CONVERGENCE OF SINGULAR INTEGRALS WITH GENERAL MEASURES

PERTTI MATTILA AND JOAN VERDERA

Abstract. We show that L^2-bounded singular integrals in metric spaces with respect to general measures and kernels converge weakly. This implies a kind of average convergence almost everywhere. For measures with zero density we prove the almost everywhere existence of principal values.

1. Introduction

Singular integrals with respect to general measures in $\mathbb{R}^n$, and also in metric spaces, have been studied widely, see, e.g., [C], [CW], [D1], [DS], [M], [P], [Ve] and [V]. In this paper our setting is a separable metric space (X, d) with a finite Borel measure μ and a Borel measurable antisymmetric kernel $K\colon X \times X \setminus \{(x, y) : x = y\} \to \mathbb{R}$. Antisymmetry means that

$$K(x, y) = -K(y, x) \text{ for } x, y \in X, \quad x \neq y.$$

Moreover, we shall assume that K is bounded in $\{(x, y) \in X \times X : d(x, y) > \delta\}$ for every $\delta > 0$. We shall also always assume that Vitali's covering theorem is valid for μ and the family of closed balls. Although this is not automatically true even when X is compact, it is true for example if $X = \mathbb{R}^n$ or μ is doubling, see, e.g., [F, Section 2.8].

The singular integral operator T associated with μ and K is formally given by

$$T(f)(x) = \int K(x, y) f(y)\, d\mu y.$$

The problem which appears already in all classical cases such as the Hilbert transform on $\mathbb{R}$, i.e., $K(x, y) = 1/(y - x)$, is that usually this integral does not exist when $x \in \operatorname{spt} \mu$, the support of μ. When μ is the Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$ and K is a standard Calderón-Zygmund kernel, this can be overcome by defining

$$T(f)(x) = \lim_{\epsilon \to 0} T_\epsilon f(x), \tag{1.1}$$

where

$$T_\epsilon(f)(x) = \int_{X \setminus B(x,\epsilon)} K(x, y) f(y)\, d\mu y.$$

2000 *Mathematics Subject Classification.* Primary 42B20.

Here $B(x,\epsilon)$ is the open ball with centre x and radius ϵ. In such a case the limit exists trivially for smooth functions due to cancellations, and by the denseness of smooth functions in $L^1(\mathcal{L}^n)$ standard techniques can be used to show that it exists almost everywhere for L^1-functions f. For general measures this approach fails. Unless μ has strong symmetry properties around points in its support there are not enough cancellations to guarantee the existence of the limit even for constant functions. However, when K is antisymmetric one often defines $T(f)$ as a distribution by

$$(1.2) \qquad (T(f),g) = (1/2)\iint K(x,y)(f(x)g(y) - f(y)g(x))\,d\mu x\,d\mu y$$

when f and g are bounded Lipschitz functions, see [C] or [D1].

A central concept in the theory of singular integrals is the boundedness in L^2. This can be formulated in several ways which all agree in the classical case of Calderón-Zygmund kernels and the Lebesgue measure. One way is to say that the distributionally defined operator T, as in (1.2), is bounded in $L^2(\mu)$ if it has a bounded extension to $L^2(\mu) \to L^2(\mu)$. Another way is to require that the truncated operators T_ϵ, $\epsilon > 0$, are uniformly bounded in $L^2(\mu)$. This agrees very generally with the boundedness in $L^2(\mu)$ of the sublinear maximal operator T^*:

$$(1.3) \qquad T^*(f)(x) = \sup_{\epsilon>0} |T_\epsilon(f)(x)|,$$

see [NTV].

A natural question is whether the L^2-boundedness forces the limit $\lim_{\epsilon\to 0} T_\epsilon(f)(x)$ to exist for μ almost all $x \in X$. One would expect this to be true at least if μ is an m-dimensional Ahlfors-David-regular measure in $\mathbb{R}^n$:

$$r^m/C \le \mu(B(x,r)) \le Cr^m \text{ for } x \in \operatorname{spt}\mu, \quad 0 < r < \operatorname{diam}(\operatorname{spt}\mu),$$

and K is the vector-valued Riesz kernel $|x-y|^{-m-1}(x-y)$. In fact, by a result of Tolsa, see [T1], this is true when $m = 1$ even for much more general measures, but the proof is based on very special relations with the kernel $x/|x|^2$ (essentially the Cauchy kernel $1/z$ for $z \in \mathbb{C} = \mathbb{R}^2$) and the so-called Menger curvature. We shall discuss some relations of this problem to rectifiability at the end of the paper. And we shall mention some kernels for which L^2-boundedness does not give the almost everywhere convergence of principal values.

In this paper we prove some substitutes for (1.1) under the L^2-boundedness:

1.4. **Theorem.** *Suppose that T^* (defined by (1.3)) is bounded in $L^2(\mu)$, that is, there exists a constant C_0 such that*

$$(1.5) \qquad \int T^*(f)^2 d\mu \le C_0 \int f^2\,d\mu$$

for $f \in L^2(\mu)$. Then the truncated operators T_ϵ converge weakly in $L^2(\mu)$, that is, there exists a bounded linear operator $T\colon L^2(\mu) \to L^2(\mu)$ such that

$$\lim_{\epsilon\to 0} \int T_\epsilon(f) g\, d\mu = \int T(f) g\, d\mu$$

for $f, g \in L^2(\mu)$. Moreover,

$$T(f)(z) = \lim_{r\to 0} \frac{1}{\mu(B(z,r))} \int_{B(z,r)} \int_{X\setminus B(z,r)} K(x,y) f(y)\, d\mu y\, d\mu x$$

for μ almost all $z \in X$.

So even if we don't know that $T(f)$ would exist as the limit of the simpler integrals $T_\epsilon(f)$, we know that it is almost everywhere the limit of the more complicated but still concrete integrals of Theorem 1.4.

Observe that with some natural estimates the limit operator T satisfies (1.2). This is so if, for example,

$$\iint |K(x,y|\, d(x,y)\, d\mu y\, d\mu x < \infty,$$

as one easily checks. In many cases also the converse in the first part of Theorem 1.4 is true. Namely, by the Banach-Steinhaus theorem the weak convergence implies that the truncated operators T_ϵ are uniformly bounded and, as said before, often this is equivalent to the L^2-boundedness of T^*.

We prove Theorem 1.4 in Section 2. We first establish the weak convergence. Then we deduce from it the average converge using Lebesgue differentiation theorem. We shall also indicate in Section 3 another way of getting the average convergence via the martingale convergence theorem.

In Section 4 we apply Theorem 1.4 to prove the following result on the existence of principal values for measures with zero density:

1.6. **Theorem.** *Suppose $X = \mathbb{R}^n$ or μ is doubling. Let $h\colon (0,\infty) \to (0,\infty)$ be an increasing function such that $\lim_{r\to 0} h(r) = 0$, $h(2r) \le Ch(r)$ for $r > 0$ and that for $x, y \in X$, $x \ne y$,*

$$|K(x,y)| \le \frac{1}{h(d(x,y))}, \tag{1.7}$$

and for $z \in X$, $z \ne x$ with $d(x,y) > 2d(y,z)$,

$$|K(x,y) - K(x,z)| \le \frac{d(y,z)}{d(x,y)h(d(x,y))}. \tag{1.8}$$

Suppose also that for all $x \in X$ and $r > 0$,

$$\mu(B(x,r)) \le h(r) \tag{1.9}$$

and for μ almost all $x \in X$,

$$\lim_{r\to 0} \frac{\mu(B(x,r))}{h(r)} = 0. \tag{1.10}$$

If $T^*: L^2(\mu) \to L^2(\mu)$ is bounded, then for $f \in L^1(\mu)$ for μ almost all $x \in X$,

$$\lim_{\epsilon \to 0} T_\epsilon(f)(x) = T(f)(x)$$

where T is the weak limit operator of Theorem 1.4.

Note that originally $T(f)$ was only defined for $f \in L^2(\mu)$, but under the assumptions of the theorem it has a unique extension to $L^1(\mu)$ because we have the weak L^1-inequality: for $t > 0$,

$$\mu(\{x \in X : |T^*(f)(x)| > t\}) \leq C||f||_1/t. \tag{1.11}$$

For the doubling measures in metric spaces this was proved in [CW] and for general measures in $\mathbb{R}^n$ in [NTV]. The assumptions on the kernels in [NTV] are not quite same as above but it is easy to check that the proofs can be modified.

Rather often the growth condition (1.9) is a consequence of the L^2 boundedness of T^* (see [D1, p. 56].

For general kernels K as above the assumption (1.10) is necessary as an example of David, which we discuss at the end of the paper, shows.

2. Proof of Theorem 1.4

Let B be a closed ball in X. We denote by χ_A the characteristic function of a set A and by A^c its complement in X. We have for all $\epsilon > 0$ (1 denotes the constant function identically 1),

$$\int T_\epsilon(1)\chi_B \, d\mu = -\int T_\epsilon(\chi_B) \, d\mu = -\int_{B^c} T_\epsilon(\chi_B) \, d\mu,$$

because by antisymmetry

$$\int_B T_\epsilon(\chi_B) \, d\mu = 0.$$

Clearly, for all $x \in B^c$ there is the limit (since B is closed)

$$T(\chi_B)(x) := \lim_{\epsilon \to 0} T_\epsilon(\chi_B)(x).$$

As $|T_\epsilon(\chi_B)| \leq T^*(\chi_B) \in L^1(\mu)$, the dominated convergence theorem yields that

$$\lim_{\epsilon \to 0} \int T_\epsilon(1)\chi_B \, d\mu = -\lim_{\epsilon \to 0} \int_{B^c} T_\epsilon(\chi_B) \, d\mu = -\int_{B^c} T(\chi_B) \, d\mu. \tag{2.1}$$

Call S the dense subspace of $L^2(\mu)$ consisting of finite linear combinations of characteristic functions of closed balls. (It is easy to verify that S is dense since we assumed Vitali's covering theorem for μ.) Fix f in $L^2(\mu)$ and take b in S extremely close to f in $L^2(\mu)$. Then for $0 < \epsilon < \delta$,

$$\int (T_\delta(1) - T_\epsilon(1)) f \, d\mu = \int (T_\delta(1) - T_\epsilon(1))(f - b) \, d\mu + \int (T_\delta(1) - T_\epsilon(1)) b \, d\mu.$$

By (2.1), the second term goes to 0 as $\delta \to 0$. For the first term we have by the Schwartz inequality and the L^2-boundedness (1.5) of T^*,

$$\begin{aligned}\left|\int (T_\delta(1) - T_\epsilon(1))(f-b)\,d\mu\right| &\le ||T_\delta(1) - T_\epsilon(1)||_2||f-b||_2\\ &\le 2||T^*(1)||_2||f-b||_2 \le 2(C_0\mu(X))^{\frac{1}{2}}||f-b||_2,\end{aligned}$$

which we can make as small as we want. This gives that the finite limit

$$\lim_{\epsilon\to 0}\int T_\epsilon(1)f\,d\mu$$

exists for all $f \in L^2(\mu)$.

Let again B be a closed ball and $f \in L^2(\mu)$. Then for $\epsilon > 0$,

$$\begin{aligned}\int T_\epsilon(\chi_B)f\,d\mu = &\int_B\int_{B\setminus B(x,\epsilon)} K(x,y)\,d\mu y f(x)\,d\mu x\\ &+ \int_{B^c}\int_{B\setminus B(x,\epsilon)} K(x,y)\,d\mu y f(x)\,d\mu x.\end{aligned}$$

Applying what we proved above to the measure $\chi_B\mu$ we conclude that the first integral converges as $\epsilon \to 0$. The second integral converges again by the dominated convergence theorem, since

$$\left|\int_{B\setminus B(x,\epsilon)} K(x,y)\,d\mu y f(x)\right| \le T^*(\chi_B)(x)|f(x)|$$

and $T^*(\chi_B)f \in L^1(\mu)$. Then also

$$\lim_{\epsilon\to 0}\int T_\epsilon(b)f\,d\mu$$

exists for all $f \in L^2(\mu), b \in S$. Arguing as above with the L^2-boundedness we find that

$$\lim_{\epsilon\to 0}\int T_\epsilon(g)f\,d\mu$$

exists for all $f, g \in L^2(\mu)$. This yields easily that there exists a bounded linear operator $T\colon L^2(\mu) \to L^2(\mu)$ such that

$$\int T(g)f\,d\mu = \lim_{\epsilon\to 0}\int T_\epsilon(g)f\,d\mu$$

for all $f, g \in L^2(\mu)$, and we have established the required weak convergence.

Let $B = B(z,r)$ be an open ball with $\mu(B) > 0$. Using the antisymmetry of K we have for all $\epsilon > 0$,

$$\begin{aligned}\int_B T_\epsilon(f\chi_{B^c})\,d\mu &= -\int f\chi_{B^c}T_\epsilon(\chi_B)\,d\mu \\ &= -\int fT_\epsilon(\chi_B)\,d\mu + \int f\chi_B T_\epsilon(\chi_B)\,d\mu \\ &= \int_B T_\epsilon(f)\,d\mu + \int_B (f - f_B)T_\epsilon(\chi_B)\,d\mu,\end{aligned}$$

where $f_B = \frac{1}{\mu(B)}\int_B f\,d\mu$ and $\int_B T_\epsilon(\chi_B)\,d\mu = 0$. Letting $\epsilon \to 0$, we obtain for the weak limit operator T,

$$\int_B T(f\chi_{B^c})\,d\mu = \int_B T(f)\,d\mu + \int_B (f - f_B)T(\chi_B)\,d\mu.$$

Dividing with $\mu(B) = \mu(B(z,r))$ and letting $r \to 0$, we have for μ almost all z for the first term of the right hand side by the Lebesgue differentiation theorem,

$$\lim_{r\to 0}\frac{1}{\mu(B(z,r))}\int_{B(z,r))} T(f)\,d\mu = T(f)(z),$$

and for the second term by the Schwartz inequality, L^2-boundedness of T and the Lebesgue differentiation theorem,

$$\lim_{r\to 0}\frac{1}{\mu(B(z,r))}\int_{B(z,r)} (f - f_{B(z,r)})T(\chi_{B(z,r)})\,d\mu = 0.$$

On the other hand,

$$T_\epsilon(f\chi_{B^c})(x) = \int_{B^c\setminus B(x,\epsilon)} K(x,y)f(y)\,d\mu y \to \int_{B^c} K(x,y)f(y)\,d\mu y$$

as $\epsilon \to 0$ for $x \in B$ with $|T_\epsilon(f\chi_{B^c})(x)| \le |T^*(f\chi_{B^c})(x)|$, and so by the dominated convergence theorem,

$$\int_B T(f\chi_{B^c})\,d\mu = \lim_{\epsilon\to 0}\int_B T_\epsilon(f\chi_{B^c})\,d\mu = \int_B\int_{B^c} K(x,y)f(y)\,d\mu y\,d\mu x.$$

Combining the above equations, we obtain

$$\lim_{r\to 0}\frac{1}{\mu(B(z,r))}\int_{B(z,r)}\int_{B(z,r)^c} K(x,y)f(y)\,d\mu y\,d\mu x = Tf(z)$$

for μ almost all $z \in X$. This proves the theorem.

For further reference we record for every ball B,

$$\int_B T(1)\,d\mu = \int_B T(\chi_{B^c})\,d\mu = -\int_{B^c} T(\chi_B)\,d\mu \tag{2.2}$$

which follows as in the above proof.

3. Martingales

We introduce a general nested system of sets. Standard examples are dyadic lattices of cubes in $\mathbb{R}^n$. For each $k \in \mathbb{N} = \{1, 2, \dots\}$ let $\mathcal{D}_k$ be a countable disjoint partition of X into μ measurable sets D such that $\mu(\partial D) = 0$. Let $\mathcal{D} = \cup_{k=1}^{\infty} \mathcal{D}_k$. We assume that the system $\{\mathcal{D}_k\}$ is nested in the sense that every $D \in \mathcal{D}_{k+1}$ is contained in some $D' \in \mathcal{D}_k$. Then every $D' \in \mathcal{D}_k$ is a disjoint union of sets in $\mathcal{D}_{k+1}$.

Suppose that T^* is bounded in $L^2(\mu)$. Let $f \in L^2(\mu)$ and $D \in \mathcal{D}_k$. As $\mu(\partial D) = 0$ we have for μ almost all $x \in D$,

$$\int_{D^c} K(x,y) f(y)\, d\mu y = \lim_{\epsilon \to 0} \int_{D^c \setminus B(x,\epsilon)} K(x,y) f(y)\, d\mu y.$$

Moreover,

$$\left| \int_{D^c \setminus B(x,\epsilon)} K(x,y) f(y)\, d\mu y \right| \leq T^*(f\chi_{D^c})(x) \leq T^*(f)(x) + T^*(f\chi_D)(x).$$

If also $g \in L^2(\mu)$ we get by the dominated convergence theorem

$$\begin{aligned} \int_D \left| \int_{D^c} K(x,y) f(y)\, d\mu y g(x) \right| d\mu x \\ \leq \int_D T^*(f)|g|\, d\mu + \int_D T^*(f\chi_D)|g|\, d\mu < \infty. \end{aligned} \tag{3.1}$$

Suppose now in addition that f is non-negative. Then by (3.1) we can define for $k \in \mathbb{N}$,

$$S_k f(z) = \left(\int_D f\, d\mu \right)^{-1} \int_D \int_{D^c} K(x,y) f(y)\, d\mu y f(x)\, d\mu x$$

when $z \in D \in \mathcal{D}_k$, where we interpret $S_k f(z) = 0$ when $z \in D \in \mathcal{D}_k$ with $\int_D f\, d\mu = 0$.

Let ν be a finite Borel measure on X such that

$$\nu(B) = \int_B f\, d\mu$$

for Borel sets $B \subset X$. Let $\mathcal{A}_k$ be the σ-algebra generated by $\mathcal{D}_k$. We shall check that $(S_k f, \mathcal{A}_k)$ is a martingale (with respect ν).

Let $D \in \mathcal{D}_k$ and let $D_1, D_2, \ldots$ be the sets in $\mathcal{D}_{k+1}$ which form the disjoint partition of D. Then

$$\begin{aligned}
\int_D S_{k+1} f \, d\nu &= \sum_i \int_{D_i} S_{k+1} f \, d\nu \\
&= \sum_i \int_{D_i} \frac{1}{\nu(D_i)} \int_{D_i} \int_{D_i^c} K(x,y) \, d\nu y \, d\nu x \, d\nu \\
&= \sum_i \int_{D_i} \int_{D_i^c} K(x,y) \, d\nu y \, d\nu x \\
&= \sum_i \int_{D_i} \sum_{j:i\neq j} \int_{D_j} K(x,y) \, d\nu y \, d\nu x \\
&\quad + \sum_i \int_{D_i} \int_{D^c} K(x,y) \, d\nu y \, d\nu x \\
&= 0 + \int_D \int_{\mathbb{R}^n \setminus D} K(x,y) \, d\nu y \, d\nu x,
\end{aligned}$$

where 0 comes from the antisymmetry of K. This gives

$$\frac{1}{\nu(D)} \int_D S_{k+1} f \, d\nu = S_k f(z) \text{ for } z \in D$$

and implies that $(S_k f, \mathcal{A}_k)$ is a martingale.

Now we check that the martingale $(S_k f, \mathcal{A}_k)$ is $L^1(\nu)$-bounded. We estimate using (3.1), the Schwartz inequality and the L^2-boundedness of T^*,

$$\begin{aligned}
\left|\int S_k f\,d\nu\right| &= \left|\sum_{D\in\mathcal{D}_k} \frac{1}{\nu(D)}\int_D\int_{D^c} K(x,y)\,d\nu y\,d\nu x\,\nu(D)\right| \\
&= \left|\sum_{D\in\mathcal{D}_k} \int_D\int_{D^c} K(x,y)f(y)\,d\mu y f(x)\,d\mu x\right| \\
&\le \sum_{D\in\mathcal{D}_k}\left(\int_D T^*(f)f\,d\mu + \int_D T^*(f\chi_D)f\,d\mu\right) \\
&\le \sum_{D\in\mathcal{D}_k}\Bigg(\left(\int_D (T^*(f)^2\,d\mu\right)^{1/2}\left(\int_D f^2\,d\mu\right)^{1/2} \\
&\qquad\qquad + \left(\int_D T^*(f\chi_D)^2\,d\mu\right)^{1/2}\left(\int_D f^2\,d\mu\right)^{1/2}\Bigg) \\
&\le \sum_{D\in\mathcal{D}_k}\left(\left(\int_D (T^*(f)^2\,d\mu\right)^{1/2} + \left(C_0\int_D f^2\,d\mu\right)^{1/2}\right)\left(\int_D f^2\,d\mu\right)^{1/2} \\
&\le \Bigg(\left(\sum_{D\in\mathcal{D}_k}\int_D (T^*(f)^2\,d\mu\right)^{1/2} \\
&\qquad\qquad + \left(\sum_{D\in\mathcal{D}_k} C_0\int_D f^2\,d\mu\right)^{1/2}\Bigg)\left(\sum_{D\in\mathcal{D}_k}\int_D f^2\,d\mu\right)^{1/2} \\
&= \left(\left(\int T^*(f)^2\,d\mu\right)^{1/2} + \left((C_0\int f^2\,d\mu\right)^{1/2}\right)\left(\int f^2\,d\mu\right)^{1/2} \\
&\le 2C_0^{1/2}\int f^2\,d\mu.
\end{aligned}$$

This proves the L^1-boundedness. Hence by the martingale convergence theorem $(S_k f(z))$ converges for μ almost all $z\in X$.

Now we assume also that

$$\lim_{k\to\infty}\sup\{\operatorname{diam}(D) : D\in\mathcal{D}_k\} = 0. \tag{3.2}$$

We define for $f\in L^2(\mu), k\in\mathbb{N}$,

$$A_k f(z) = \frac{1}{\mu(D)}\int_D\int_{D^c} K(x,y)f(y)\,d\mu y\,d\mu x \text{ when } z\in D\in\mathcal{D}_k,$$

where $A_kf(z) = 0$ if $\mu(D) = 0$. Using the convergence of $(S_kf(z))$ we shall now verify that for $f \in L^2(\mu)$ there exists the finite limit

$$Tf(z) = \lim_{k\to\infty} A_kf(z) \tag{3.3}$$

for μ almost all $z \in X$. Clearly, we may assume that f is non-negative. Moreover, since $A_k(f) = A_k(f+1) - A_k(1)$, we may assume that $f \geq 1$. To prove (3.3) for such an f, write $f_D = \frac{1}{\mu(D)}\int_D f\,d\mu$ for $D \in \mathcal{D}_k$ with $\mu(D) > 0$. Then by (3.1), the Schwartz inequality and (1.5) we have for $z \in D$,

$$\begin{aligned}
&|S_kf(z) - A_kf(z)| \\
&= \left|\left(\int_D f\,d\mu\right)^{-1}\int_D\int_{D^c} K(x,y)f(y)\,d\mu y\,(f(x) - f_D)\,d\mu x\right| \\
&\leq \frac{1}{\mu(D)}\left(\int_D T^*(f)|f - f_D|\,d\mu + \int_D T^*(f\chi_D)|f - f_D|\,d\mu\right) \\
&\leq \frac{1}{\mu(D)}\left(\left(\int_D T^*(f)^2\,d\mu\right)^{1/2}\right. \\
&\qquad\qquad \left.+\left(\int_D T^*(f\chi_D)^2\,d\mu\right)^{1/2}\right)\left(\int_D (f - f_D)^2\,d\mu\right)^{1/2} \\
&\leq \left(\frac{1}{\mu(D)}\left(2\int_D (T^*(f)^2 + C_0f^2)\,d\mu\right)^{1/2}\left(\frac{1}{\mu(D)}\int_D (f - f_D)^2\,d\mu\right)^{1/2}\right).
\end{aligned}$$

Here for μ almost all $z \in X$ as $k \to \infty$, the first factor goes to $2^{1/2}\,(T^*(f)(z)^2 + C_0f(z)^2)^{1/2}$, and the second goes to 0. Hence $S_kf(z) - A_kf(z) \to 0$, which proves (3.3) for non-negative functions $f \in L^2(\mu)$ and of course then also for all $f \in L^2(\mu)$. Moreover, $T\colon L^2(\mu) \to L^2(\mu)$ is bounded.

To get from this the average convergence with balls one needs to approximate balls with nested systems. At least in $\mathbb{R}^n$ this approximation procedure can be done with dyadic cubes. The argument is quite technical and will be omitted.

4. Proof of Theorem 1.6

We shall first make two reductions using the weak type inequality (1.11). Firstly, we may assume that $f = 1$. To see this, note that we may of course assume that f is non-negative. Bounded functions f such that $f > \delta$ for some $\delta > 0$ are dense in the space of non-negative $L^1(\mu)$-functions, whence standard techniques (as for (4.1) below) allow us to assume that f is such a function. Replacing μ by $f\mu$ gives then the reduction to $f = 1$.

Secondly, we may assume the uniform condition

$$\mu(B(x,r)) \leq \eta(r)h(r) \leq h(r) \text{ for } x \in X, \quad r > 0, \tag{4.1}$$

where η is a non-decreasing function such that $\eta(r) \to 0$ as $r \to 0$. To see this, we use Egoroff's theorem to select closed sets $E_k, k = 1, 2, \ldots$, such that $\mu(X \setminus E_k) < 1/k$ and $\mu(B(x,r))/h(r) \to 0$ as $r \to 0$ uniformly on E_k. Then using (1.11) we have for all $t > 0$,

$$\begin{aligned}
&\mu(\{x : \limsup_{\epsilon,\delta\to 0} |T_\epsilon(1)(x) - T_\delta(1)(x)| > t\}) \\
&= \mu(\{x : \limsup_{\epsilon,\delta\to 0} |T_\epsilon(1 - \chi_{E_k})(x) - T_\delta(1 - \chi_{E_k}(x)| > t\}) \\
&\leq \mu(\{x : T^*(1 - \chi_{E_k})(x) > t/2\}) \\
&\leq C\mu(X \setminus E_k)/t,
\end{aligned}$$

provided the limit $\lim_{\epsilon\to 0} T_\epsilon(\chi_{E_k})(x)$ exists for μ almost all $x \in E_k$. (It exists also for all $x \in E_k^c$ since E_k is closed.) That is, if we have the convergence for the measures $\chi_{E_k}\mu$, which satisfy (4.1), we have it also for μ. Then it is easy to check that the limit must be $T(1)(x)$ μ almost everywhere.

Thus it is enough to prove that $\lim_{\epsilon\to 0} T_\epsilon(1)(a) = T(1)(a)$ for μ almost all $a \in X$ assuming (4.1). It is enough to consider points $a \in X$ such that

$$T(1)(a) = \lim_{\epsilon\to 0} \frac{1}{\mu(B(a,\epsilon))} \int_{B(a,\epsilon)} T(1)\, d\mu.$$

Let $0 < \delta < 1/2$ and choose $p > 1/\delta$. Using (2.2) we can write for $\epsilon > 0$,

$$\begin{aligned}
\phi(\epsilon) &:= T_\epsilon(1)(a) - \frac{1}{\mu(B(a,\epsilon))} \int_{B(a,\epsilon)} T(1)\, d\mu \\
&= \int_{B(a,p\epsilon)\setminus B(a,\epsilon)} K(a,x)\, d\mu x \\
&\quad + \int_{B(a,p\epsilon)^c} K(a,x)\, d\mu x + \frac{1}{\mu(B(a,\epsilon))} \int_{B(a,p\epsilon)^c} T(\chi_{B(a,\epsilon)})\, d\mu \\
&\quad + \frac{1}{\mu(B(a,\epsilon))} \int_{B(a,p\epsilon)\setminus B(a,\epsilon)} T(\chi_{B(a,\epsilon)})\, d\mu \\
&= \phi_1(\epsilon) + \phi_2(\epsilon) + \phi_3(\epsilon),
\end{aligned}$$

where

$$\phi_1(\epsilon) = \int_{B(a,p\epsilon)\setminus B(a,\epsilon)} K(a,x)\, d\mu x,$$

$$\phi_2(\epsilon) = \int_{B(a,p\epsilon)^c} K(a,x)\, d\mu x + \frac{1}{\mu(B(a,\epsilon))} \int_{B(a,p\epsilon)^c} T(\chi_{B(a,\epsilon)})\, d\mu$$

and

$$\phi_3(\epsilon) = \frac{1}{\mu(B(a,\epsilon))}\int_{B(a,p\epsilon)\setminus B(a,\epsilon)} T(\chi_{B(a,\epsilon)})\,d\mu.$$

The first term ϕ_1 is easy to estimate:

$$\begin{aligned}|\phi_1(\epsilon)| &= \left|\int_{B(a,p\epsilon)\setminus B(a,\epsilon)} K(a,x)\,d\mu x\right| \\ &\leq \frac{\mu(B(a,p\epsilon))}{h(\epsilon)} \leq C_p \frac{\mu(B(a,p\epsilon))}{h(p\epsilon)} < \delta\end{aligned}$$

by (1.7) and (1.10) for sufficiently small ϵ. Here and later C_q for $q > 1$ denotes a constant such that $h(qr) \leq C_q h(r)$ for $r > 0$. We estimate ϕ_2 using (1.8) and (4.1),

$$\begin{aligned}&|\phi_2(\epsilon)| \\ &= \left|\int_{B(a,p\epsilon)^c} K(a,x)\,d\mu x + \frac{1}{\mu(B(a,\epsilon))}\int_{B(a,p\epsilon)^c} T(\chi_{B(a,\epsilon)})\,d\mu\right| \\ &= \left|\frac{1}{\mu(B(a,\epsilon))}\int_{B(a,\epsilon)}\left(\int_{B(a,p\epsilon)^c} K(a,x)\,d\mu x - \int_{B(a,p\epsilon)^c} K(y,x)\,d\mu x\right)\,d\mu y\right| \\ &\leq \frac{1}{\mu(B(a,\epsilon))}\int_{B(a,\epsilon)}\int_{B(a,p\epsilon)^c} \frac{d(a,y)}{d(a,x)h(d(a,x))}\,d\mu x\,d\mu y \\ &\leq \epsilon\sum_{i=0}^{\infty}\frac{\mu(B(a,2^{i+1}p\epsilon))}{2^i p\epsilon h(2^i p\epsilon)} \leq \sum_{i=0}^{\infty}\frac{\mu(B(a,2^{i+1}p\epsilon))}{2^i p C_2^{-1} h(2^{i+1}p\epsilon)} \leq 2C_2/p < 2C_2\delta.\end{aligned}$$

To estimate ϕ_3 we first show that at almost every point μ is doubling at some small scales. Then we only need to treat the case $X = \mathbb{R}^n$. More precisely, let $C > 2C_2$ be a constant and let F be the set of those $a \in \mathbb{R}^n$ for which there exists ϵ, $0 < \epsilon < 1$, such that

$$\mu(B(a, 2^{1-k}\epsilon)) \geq C\mu(B(a, 2^{-k}\epsilon) \text{ for } k = 0, 1, \ldots .$$

We also assume that $C > 2^{n+1}$. We show now that $\mu(F) = 0$. To prove this we may assume that the support of μ is bounded, say $\operatorname{spt}\mu \subset B(0,R)$. For $a \in F$ let $\epsilon = \epsilon(a)$ be as above. Fix a big positive integer m and pick for each $a \in F$ an integer $k(a) \geq m$ such that for $k \geq k(a)$,

$$C^{-k} \leq (2^{-k}\epsilon(a))^{n+1}.$$

By Vitali's covering theorem we can find disjoint balls $B(a_i, 2^{-k_i}\epsilon_i) \subset B(0,R)$ with $\epsilon_i = \epsilon(a_i)$ and $k_i \geq k(a_i)$ which cover μ almost all of F.

Then

$$\mu(F) \le \sum_i \mu(B(a_i, 2^{-k_i}\epsilon_i)) \le \sum_i C^{-k_i}\mu(B(a,\epsilon_i))$$
$$\le \sum_i (2^{-k_i}\epsilon_i)^{n+1}\mu(\mathbb{R}^n) \le R^n 2^{-m}\mu(\mathbb{R}^n).$$

Letting $m \to \infty$ we get $\mu(F) = 0$.

Let now $a \in F^c$ and $0 < \epsilon < 1$. Then there is $k = 0, 1, \ldots,$ such that

$$\mu(B(a, 2^{1-k}\epsilon)) \le C\mu(B(a, 2^{-k}\epsilon))$$

and

$$\mu(B(a, 2^{1-j}\epsilon)) \ge C\mu(B(a, 2^{-j}\epsilon) \text{ for } j = 0, \ldots, k-1,$$

whence

$$\mu(B(a, 2^{-j}\epsilon)) \le C^{-j}\mu(B(a,\epsilon) \text{ for } j = 0, \ldots, k-1.$$

Let $\epsilon_1 = 2^{-k}\epsilon$. Then $\mu(B(a, 2\epsilon_1)) \le C\mu(B(a,\epsilon_1))$ and, since $C > 2C_2$, we get

$$|T_\epsilon(1)(a) - T_{\epsilon_1}(1)(a)| \le \sum_{j=1}^{k} |T_{2^{1-j}\epsilon}(1)(a) - T_{2^{-j}\epsilon}(1)(a)|$$
$$\le \sum_{j=1}^{k} \int_{B(a,2^{1-j}\epsilon)\setminus B(a,2^{-j}\epsilon)} |K(a,x)|\, d\mu x$$
$$\le \sum_{j=1}^{k} \frac{\mu(B(a, 2^{1-j}\epsilon))}{h(2^{-j}\epsilon)}$$
$$\le \sum_{j=1}^{k} \frac{C^{1-j}\mu(B(a,\epsilon))}{C_2^{-j}h(\epsilon)} \le C\,\eta(\epsilon)\sum_{j=1}^{k} 2^{-j} \le C\,\eta(\epsilon) < \delta$$

when ϵ is small enough. Consequently,

$$|\phi(\epsilon) - \phi(\epsilon_1)| \le |T_\epsilon(1)(a) - T_{\epsilon_1}(1)(a)|$$
$$+ \left| \frac{1}{\mu(B(a,\epsilon))}\int_{B(a,\epsilon)} T(1)\, d\mu - \frac{1}{\mu(B(a,\epsilon_1))}\int_{B(a,\epsilon_1)} T(1)\, d\mu \right| < \delta$$

when ϵ is small enough. Now we estimate the average of $|\phi_3(t)|$ over $[\epsilon_1, 2\epsilon_1]$ by

$$\begin{aligned}
&\frac{1}{\epsilon_1}\int_{\epsilon_1}^{2\epsilon_1} |\phi_3(t)|\, dt \\
&\leq \frac{1}{\epsilon_1}\int_{\epsilon_1}^{2\epsilon_1} \frac{1}{\mu(B(a,t))}\int_{B(a,pt)\setminus B(a,t)}\int_{B(a,t)} |K(x,y)|\, d\mu x\, d\mu y\, dt \\
&= \frac{1}{\epsilon_1}\iiint_A \frac{1}{\mu(B(a,t))}|K(x,y)|\, d\mu x\, d\mu y\, dt
\end{aligned}$$

where

$$\begin{aligned}
A &= \{(x,y,t) : d(x,a) < t \leq d(y,a) < pt,\ \epsilon_1 \leq t \leq 2\epsilon_1\} \\
&\subset \{(x,y,t) : d(x,a) < 2\epsilon_1,\ d(y,a) < 2p\epsilon_1,\ d(x,a) < t \leq d(y,a)\}.
\end{aligned}$$

Thus by Fubini's theorem, (1.7) and (4.1),

$$\begin{aligned}
&\frac{1}{\epsilon_1}\int_{\epsilon_1}^{2\epsilon_1} |\phi_3(t)|\, dt \\
&\leq \frac{1}{\epsilon_1\mu(B(a,\epsilon_1))}\int_{B(a,2p\epsilon_1)}\int_{B(a,2\epsilon_1)} |K(x,y)| \int_{d(x,a)}^{d(y,a)} dt\, d\mu x\, d\mu y \\
&= \frac{1}{\epsilon_1\mu(B(a,\epsilon_1))}\int_{B(a,2\epsilon_1)}\int_{B(a,2p\epsilon_1)} |K(x,y)|(d(y,a) - d(x,a))\, d\mu y\, d\mu x \\
&\leq \frac{1}{\epsilon_1\mu(B(a,\epsilon_1))}\int_{B(a,2\epsilon_1)}\int_{B(x,2(p+1)\epsilon_1)} |K(x,y)|\, d(x,y)\, d\mu y\, d\mu x \\
&\leq \frac{1}{\epsilon_1\mu(B(a,\epsilon_1))}\int_{B(a,2\epsilon_1)}\sum_{i=0}^{\infty}\int_{B(x,2^{1-i}(p+1)\epsilon_1)\setminus B(x,2^{-i}(p+1)\epsilon_1)} |K(x,y)|\, d(x,y)\, d\mu y\, d\mu x \\
&\leq \frac{1}{\epsilon_1\mu(B(a,\epsilon_1))}\int_{B(a,2\epsilon_1)}\sum_{i=0}^{\infty}\frac{2^{1-i}(p+1)\epsilon_1\mu(B(x,2^{1-i}(p+1)\epsilon_1))}{h(2^{-i}(p+1)\epsilon_1)}\, d\mu x \\
&\leq \frac{1}{\epsilon_1\mu(B(a,\epsilon_1))}\sum_{i=0}^{\infty}\frac{2^{1-i}(p+1)\epsilon_1\eta(2^{1-i}(p+1)\epsilon_1)h(2^{1-i}(p+1)\epsilon_1))}{h(2^{-i}(p+1)\epsilon_1)}\mu(B(a,2\epsilon_1)) \\
&\leq \frac{4C_2(p+1)\eta(2(p+1)\epsilon_1))\mu(B(a,2\epsilon_1))}{\mu(B(a,\epsilon_1))} \leq 4CC_2(p+1)\eta(2(p+1)\epsilon) < \delta.
\end{aligned}$$

when ϵ is small enough. So there is $\epsilon_2, \epsilon_1 \leq \epsilon_2 \leq 2\epsilon_1$, such that $|\phi_3(\epsilon_2)| < \delta$. Then $|\phi(\epsilon_1) - \phi(\epsilon_2)| < \delta$ as above and so

$$\begin{aligned}
|\phi(\epsilon)| &\leq |\phi(\epsilon) - \phi(\epsilon_1)| + |\phi(\epsilon_1) - \phi(\epsilon_2)| + |\phi(\epsilon_2)| \\
&< 2\delta + |\phi_1(\epsilon_2)| + |\phi_2(\epsilon_2)| + |\phi_3(\epsilon_2)| < (4 + 2C_2)\delta.
\end{aligned}$$

This completes the proof of Theorem 1.6.

5. Remarks on rectifiability

One motivation for the developments in this paper was to find some new insight to the following problem:

Let m be an integer, $0 < m < n$, and let μ be an m-dimensional Ahlfors-David-regular Borel measure on $\mathbb{R}^n$, as in Section 1. For $i = 1, 2, \ldots, n$ let T_i^* be the maximal operator related to μ and the kernel $|x-y|^{-m-1}(x_i - y_i)$. Suppose that each T_i^* is bounded in $L^2(\mu)$. Does μ have to be rectifiable, or even uniformly rectifiable in the sense of David and Semmes?

By the rectifiability of μ we mean that there are m-dimensional C^1-surfaces $M_1, M_2, \ldots$ such that $\mu(\mathbb{R}^n \setminus \cup_i M_i) = 0$. For the definitions of uniform rectifiability, see [DS].

If $m = 1$, the answer to the above question is yes by [MMV], and the regularity assumptions on μ can be considerably relaxed, see [T2]. The problem is open for $m \geq 2$.

It was shown in [MPr], see also [M], that the rectifiability of an Ahlfors-David-regular measure μ follows from the existence of the principal values

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B(x,\epsilon)} |x-y|^{-m-1}(x_i - y_i)\, d\mu y, \quad i = 1, \ldots, n,$$

for μ almost all $x \in \mathbb{R}^n$. But it is not known if the L^2-boundedness implies the above almost everywhere convergence. Thus Theorem 1.4 is a kind of replacement for this. Unfortunately we don't know if the almost everywhere convergence of the averages of Theorem 1.4 implies rectifiablity, nor do we know if it implies the almost everywhere existence of the principal values in this particular case.

These questions are also related to geometric properties of removable sets of bounded analytic functions in $\mathbb{C}$, see [MMV], [P] and [T3], and of Lipschitz harmonic functions in $\mathbb{R}^n$, see [MP].

The L^2-boundedness does not always imply the almost everywhere existence of principal values in the setting of Theorem 1.4. This can be seen by considering a standard example of a purely unrectifiable 1-dimensional Ahlfors-David-regular set in the plane which is the Cantor set obtained by starting with the unit square, taking four squares of side-length 1/4 inside it in its corners, then taking the squares of side-length 1/16 in the corners of these, and so on. The final Cantor set C is the compact set inside all these squares of all generations. In [D2] David constructed a 1-dimensional odd Calderón-Zygmund kernel K such that the operator T^* related to K is bounded in $L^2(\mu)$ where μ is the natural (1-dimensional Hausdorff) measure on C. However, it is easy to check that the principal values

$$\lim_{\epsilon \to 0} \int_{B(x,\epsilon)^c} K(x-y)\, d\mu y$$

fail to exist at μ almost all points $x \in \mathbb{R}^2$.

In [H2] Huovinen considered homogeneous kernels such as

$$K(z) = Re(z/|z|^2 - z^3/|z|^4)$$

for $z \in \mathbb{C}$. He showed that there exist purely unrectifiable 1-dimensional Ahlfors-David-regular sets on which for such a kernel the principal values exist almost everywhere and the related operator is bounded in L^2 on some subset of positive measure. On the other hand, he showed in [H1] that for the kernels $z^{2k-1}/|z|^{2k}$, $k = 1, 2, \dots$, and their linear combinations the almost everywhere convergence of principal values on 1-dimensional AD-regular sets implies their rectifiability.

Acknowledgements. The authors are grateful to the "Centre de Recerca Matemàtica" for the perfect organization of a special trimester in Fourier Analysis during the spring of 2006. The event offered an excellent opportunity to work on the paper. The second named author was partially supported by grants 2005SGR00774 (Generalitat de Catalunya), MTM2004-00519 and HF2004-0208.

References

[C] M. Christ: *Lectures on Singular Integral Operators*, Regional Conference Series in Mathematics 77, Amer. Math. Soc., 1990.

[CW] R. R. Coifman and G. Weiss: *Analyse harmonique non-commutative sur certains espaces homogenes*, Lecture Notes in Mathematics 242, Springer-Verlag, 1971.

[D1] G. David: *Wavelets and Singular Integrals on Curves and Surfaces*, Lecture Notes in Mathematics 1465, Springer-Verlag, 1991.

[D2] G. David: Des intégrales singulières sur un ensemble de Cantor, *C. R. Acad. Sci. Paris Sér. I Math.* **332** (2001), 391–396.

[DS] G. David and S. Semmes: *Analysis of and on Uniformly Rectifiable Sets*, Surveys and Monograhs 38, Amer. Math. Soc., 1993.

[F] H. Federer: *Geometric Measure Theory*, Springer-Verlag, 1969.

[H1] P. Huovinen: Singular integrals and rectifiablity of measures in the plane, *Ann. Acad. Sci. Fenn. Math. Diss.* **109** (1997), 1–63.

[H2] P. Huovinen: A nicely behaved singular integral on a purely unrectifiable set, *Proc. Amer. Math. Soc.* **129** (2001), 3345–3351.

[M] P. Mattila: *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.

[MMV] P. Mattila, M.S. Melnikov and J. Verdera: The Cauchy integral, analytic capacity, and uniform rectifiability, *Ann. of Math.* **144** (1996), 127–136.

[MP] P. Mattila and P.V. Paramonov: On geometric properties of harmonic Lip_1-capacity, *Pacific J. Math.* **171** (1995), 469–491.

[MPr] P. Mattila and D. Preiss: Rectifiable measures in $\mathbb{R}^n$ and existence of principal values for singular integrals, *J. London Math. Soc.* **52** (1994), 482–496.

[NTV] F. Nazarov, S. Treil and A. Volberg: Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces, *International Math. Research Notices* No. 9 (1998), 463–487.

[P] H. Pajot: *Analytic Capacity, Rectifiablitity, Menger Curvature and the Cauchy Integral*, Lecture Notes in Mathematics 1799, Springer-Verlag, 2002.

[T1] X. Tolsa: Cotlar's inequality and the existence of principal values for the Cauchy integral without doubling condition, *J. Reine Angew. Math.* **502** (1998), 199–235.

[T2] X. Tolsa: Principal values for the Cauchy integral and rectifiability *Proc. Amer. Math. Soc* **128** (2000), 2111–2119.

[T3] X. Tolsa: Painlevés problem and the semiadditivity of the analytic capacity, *Acta Math.* **190** (2003), 105–149.

[Ve] J. Verdera: The fall of the doubling condition in Calderón-Zygmund theory, Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), *Publ. Mat.* Vol. Extra (2002), 275–292.

[V] A. Volberg: *Calderón-Zygmund Capacities and Operators on Nonhomogeneous Spaces*, Regional Conference Series in Mathematics 100, Amer. Math. Soc., 2002.

Pertti Mattila
Department of Mathematics and Statistics
P.O. Box 68
FI-00014 University of Helsinki
Finland
E-mail: `Pertti.Mattila@Helsinki.fi`

Joan Verdera
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona
Catalonia
E-mail: `jvm@mat.uab.cat`