathbb{P}}\left(\int_0^T (\kappa f)(s)dM_s, M_T\right) \\ &=\; \int_0^T (\kappa f)(s)d\langle M\rangle_s,\ f\in\Lambda_T(X). \end{aligned} \tag{4.12}$$

By combing (4.9) and (4.12), and then using (4.11), we obtain that

$$\eta^X f \;=\; \left(\kappa^{-1}\eta^M\kappa\right)f,\ f\in\Lambda_T(X). \tag{4.13}$$

Also, by combining (2.3), (4.13), (4.7) and (2.4), we have that

$$\widehat{X}_t^T \;=\; \int_0^T \overline{k(t,\cdot)}(s)d\widehat{M}_s^T,\ a.s.,\ t\in[0,T]. \tag{4.14}$$

Example 4.1. If $X = B^H$, then (4.1) is satisfied. We have that

$$k^*(t,s) = 1_{[0,t)}(s) + \left(\frac{\sin(\pi(H-\frac{1}{2}))}{\pi} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} \int_t^T \frac{u^{H-\frac{1}{2}}(u-t)^{H-\frac{1}{2}}}{u-s} du \right) 1_{[0,t)}(s),$$

$s, t \in [0,T]$. See [15] for a full proof.

The following result generalizes Lemma 3.1:

Lemma 4.2. *1. For $i \in \{1,2\}$ and $f \in \Lambda_T(X)$, let*

$$\alpha^{X,i} f \;:=\; \left(\kappa^{-1}\alpha^{M,i}\kappa\right) f \tag{4.15}$$

and

$$\beta^{X,i} f \;:=\; \left(\kappa^{-1}\beta^{M,i}\kappa\right) f. \tag{4.16}$$

Then $\alpha^{X,i} : \Lambda_{T,0}(X) \to \Lambda_T(X)$ and $\beta^{X,i} : \Lambda_T(X) \to \Lambda_{T,0}(X)$ are isometric isomorphisms with $\left(\alpha^{X,i}\right)^{-1} = \beta^{X,i}$.
2. For $i \in \{1,2\}$, we have that

$$\alpha^{X,i}\eta^X f \;=\; \alpha^{X,i} f, \;\; f \in \Lambda_T(X), \tag{4.17}$$

and

$$\eta^X \beta^{X,i} f \;=\; \beta^{X,i} f, \;\; f \in \Lambda_T(X). \tag{4.18}$$

Also,

$$\beta^{X,i}\alpha^{X,i} f = \eta^X f, \;\; f \in \Lambda_T(X). \tag{4.19}$$

Proof. 1. Follows straightforward from part 2 of Lemma 3.1.
2. Follows from part 3 of Lemma 3.1 by using (4.13). □

Remark 4.3. 1. Let $t \in (0,T]$ and $s \in [t,T)$. By combining part 1 of Remark 3.2 and (4.11), it follows that $\left(\alpha^{X,2}1_{[0,t)}\right)(s) = 0$ and $\left(\beta^{X,2}1_{[0,t)}\right)(s) = c_M(k(t,\cdot),t)$.
2. From part 2 of Remark 3.2 and (4.11), we obtain that $\alpha^{X,i}c \equiv 0$, $c \in \mathcal{C}_T$, $i \in \{1,2\}$.

The generalization of Theorem 3.3 is straightforward:

Theorem 4.4. *Let $i \in \{1,2\}$. The transformation defined by*

$$\mathcal{T}_t^{(i)}(X) \;:=\; \int_0^T \left(\beta^{X,i}1_{[0,t)}\right)(s) dX_s, \;\; t \in [0,T], \tag{4.20}$$

is measure-preserving, i.e. $\mathcal{T}^{(i)}(X)$ is a centered Gaussian process with $\mathcal{T}_0^{(i)}(X) = 0$, a.s., and $R^{\mathcal{T}^{(i)}(X)} = R^X$. Furthermore, the process

$$\mathcal{B}_t^{(i)}(X) \;:=\; \int_0^T \left(\alpha^{X,i}1_{[0,t)}\right)(s) dX_s, \;\; t \in [0,T], \tag{4.21}$$

is a bridge of X.

Proof. By construction, $\mathcal{T}^{(i)}(X)$ and $\mathcal{B}^{(i)}(X)$ are centered Gaussian processes. By using part 1 of Lemma 4.2, we obtain that

$$\mathrm{Cov}_{\mathbb{P}}\left(\mathcal{T}_s^{(i)}(X), \mathcal{T}_t^{(i)}(X)\right) \;=\; \mathrm{Cov}_{\mathbb{P}}(X_s, X_t), \;\; s,t \in [0,T].$$

Moreover, by combining (4.17), part 1 of Lemma 4.2 and (2.3), we obtain that

$$\mathrm{Cov}_{\mathbb{P}}\left(\mathcal{B}_s^{(i)}(X), \mathcal{B}_t^{(i)}(X)\right) \;=\; \mathrm{Cov}_{\mathbb{P}}\left(X_s^T, X_t^T\right), \;\; s,t \in [0,T].$$

□

Remark 4.5. Lemma 4.2 and Theorem 4.4 hold true also without assumptions (1.6) and (4.1).

Next, we want to explicitly evaluate the functions $\alpha^{X,i}1_{[0,t)}$ and $\beta^{X,i}1_{[0,t)}$ in definitions (4.21) and (4.20), respectively. For this purpose, we recall the notion of the Bochner integral in a Banach space: Let $(Q, \mathcal{Q}, \mu)$ be a measure space with a finite signed measure and let B be a Banach space with norm $|\cdot|_B$. The Bochner integral in B of an indicator function $f(q) := g \cdot 1_A(q)$, $q \in Q$, where $A \in \mathcal{Q}$ and $g \in B$, is given by $\int_Q f(q)d\mu(q) := g \cdot \mu(A)$. A measurable map $f : (Q, \mathcal{Q}) \to B$, $q \mapsto f(q)$, is Bochner integrable if $\int_Q |f(q)|_B d\mu(q) < \infty$. The space of Bochner integrable functions is a Banach space. The Bochner integral in B of a Bochner integrable function is defined as the unique continuous, linear extension of $\int_Q \cdot \, d\mu$ from the set of indicator functions to B. By construction, the Bochner integral commutes with bounded linear maps: if $f : Q \to B$ is Bochner integrable, B' is a Banach space and $A : B \to B'$ is a bounded linear map, then $Af : Q \to B'$ is Bochner integrable, and

$$A \int_Q f(q)d\mu(q) \;=\; \int_Q Af(q)d\mu(q).$$

If the elements of B are functions from $[0, T]$ to $\mathbb{R}$, then we denote $\int_Q f(q, s)d\mu(q) := \left(\int_Q f(q)d\mu(q)\right)(s)$, $s \in [0, T]$. For details on Bochner integration, see [11].

We can state and prove the following Lemma:

Lemma 4.6. *Let $z \in (0, T]$ and $g \in L^2_T(M)$.*
1. If g is absolutely continuous on $[a, z]$ for every $a > 0$, then

$$\left(\kappa^{-1} g 1_{[0,z)}\right)(s) = k^*(z, s)g(z) - \int_0^z k^*(u, s)dg(u), \;\; \lambda\text{-a.e. } s \in (0, T),$$

where the integral is a Bochner integral in $\Lambda_T(X)$.
2. If g is absolutely continuous on $[0, b]$ for every $b < z$, then

$$\left(\kappa^{-1} g 1_{[0,z)}\right)(s) = \int_0^z \left(k^*(z, s) - k^*(u, s)\right) dg(u) + k^*(z, s)g(0), \;\; \lambda\text{-a.e. } s \in (0, T),$$

where the integral is a Bochner integral in $\Lambda_T(X)$.

Proof. 1. We have that

$$\int_0^z 1_{[0,u)}(s)dg(u) \;=\; (g(z) - g(s))1_{[0,z)}(s), \;\; \lambda\text{-a.e. } s \in (0, T),$$

where the integral is a Bochner integral in $L^2_T(M)$. By combining this, the fact that $\kappa^{-1} : L^2_T(M) \to \Lambda_T(X)$ is a bounded linear map and Remark 2.1, we obtain that

$$\begin{aligned}
\left(\kappa^{-1} g 1_{[0,z)}\right)(s) &= \left(\kappa^{-1} g(z) 1_{[0,z)}\right)(s) \;-\; \left(\kappa^{-1}(g(z) - g)1_{[0,z)}\right)(s) \\
&= g(z)k^*(z, s) \;-\; \left(\kappa^{-1} \int_0^z 1_{[0,u)}dg(u)\right)(s) \\
&= g(z)k^*(z, s) \;-\; \left(\int_0^z \kappa^{-1} 1_{[0,u)}dg(u)\right)(s) \\
&= g(z)k^*(z, s) \;-\; \int_0^z k^*(u, s)dg(u), \;\; \lambda\text{-a.e. } s \in (0, T).
\end{aligned}$$

2. Similarly, we have that

$$\int_0^z \left(1_{[0,z)}(s) - 1_{[0,u)}(s)\right) dg(u) \;=\; (g(s) \;-\; g(0))\, 1_{[0,z)}(s), \;\; \lambda\text{-a.e. } s \in (0, T),$$

where the integral is a Bochner integral in $L_T^2(M)$. Hence,

$$\begin{aligned}
\left(\kappa^{-1} g 1_{[0,z)}\right)(s) &= \left(\kappa^{-1}\left(g-g(0)\right)1_{[0,z)}\right)(s) \;+\; \left(\kappa^{-1} g(0) 1_{[0,z)}\right)(s) \\
&= \left(\kappa^{-1}\int_0^z \left(1_{[0,z)}-1_{[0,u)}\right) dg(u)\right)(s) \;+\; g(0)k^*(z,s) \\
&= \left(\int_0^z \kappa^{-1}(1_{[0,z)}-1_{[0,u)})dg(u)\right)(s) \;+\; g(0)k^*(z,s) \\
&= \int_0^z \left(k^*(z,s)-k^*(u,s)\right) dg(u) + g(0)k^*(z,s),\ \lambda\text{-}a.e.\ s\in(0,T).
\end{aligned}$$

□

Let $t\in(0,T]$. The function $\mathcal{H}^{M,1}k(t,\cdot)$ is absolutely continuous with respect to $\langle M\rangle_\cdot$, or equivalently, with respect to $\lambda(\cdot)$, on $[a,T]$ for every $a>0$. By combining part 1 of Lemma 4.6 with $z=T$ and (4.10), we obtain that

$$\begin{aligned}
\left(\alpha^{X,1}1_{[0,t)}\right)(s) &= 1_{[0,t)}(s) - \frac{1}{\langle M\rangle_T}\int_0^t k(t,u)d\langle M\rangle_u \;+\; \int_0^t k^*(u,s)\frac{k(t,u)}{\langle M\rangle_u}d\langle M\rangle_u \\
&\quad - \int_0^T \frac{k^*(u,s)}{\langle M\rangle_u^2}\int_0^u k(t,v)d\langle M\rangle_v d\langle M\rangle_u,\ \lambda\text{-}a.e.\ s\in(0,T). \quad (4.22)
\end{aligned}$$

Let $t\in(0,T)$. The function $\mathcal{H}^{M,2}k(t,\cdot)$ is absolutely continuous on $[0,T]$. Moreover, $\left(\mathcal{H}^{M,2}k(t,\cdot)\right)(s)=0,\ s\geq t$. By using part 1 of Lemma 4.6 with $z=t$, we obtain that

$$\begin{aligned}
\left(\alpha^{X,2}1_{[0,t)}\right)(s) &= 1_{[0,t)}(s) - \int_0^t k^*(u,s)\frac{k(t,u)}{\langle M\rangle_{T,u}}d\langle M\rangle_u \\
&\quad + \int_0^t \frac{k^*(u,s)}{\langle M\rangle_{T,u}^2}\int_u^t k(t,v)d\langle M\rangle_v d\langle M\rangle_u,\ \lambda\text{-}a.e.\ s\in(0,T). \quad (4.23)
\end{aligned}$$

Let $t\in(0,T]$. The function $\mathcal{H}^{M,1,*}k(t,\cdot)$ is absolutely continuous on $[a,T]$ for every $a>0$, and $\left(\mathcal{H}^{M,1,*}k(t,\cdot)\right)(s)=0,\ s\geq t$. By using part 1 of Lemma 4.6 with $z=t$, we obtain that

$$\left(\beta^{X,1}1_{[0,t)}\right)(s) \;=\; 1_{[0,t)}(s) - \int_0^t k^*(u,s)\frac{k(t,u)}{\langle M\rangle_u}d\langle M\rangle_u,\ \lambda\text{-}a.e.\ s\in(0,T). \quad (4.24)$$

Let $t\in(0,T]$. The function $\mathcal{H}^{M,2,*}k(t,\cdot)$ is absolutely continuous on $[0,b]$ for every $b<T$. By using part 2 of Lemma 4.6 with $z=T$ and (4.10), we have that

$$\left(\beta^{X,2}1_{[0,t)}\right)(s) = 1_{[0,t)}(s) - \int_0^t \left(1-k^*(u,s)\right)\frac{k(t,u)}{\langle M\rangle_{T,u}}d\langle M\rangle_u,\ \lambda\text{-}a.e.\ s\in(0,T). \quad (4.25)$$

4.2 Pathwise relations for the transformed processes

By combining (4.20), (4.16), (4.7) and (3.5), we obtain that

$$\mathcal{T}_t^{(i)}(X) \;=\; \int_0^t k(t,s)d\mathcal{T}_s^{(i)}(M),\ a.s.,\ t\in[0,T],\ i\in\{1,2\}, \quad (4.26)$$

i.e. $\mathcal{T}^{(i)}(M)$ is the prediction martingale of $\mathcal{T}_T^{(i)}(X)$ with respect to $\mathbb{F}_T^{\mathcal{T}^{(i)}(X)}$. Clearly, we can evaluate $\mathcal{T}^{(i)}(X)$ explicitly, by using (4.24) and (4.25). However, in order to obtain expressions corresponding to (3.7) and (3.8), we combine (4.20), (4.7), the stochastic Fubini theorem, and (4.8), and obtain that

$$\mathcal{T}_t^{(1)}(X) = X_t - \int_0^t \left(\int_0^u k^*(u,s) dX_s \right) \frac{k(t,u)}{\langle M \rangle_u} d\langle M \rangle_u, \ a.s., \ t \in [0,T].$$

In the same way, we have that

$$\mathcal{T}_t^{(2)}(X) = X_t - \int_0^t \left(X_T - \int_0^u k^*(u,s) dX_s \right) \frac{k(t,u)}{\langle M \rangle_{T,u}} d\langle M \rangle_u, \ a.s., \ t \in [0,T]. \tag{4.27}$$

On the one hand, by combining (4.20), (4.18) and (2.4), we have that

$$\mathcal{T}_t^{(i)}(X) \quad = \quad \int_0^T \left(\overline{\beta^{X,i} 1_{[0,t)}} \right)(s) d\widehat{X}_s^T, \ a.s., \ t \in [0,T], \ i \in \{1,2\}. \tag{4.28}$$

On the other hand, by combining (2.3), (4.19), and (4.20), we obtain the reciprocal

$$\widehat{X}_t^T \quad = \quad \int_0^T \left(\alpha^{X,i} 1_{[0,t)} \right)(s) d\mathcal{T}_s^{(i)}(X), \ a.s., \ t \in [0,T], \ i \in \{1,2\}. \tag{4.29}$$

From (4.28) and (4.29), we have that

$$\Gamma_T\Big(\mathcal{T}^{(i)}(X)\Big) \ = \ \Gamma_T\Big(\widehat{X}^T\Big), \ i \in \{1,2\}. \tag{4.30}$$

For $i = 2$, it follows from part 1 of Remark 4.3, that

$$\Gamma_t\Big(\mathcal{T}^{(2)}(X)\Big) \ = \ \Gamma_t\Big(\widehat{X}^T\Big), \ t \in [0,T],$$

i.e.

$$\mathbb{F}_T^{\mathcal{T}^{(2)}(X)} \ = \ \mathbb{F}_T^{\widehat{X}^T}.$$

By combining (4.27), (2.5) and (4.12), we obtain that

$$\mathcal{T}_t^{(2)}(X) \ = \ \widehat{X}_t^T \ + \ \int_0^t \int_0^T \overline{k^*(u,\cdot)}(s) d\widehat{X}_s^T \frac{k(t,u)}{\langle M \rangle_{T,u}} d\langle M \rangle_u, \ a.s., \ t \in [0,T].$$

Note that, in contrast to $\widehat{M}^T$, the process $\widehat{X}^T$ is *not* the solution of a linear stochastic differential equation (unless $X = M$). By combining (4.21), (4.15), (4.7) and (3.6), we obtain that

$$\mathcal{B}_t^{(i)}(X) \quad = \quad \int_0^T \overline{k(t,\cdot)}(s) d\mathcal{B}_s^{(i)}(M), \ a.s., \ t \in [0,T], \ i \in \{1,2\}. \tag{4.31}$$

By using (4.22), we obtain that

$$\begin{aligned}
\mathcal{B}_t^{(1)}(X) \quad = \quad & X_t - \frac{X_T}{\langle M \rangle_T} \int_0^t k(t,u) d\langle M \rangle_u \ + \ \int_0^t \left(\int_0^t k^*(u,s) \frac{k(t,u)}{\langle M \rangle_u} d\langle M \rangle_u \right) dX_s \\
& - \int_0^T \left(\int_0^T \frac{k^*(u,s)}{\langle M \rangle_u^2} \int_0^u k(t,v) d\langle M \rangle_v d\langle M \rangle_u \right) dX_s, \ a.s., \ t \in (0,T].
\end{aligned}$$

Similarly, by using (4.23), we have that

$$\begin{aligned}
\mathcal{B}_t^{(2)}(X) \quad = \quad & X_t - \int_0^t \left(\int_0^t k^*(u,s) \frac{k(t,u)}{\langle M \rangle_{T,u}} d\langle M \rangle_u \right) dX_s \\
& + \int_0^t \left(\int_0^t \frac{k^*(u,s)}{\langle M \rangle_{T,u}^2} \int_u^t k(t,v) d\langle M \rangle_v d\langle M \rangle_u \right) dX_s, \ a.s., \ t \in [0,T).
\end{aligned}$$

By combining part 1 of Lemma 4.2 and (4.21), we have that

$$X_t \;=\; \int_0^T \left(\overline{\beta^{X,i}1_{[0,t)}}\right)(s)\mathcal{B}_s^{(i)}(X),\ a.s.,\ t\in[0,T],\ i\in\{1,2\}. \tag{4.32}$$

By comparing identities (4.21) and (4.32) with identities (4.29) and (4.28), we conclude that bridge $\mathcal{B}^{(i)}(X)$ is related to the process X in the same way, as the bridge $\widehat{X}^T$ is related to the process $\mathcal{T}^{(i)}(X)$. Therefore,

$$\Gamma_T\left(\mathcal{B}^{(i)}(X)\right) \;=\; \Gamma_T(X),\ i\in\{1,2\}. \tag{4.33}$$

Furthermore,

$$\Gamma_t\left(\mathcal{B}^{(2)}(X)\right) \;=\; \Gamma_t(X),\ t\in[0,T],$$

i.e.

$$\mathcal{F}_T^{\mathcal{B}^{(2)}(X)} \;=\; \mathbb{F}_T^X.$$

Remark 4.7. $\mathcal{B}^{(2)}(X)$ is the dynamic bridge of X, and has been introduced in [4].

From (4.29) and (4.32) we obtain the following:

Lemma 4.8. *Let $i\in\{1,2\}$. Then*

$$\widehat{X}_t^T \;=\; \mathcal{B}_t^{(i)}\left(\mathcal{T}^{(i)}(X)\right),\ a.s.,\ t\in[0,T].$$

Furthermore,

$$X_t \;=\; \mathcal{T}_t^{(i)}\left(\mathcal{B}^{(i)}(X)\right),\ a.s.,\ t\in[0,T].$$

Remark 4.9. 1. From (4.30), we have that $\mathcal{F}_T^{\mathcal{T}^{(i)}(X)} \subsetneq \mathcal{F}_T^X$, so $\mathcal{T}^{(i)}$ is injective but not surjective.
2. From (4.33), it follows that $\mathcal{F}_T^{\mathcal{B}^{(i)}(X)} = \mathcal{F}_T^X$, and so $\mathcal{B}^{(i)}$ is bijective.

4.3 Connection between the cases $i=1$ and $i=2$

Consider (4.5), and let

$$\begin{aligned}X_t^{\mathcal{S}} &:= \int_0^t k(t,s)d\mathcal{S}_s(M)\\ &= \int_0^t \left(\frac{z_X(t,s)}{z_X(T,s)}z_X(T,T-s)\right)d\mathcal{S}_s(W),\ a.s.,\ t\in[0,T],\end{aligned}$$

where $\mathcal{S}$ is defined as in (3.17). Note that $X^{\mathcal{S}}$ is well-defined by part 1 of Remark 1.1. Clearly, $X^{\mathcal{S}}$ is a continuous Volterra Gaussian process with a non-degenerate Volterra kernel, and $\mathcal{S}(M)$ is the prediction martingale of $X_T^{\mathcal{S}}$ with respect to $\mathbb{F}_T^{X^{\mathcal{S}}} = \mathbb{F}_T^{\mathcal{S}(M)}$. By combining (4.26), (3.20) and again (4.26), we obtain that

$$\left(\mathcal{T}^{(1)}(X)\right)_t^{\mathcal{S}} \;=\; \mathcal{T}_t^{(2)}\left(X^{\mathcal{S}}\right),\ a.s.,\ t\in[0,T].$$

Clearly,

$$R^{\mathcal{S}(X)}(T,T-t) \;=\; R^X(T,T)-R^X(T,t),\ t\in[0,T]. \tag{4.34}$$

By using (4.34) and (3.18), we obtain that

$$\mathcal{S}_t\left(\widehat{X}^T\right) \;=\; \widehat{\mathcal{S}(X)}_t^T,\ a.s.,\ t\in[0,T]. \tag{4.35}$$

Consider (4.14) and let

$$\left(\widehat{X}^T\right)_t^{\mathcal{S}} \;:=\; \int_0^T \overline{k(t,\cdot)}(s)d\mathcal{S}_s\left(\widehat{M}^T\right),\; t\in[0,T]. \tag{4.36}$$

Due to (4.35), the right-hand side of (4.36) is a Wiener integral with respect to $\widehat{\mathcal{S}(M)}^T$. The process $\left(\widehat{X}^T\right)^{\mathcal{S}}$ is well-defined due to part 1 of Remark 1.1. In particular, from (4.14) it follows that

$$\left(\widehat{X}^T\right)_t^{\mathcal{S}} \;=\; \widehat{X_t^{\mathcal{S}}}^T,\; a.s.,\; t\in[0,T].$$

By combining (4.31), (3.21) and again (4.31), we have that

$$\left(\mathcal{B}^{(1)}(X)\right)_t^{\mathcal{S}} \;=\; \mathcal{B}_t^{(2)}\left(X^{\mathcal{S}}\right),\; a.s.,\; t\in[0,T].$$

Remark 4.10. If X has stationary increments, then $\mathcal{S}(X)\overset{d}{=}X$, and hence $\mathcal{S}(X)$ is a Volterra Gaussian process. In general, however, it is not clear, whether $\mathcal{S}(X)$ is a Volterra Gaussian process.

5 A Fourier-Laguerre series expansion

Let $(M_t)_{t\in[0,\infty)}$ be a continuous Gaussian martingale with $M_0=0$, a.s., such that $\langle M\rangle_\cdot$ is strictly increasing and

$$\lim_{t\to\infty}\langle M\rangle_t \;=\; \infty. \tag{5.1}$$

As before, we assume that the underlying space $(\Omega,\mathcal{F},\mathbb{P})$ is the coordinate space of M, where the definition of *coordinate space* is similar as in the introduction, with $[0,\infty)$ replacing $[0,T]$. Hence $\mathcal{F}=\mathcal{F}_\infty^M:=\sigma(M_t\,|\,t\in[0,\infty))$. Let

$$\Gamma_\infty(M) \;:=\; \overline{\operatorname{span}\{M_t\,|\,t\in[0,\infty)\}}$$

and

$$\Gamma_{[T,\infty)}(M) \;:=\; \overline{\operatorname{span}\{M_t-M_s\,|\,s,t\in[T,\infty)\}}$$

denote the first Wiener chaoses of M over $[0,\infty)$ and $[T,\infty)$, respectively. By (5.1), there exists a standard Brownian motion $(W_t)_{t\in[0,\infty)}$ such that $M_t=W_{\langle M\rangle_t}$, *a.s.*, $t\in[0,\infty)$ (see [10], Theorem 4.6, p. 174). From the strong law of large numbers for standard Brownian motion, it follows that

$$\lim_{t\to\infty}\frac{M_t}{\langle M\rangle_t} \;=\; \lim_{t\to\infty}\frac{W_t}{t} \;=\; 0,\; a.s. \tag{5.2}$$

From (3.7), we see that transformation $\mathcal{T}^{(1)}$ does not depend on T, hence we can write

$$\mathcal{T}_t(M) \;:=\; \mathcal{T}_t^{(1)}(M) \;=\; M_t-\int_0^t\frac{M_s}{\langle M\rangle_s}dM_s,\; a.s.,\; t\in[0,\infty).$$

By using the stochastic Fubini theorem, we have that

$$\mathcal{T}_t(M) \;=\; \int_0^\infty\left(\beta_\infty^{M,1}1_{[0,t)}\right)(s)dM_s,\; a.s.,\; t\in[0,\infty),$$

where

$$\begin{array}{rcl}\beta_\infty^{M,1}\ :\ L^2([0,\infty),d\langle M\rangle_\cdot) & \to & L^2([0,\infty),d\langle M\rangle_\cdot)\\ f(\cdot) & \mapsto & f(\cdot)-\displaystyle\int_\cdot^\infty \frac{f(u)}{\langle M\rangle_u}d\langle M\rangle_u.\end{array}$$

By combining the classical Fubini theorem and (5.1), we obtain that $\beta_\infty^{M,1}$ is an isometric isomorphism with inverse

$$\begin{array}{rcl}\alpha_\infty^{M,1}\ :\ L^2([0,\infty),d\langle M\rangle_\cdot) & \to & L^2([0,\infty),d\langle M\rangle_\cdot)\\ f(\cdot) & \mapsto & f(\cdot)-\displaystyle\frac{1}{\langle M\rangle_\cdot}\int_0^\cdot f(u)d\langle M\rangle_u.\end{array}$$

Hence, the measure-preserving transformation $\mathcal{T}$ is an automorphism with inverse

$$\mathcal{T}_t^{-1}(M)=\int_0^\infty\left(\alpha_\infty^{M,1}1_{[0,t)}\right)(s)dM_s=-\langle M\rangle_t\int_t^\infty\frac{1}{\langle M\rangle_s}dM_s,\ a.s.,\ t\in(0,\infty). \tag{5.3}$$

In particular,

$$\Gamma_\infty\left(\mathcal{T}^{-1}(M)\right)\ =\ \Gamma_\infty(M). \tag{5.4}$$

Remark 5.1. For the (on T dependent) bridge of M in (3.14), it holds by using (5.3), that

$$\lim_{\substack{T\to\infty\\ T>t}}\ \mathcal{B}_t^{(1)}(M)\ =\ \mathcal{T}_t^{-1}(M),\ a.s.,\ t\in(0,\infty).$$

Let $T>0$ be fixed. M has independent increments, hence

$$\Gamma_\infty(M)\ =\ \Gamma_T(M)\ \oplus\ \Gamma_{[T,\infty)}(M). \tag{5.5}$$

On the one hand, from (1.8) and (3.11), it follows that

$$\Gamma_T(M)\ =\ \Gamma_T\left(\mathcal{T}(M)\right)\ \oplus\ \operatorname{span}\{M_T\}. \tag{5.6}$$

On the other hand, by using (5.4) and (5.6) with $\mathcal{T}^{-1}(M)$ instead of M, we have that

$$\Gamma_{[T,\infty)}(M)\ \ =\ \ \Gamma_{[T,\infty)}\left(\mathcal{T}^{-1}(M)\right)\ \oplus\ \operatorname{span}\left\{\mathcal{T}_T^{-1}(M)\right\}. \tag{5.7}$$

By combining (5.5), (5.6) and (5.7), and iterating this decomposing procedure, we obtain that $\{\mathcal{T}_T^n(M)\}_{n\in\mathbb{Z}}$ is an orthogonal system in $\Gamma_\infty(M)$. From the following two-sided Fourier-Laguerre series expansion, it follows that this system is complete:

Theorem 5.2. *Let $Z:=\int_0^\infty f(s)dM_s\in\Gamma_\infty(M)$. Then*

$$Z\ =\ L^2(\mathbb{P})\text{-}\sum_{n\in\mathbb{Z}}\left(\int_0^\infty f(s)\mathcal{L}_n^{M,T}(s)d\langle M\rangle_s\right)\cdot\varepsilon_n^{M,T},$$

where the sequence $\left\{\varepsilon_n^{M,T}\right\}_{n\in\mathbb{Z}}:=\left\{\frac{\mathcal{T}_T^n(M)}{\sqrt{\langle M\rangle_T}}\right\}_{n\in\mathbb{Z}}$ is i.i.d. with $\varepsilon_0^{M,T}\sim\mathcal{N}(0,1)$, and

$$\mathcal{L}_n^{M,T}(s)\ :=\begin{cases}\frac{1}{\sqrt{\langle M\rangle_T}}L_n\left(\ln\frac{\langle M\rangle_T}{\langle M\rangle_s}\right)1_{(0,T)}(s), & n\in\mathbb{N}_0;\\ \\ -\frac{\sqrt{\langle M\rangle_T}}{\langle M\rangle_s}L_{-n-1}\left(\ln\frac{\langle M\rangle_s}{\langle M\rangle_T}\right)1_{(T,\infty)}(s), & n\in-\mathbb{N},\end{cases}$$

with $L_n(x):=\sum_{k=0}^n\binom{n}{k}\frac{1}{k!}(-x)^k$ denoting the n-th Laguerre polynomial, $n\in\mathbb{N}_0$.

Proof. First, we show that $\left\{\varepsilon_n^{M,T}\right\}_{n\in\mathbb{N}_0}$ is a Hilbert basis of $\Gamma_T(M)$. By iterating $\mathcal{T}$ and using the stochastic Fubini theorem, we have that

$$\begin{aligned}
\mathcal{T}_T^n(M) &= M_T + \sum_{k=1}^n (-1)^k \binom{n}{k} \int_0^T \int_0^{t_{k-1}} \cdots \int_0^{t_1} M_s \frac{d\langle M\rangle_s}{\langle M\rangle_s} \frac{d\langle M\rangle_{t_1}}{\langle M\rangle_{t_1}} \cdots \frac{d\langle M\rangle_{t_{k-1}}}{\langle M\rangle_{t_{k-1}}} \\
&= M_T + \sum_{k=1}^n \frac{(-1)^k}{k!} \binom{n}{k} \int_0^T \ln^k \left(\frac{\langle M\rangle_T}{\langle M\rangle_s}\right) dM_s \\
&= \int_0^T L_n \left(\ln \frac{\langle M\rangle_T}{\langle M\rangle_s}\right) dM_s \\
&= \sqrt{\langle M\rangle_T} \int_0^T \mathcal{L}_n^{M,T}(s) dM_s, a.s., \ n \in \mathbb{N}_0. \qquad (5.8)
\end{aligned}$$

It is well-known, that

$$\{L_n(x), x \in [0,\infty)\}_{n\in\mathbb{N}_0} \text{ is a Hilbert basis of } L^2([0,\infty), e^{-x}dx). \qquad (5.9)$$

By substitution, we obtain that $\left\{\mathcal{L}_n^{M,T}(s), s \in (0,T)\right\}_{n\in\mathbb{N}_0}$ is a Hilbert basis of $L_T^2(M)$. The claim follows from (5.8) and the Wiener isometry between $L_T^2(M)$ and $\Gamma_T(M)$. Second, we show that $\{\varepsilon_n^{M,T}\}_{n\in-\mathbb{N}}$ is a Hilbert basis of $\Gamma_{[T,\infty)}(M)$. By using (5.3), partial integration and (5.2), we have that

$$\mathcal{T}_t^{-1}(M) \ = \ M_t \ - \ \langle M\rangle_t \cdot \mathcal{J}_t^1(M), \ a.s., \ t \in (0,\infty), \qquad (5.10)$$

where

$$\mathcal{J}_t^k(M) := \int_t^\infty \int_{t_{k-1}}^\infty \cdots \int_{t_1}^\infty \frac{M_s}{\langle M\rangle_s} \frac{d\langle M\rangle_s}{\langle M\rangle_s} \frac{d\langle M\rangle_{t_1}}{\langle M\rangle_{t_1}} \cdots \frac{d\langle M\rangle_{t_{k-1}}}{\langle M\rangle_{t_{k-1}}}, \ k \in \mathbb{N}.$$

By combining (5.3) and (5.10), using the stochastic Fubini theorem, and iterating, we obtain that

$$\begin{aligned}
\mathcal{J}_T^k(M) &= \int_T^\infty \int_{t_{k-1}}^\infty \cdots \int_{t_2}^\infty \left(\frac{M_{t_1}}{\langle M\rangle_{t_1}} + \int_{t_1}^\infty \frac{dM_s}{\langle M\rangle_s}\right) \frac{d\langle M\rangle_{t_1}}{\langle M\rangle_{t_1}} \cdots \frac{d\langle M\rangle_{t_{k-1}}}{\langle M\rangle_{t_{k-1}}} \\
&= \mathcal{J}_T^{k-1}(M) \ + \ \int_T^\infty \frac{1}{(k-1)!} \ln^{k-1}\left(\frac{\langle M\rangle_s}{\langle M\rangle_T}\right) \frac{dM_s}{\langle M\rangle_s} \ = \ \cdots \\
&= \mathcal{J}_T^1(M) \ + \ \sum_{j=1}^{k-1} \int_T^\infty \frac{1}{j!} \ln^j\left(\frac{\langle M\rangle_s}{\langle M\rangle_T}\right) \frac{dM_s}{\langle M\rangle_s}, \ a.s., \ k \geq 2. \qquad (5.11)
\end{aligned}$$

From the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, $1 < k \leq n$, it follows that

$$\sum_{k=j+1}^n (-1)^{k+j-1} \binom{n}{k} \ = \ \binom{n-1}{j}, \ 0 \leq j < n. \qquad (5.12)$$

By using (5.10) and iterating, then using (5.11), again (5.10) and (5.12), we obtain that

$$
\begin{aligned}
\mathcal{T}_T^{-n}(M) &= M_T + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \langle M \rangle_T \cdot \mathcal{J}_T^k(M) \\
&= M_T + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \langle M \rangle_T \left(\mathcal{J}_T^1(M) + \sum_{j=1}^{k-1} \int_T^\infty \frac{1}{j!} \ln^j \left(\frac{\langle M \rangle_s}{\langle M \rangle_T} \right) \frac{dM_s}{\langle M \rangle_s} \right) \\
&= \sum_{k=1}^{n} (-1)^k \binom{n}{k} \langle M \rangle_T \left(\int_T^\infty \frac{1}{\langle M \rangle_s} dM_s \; + \; \sum_{j=1}^{k-1} \int_T^\infty \frac{1}{j!} \ln^j \left(\frac{\langle M \rangle_s}{\langle M \rangle_T} \right) \frac{dM_s}{\langle M \rangle_s} \right) \\
&= \langle M \rangle_T \int_T^\infty \left(-1 + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \sum_{j=1}^{k-1} \frac{1}{j!} \ln^j \left(\frac{\langle M \rangle_s}{\langle M \rangle_T} \right) \right) \frac{dM_s}{\langle M \rangle_s} \\
&= -\langle M \rangle_T \int_T^\infty \left(1 + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} (-1)^{k-1+j} \binom{n}{k} \frac{1}{j!} \ln^j \left(\frac{\langle M \rangle_s}{\langle M \rangle_T} \right) (-1)^j \right) \frac{dM_s}{\langle M \rangle_s} \\
&= -\langle M \rangle_T \int_T^\infty L_{n-1} \left(\ln \frac{\langle M \rangle_s}{\langle M \rangle_T} \right) \frac{dM_s}{\langle M \rangle_s} \\
&= \sqrt{\langle M \rangle_T} \int_T^\infty \mathcal{L}_{-n}^{M,T}(s) dM_s, \; a.s., \; n \in \mathbb{N}. \qquad (5.13)
\end{aligned}
$$

From (5.9), we obtain by substitution, that $\left\{\mathcal{L}_n^{M,T}(s), s \in (T,\infty)\right\}_{n\in -\mathbb{N}}$ is a Hilbert basis of $L^2([T,\infty), d\langle M\rangle.)$. By combining (5.13) and the Wiener isometry between $L^2([T,\infty), d\langle M\rangle.)$ and $\Gamma_{[T,\infty)}(M)$, we obtain the claim. Third, by combining (5.5) with these results, we obtain that $\left\{\varepsilon_n^{M,T}\right\}_{n\in\mathbb{Z}}$ is a Hilbert basis of $\Gamma_\infty(M)$. A Fourier expansion yields

$$
\begin{aligned}
Z &= L^2(\mathbb{P})\text{-}\sum_{n\in\mathbb{Z}} \mathrm{Cov}_{\mathbb{P}}\left(Z, \varepsilon_n^{M,T}\right) \cdot \varepsilon_n^{M,T} \\
&= L^2(\mathbb{P})\text{-}\sum_{n\in\mathbb{Z}} \left(\int_0^\infty f(s) \mathcal{L}_n^{M,T}(s) d\langle M \rangle_s \right) \cdot \varepsilon_n^{M,T}. \quad \square
\end{aligned}
$$

As a special case, we obtain the following one-sided Fourier-Laguerre series expansion. For $M = W$, it was shown in [8].

Corollary 5.3. *We have that*

$$
M_t = L^2(\mathbb{P})\text{-}\sum_{n\in\mathbb{N}_0} \sqrt{\langle M \rangle_T} \cdot \zeta_n \left(\ln \frac{\langle M \rangle_T}{\langle M \rangle_t} \right) \cdot \varepsilon_n^{M,T}, \; t \in (0,T],
$$

where $\zeta_n(y) := \int_y^\infty L_n(x) e^{-x} dx$, $y \in [0,\infty)$.

Proof. Set $f := 1_{[0,t)}$ in Theorem 5.2. $\square$

Remark 5.4. Clearly, we have that

$$
\mathcal{F}_T^M = \bigvee_{n\in\mathbb{N}_0} \sigma\left(\mathcal{T}_T^n(M)\right)
$$

and

$$
\mathcal{F}_{[T,\infty)}^M := \sigma(M_t - M_s \,|\, s,t \in [T,\infty)) = \bigvee_{n\in -\mathbb{N}} \sigma\left(\mathcal{T}_T^n(M)\right).
$$

Furthermore,

$$\mathcal{F} \;=\; \bigvee_{n\in\mathbb{Z}} \sigma\left(\mathcal{T}_T^n(M)\right).$$

Recall that an automorphism $\mathcal{T}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a *Kolmogorov automorphism*, if there exists a σ-algebra $\mathcal{A} \subseteq \mathcal{F}$, such that $\mathcal{T}^{-1}\mathcal{A} \subseteq \mathcal{A}$, $\vee_{n\in\mathbb{Z}}\mathcal{T}^n\mathcal{A} = \mathcal{F}$ and $\cap_{n\in\mathbb{N}_0}\mathcal{T}^{-n}\mathcal{A} = \{\Omega, \emptyset\}$. It is straightforward to see, that $\mathcal{T}$ and $\mathcal{T}^{-1}$ are Kolmogorov automorphisms with $\mathcal{A} = \mathcal{F}^M_{[T,\infty)}$ and $\mathcal{A} = \mathcal{F}^M_T$, respectively. Hence, $\mathcal{T}$ and $\mathcal{T}^{-1}$ are strongly mixing and hence ergodic (see [13], Proposition 5.11 and Proposition 5.9).

Acknowledgements. Thanks are due to my supervisor Esko Valkeila for motivating the writing of this article, for good questions and helpful comments. I thank Giovanni Peccati for a particularly motivating discussion. I am indebted to the Finnish Graduate School in Stochastics (FGSS) for financial support.

References

[1] Alòs, E., Mazet, O., Nualart, D., Stochastic Calculus with respect to Gaussian Processes. Annals of Probability 29(2), 766-801, 2001.

[2] Baudoin, F., Coutin, L., Volterra Bridges and non-canonical Representations. To appear in Markov Processes and Related Fields, 2006.

[3] Baudoin, F., Nualart, D., Equivalence of Volterra processes. Stochastic Processes and their Applications 107, 327-350, 2003.

[4] Gasbarra, D., Sottinen, T., Valkeila, E., Gaussian Bridges. To appear in Proceedings of the Abel Symposium 2005.

[5] Hardy, G.H., Littlewood, J.E., Pólya, G., Inequalities. Cambridge University Press, 1934.

[6] Hida, T., Hitsuda, M., Gaussian Processes. American Mathematical Society. Translations of Mathematical Monographs, 1976.

[7] Huang, S.T., Cambanis, S., Stochastic and Multiple Wiener Integrals for Gaussian Processes. Annals of Probability 6(4), 585-614, 1978.

[8] Jeulin, T., Yor, M., Filtration des ponts browniens et équations différentielles stochastiques linéaires. Séminaire de probabilités de Strasbourg 24, 227-265, 1990.

[9] Jost, C., Transformation formulas for fractional Brownian motion. Stochastic Processes and their Applications 116, 1341-1357, 2006.

[10] Karatzas, I., Shreve, S.E., Brownian Motion and Stochastic Calculus, Second Edition. Springer New York, 1991.

[11] Mikkola, K., Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations. Research report A 452, Helsinki University of Technology, 2002.

[12] Norros, I., Valkeila, E., Virtamo, J., An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. Bernoulli 5(4), 571-587, 1999.

[13] Peccati, G., Explicit formulae for time-space Brownian chaos. Bernoulli 9(1), 25-48, 2003.

[14] Petersen, K., Ergodic theory. Cambridge University Press, 1983.

[15] Pipiras, V., Taqqu, M.S., Are classes of deterministic integrands for fractional Brownian motion on an interval complete? Bernoulli 7(6), 873-897, 2001.

[16] Yor, M., Some Aspects of Brownian motion, Part 1: Some Special Functionals. Lecture Notes in Mathematics, Birkhäuser Verlag, 1992.

Measure-preserving transformations of Volterra Gaussian processes and related bridges

Céline Jost*

Abstract

We consider Volterra Gaussian processes on $[0, T]$, where $T > 0$ is a fixed time horizon. These are processes of type $X_t = \int_0^t z_X(t, s)dW_s,\ t \in [0, T]$, where z_X is a square-integrable kernel, and W is a standard Brownian motion. An example is fractional Brownian motion. By using classical techniques from operator theory, we derive measure-preserving transformations of X, and their inherently related bridges of X. As a closely connected result, we obtain a Fourier-Laguerre series expansion for the first Wiener chaos of a Gaussian martingale over $[0, \infty)$.

MSC: 60G15; 37A05; 42C10; 60G44

Keywords: Volterra Gaussian process; Measure-preserving transformation; Bridge; Series expansion; Gaussian martingale; Fractional Brownian motion

1 Introduction

Fix $T > 0$, and let $(X_t)_{t\in[0,T]}$ be a continuous Volterra Gaussian process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Hence, there exist a kernel $z_X \in L^2\left([0, T]^2\right)$ which is *Volterra*, i.e. $z_X(t, s) = 0,\ s \geq t$, and a standard Brownian motion $(W_t)_{t\in[0,T]}$, such that

$$X_t \ = \ \int_0^t z_X(t, s)dW_s,\ a.s.,\ t \in [0, T]. \tag{1.1}$$

Clearly, $X_0 = 0$, a.s., X is centered, and

$$R^X(s, t) \ = \ \int_0^{s\wedge t} z_X(t, u)z_X(s, u)du,\ s, t \in [0, T], \tag{1.2}$$

where $R^Y(s, t) := \mathrm{Cov}_{\mathbb{P}}\left(Y_s, Y_t\right),\ s, t \in [0, T]$, for a general Gaussian process Y. We assume that z_X is *non-degenerate*, meaning that the family $\{z_X(t, \cdot)\,|\,t \in (0, T]\}$ is linearly independent and generates a dense subspace of $L^2([0, T])$. On the one hand, it follows from this, that R^X is *non-degenerate on* $(0, T]$, meaning that the matrices $\left\{R^X(t_i, t_j)\right\}_{i,j=1,\ldots,n}$, where $0 < t_1 < \ldots < t_n \leq T$ and $n \in \mathbb{N}$, are positive definite. On the other hand, it follows that

$$\Gamma_t(X) \ = \ \Gamma_t(W),\ t \in [0, T], \tag{1.3}$$

i.e.

$$\mathbb{F}_T^X \ = \ \mathbb{F}_T^W. \tag{1.4}$$

*Department of Mathematics and Statistics, P.O. Box 68 (Gustaf Hällströmin katu 2b), 00014 University of Helsinki, Finland. E-mail: celine.jost@helsinki.fi

Here, $\Gamma_T(Y) := \overline{\mathrm{span}\{Y_t \,|\, t\in[0,T]\}} \subseteq L^2(\mathbb{P})$ and $\mathbb{F}_T^Y := \left(\mathcal{F}_t^Y\right)_{t\in[0,T]}$ denote the first Wiener chaos and the completed natural filtration of the Gaussian process Y with $Y_0 = 0$, a.s., over $[0,T]$, respectively. Moreover, we assume that

$$z_X(T,\cdot) \;\neq\; 0,\; \lambda\text{-}a.e., \tag{1.5}$$

where λ denotes Lebesgue measure, and

$$z_X(\cdot,s) \text{ has bounded variation on } [u,T] \text{ for all } u\in(s,T). \tag{1.6}$$

Processes of this type are a natural generalization of the nowadays, in connection with finance and telecommunications, extensively studied *fractional Brownian motion with Hurst index* $H\in(0,1)$. Fractional Brownian motion, denoted by B^H, is the unique (in the sense of equality of finite dimensional distributions) centered, H-self-similar Gaussian process with stationary increments. For $H=\frac{1}{2}$, fractional Brownian motion is standard Brownian motion. We have that

$$z_{B^H}(t,s) = \frac{C(H)}{\Gamma\left(H+\frac{1}{2}\right)}(t-s)^{H-\frac{1}{2}}{}_2F_1\left(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, \frac{s-t}{s}\right)1_{[0,t)}(s),$$

$s,t\in[0,T]$, where Γ denotes the gamma function, $C(H) := \left(\frac{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-H\right)}{\Gamma(2-2H)}\right)^{\frac{1}{2}}$ and ${}_2F_1$ is Gauss' hypergeometric function. Moreover,

$$R^{B^H}(s,t) \;=\; \frac{1}{2}\left(s^{2H}+t^{2H}-|t-s|^{2H}\right),\; s,t\in[0,T].$$

For a Gaussian process $(Y_t)_{t\in[0,T]}$ with $Y_0=0$, a.s., recall that the process $\left(Y_t^T\right)_{t\in[0,T]}$ is a *bridge of* Y *(from* 0 *to* 0 *on* $[0,T]$*)* if

$$\mathrm{Law}_{\mathbb{P}}\left(Y^T\right) \;=\; \mathrm{Law}_{\mathbb{P}}\left(Y \,|\, Y_T = 0\right).$$

Clearly, Y^T is a Gaussian process with

$$Y_0^T \;=\; Y_T^T \;=\; 0,\; a.s.,$$

i.e. it is tied to 0 at both ends. Therefore, the bridge is an intuitive concept for interpolation. It is well-known, that the continuous process

$$\widehat{X}_t^T \;:=\; X_t \;-\; \frac{R^X(t,T)}{R^X(T,T)}X_T,\; t\in[0,T], \tag{1.7}$$

is a bridge of X, that satisfies

$$\Gamma_T\left(\widehat{X}^T\right) \;\oplus\; \mathrm{span}\{X_T\} \;=\; \Gamma_T(X). \tag{1.8}$$

(see [4], Proposition 4). $\widehat{X}^T$ is called the *anticipative* bridge of X, since

$$\mathbb{F}_T^{\widehat{X}^T} \vee \sigma(X_T) \;=\; \mathbb{F}_T^X \vee \sigma(X_T), \tag{1.9}$$

i.e. not the natural, but the initially $\sigma(X_T)$-enlarged filtrations of X and $\widehat{X}^T$ coincide. Note that, more generally, $\widehat{Y}^T$ is a bridge of Y for every Gaussian process $(Y_t)_{t\in[0,T]}$ with $Y_0=0$, a.s., and $R^Y(T,T)>0$.

We implicitly assume, that $(\Omega,\mathcal{F},\mathbb{P}) = \left(C([0,T]), B([0,T]), \mathbb{P}^X\right)$, where $C([0,T]) := \{\omega : [0,T]\to\mathbb{R} \,|\, \omega \text{ is continuous}\}$, $\mathcal{B}([0,T])$ is the Borel-σ-algebra on $C([0,T])$

equipped with the norm of uniform convergence, and $\mathbb{P}^X$ is the probability measure, with respect to which the *coordinate process* $X_t(\omega) := \omega(t)$, $\omega \in C([0,T])$, $t \in [0,T]$, is a centered Gaussian process with covariance function R^X. Then $(\Omega, \mathcal{F}, \mathbb{P})$ is called the *coordinate space of* X. Note that $\mathcal{F} = \mathcal{F}_T^X$. The measurable map

$$\begin{array}{rcl} \mathcal{T} \ : \ (\Omega, \mathcal{F}, \mathbb{P}) & \to & (\Omega, \mathcal{F}, \mathbb{P}) \\ X(\omega) & \mapsto & \mathcal{T}(X(\omega)) \end{array}$$

is a *measure-preserving transformation on* $(\Omega, \mathcal{F}, \mathbb{P})$, if it is a morphism, i.e. if $\mathbb{P}^{\mathcal{T}} = \mathbb{P}$, or equivalently, if $\mathcal{T}(X) \stackrel{d}{=} X$, where $\stackrel{d}{=}$ denotes equality of finite dimensional distributions. The measurable map

$$\begin{array}{rcl} \mathcal{B} \ : \ (\Omega, \mathcal{F}, \mathbb{P}) & \to & (\Omega, \mathcal{F}, \mathbb{P}) \\ X(\omega) & \mapsto & \mathcal{B}(X(\omega)) \end{array}$$

is a *bridge transformation on* $(\Omega, \mathcal{F}, \mathbb{P})$, if $\mathcal{B}(X) \stackrel{d}{=} X^T$. Note that, in particular, $\mathcal{T}, \mathcal{B} : C([0,T]) \to C([0,T])$ are linear maps.

In this work, we derive measure-preserving transformations, such that

$$\Gamma_T(\mathcal{T}(X)) \ = \ \Gamma_T\left(\widehat{X}^T\right).$$

As a naturally related problem, we derive bridge transformations, that satisfy

$$\Gamma_T(\mathcal{B}(X)) \ = \ \Gamma_T(X).$$

We define $\mathcal{T}(X)$ and $\mathcal{B}(X)$ as Wiener integral processes with respect to X, where the integrand kernels are based on suitable isometric operators on the Wiener integrand space of X. This method follows ideas of Jeulin and Yor (see [8]) and Peccati (see [13]), where the case $X = W$ is considered. If X is a martingale, then due to independence of increments, the operators and the resulting transformations are simple. However, if X is not a martingale, then operators and obtained transformations are technically more involved. In fact, the operators are composed by the corresponding operators for the prediction martingale of X_T, and the isometry between the Wiener integrand spaces of X and the prediction martingale, respectively. From these results, we obtain in particular an alternative, purely operator theoretic derivation for the *dynamic* bridge of X, as introduced in [4]. Furthermore, by using the same operator theoretic methods and as a closely connected result, we obtain a Fourier-Laguerre series expansion for the first Wiener chaos of a Gaussian martingale over $[0, \infty)$.

Remark 1.1. 1. Representation (1.1) is unique in the following sense: assume that there exist another non-degenerate Volterra kernel z'_X and another standard Brownian motion W', such that $X_t = \int_0^t z'_X(t,s)dW'_s$, a.s., $t \in [0,T]$. Then, it follows from (1.3), that $\Gamma_t(W) = \Gamma_t(W')$, $t \in [0,T]$, i.e. W and W' are indistinguishable. In particular,

$$0 \ = \ \mathrm{E}_{\mathbb{P}}(X_t - X_t)^2 \ = \ \int_0^t \left(z_X(t,s) - z'_X(t,s)\right)^2 ds, \ t \in [0,T],$$

which implies that $z_X(t,\cdot) = z'_X(t,\cdot)$, λ-a.e., $t \in [0,T]$.
2. Although the covariance function R^X is continuous on $[0,T]^2$, it is generally not true, that $z_X(\cdot, s)$ is continuous for all $s \in [0,T]$. For example, for the fractional Brownian motion with $H < \frac{1}{2}$, the function $z_{B^H}(\cdot, s)$ is discontinuous in $t = s$ for every $s \in (0,T)$.
3. (1.5) is not a necessary condition for the non-degeneracy of a Volterra kernel, as the example $z_X(t,s) := 1_{[\frac{t}{2},t)}(s)$, $s,t \in [0,T]$, shows.

The article is organized as follows. Section 2 is dedicated to Wiener integration with respect to X and X^T, respectively. In Section 3, we derive the measure-preserving and bridge transformations in the case, when X is a martingale. These results are generalized to Volterra Gaussian processes in Section 4. In Section 5, we derive the series expansion.

2 Wiener integrals

First, we review the construction of abstract and time domain Wiener integrals with respect to the Volterra Gaussian process X. Second, we define abstract Wiener integrals with respect to the bridge X^T, and explain their relation to Wiener integrals with respect to X.

2.1 Wiener integrals with respect to X

Let $\mathcal{E}_T := \operatorname{span}\{1_{[0,t)} \,|\, t \in (0,T]\}$ be the space of elementary functions on $[0,T]$. The Hilbert space of *abstract Wiener integrands of X on* $[0,T]$, denoted by $\Lambda_T(X)$, is defined as the completion of $\mathcal{E}_T$ with respect to the scalar product

$$(1_{[0,s)}, 1_{[0,t)})_X \;:=\; R^X(s,t),\; s,t \in (0,T].$$

Hence, $f \in \Lambda_T(X)$ is an equivalence class of Cauchy sequences $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{E}_T^{\mathbb{N}}$, where $\{f_n\}_{n\in\mathbb{N}} \sim \{g_n\}_{n\in\mathbb{N}} :\Leftrightarrow (f_n - g_n, f_n - g_n)_X \to 0,\; n \to \infty$. The scalar product on $\Lambda_T(X)$ is given by

$$(f,g)_{\Lambda_T(X)} := \lim_{n\to\infty} (f_n, g_n)_X,\; f,g \in \Lambda_T(X),\; \{f_n\}_{n\in\mathbb{N}} \in f,\; \{g_n\}_{n\in\mathbb{N}} \in g,$$

and induces the norm $|\cdot|_{\Lambda_T(X)}$. The isometric isomorphism defined by

$$\begin{array}{rcl} I_T^X \;:\; \Lambda_T(X) & \to & \Gamma_T(X) \\ 1_{[0,t)} & \mapsto & X_t,\; t \in (0,T], \end{array}$$

is called the *abstract Wiener integral with respect to X on* $[0,T]$. By definition, $\int_0^T f(s)dX_s := I_T^X(f)$ is centered, Gaussian and $\mathrm{E}_{\mathbb{P}}\left(\int_0^T f(s)dX_s\right)^2 = |f|^2_{\Lambda_T(X)}$ for all $f \in \Lambda_T(X)$.

By combining (1.1), the so-called *time domain representation of* X, with the standard Wiener integral, we obtain a subspace of $\Lambda_T(X)$, whose elements can be identified with functions. For this purpose, define a linear operator

$$\left(\mathbf{K}^X f\right)(s) \;:=\; f(s)z_X(T,s) \;+\; \int_s^T \left(f(u) - f(s)\right) z_X(du,s),\; s \in (0,T).$$

By (1.6), the integral is well-defined for a suitably large class of functions. Clearly, $\mathbf{K}^X$ extends the linear isometry

$$\begin{array}{rcl} (\mathcal{E}_T, (\cdot\,,\cdot)_X) & \to & L^2([0,T]) \\ 1_{[0,t)} & \mapsto & z_X(t,\cdot),\; t \in (0,T]. \end{array}$$

We define the space of *time domain Wiener integrands of X on* $[0,T]$ by

$$\check{\Lambda}_T(X) \;:=\; \left\{ f : [0,T] \to \mathbb{R} \;\middle|\; \mathbf{K}^X f \text{ is well-defined and } \int_0^T \left(\mathbf{K}^X f\right)^2(s)ds < \infty \right\}$$

with scalar product

$$(f,g)_{\check{\Lambda}_T(X)} \;:=\; \left(\mathbf{K}^X f, \mathbf{K}^X g\right)_{L^2([0,T])},\; f,g \in \check{\Lambda}_T(X).$$

By (1.2), we have that

$$(1_{[0,s)}, 1_{[0,t)})_{\check{\Lambda}_T(X)} \;=\; (1_{[0,s)}, 1_{[0,t)})_{\Lambda_T(X)},\;\; s,t \in [0,T].$$

Let $f \in \Lambda_T(X)$. If there exists $\check{f} \in \check{\Lambda}_T(X)$, such that

$$\left(f, 1_{[0,t)}\right)_{\Lambda_T(X)} \;=\; \left(\check{f}, 1_{[0,t)}\right)_{\check{\Lambda}_T(X)} \text{ for all } t \in (0,T],$$

then we identify f and $\check{f}$. $\mathcal{E}_T$ is dense in $\check{\Lambda}_T(X)$, hence $\check{\Lambda}_T(X) \subseteq \Lambda_T(X)$. However, in general, $\check{\Lambda}_T(X) \neq \Lambda_T(X)$. The restriction $I_T^X|_{\check{\Lambda}_T(X)}$ is called the *time domain Wiener integral with respect to* X *on* $[0,T]$.

Remark 2.1. Due to the non-degeneracy of z_X, we have that $|f|_{\check{\Lambda}_T(X)} = 0$ if and only if $f = 0$, λ-a.e.

Remark 2.2. From (1.3), it follows that (1.1) has a reciprocal in the sense that there exists a Volterra kernel z_X^*, with $z_X^*(t,\cdot) \in \Lambda_T(X)$, $t \in (0,T]$, such that

$$W_t \;=\; \int_0^t z_X^*(t,s)dX_s,\;\; a.s.,\;\; t \in [0,T].$$

For $t \in (0,T]$, this abstract Wiener integral is a time domain Wiener integral if and only if there exists $f \in \check{\Lambda}_T(X)$, such that $\mathbf{K}^X f = 1_{[0,t)}$, λ-a.e.

Remark 2.3. If R^X is of bounded variation, then it determines a finite signed measure on $[0,T]^2$. Hence, one can define an alternative space of Wiener integrands of X on $[0,T]$ by

$$|\Lambda_T(X)| \;:=\; \left\{ f : [0,T] \to \mathbb{R} \;\middle|\; \int_0^T \int_0^T |f(s)||f(t)|\left|R^X\right|(dt,ds) < \infty \right\},$$

where $\left|R^X\right|$ denotes the measure of total variation of R^X, and the integral is a Lebesgue-Stieltjes integral. The corresponding scalar product is given by

$$(f,g)_{|\Lambda_T(X)|} \;:=\; \int_0^T \int_0^T f(t)g(s)R^X(dt,ds),\;\; f,g \in |\Lambda_T(X)|.$$

For details, see [7].

Example 2.4. For the fractional Brownian motion holds that (see [1], p.797-800)

$$\left(\mathbf{K}^{B^H} f\right)(s) \;=\; C(H)s^{\frac{1}{2}-H}\left(\mathcal{I}_{T-}^{H-\frac{1}{2}}\, \cdot^{H-\frac{1}{2}} f\right)(s),\;\; s \in (0,T),$$

where $\mathcal{I}_{T-}^{H-\frac{1}{2}}$ denotes the right-sided Riemann-Liouville fractional integral operator of order $H - \frac{1}{2}$ over $[0,T]$. Furthermore,

$$z_{B^H}^*(t,s) \;=\; \frac{C(H)^{-1}}{\Gamma\left(\frac{3}{2}-H\right)}(t-s)^{\frac{1}{2}-H}{}_2F_1\left(\frac{1}{2}-H, \frac{1}{2}-H, \frac{3}{2}-H, \frac{s-t}{s}\right)1_{[0,t)}(s),$$

$s,t \in [0,T]$. We have that $\check{\Lambda}_T\left(B^H\right) = \Lambda_T\left(B^H\right)$ if and only if $H \leq \frac{1}{2}$. Clearly, R^{B^H} is of bounded variation if and only if $H \geq \frac{1}{2}$. Then, $z_{B^H}^*(t,\cdot) \in \left|\Lambda_T\left(B^H\right)\right| \subseteq \check{\Lambda}_T\left(B^H\right)$, $t \in (0,T]$. See [15] for full proofs.

2.2 Wiener integrals with respect to X^T

From (1.7), it follows that X^T is centered and

$$R^{X^T}(s,t) \;=\; R^X(s,t) \;-\; \frac{R^X(s,T)R^X(t,T)}{R^X(T,T)},\;\; s,t\in[0,T]. \tag{2.1}$$

Hence, R^{X^T} is non-degenerate on $(0,T)$, and $R^{X^T}(T,T)=0$. In order to define a scalar product based on R^{X^T}, let

$$\mathcal{C}_T \;:=\; \{f:[0,T]\to\mathbb{R}\mid f \text{ is constant } \lambda\text{-}a.e.\}$$

and consider the quotient space $\mathcal{E}_T/\mathcal{C}_T$. Clearly, $1_{[0,T)}\sim 0 \bmod \mathcal{C}_T$. The Hilbert space of *abstract Wiener integrands of* X^T *on* $[0,T]$, denoted by $\Lambda_T\big(X^T\big)$, is defined as the completion of $\mathcal{E}_T/\mathcal{C}_T$ with respect to the scalar product

$$\big(\overline{1_{[0,s)}},\overline{1_{[0,t)}}\big)_{X^T} \;:=\; R^{X^T}(s,t),\;\; s,t\in(0,T].$$

Thus, $\overline{f}\in\Lambda_T\big(X^T\big)$ is an equivalence class of Cauchy sequences $\big\{\overline{f_n}\big\}_{n\in\mathbb{N}}\in(\mathcal{E}_T/\mathcal{C}_T)^{\mathbb{N}}$, where $\big\{\overline{f_n}\big\}_{n\in\mathbb{N}}\sim\{\overline{g_n}\}_{n\in\mathbb{N}} :\Leftrightarrow \big(\overline{f_n-g_n},\overline{f_n-g_n}\big)_{X^T}\to 0,\, n\to\infty$. The scalar product on $\Lambda_T\big(X^T\big)$ is given by

$$\big(\overline{f},\overline{g}\big)_{\Lambda_T(X^T)} := \lim_{n\to\infty}\big(\overline{f_n},\overline{g_n}\big)_{X^T},\;\; \overline{f},\overline{g}\in\Lambda_T\big(X^T\big),\;\; \big\{\overline{f_n}\big\}_{n\in\mathbb{N}}\in\overline{f},\;\; \{\overline{g_n}\}_{n\in\mathbb{N}}\in\overline{g},$$

with induced norm $|\cdot|_{\Lambda_T(X^T)}$. The isometric isomorphism defined by

$$\begin{aligned} I_T^{X^T} \;:\; \Lambda_T\big(X^T\big) \;&\to\; \Gamma_T\big(X^T\big)\\ \overline{1_{[0,t)}} \;&\mapsto\; X_t^T,\;\; t\in(0,T], \end{aligned}$$

is called the *abstract Wiener integral with respect to* X^T *on* $[0,T]$. We have that $\int_0^T\overline{f}(s)dX_s^T := I_T^{X^T}\big(\overline{f}\big)$ is centered, Gaussian and $\mathrm{E}_{\mathbb{P}}\Big(\int_0^T\overline{f}(s)dX_s^T\Big)=\big|\overline{f}\big|^2_{\Lambda_T(X^T)}$ for all $\overline{f}\in\Lambda_T\big(X^T\big)$.

Let

$$\begin{aligned} \eta^X \;:\; \Lambda_T(X) \;&\to\; \Lambda_T(X)\\ f(\cdot) \;&\mapsto\; f(\cdot) \;-\; \frac{\big(f,1_{[0,T)}\big)_{\Lambda_T(X)}}{R^X(T,T)}. \end{aligned}$$

η^X is the linear orthoprojection from $\Lambda_T(X)$ onto the closed subspace

$$\Lambda_{T,0}(X) \;:=\; \Big\{f\in\Lambda_T(X)\;\Big|\;\big(f,1_{[0,T)}\big)_{\Lambda_T(X)}=0\Big\}$$

along $\Lambda_{T,0}(X)^\perp=\mathcal{C}_T$. In particular, $\ker\big(\eta^X\big)=\mathcal{C}_T$ and

$$\eta^X(\Lambda_T(X)) \;=\; \Lambda_{T,0}(X). \tag{2.2}$$

Thus, the map

$$\begin{aligned} \overline{\eta^X} \;:\; \Lambda_T(X)/\mathcal{C}_T \;&\to\; \Lambda_{T,0}(X)\\ \overline{f} \;&\mapsto\; \eta^X f \end{aligned}$$

is an isomorphism. From (2.1), we obtain that

$$\big(\overline{1_{[0,s)}},\overline{1_{[0,t)}}\big)_{X^T} \;=\; \big(\eta^X 1_{[0,s)},\eta^X 1_{[0,t)}\big)_X,\;\; s,t\in(0,T].$$

Hence,

$$\Lambda_T\big(X^T\big) \;=\; \Lambda_T(X)/\mathcal{C}_T,$$

i.e.

$$\overline{f} \in \Lambda_T\left(X^T\right) \;\Leftrightarrow\; f \in \Lambda_T(X).$$

From (1.7), it follows that

$$\widehat{X}_t^T \;=\; \int_0^T \left(\eta^X 1_{[0,t)}\right)(s)dX_s,\ a.s.,\ t \in [0,T]. \tag{2.3}$$

Hence,

$$\int_0^T \overline{f}(s)d\widehat{X}_s^T \;=\; \int_0^T \left(\eta^X f\right)(s)dX_s,\ a.s.,\ f \in \Lambda_T(X). \tag{2.4}$$

From (1.8), we see that that (2.3) does not have a reciprocal, i.e. X_t can not be written as an abstract Wiener integral with respect to $\widehat{X}^T$. By using (2.4), we can write (1.8) as follows:

$$\int_0^T f(s)dX_s \;=\; \int_0^T \overline{f}(s)d\widehat{X}_s^T \;+\; \frac{\left(f, 1_{[0,T)}\right)_{\Lambda_T(X)}}{R^X(T,T)} \cdot X_T,\ f \in \Lambda_T(X). \tag{2.5}$$

By combining (2.4) and (2.2), we obtain the following:

Lemma 2.5. *We have that*

$$\Gamma_T\left(\widehat{X}^T\right) \;=\; I_T^X\left(\Lambda_{T,0}(X)\right).$$

3 Gaussian martingales

We consider the special case when X is an $\mathbb{F}_T^X$-martingale (or equivalently, an $\mathbb{F}_T^W$-martingale). This corresponds to a Volterra kernel of type $z_X(t,s) = z_M(s)1_{[0,t)}(s)$, $s,t \in [0,T]$, where $z_M \in L^2([0,T])$ and $z_M \neq 0$, λ-a.e. For convenience, we write $X = M$. Let $\langle M\rangle_\cdot := \int_0^\cdot z_M^2(s)ds$ denote the quadratic variation or variance function of M. Then $\langle M\rangle_\cdot$ is strictly increasing on $[0,T]$. We denote

$$\langle M\rangle_{T,t} \;:=\; \langle M\rangle_T \;-\; \langle M\rangle_t,\ t \in [0,T].$$

We derive two types of measure-preserving and bridge transformations, present some pathwise relations between the transformed processes, and show how the types a connected. For the case $M = W$, large parts of the results have been obtained in [8] and [13], although not in the present form and not so detailed. The generalization from W to M can be obtained by using the fact that $M \overset{d}{=} W_{\langle M\rangle_\cdot}$. However, for convenience, we provide independent and complete proofs.

3.1 Measure-preserving and bridge transformations

Clearly, we have that

$$\Lambda_T(M) \;=\; L_T^2(M) \;:=\; L^2([0,T], d\langle M\rangle_\cdot).$$

Furthermore,

$$\Lambda_{T,0}(M) \;=\; L_{T,0}^2(M) \;:=\; \left\{ f \in L_T^2(M) \;\middle|\; \int_0^T f(s)d\langle M\rangle_s = 0 \right\}$$

and

$$\left(\eta^M f\right)(\cdot) \;=\; f(\cdot) \;-\; \frac{1}{\langle M\rangle_T}\int_0^T f(s)d\langle M\rangle_s,\ f \in L_T^2(M).$$

Lemma 3.1. *1. For $f \in L_T^2(M)$, let*

$$\left(\mathcal{H}^{M,1} f\right)(s) \;:=\; \frac{1}{\langle M\rangle_s} \int_0^s f(u) d\langle M\rangle_u,\;\; s \in (0,T],$$

and

$$\left(\mathcal{H}^{M,2} f\right)(s) \;:=\; \frac{1}{\langle M\rangle_{T,s}} \int_s^T f(u) d\langle M\rangle_u,\;\; s \in [0,T).$$

The Hardy type operators $\mathcal{H}^{M,1}$ and $\mathcal{H}^{M,2}$ are bounded endomorphisms on $L_T^2(M)$ with adjoints

$$\left(\mathcal{H}^{M,1,*} f\right)(s) \;=\; \int_s^T \frac{f(u)}{\langle M\rangle_u} d\langle M\rangle_u,\;\; s \in (0,T],$$

and

$$\left(\mathcal{H}^{M,2,*} f\right)(s) \;=\; \int_0^s \frac{f(u)}{\langle M\rangle_{T,u}} d\langle M\rangle_u,\;\; s \in [0,T),$$

respectively.
2. For $i \in \{1,2\}$, let furthermore

$$\alpha^{M,i} f \;:=\; f \;-\; \mathcal{H}^{M,i} f$$

and

$$\beta^{M,i} f \;:=\; f \;-\; \mathcal{H}^{M,i,*} f.$$

Then $\alpha^{M,i} : L_{T,0}^2(M) \to L_T^2(M)$ and $\beta^{M,i} : L_T^2(M) \to L_{T,0}^2(M)$ are isometric isomorphisms with $\left(\alpha^{M,i}\right)^{-1} = \beta^{M,i}$.
3. For $i \in \{1,2\}$, we have that

$$\alpha^{M,i} \eta^M f \;=\; \alpha^{M,i} f,\;\; f \in\; L_T^2(M), \tag{3.1}$$

and

$$\eta^M \beta^{M,i} f \;=\; \beta^{M,i} f,\;\; f \in\; L_T^2(M). \tag{3.2}$$

Also,

$$\beta^{M,i} \alpha^{M,i} f \;=\; \eta^M f,\;\; f \in L_T^2(M). \tag{3.3}$$

Proof. 1. From Hardy's inequality (see [5], Theorem 327, p. 240), it follows that

$$\int_0^1 \frac{1}{x^2} \left(\int_0^x g(z) dz\right)^2 dx \leq\; 4 \int_0^1 g^2(x) dx,\;\; g \in L^2([0,1]). \tag{3.4}$$

By using (3.4) with $g(z) := f\left(y^{-1}(z)\right)$ for $i = 1$, and $g(z) := f\left(y^{-1}(1-z)\right)$ for $i = 2$, where $y(u) := \frac{\langle M\rangle_u}{\langle M\rangle_T}$, we obtain that $\left|\mathcal{H}^{M,i} f\right|^2_{L_T^2(M)} \leq 4 |f|^2_{L_T^2(M)}$, $f \in L_T^2(M)$. From Fubini's theorem, it follows that $\mathcal{H}^{M,i,*}$ is the adjoint of $\mathcal{H}^{M,i}$.
2. By using Fubini's theorem and splitting integrals, we have that $\alpha^{M,i} : L_{T,0}^2(M) \to L_T^2(M)$ is the inverse of $\beta^{M,i} : L_T^2(M) \to L_{T,0}^2(M)$. Moreover, from part 1, it follows that $\alpha^{M,i}$ is the adjoint of $\beta^{M,i}$. So $\alpha^{M,i}$ and $\beta^{M,i}$ are unitary and hence isometric.
3. A straightforward calculation yields (3.1). (3.2) follows from part 2. By using Fubini's theorem and splitting integrals, we obtain (3.3). □

Remark 3.2. 1. Let $f \in L_T^2(M)$. For $t \in (0,T]$ and $s \in [t,T)$, we have that $\left(\alpha^{M,2} f 1_{[0,t)}\right)(s) = 0$ and $\left(\beta^{M,2} f 1_{[0,t)}\right)(s) = c_M(f,t)$, where c_M is a function independent of s.
2. It holds that $\alpha^{M,i} c \equiv 0$, $c \in \mathcal{C}_T$, $i \in \{1,2\}$.

We can state and prove the following Theorem:

Theorem 3.3. *Let $i \in \{1,2\}$. The transformation defined by*

$$\mathcal{T}_t^{(i)}(M) \ := \ \int_0^T \left(\beta^{M,i} 1_{[0,t)}\right)(s) dM_s, \ t \in [0,T], \tag{3.5}$$

is measure-preserving, i.e. $\mathcal{T}^{(i)}(M)$ is an $\mathbb{F}_T^{\mathcal{T}^{(i)}(M)}$-martingale with $\mathcal{T}_0^{(i)}(M) = 0$, a.s., and $\langle \mathcal{T}^{(i)}(M)\rangle_\cdot = \langle M\rangle_\cdot$. Furthermore, the process

$$\mathcal{B}_t^{(i)}(M) \ := \ \int_0^T \left(\alpha^{M,i} 1_{[0,t)}\right)(s) dM_s, \ t \in [0,T], \tag{3.6}$$

is a bridge of M.

Proof. Clearly, $\mathcal{T}^{(i)}(M)$ and $\mathcal{B}^{(i)}(M)$ are centered Gaussian processes. From part 2 of Lemma 3.1, we obtain that

$$\mathrm{Cov}_{\mathbb{P}}\left(\mathcal{T}_s^{(i)}(M), \mathcal{T}_t^{(i)}(M)\right) \ = \ \mathrm{Cov}_{\mathbb{P}}(M_s, M_t), \ s,t \in [0,T].$$

Furthermore, by combining (3.1), part 2 of Lemma 3.1 and (2.3), we obtain that

$$\mathrm{Cov}_{\mathbb{P}}\left(\mathcal{B}_s^{(i)}(M), \mathcal{B}_t^{(i)}(M)\right) \ = \ \mathrm{Cov}_{\mathbb{P}}\left(M_s^T, M_t^T\right), \ s,t \in [0,T].$$

□

3.2 Pathwise relations for the transformed processes

By using the stochastic Fubini theorem, we have that

$$\mathcal{T}_t^{(1)}(M) \ = \ M_t \ - \ \int_0^t \frac{M_s}{\langle M\rangle_s} d\langle M\rangle_s, \ a.s., \ t \in [0,T]. \tag{3.7}$$

Similarly, we obtain that

$$\mathcal{T}_t^{(2)}(M) \ = \ M_t \ - \ \int_0^t \frac{M_T - M_s}{\langle M\rangle_{T,s}} d\langle M\rangle_s, \ a.s., \ t \in [0,T]. \tag{3.8}$$

On the one hand, by combining (3.5), (3.2) and (2.4), we have that

$$\mathcal{T}_t^{(i)}(M) \ = \ \int_0^T \left(\overline{\beta^{M,i} 1_{[0,t)}}\right)(s) d\widehat{M}_s^T, \ a.s., \ t \in [0,T], \ i \in \{1,2\}. \tag{3.9}$$

On the other hand, by combining (2.3), (3.3) and (3.5), we obtain the reciprocal

$$\widehat{M}_t^T \ = \ \int_0^T \left(\alpha^{M,i} 1_{[0,t)}\right)(s) d\mathcal{T}_s^{(i)}(M), \ a.s., \ t \in [0,T], \ i \in \{1,2\}. \tag{3.10}$$

Clearly, from (3.9) and (3.10), it follows that

$$\Gamma_T\left(\mathcal{T}^{(i)}(M)\right) \ = \ \Gamma_T\left(\widehat{M}^T\right), \ i \in \{1,2\}. \tag{3.11}$$

For $i = 2$, it follows more precisely from part 1 of Remark 3.2, that

$$\Gamma_t\left(\mathcal{T}^{(2)}(M)\right) \ = \ \Gamma_t\left(\widehat{M}^T\right), \ t \in [0,T], \tag{3.12}$$

i.e.

$$\mathbb{F}_T^{\mathcal{T}^{(2)}(M)} \ = \ \mathbb{F}_T^{\widehat{M}^T}. \tag{3.13}$$

From (3.13), we obtain that $\mathcal{T}^{(2)}(M)$ is an $\mathbb{F}_T^{\widehat{M}^T}$-martingale. Furthermore, by combining (3.11) and (1.8), we have that $\mathcal{T}^{(2)}(M)$ is orthogonal to M_T, and thus independent of M_T. Hence, $\mathcal{T}^{(2)}(M)$ is also an $\left(\mathbb{F}_T^{\widehat{M}^T} \vee \sigma(M_T)\right)$-martingale and so, by using (1.9), an $\left(\mathbb{F}_T^M \vee \sigma(M_T)\right)$-martingale. From (3.8), it is then easy to see, that $\mathcal{T}^{(2)}(M)$ is the martingale component in the $\left(\mathbb{F}_T^M \vee \sigma(M_T)\right)$-semimartingale decomposition of M.

By combining (3.8) and (1.7), we obtain that

$$\mathcal{T}_t^{(2)}(M) \;=\; \widehat{M}_t^T \;+\; \int_0^t \frac{\widehat{M}_s^T}{\langle M\rangle_{T,s}} d\langle M\rangle_s,\ a.s.,\ t \in [0,T].$$

In particular, $\widehat{M}^T$ is the unique solution of the linear stochastic differential equation

$$d\widehat{M}_t^T \;=\; d\mathcal{T}_t^{(2)}(M) \;-\; \frac{\widehat{M}_t^T}{\langle M\rangle_{T,t}} d\langle M\rangle_t, \qquad \widehat{M}_0^T \;=\; 0.$$

A straightforward calculation yields

$$\mathcal{B}_t^{(1)}(M) \;=\; -\langle M\rangle_t \int_t^T \frac{1}{\langle M\rangle_s} dM_s,\ a.s.,\ t \in (0,T]. \tag{3.14}$$

Similarly,

$$\mathcal{B}_t^{(2)}(M) \;=\; \langle M\rangle_{T,t} \int_0^t \frac{dM_s}{\langle M\rangle_{T,s}},\ a.s.,\ t \in [0,T).$$

By combining part 2 of Lemma 3.1 and (3.6), we obtain that

$$M_t \;=\; \int_0^T \left(\overline{\beta^{M,i} 1_{[0,t)}}\right)(s) d\mathcal{B}_s^{(i)}(M),\ a.s.,\ t \in [0,T],\ i \in \{1,2\}. \tag{3.15}$$

By comparing identities (3.6) and (3.15) with identities (3.10) and (3.9), we observe that the bridge $\mathcal{B}^{(i)}(M)$ is related to the martingale M in the same way, as the bridge $\widehat{M}^T$ is related to the martingale $\mathcal{T}^{(i)}(M)$. Hence, from (3.11), we obtain that

$$\Gamma_T\Big(\mathcal{B}^{(i)}(M)\Big) \;=\; \Gamma_T(M),\ i \in \{1,2\}. \tag{3.16}$$

Moreover, from (3.12), we have that

$$\Gamma_t\Big(\mathcal{B}^{(2)}(M)\Big) \;=\; \Gamma_t(M),\ t \in [0,T],$$

i.e.

$$\mathbb{F}_T^{\mathcal{B}^{(2)}(M)} \;=\; \mathbb{F}_T^M.$$

From (3.10) and (3.15) we obtain the following:

Lemma 3.4. *Let $i \in \{1,2\}$. Then*

$$\widehat{M}_t^T \;=\; \mathcal{B}_t^{(i)}\left(\mathcal{T}^{(i)}(M)\right),\ a.s.,\ t \in [0,T].$$

Furthermore,

$$M_t \;=\; \mathcal{T}_t^{(i)}\left(\mathcal{B}^{(i)}(M)\right),\ a.s.,\ t \in [0,T].$$

Remark 3.5. 1. From (3.11), we have that $\mathcal{F}_T^{\mathcal{T}^{(i)}(M)} \subsetneq \mathcal{F}_T^M$. Hence, $\mathcal{T}^{(i)}$ is injective, but not surjective.
2. From (3.16), it follows that $\mathcal{F}_T^{\mathcal{B}^{(i)}(M)} = \mathcal{F}_T^M$. Hence, $\mathcal{B}^{(i)}$ is bijective.

3.3 Connection between the cases $i = 1$ and $i = 2$

Let $(Y_t)_{t\in[0,T]}$ be a Gaussian process with $Y_0 = 0$, a.s., and let

$$\mathcal{S}_t(Y) \ := \ Y_T - Y_{T-t}, \ t \in [0,T], \tag{3.17}$$

be the time-reversion of the process Y. Clearly, we have that $\mathcal{S}_0(Y) = 0$, a.s., and

$$\mathcal{S}_T(Y) \ = \ Y_T, \ a.s. \tag{3.18}$$

Furthermore, $\Gamma_T(\mathcal{S}(Y)) = \Gamma_T(Y)$ and $\mathcal{S}_t^2(Y) = Y_t$, a.s., $t \in [0,T]$.

It is easy to show, that $\mathcal{S}(M)$ is a continuous $\mathbb{F}_T^{\mathcal{S}(M)}$-martingale with

$$\langle \mathcal{S}(M)\rangle_t \ = \ \langle M\rangle_{T,T-t}, \ t \in [0,T]. \tag{3.19}$$

Hence $\langle \mathcal{S}(M)\rangle_\cdot$ is strictly increasing on $[0,T]$. Note that $\mathcal{S}$ is a measure-preserving transformation if and only if M has stationary increments, i.e. if and only if $M = W$. By using (3.19), we obtain that

$$\left(\beta^{M,1}1_{[T-t,T)}\right)(\cdot) \ = \ \left(\beta^{\mathcal{S}(M),2}1_{[0,t)}\right)(T-\cdot), \ \lambda\text{-}a.e., \ t \in (0,T].$$

It follows from this, that

$$\mathcal{S}_t\left(\mathcal{T}^{(1)}(M)\right) \ = \ \mathcal{T}_t^{(2)}\left(\mathcal{S}(M)\right), \ a.s., \ t \in [0,T]. \tag{3.20}$$

Similarly, we have that

$$\left(\alpha^{M,1}1_{[T-t,T)}\right)(\cdot) \ = \ \left(\alpha^{\mathcal{S}(M),2}1_{[0,t)}\right)(T-\cdot), \ \lambda\text{-}a.e., \ t \in (0,T].$$

Therefore,

$$\mathcal{S}_t\left(\mathcal{B}^{(1)}(M)\right) \ = \ \mathcal{B}_t^{(2)}\left(\mathcal{S}(M)\right), \ a.s., \ t \in [0,T]. \tag{3.21}$$

4 Volterra Gaussian processes

We generalize the results of Section 3 to the case, when X is a continuous Volterra Gaussian processes with a non-degenerate Volterra kernel satisfying (1.5), (1.6) and that

$$\mathbf{K}^X f = z_X(T,\cdot)1_{[0,t)} \text{ has a solution in } \check{\Lambda}_T(X) \text{ for all } t \in (0,T], \tag{4.1}$$

which we denote by $k^*(t,\cdot) := k^*_{X,T}(t,\cdot)$, $t \in (0,T]$. We proceed similarly as in Section 3.

4.1 Measure-preserving and bridge transformations

Let

$$M_t \ := \ M_t(X,T) \ := \ \mathrm{E}_{\mathbb{P}}\left(X_T \middle| \mathcal{F}_t^X\right), \ t \in [0,T],$$

denote the *prediction martingale of X_T with respect to $\mathbb{F}_T^X$*. By definition,

$$M_T \ = \ X_T, \ a.s. \tag{4.2}$$

Clearly, by (1.4) we have that

$$M_t \ = \ \int_0^t z_X(T,s)dW_s, \ a.s., \ t \in [0,T]. \tag{4.3}$$

From (4.3) and (1.5), it follows that M is a continuous Gaussian martingale with $M_0 = 0$, a.s., and the quadratic variation function $\langle M \rangle_\cdot = \int_0^\cdot z_X^2(T,s)ds$ is strictly increasing on $[0,T]$. In particular, all results from Section 3 hold true for M. We have that $\Gamma_t(M) = \Gamma_t(W)$, $t \in [0,T]$, and hence, from (1.3), it follows that

$$\Gamma_t(M) \;=\; \Gamma_t(X),\; t\in[0,T]. \tag{4.4}$$

More precisely, we have that

$$X_t \;=\; \int_0^t k(t,s)dM_s,\; a.s.,\; t\in[0,T], \tag{4.5}$$

where the Volterra kernel is given by

$$k(t,s) \;:=\; k_{X,T}(t,s) \;:=\; \frac{z_X(t,s)}{z_X(T,s)},\; s,t\in[0,T]. \tag{4.6}$$

Let $\kappa := \kappa^{X,T}$ be the isometric isomorphism defined by

$$\begin{aligned}\kappa \;:\; \Lambda_T(X) &\;\to\; L^2_T(M)\\ 1_{[0,t)} &\;\mapsto\; k(t,\cdot),\; t\in(0,T].\end{aligned}$$

Then

$$\int_0^T f(s)dX_s \;=\; \int_0^T (\kappa f)(s)dM_s,\; a.s.,\; f\in\Lambda_T(X). \tag{4.7}$$

Also, from (4.3) and (4.1), it follows that

$$M_t \;=\; \int_0^T k^*(t,s)dX_s,\; a.s.,\; t\in[0,T]. \tag{4.8}$$

Hence

$$\kappa^{-1}1_{[0,t)} \;=\; k^*(t,\cdot),\; t\in(0,T].$$

From (4.4), it follows that k^* is a Volterra kernel. Clearly, by using (4.2), we have that

$$R^X(T,T) \;=\; \langle M\rangle_T \tag{4.9}$$

and

$$k^*(T,\cdot) \;\equiv\; 1. \tag{4.10}$$

From (4.6) and (4.10), it follows that

$$\kappa(c) \;=\; \kappa^{-1}(c) \;=\; c,\; c\in\mathcal{C}_T. \tag{4.11}$$

Moreover, (4.7) and (4.2) imply that

$$\begin{aligned}\left(f, 1_{[0,T)}\right)_{\Lambda_T(X)} &\;=\; \mathrm{Cov}_{\mathbb{P}}\left(\int_0^T f(s)dX_s, X_T\right)\\ &\;=\; \mathrm{Cov}_{\mathbb{P}}\left(\int_0^T (\kappa f)(s)dM_s, M_T\right)\\ &\;=\; \int_0^T (\kappa f)(s)d\langle M\rangle_s,\; f\in\Lambda_T(X).\end{aligned} \tag{4.12}$$

By combing (4.9) and (4.12), and then using (4.11), we obtain that

$$\eta^X f \;=\; \left(\kappa^{-1}\eta^M\kappa\right)f,\; f\in\Lambda_T(X). \tag{4.13}$$

Also, by combining (2.3), (4.13), (4.7) and (2.4), we have that

$$\widehat{X}_t^T \;=\; \int_0^T \overline{k(t,\cdot)}(s)d\widehat{M}_s^T,\; a.s.,\; t\in[0,T]. \tag{4.14}$$

Example 4.1. If $X = B^H$, then (4.1) is satisfied. We have that

$$k^*(t,s) = 1_{[0,t)}(s) + \left(\frac{\sin(\pi(H-\frac{1}{2}))}{\pi} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} \int_t^T \frac{u^{H-\frac{1}{2}}(u-t)^{H-\frac{1}{2}}}{u-s} du \right) 1_{[0,t)}(s),$$

$s, t \in [0,T]$. See [15] for a full proof.

The following result generalizes Lemma 3.1:

Lemma 4.2. *1. For $i \in \{1,2\}$ and $f \in \Lambda_T(X)$, let*

$$\alpha^{X,i} f \ := \ \left(\kappa^{-1}\alpha^{M,i}\kappa\right) f \tag{4.15}$$

and

$$\beta^{X,i} f \ := \ \left(\kappa^{-1}\beta^{M,i}\kappa\right) f. \tag{4.16}$$

Then $\alpha^{X,i} : \Lambda_{T,0}(X) \to \Lambda_T(X)$ and $\beta^{X,i} : \Lambda_T(X) \to \Lambda_{T,0}(X)$ are isometric isomorphisms with $\left(\alpha^{X,i}\right)^{-1} = \beta^{X,i}$.
2. For $i \in \{1,2\}$, we have that

$$\alpha^{X,i}\eta^X f \ = \ \alpha^{X,i} f, \ f \in \Lambda_T(X), \tag{4.17}$$

and

$$\eta^X \beta^{X,i} f \ = \ \beta^{X,i} f, \ f \in \Lambda_T(X). \tag{4.18}$$

Also,

$$\beta^{X,i}\alpha^{X,i} f = \eta^X f, \ f \in \Lambda_T(X). \tag{4.19}$$

Proof. 1. Follows straightforward from part 2 of Lemma 3.1.
2. Follows from part 3 of Lemma 3.1 by using (4.13). □

Remark 4.3. 1. Let $t \in (0,T]$ and $s \in [t,T)$. By combining part 1 of Remark 3.2 and (4.11), it follows that $\left(\alpha^{X,2} 1_{[0,t)}\right)(s) = 0$ and $\left(\beta^{X,2} 1_{[0,t)}\right)(s) = c_M(k(t,\cdot),t)$.
2. From part 2 of Remark 3.2 and (4.11), we obtain that $\alpha^{X,i} c \equiv 0$, $c \in \mathcal{C}_T$, $i \in \{1,2\}$.

The generalization of Theorem 3.3 is straightforward:

Theorem 4.4. *Let $i \in \{1,2\}$. The transformation defined by*

$$\mathcal{T}_t^{(i)}(X) \ := \ \int_0^T \left(\beta^{X,i} 1_{[0,t)}\right)(s) dX_s, \ t \in [0,T], \tag{4.20}$$

is measure-preserving, i.e. $\mathcal{T}^{(i)}(X)$ is a centered Gaussian process with $\mathcal{T}_0^{(i)}(X) = 0$, a.s., and $R^{\mathcal{T}^{(i)}(X)} = R^X$. Furthermore, the process

$$\mathcal{B}_t^{(i)}(X) \ := \ \int_0^T \left(\alpha^{X,i} 1_{[0,t)}\right)(s) dX_s, \ t \in [0,T], \tag{4.21}$$

is a bridge of X.

Proof. By construction, $\mathcal{T}^{(i)}(X)$ and $\mathcal{B}^{(i)}(X)$ are centered Gaussian processes. By using part 1 of Lemma 4.2, we obtain that

$$\mathrm{Cov}_{\mathbb{P}}\left(\mathcal{T}_s^{(i)}(X), \mathcal{T}_t^{(i)}(X)\right) \ = \ \mathrm{Cov}_{\mathbb{P}}(X_s, X_t), \ s,t \in [0,T].$$

Moreover, by combining (4.17), part 1 of Lemma 4.2 and (2.3), we obtain that

$$\mathrm{Cov}_{\mathbb{P}}\left(\mathcal{B}_s^{(i)}(X), \mathcal{B}_t^{(i)}(X)\right) \ = \ \mathrm{Cov}_{\mathbb{P}}\left(X_s^T, X_t^T\right), \ s,t \in [0,T].$$

□

Remark 4.5. Lemma 4.2 and Theorem 4.4 hold true also without assumptions (1.6) and (4.1).

Next, we want to explicitly evaluate the functions $\alpha^{X,i}1_{[0,t)}$ and $\beta^{X,i}1_{[0,t)}$ in definitions (4.21) and (4.20), respectively. For this purpose, we recall the notion of the Bochner integral in a Banach space: Let $(Q,\mathcal{Q},\mu)$ be a measure space with a finite signed measure and let B be a Banach space with norm $|\cdot|_B$. The Bochner integral in B of an indicator function $f(q) := g\cdot 1_A(q)$, $q\in Q$, where $A\in\mathcal{Q}$ and $g\in B$, is given by $\int_Q f(q)d\mu(q) := g\cdot\mu(A)$. A measurable map $f:(Q,\mathcal{Q})\to B$, $q\mapsto f(q)$, is Bochner integrable if $\int_Q |f(q)|_B d\mu(q) < \infty$. The space of Bochner integrable functions is a Banach space. The Bochner integral in B of a Bochner integrable function is defined as the unique continuous, linear extension of $\int_Q \cdot\, d\mu$ from the set of indicator functions to B. By construction, the Bochner integral commutes with bounded linear maps: if $f:Q\to B$ is Bochner integrable, B' is a Banach space and $A:B\to B'$ is a bounded linear map, then $Af:Q\to B'$ is Bochner integrable, and

$$A\int_Q f(q)d\mu(q) \;=\; \int_Q Af(q)d\mu(q).$$

If the elements of B are functions from $[0,T]$ to $\mathbb{R}$, then we denote $\int_Q f(q,s)d\mu(q) := \left(\int_Q f(q)d\mu(q)\right)(s)$, $s\in[0,T]$. For details on Bochner integration, see [11].

We can state and prove the following Lemma:

Lemma 4.6. *Let $z\in(0,T]$ and $g\in L^2_T(M)$.*
1. If g is absolutely continuous on $[a,z]$ for every $a>0$, then

$$\left(\kappa^{-1}g1_{[0,z)}\right)(s) = k^*(z,s)g(z) - \int_0^z k^*(u,s)dg(u),\ \lambda\text{-a.e. } s\in(0,T),$$

where the integral is a Bochner integral in $\Lambda_T(X)$.
2. If g is absolutely continuous on $[0,b]$ for every $b<z$, then

$$\left(\kappa^{-1}g1_{[0,z)}\right)(s) = \int_0^z \left(k^*(z,s)-k^*(u,s)\right)dg(u) + k^*(z,s)g(0),\ \lambda\text{-a.e. } s\in(0,T),$$

where the integral is a Bochner integral in $\Lambda_T(X)$.

Proof. 1. We have that

$$\int_0^z 1_{[0,u)}(s)dg(u) \;=\; (g(z)-g(s))1_{[0,z)}(s),\ \lambda\text{-a.e. } s\in(0,T),$$

where the integral is a Bochner integral in $L^2_T(M)$. By combining this, the fact that $\kappa^{-1}: L^2_T(M)\to\Lambda_T(X)$ is a bounded linear map and Remark 2.1, we obtain that

$$\begin{aligned}
\left(\kappa^{-1}g1_{[0,z)}\right)(s) &= \left(\kappa^{-1}g(z)1_{[0,z)}\right)(s) \;-\; \left(\kappa^{-1}(g(z)-g)1_{[0,z)}\right)(s)\\
&= g(z)k^*(z,s) \;-\; \left(\kappa^{-1}\int_0^z 1_{[0,u)}dg(u)\right)(s)\\
&= g(z)k^*(z,s) \;-\; \left(\int_0^z \kappa^{-1}1_{[0,u)}dg(u)\right)(s)\\
&= g(z)k^*(z,s) \;-\; \int_0^z k^*(u,s)dg(u),\ \lambda\text{-a.e. } s\in(0,T).
\end{aligned}$$

2. Similarly, we have that

$$\int_0^z \left(1_{[0,z)}(s)-1_{[0,u)}(s)\right)dg(u) \;=\; (g(s)\;-\;g(0))\,1_{[0,z)}(s),\ \lambda\text{-a.e. } s\in(0,T),$$

where the integral is a Bochner integral in $L^2_T(M)$. Hence,

$$\begin{aligned}\left(\kappa^{-1}g1_{[0,z)}\right)(s) &= \left(\kappa^{-1}\left(g-g(0)\right)1_{[0,z)}\right)(s) \;+\; \left(\kappa^{-1}g(0)1_{[0,z)}\right)(s)\\ &= \left(\kappa^{-1}\int_0^z \left(1_{[0,z)}-1_{[0,u)}\right)dg(u)\right)(s) \;+\; g(0)k^*(z,s)\\ &= \left(\int_0^z \kappa^{-1}(1_{[0,z)}-1_{[0,u)})dg(u)\right)(s) \;+\; g(0)k^*(z,s)\\ &= \int_0^z \left(k^*(z,s)-k^*(u,s)\right)dg(u)+g(0)k^*(z,s),\;\; \lambda\text{-}a.e.\ s\in(0,T).\end{aligned}$$

□

Let $t\in(0,T]$. The function $\mathcal{H}^{M,1}k(t,\cdot)$ is absolutely continuous with respect to $\langle M\rangle_\cdot$, or equivalently, with respect to $\lambda(\cdot)$, on $[a,T]$ for every $a>0$. By combining part 1 of Lemma 4.6 with $z=T$ and (4.10), we obtain that

$$\begin{aligned}\left(\alpha^{X,1}1_{[0,t)}\right)(s) &= 1_{[0,t)}(s)-\frac{1}{\langle M\rangle_T}\int_0^t k(t,u)d\langle M\rangle_u \;+\; \int_0^t k^*(u,s)\frac{k(t,u)}{\langle M\rangle_u}d\langle M\rangle_u\\ &\quad -\int_0^T\frac{k^*(u,s)}{\langle M\rangle_u^2}\int_0^u k(t,v)d\langle M\rangle_v d\langle M\rangle_u,\;\; \lambda\text{-}a.e.\ s\in(0,T).\end{aligned}\tag{4.22}$$

Let $t\in(0,T)$. The function $\mathcal{H}^{M,2}k(t,\cdot)$ is absolutely continuous on $[0,T]$. Moreover, $\left(\mathcal{H}^{M,2}k(t,\cdot)\right)(s)=0,\ s\geq t$. By using part 1 of Lemma 4.6 with $z=t$, we obtain that

$$\begin{aligned}\left(\alpha^{X,2}1_{[0,t)}\right)(s) &= 1_{[0,t)}(s)-\int_0^t k^*(u,s)\frac{k(t,u)}{\langle M\rangle_{T,u}}d\langle M\rangle_u\\ &\quad +\int_0^t\frac{k^*(u,s)}{\langle M\rangle_{T,u}^2}\int_u^t k(t,v)d\langle M\rangle_v d\langle M\rangle_u,\;\; \lambda\text{-}a.e.\ s\in(0,T).\end{aligned}\tag{4.23}$$

Let $t\in(0,T]$. The function $\mathcal{H}^{M,1,*}k(t,\cdot)$ is absolutely continuous on $[a,T]$ for every $a>0$, and $\left(\mathcal{H}^{M,1,*}k(t,\cdot)\right)(s)=0,\ s\geq t$. By using part 1 of Lemma 4.6 with $z=t$, we obtain that

$$\left(\beta^{X,1}1_{[0,t)}\right)(s) \;=\; 1_{[0,t)}(s)-\int_0^t k^*(u,s)\frac{k(t,u)}{\langle M\rangle_u}d\langle M\rangle_u,\;\; \lambda\text{-}a.e.\ s\in(0,T).\tag{4.24}$$

Let $t\in(0,T]$. The function $\mathcal{H}^{M,2,*}k(t,\cdot)$ is absolutely continuous on $[0,b]$ for every $b<T$. By using part 2 of Lemma 4.6 with $z=T$ and (4.10), we have that

$$\left(\beta^{X,2}1_{[0,t)}\right)(s)=1_{[0,t)}(s)-\int_0^t\left(1-k^*(u,s)\right)\frac{k(t,u)}{\langle M\rangle_{T,u}}d\langle M\rangle_u,\;\; \lambda\text{-}a.e.\ s\in(0,T).\tag{4.25}$$

4.2 Pathwise relations for the transformed processes

By combining (4.20), (4.16), (4.7) and (3.5), we obtain that

$$\mathcal{T}_t^{(i)}(X) \;=\; \int_0^t k(t,s)d\mathcal{T}_s^{(i)}(M),\;\; a.s.,\;\; t\in[0,T],\;\; i\in\{1,2\},\tag{4.26}$$

i.e. $\mathcal{T}^{(i)}(M)$ is the prediction martingale of $\mathcal{T}_T^{(i)}(X)$ with respect to $\mathbb{F}_T^{\mathcal{T}^{(i)}(X)}$. Clearly, we can evaluate $\mathcal{T}^{(i)}(X)$ explicitly, by using (4.24) and (4.25). However, in order to obtain expressions corresponding to (3.7) and (3.8), we combine (4.20), (4.7), the stochastic Fubini theorem, and (4.8), and obtain that

$$\mathcal{T}_t^{(1)}(X) = X_t - \int_0^t \left(\int_0^u k^*(u,s) dX_s \right) \frac{k(t,u)}{\langle M \rangle_u} d\langle M \rangle_u, \ a.s., \ t \in [0,T].$$

In the same way, we have that

$$\mathcal{T}_t^{(2)}(X) = X_t - \int_0^t \left(X_T - \int_0^u k^*(u,s) dX_s \right) \frac{k(t,u)}{\langle M \rangle_{T,u}} d\langle M \rangle_u, \ a.s., \ t \in [0,T]. \tag{4.27}$$

On the one hand, by combining (4.20), (4.18) and (2.4), we have that

$$\mathcal{T}_t^{(i)}(X) \;=\; \int_0^T \left(\overline{\beta^{X,i} 1_{[0,t)}} \right)(s) d\widehat{X}_s^T, \ a.s., \ t \in [0,T], \ i \in \{1,2\}. \tag{4.28}$$

On the other hand, by combining (2.3), (4.19), and (4.20), we obtain the reciprocal

$$\widehat{X}_t^T \;=\; \int_0^T \left(\alpha^{X,i} 1_{[0,t)} \right)(s) d\mathcal{T}_s^{(i)}(X), \ a.s., \ t \in [0,T], \ i \in \{1,2\}. \tag{4.29}$$

From (4.28) and (4.29), we have that

$$\Gamma_T\Big(\mathcal{T}^{(i)}(X)\Big) \;=\; \Gamma_T\Big(\widehat{X}^T\Big), \ i \in \{1,2\}. \tag{4.30}$$

For $i = 2$, it follows from part 1 of Remark 4.3, that

$$\Gamma_t\Big(\mathcal{T}^{(2)}(X)\Big) \;=\; \Gamma_t\Big(\widehat{X}^T\Big), \ t \in [0,T],$$

i.e.

$$\mathbb{F}_T^{\mathcal{T}^{(2)}(X)} \;=\; \mathbb{F}_T^{\widehat{X}^T}.$$

By combining (4.27), (2.5) and (4.12), we obtain that

$$\mathcal{T}_t^{(2)}(X) \;=\; \widehat{X}_t^T \;+\; \int_0^t \int_0^T \overline{k^*(u,\cdot)}(s) d\widehat{X}_s^T \frac{k(t,u)}{\langle M \rangle_{T,u}} d\langle M \rangle_u, \ a.s., \ t \in [0,T].$$

Note that, in contrast to $\widehat{M}^T$, the process $\widehat{X}^T$ is *not* the solution of a linear stochastic differential equation (unless $X = M$). By combining (4.21), (4.15), (4.7) and (3.6), we obtain that

$$\mathcal{B}_t^{(i)}(X) \;=\; \int_0^T \overline{k(t,\cdot)}(s) d\mathcal{B}_s^{(i)}(M), \ a.s., \ t \in [0,T], \ i \in \{1,2\}. \tag{4.31}$$

By using (4.22), we obtain that

$$\begin{aligned}\mathcal{B}_t^{(1)}(X) \;=\;& X_t - \frac{X_T}{\langle M \rangle_T} \int_0^t k(t,u) d\langle M \rangle_u \;+\; \int_0^t \left(\int_0^t k^*(u,s) \frac{k(t,u)}{\langle M \rangle_u} d\langle M \rangle_u \right) dX_s \\ &- \int_0^T \left(\int_0^T \frac{k^*(u,s)}{\langle M \rangle_u^2} \int_0^u k(t,v) d\langle M \rangle_v d\langle M \rangle_u \right) dX_s, \ a.s., \ t \in (0,T].\end{aligned}$$

Similarly, by using (4.23), we have that

$$\begin{aligned}\mathcal{B}_t^{(2)}(X) \;=\;& X_t - \int_0^t \left(\int_0^t k^*(u,s) \frac{k(t,u)}{\langle M \rangle_{T,u}} d\langle M \rangle_u \right) dX_s \\ &+ \int_0^t \left(\int_0^t \frac{k^*(u,s)}{\langle M \rangle_{T,u}^2} \int_u^t k(t,v) d\langle M \rangle_v d\langle M \rangle_u \right) dX_s, \ a.s., \ t \in [0,T).\end{aligned}$$

By combining part 1 of Lemma 4.2 and (4.21), we have that

$$X_t \;=\; \int_0^T \left(\overline{\beta^{X,i}1_{[0,t)}}\right)(s)\mathcal{B}_s^{(i)}(X),\ a.s.,\ t\in[0,T],\ i\in\{1,2\}. \tag{4.32}$$

By comparing identities (4.21) and (4.32) with identities (4.29) and (4.28), we conclude that bridge $\mathcal{B}^{(i)}(X)$ is related to the process X in the same way, as the bridge $\widehat{X}^T$ is related to the process $\mathcal{T}^{(i)}(X)$. Therefore,

$$\Gamma_T\left(\mathcal{B}^{(i)}(X)\right) \;=\; \Gamma_T(X),\ i\in\{1,2\}. \tag{4.33}$$

Furthermore,

$$\Gamma_t\left(\mathcal{B}^{(2)}(X)\right) \;=\; \Gamma_t(X),\ t\in[0,T],$$

i.e.

$$\mathcal{F}_T^{\mathcal{B}^{(2)}(X)} \;=\; \mathbb{F}_T^X.$$

Remark 4.7. $\mathcal{B}^{(2)}(X)$ is the dynamic bridge of X, and has been introduced in [4].

From (4.29) and (4.32) we obtain the following:

Lemma 4.8. *Let $i\in\{1,2\}$. Then*

$$\widehat{X}_t^T \;=\; \mathcal{B}_t^{(i)}\left(\mathcal{T}^{(i)}(X)\right),\ a.s.,\ t\in[0,T].$$

Furthermore,

$$X_t \;=\; \mathcal{T}_t^{(i)}\left(\mathcal{B}^{(i)}(X)\right),\ a.s.,\ t\in[0,T].$$

Remark 4.9. 1. From (4.30), we have that $\mathcal{F}_T^{\mathcal{T}^{(i)}(X)} \subsetneq \mathcal{F}_T^X$, so $\mathcal{T}^{(i)}$ is injective but not surjective.
2. From (4.33), it follows that $\mathcal{F}_T^{\mathcal{B}^{(i)}(X)} = \mathcal{F}_T^X$, and so $\mathcal{B}^{(i)}$ is bijective.

4.3 Connection between the cases $i=1$ and $i=2$

Consider (4.5), and let

$$\begin{aligned} X_t^{\mathcal{S}} \;&:=\; \int_0^t k(t,s)d\mathcal{S}_s(M) \\ &=\; \int_0^t \left(\frac{z_X(t,s)}{z_X(T,s)} z_X(T,T-s)\right) d\mathcal{S}_s(W),\ a.s.,\ t\in[0,T], \end{aligned}$$

where $\mathcal{S}$ is defined as in (3.17). Note that $X^{\mathcal{S}}$ is well-defined by part 1 of Remark 1.1. Clearly, $X^{\mathcal{S}}$ is a continuous Volterra Gaussian process with a non-degenerate Volterra kernel, and $\mathcal{S}(M)$ is the prediction martingale of $X_T^{\mathcal{S}}$ with respect to $\mathbb{F}_T^{X^{\mathcal{S}}} = \mathbb{F}_T^{\mathcal{S}(M)}$. By combining (4.26), (3.20) and again (4.26), we obtain that

$$\left(\mathcal{T}^{(1)}(X)\right)_t^{\mathcal{S}} \;=\; \mathcal{T}_t^{(2)}\left(X^{\mathcal{S}}\right),\ a.s.,\ t\in[0,T].$$

Clearly,

$$R^{\mathcal{S}(X)}(T,T-t) \;=\; R^X(T,T) - R^X(T,t),\ t\in[0,T]. \tag{4.34}$$

By using (4.34) and (3.18), we obtain that

$$\mathcal{S}_t\left(\widehat{X}^T\right) \;=\; \widehat{\mathcal{S}(X)}_t^T,\ a.s.,\ t\in[0,T]. \tag{4.35}$$

Consider (4.14) and let

$$\left(\widehat{X}^T\right)_t^{\mathcal{S}} \;:=\; \int_0^T \overline{k(t,\cdot)}(s)d\mathcal{S}_s\left(\widehat{M}^T\right),\; t\in[0,T]. \tag{4.36}$$

Due to (4.35), the right-hand side of (4.36) is a Wiener integral with respect to $\widehat{\mathcal{S}(M)}^T$. The process $\left(\widehat{X}^T\right)^{\mathcal{S}}$ is well-defined due to part 1 of Remark 1.1. In particular, from (4.14) it follows that

$$\left(\widehat{X}^T\right)_t^{\mathcal{S}} \;=\; \widehat{X_t^{\mathcal{S}}}^T,\; a.s.,\; t\in[0,T].$$

By combining (4.31), (3.21) and again (4.31), we have that

$$\left(\mathcal{B}^{(1)}(X)\right)_t^{\mathcal{S}} \;=\; \mathcal{B}_t^{(2)}\left(X^{\mathcal{S}}\right),\; a.s.,\; t\in[0,T].$$

Remark 4.10. If X has stationary increments, then $\mathcal{S}(X)\overset{d}{=}X$, and hence $\mathcal{S}(X)$ is a Volterra Gaussian process. In general, however, it is not clear, whether $\mathcal{S}(X)$ is a Volterra Gaussian process.

5 A Fourier-Laguerre series expansion

Let $(M_t)_{t\in[0,\infty)}$ be a continuous Gaussian martingale with $M_0=0$, a.s., such that $\langle M\rangle_\cdot$ is strictly increasing and

$$\lim_{t\to\infty}\langle M\rangle_t \;=\; \infty. \tag{5.1}$$

As before, we assume that the underlying space $(\Omega,\mathcal{F},\mathbb{P})$ is the coordinate space of M, where the definition of *coordinate space* is similar as in the introduction, with $[0,\infty)$ replacing $[0,T]$. Hence $\mathcal{F}=\mathcal{F}_\infty^M:=\sigma(M_t\,|\,t\in[0,\infty))$. Let

$$\Gamma_\infty(M) \;:=\; \overline{\operatorname{span}\{M_t\,|\,t\in[0,\infty)\}}$$

and

$$\Gamma_{[T,\infty)}(M) \;:=\; \overline{\operatorname{span}\{M_t-M_s\,|\,s,t\in[T,\infty)\}}$$

denote the first Wiener chaoses of M over $[0,\infty)$ and $[T,\infty)$, respectively. By (5.1), there exists a standard Brownian motion $(W_t)_{t\in[0,\infty)}$ such that $M_t=W_{\langle M\rangle_t}$, *a.s.*, $t\in[0,\infty)$ (see [10], Theorem 4.6, p. 174). From the strong law of large numbers for standard Brownian motion, it follows that

$$\lim_{t\to\infty}\frac{M_t}{\langle M\rangle_t} \;=\; \lim_{t\to\infty}\frac{W_t}{t} \;=\; 0,\; a.s. \tag{5.2}$$

From (3.7), we see that transformation $\mathcal{T}^{(1)}$ does not depend on T, hence we can write

$$\mathcal{T}_t(M) \;:=\; \mathcal{T}_t^{(1)}(M) \;=\; M_t-\int_0^t\frac{M_s}{\langle M\rangle_s}dM_s,\; a.s.,\; t\in[0,\infty).$$

By using the stochastic Fubini theorem, we have that

$$\mathcal{T}_t(M) \;=\; \int_0^\infty\left(\beta_\infty^{M,1}1_{[0,t)}\right)(s)dM_s,\; a.s.,\; t\in[0,\infty),$$

where

$$\begin{array}{rcl}\beta_\infty^{M,1} \ : \ L^2([0,\infty), d\langle M\rangle_\cdot) & \to & L^2([0,\infty), d\langle M\rangle_\cdot) \\ f(\cdot) & \mapsto & f(\cdot) - \displaystyle\int_\cdot^\infty \frac{f(u)}{\langle M\rangle_u} d\langle M\rangle_u.\end{array}$$

By combining the classical Fubini theorem and (5.1), we obtain that $\beta_\infty^{M,1}$ is an isometric isomorphism with inverse

$$\begin{array}{rcl}\alpha_\infty^{M,1} \ : \ L^2([0,\infty), d\langle M\rangle_\cdot) & \to & L^2([0,\infty), d\langle M\rangle_\cdot) \\ f(\cdot) & \mapsto & f(\cdot) - \displaystyle\frac{1}{\langle M\rangle_\cdot}\int_0^\cdot f(u) d\langle M\rangle_u.\end{array}$$

Hence, the measure-preserving transformation $\mathcal{T}$ is an automorphism with inverse

$$\mathcal{T}_t^{-1}(M) = \int_0^\infty \left(\alpha_\infty^{M,1} 1_{[0,t)}\right)(s) dM_s = -\langle M\rangle_t \int_t^\infty \frac{1}{\langle M\rangle_s} dM_s, \ a.s., \ t \in (0,\infty). \tag{5.3}$$

In particular,

$$\Gamma_\infty\left(\mathcal{T}^{-1}(M)\right) \ = \ \Gamma_\infty(M). \tag{5.4}$$

Remark 5.1. For the (on T dependent) bridge of M in (3.14), it holds by using (5.3), that

$$\lim_{\substack{T \to \infty \\ T > t}} \mathcal{B}_t^{(1)}(M) \ = \ \mathcal{T}_t^{-1}(M), \ a.s., \ t \in (0,\infty).$$

Let $T > 0$ be fixed. M has independent increments, hence

$$\Gamma_\infty(M) \ = \ \Gamma_T(M) \ \oplus \ \Gamma_{[T,\infty)}(M). \tag{5.5}$$

On the one hand, from (1.8) and (3.11), it follows that

$$\Gamma_T(M) \ = \ \Gamma_T\left(\mathcal{T}(M)\right) \ \oplus \ \mathrm{span}\{M_T\}. \tag{5.6}$$

On the other hand, by using (5.4) and (5.6) with $\mathcal{T}^{-1}(M)$ instead of M, we have that

$$\Gamma_{[T,\infty)}(M) \ = \ \Gamma_{[T,\infty)}\left(\mathcal{T}^{-1}(M)\right) \ \oplus \ \mathrm{span}\left\{\mathcal{T}_T^{-1}(M)\right\}. \tag{5.7}$$

By combining (5.5), (5.6) and (5.7), and iterating this decomposing procedure, we obtain that $\{\mathcal{T}_T^n(M)\}_{n\in\mathbb{Z}}$ is an orthogonal system in $\Gamma_\infty(M)$. From the following two-sided Fourier-Laguerre series expansion, it follows that this system is complete:

Theorem 5.2. *Let $Z := \int_0^\infty f(s) dM_s \in \Gamma_\infty(M)$. Then*

$$Z \ = \ L^2(\mathbb{P})\text{-}\sum_{n\in\mathbb{Z}} \left(\int_0^\infty f(s)\mathcal{L}_n^{M,T}(s) d\langle M\rangle_s\right) \cdot \varepsilon_n^{M,T},$$

where the sequence $\left\{\varepsilon_n^{M,T}\right\}_{n\in\mathbb{Z}} := \left\{\frac{\mathcal{T}_T^n(M)}{\sqrt{\langle M\rangle_T}}\right\}_{n\in\mathbb{Z}}$ is i.i.d. with $\varepsilon_0^{M,T} \sim \mathcal{N}(0,1)$, and

$$\mathcal{L}_n^{M,T}(s) \ := \begin{cases} \frac{1}{\sqrt{\langle M\rangle_T}} L_n\left(\ln\frac{\langle M\rangle_T}{\langle M\rangle_s}\right) 1_{(0,T)}(s), & n \in \mathbb{N}_0; \\ \\ -\frac{\sqrt{\langle M\rangle_T}}{\langle M\rangle_s} L_{-n-1}\left(\ln\frac{\langle M\rangle_s}{\langle M\rangle_T}\right) 1_{(T,\infty)}(s), & n \in -\mathbb{N}, \end{cases}$$

with $L_n(x) := \sum_{k=0}^n \binom{n}{k}\frac{1}{k!}(-x)^k$ denoting the n-th Laguerre polynomial, $n \in \mathbb{N}_0$.

Proof. First, we show that $\left\{\varepsilon_n^{M,T}\right\}_{n\in\mathbb{N}_0}$ is a Hilbert basis of $\Gamma_T(M)$. By iterating $\mathcal{T}$ and using the stochastic Fubini theorem, we have that

$$\begin{aligned}
\mathcal{T}_T^n(M) &= M_T + \sum_{k=1}^n (-1)^k \binom{n}{k} \int_0^T \int_0^{t_{k-1}} \cdots \int_0^{t_1} M_s \frac{d\langle M\rangle_s}{\langle M\rangle_s} \frac{d\langle M\rangle_{t_1}}{\langle M\rangle_{t_1}} \cdots \frac{d\langle M\rangle_{t_{k-1}}}{\langle M\rangle_{t_{k-1}}} \\
&= M_T + \sum_{k=1}^n \frac{(-1)^k}{k!} \binom{n}{k} \int_0^T \ln^k \left(\frac{\langle M\rangle_T}{\langle M\rangle_s}\right) dM_s \\
&= \int_0^T L_n \left(\ln \frac{\langle M\rangle_T}{\langle M\rangle_s}\right) dM_s \\
&= \sqrt{\langle M\rangle_T} \int_0^T \mathcal{L}_n^{M,T}(s) dM_s, a.s., \ n \in \mathbb{N}_0. \qquad (5.8)
\end{aligned}$$

It is well-known, that

$$\{L_n(x), x \in [0,\infty)\}_{n\in\mathbb{N}_0} \text{ is a Hilbert basis of } L^2([0,\infty), e^{-x}dx). \qquad (5.9)$$

By substitution, we obtain that $\left\{\mathcal{L}_n^{M,T}(s), s \in (0,T)\right\}_{n\in\mathbb{N}_0}$ is a Hilbert basis of $L_T^2(M)$. The claim follows from (5.8) and the Wiener isometry between $L_T^2(M)$ and $\Gamma_T(M)$. Second, we show that $\{\varepsilon_n^{M,T}\}_{n\in-\mathbb{N}}$ is a Hilbert basis of $\Gamma_{[T,\infty)}(M)$. By using (5.3), partial integration and (5.2), we have that

$$\mathcal{T}_t^{-1}(M) \ = \ M_t \ - \ \langle M\rangle_t \cdot \mathcal{J}_t^1(M), \ a.s., \ t \in (0,\infty), \qquad (5.10)$$

where

$$\mathcal{J}_t^k(M) := \int_t^\infty \int_{t_{k-1}}^\infty \cdots \int_{t_1}^\infty \frac{M_s}{\langle M\rangle_s} \frac{d\langle M\rangle_s}{\langle M\rangle_s} \frac{d\langle M\rangle_{t_1}}{\langle M\rangle_{t_1}} \cdots \frac{d\langle M\rangle_{t_{k-1}}}{\langle M\rangle_{t_{k-1}}}, \ k \in \mathbb{N}.$$

By combining (5.3) and (5.10), using the stochastic Fubini theorem, and iterating, we obtain that

$$\begin{aligned}
\mathcal{J}_T^k(M) &= \int_T^\infty \int_{t_{k-1}}^\infty \cdots \int_{t_2}^\infty \left(\frac{M_{t_1}}{\langle M\rangle_{t_1}} + \int_{t_1}^\infty \frac{dM_s}{\langle M\rangle_s}\right) \frac{d\langle M\rangle_{t_1}}{\langle M\rangle_{t_1}} \cdots \frac{d\langle M\rangle_{t_{k-1}}}{\langle M\rangle_{t_{k-1}}} \\
&= \mathcal{J}_T^{k-1}(M) \ + \ \int_T^\infty \frac{1}{(k-1)!} \ln^{k-1} \left(\frac{\langle M\rangle_s}{\langle M\rangle_T}\right) \frac{dM_s}{\langle M\rangle_s} \ = \ \cdots \\
&= \mathcal{J}_T^1(M) \ + \ \sum_{j=1}^{k-1} \int_T^\infty \frac{1}{j!} \ln^j \left(\frac{\langle M\rangle_s}{\langle M\rangle_T}\right) \frac{dM_s}{\langle M\rangle_s}, \ a.s., \ k \geq 2. \qquad (5.11)
\end{aligned}$$

From the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, $1 < k \leq n$, it follows that

$$\sum_{k=j+1}^n (-1)^{k+j-1} \binom{n}{k} \ = \ \binom{n-1}{j}, \ 0 \leq j < n. \qquad (5.12)$$

By using (5.10) and iterating, then using (5.11), again (5.10) and (5.12), we obtain that

$$
\begin{aligned}
\mathcal{T}_T^{-n}(M) &= M_T + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \langle M\rangle_T \cdot \mathcal{J}_T^k(M) \\
&= M_T + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \langle M\rangle_T \left(\mathcal{J}_T^1(M) + \sum_{j=1}^{k-1} \int_T^\infty \frac{1}{j!} \ln^j \left(\frac{\langle M\rangle_s}{\langle M\rangle_T} \right) \frac{dM_s}{\langle M\rangle_s} \right) \\
&= \sum_{k=1}^{n} (-1)^k \binom{n}{k} \langle M\rangle_T \left(\int_T^\infty \frac{1}{\langle M\rangle_s} dM_s \; + \sum_{j=1}^{k-1} \int_T^\infty \frac{1}{j!} \ln^j \left(\frac{\langle M\rangle_s}{\langle M\rangle_T} \right) \frac{dM_s}{\langle M\rangle_s} \right) \\
&= \langle M\rangle_T \int_T^\infty \left(-1 + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \sum_{j=1}^{k-1} \frac{1}{j!} \ln^j \left(\frac{\langle M\rangle_s}{\langle M\rangle_T} \right) \right) \frac{dM_s}{\langle M\rangle_s} \\
&= -\langle M\rangle_T \int_T^\infty \left(1 + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} (-1)^{k-1+j} \binom{n}{k} \frac{1}{j!} \ln^j \left(\frac{\langle M\rangle_s}{\langle M\rangle_T} \right) (-1)^j \right) \frac{dM_s}{\langle M\rangle_s} \\
&= -\langle M\rangle_T \int_T^\infty L_{n-1} \left(\ln \frac{\langle M\rangle_s}{\langle M\rangle_T} \right) \frac{dM_s}{\langle M\rangle_s} \\
&= \sqrt{\langle M\rangle_T} \int_T^\infty \mathcal{L}_{-n}^{M,T}(s) dM_s, \; a.s., \; n \in \mathbb{N}.
\end{aligned}
\tag{5.13}
$$

From (5.9), we obtain by substitution, that $\left\{\mathcal{L}_n^{M,T}(s), s \in (T,\infty)\right\}_{n\in -\mathbb{N}}$ is a Hilbert basis of $L^2([T,\infty), d\langle M\rangle_\cdot)$. By combining (5.13) and the Wiener isometry between $L^2([T,\infty), d\langle M\rangle_\cdot)$ and $\Gamma_{[T,\infty)}(M)$, we obtain the claim. Third, by combining (5.5) with these results, we obtain that $\left\{\varepsilon_n^{M,T}\right\}_{n\in\mathbb{Z}}$ is a Hilbert basis of $\Gamma_\infty(M)$. A Fourier expansion yields

$$
\begin{aligned}
Z &= L^2(\mathbb{P})\text{-}\sum_{n\in\mathbb{Z}} \mathrm{Cov}_{\mathbb{P}} \left(Z, \varepsilon_n^{M,T} \right) \cdot \varepsilon_n^{M,T} \\
&= L^2(\mathbb{P})\text{-}\sum_{n\in\mathbb{Z}} \left(\int_0^\infty f(s) \mathcal{L}_n^{M,T}(s) d\langle M\rangle_s \right) \cdot \varepsilon_n^{M,T}. \quad \square
\end{aligned}
$$

As a special case, we obtain the following one-sided Fourier-Laguerre series expansion. For $M = W$, it was shown in [8].

Corollary 5.3. *We have that*

$$
M_t = L^2(\mathbb{P})\text{-}\sum_{n\in\mathbb{N}_0} \sqrt{\langle M\rangle_T} \cdot \zeta_n \left(\ln \frac{\langle M\rangle_T}{\langle M\rangle_t} \right) \cdot \varepsilon_n^{M,T}, \; t \in (0,T],
$$

where $\zeta_n(y) := \int_y^\infty L_n(x) e^{-x} dx, \; y \in [0,\infty)$.

Proof. Set $f := 1_{[0,t)}$ in Theorem 5.2. $\square$

Remark 5.4. Clearly, we have that

$$
\mathcal{F}_T^M = \bigvee_{n\in\mathbb{N}_0} \sigma\left(\mathcal{T}_T^n(M)\right)
$$

and

$$
\mathcal{F}_{[T,\infty)}^M := \sigma(M_t - M_s \,|\, s,t \in [T,\infty)) = \bigvee_{n\in -\mathbb{N}} \sigma\left(\mathcal{T}_T^n(M)\right).
$$

Furthermore,

$$\mathcal{F} \;=\; \bigvee_{n\in\mathbb{Z}} \sigma\left(\mathcal{T}_T^n(M)\right).$$

Recall that an automorphism $\mathcal{T}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a *Kolmogorov automorphism*, if there exists a σ-algebra $\mathcal{A} \subseteq \mathcal{F}$, such that $\mathcal{T}^{-1}\mathcal{A} \subseteq \mathcal{A}$, $\vee_{n\in\mathbb{Z}}\mathcal{T}^n\mathcal{A} = \mathcal{F}$ and $\cap_{n\in\mathbb{N}_0}\mathcal{T}^{-n}\mathcal{A} = \{\Omega, \emptyset\}$. It is straightforward to see, that $\mathcal{T}$ and $\mathcal{T}^{-1}$ are Kolmogorov automorphisms with $\mathcal{A} = \mathcal{F}^M_{[T,\infty)}$ and $\mathcal{A} = \mathcal{F}^M_T$, respectively. Hence, $\mathcal{T}$ and $\mathcal{T}^{-1}$ are strongly mixing and hence ergodic (see [13], Proposition 5.11 and Proposition 5.9).

Acknowledgements. Thanks are due to my supervisor Esko Valkeila for motivating the writing of this article, for good questions and helpful comments. I thank Giovanni Peccati for a particularly motivating discussion. I am indebted to the Finnish Graduate School in Stochastics (FGSS) for financial support.

References

[1] Alòs, E., Mazet, O., Nualart, D., Stochastic Calculus with respect to Gaussian Processes. Annals of Probability 29(2), 766-801, 2001.

[2] Baudoin, F., Coutin, L., Volterra Bridges and non-canonical Representations. To appear in Markov Processes and Related Fields, 2006.

[3] Baudoin, F., Nualart, D., Equivalence of Volterra processes. Stochastic Processes and their Applications 107, 327-350, 2003.

[4] Gasbarra, D., Sottinen, T., Valkeila, E., Gaussian Bridges. To appear in Proceedings of the Abel Symposium 2005.

[5] Hardy, G.H., Littlewood, J.E., Pólya, G., Inequalities. Cambridge University Press, 1934.

[6] Hida, T., Hitsuda, M., Gaussian Processes. American Mathematical Society. Translations of Mathematical Monographs, 1976.

[7] Huang, S.T., Cambanis, S., Stochastic and Multiple Wiener Integrals for Gaussian Processes. Annals of Probability 6(4), 585-614, 1978.

[8] Jeulin, T., Yor, M., Filtration des ponts browniens et équations différentielles stochastiques linéaires. Séminaire de probabilités de Strasbourg 24, 227-265, 1990.

[9] Jost, C., Transformation formulas for fractional Brownian motion. Stochastic Processes and their Applications 116, 1341-1357, 2006.

[10] Karatzas, I., Shreve, S.E., Brownian Motion and Stochastic Calculus, Second Edition. Springer New York, 1991.

[11] Mikkola, K., Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations. Research report A 452, Helsinki University of Technology, 2002.

[12] Norros, I., Valkeila, E., Virtamo, J., An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. Bernoulli 5(4), 571-587, 1999.

[13] Peccati, G., Explicit formulae for time-space Brownian chaos. Bernoulli 9(1), 25-48, 2003.

[14] Petersen, K., Ergodic theory. Cambridge University Press, 1983.

[15] Pipiras, V., Taqqu, M.S., Are classes of deterministic integrands for fractional Brownian motion on an interval complete? Bernoulli 7(6), 873-897, 2001.

[16] Yor, M., Some Aspects of Brownian motion, Part 1: Some Special Functionals. Lecture Notes in Mathematics, Birkhäuser Verlag, 1992.