COMPACT DIFFERENCES OF COMPOSITION OPERATORS ON BLOCH AND LIPSCHITZ SPACES

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Abstract. We consider the difference $T = C_\phi - C_\psi$ of two analytic composition operators in the unit disc. We characterize the compactness and weak compactness of T on the standard Bloch space, improving an earlier result by Hosokawa and Ohno. We also characterize the compactness and weak compactness of T on analytic Lipschitz spaces. These characterizations are derived from a general result dealing with differences of weighted composition operators on weighted Banach spaces of analytic functions. We also make complementary remarks on the compactness properties of a single composition operator on the Lipschitz spaces.

1. Introduction

Let $\mathbb{D}$ be the unit disc of the complex plane and assume that $\phi : \mathbb{D} \to \mathbb{D}$ is an analytic map. Then the composition operator C_ϕ taking f to $f \circ \phi$ is a linear operator on $H(\mathbb{D})$, the space of all analytic functions on $\mathbb{D}$. During the past few decades much effort has been devoted to the research of such operators on a variety of Banach spaces of analytic functions. The general idea has been to explain the operator-theoretic behaviour of C_ϕ, such as compactness and spectra, in terms of the function-theoretic properties of the symbol ϕ. We refer to the book by Cowen and MacCluer [2] for a rather comprehensive overview of the field as of the early 1990s.

Several authors have also studied the mapping properties of the difference of two composition operators, i.e. an operator of the form

$$T = C_\phi - C_\psi$$

where ϕ and ψ are two analytic self-maps of $\mathbb{D}$. The primary motivation for this has been the desire to understand the topological structure of the whole set of composition operators acting on a given function space. Most papers in this area have focused on (weighted) Bergman and Dirichlet spaces and especially the Hardy space H^2; see e.g. [10], [23], [21], [17], [9] and [18]. However, some classical non-reflexive spaces have also been considered lately. In [11] MacCluer, Ohno and Zhao described compact differences and connected components of composition operators on H^∞. Their work was extended to the setting of weighted composition operators by Hosokawa, Izuchi and Ohno [7]. Lastly, Hosokawa and Ohno [8] studied the same questions on the Bloch and little Bloch spaces of the disc.

The present paper continues this line of research. We will study the compactness and weak compactness of T primarily on the Bloch-type spaces $\mathcal{B}^\alpha$ consisting of all

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analytic functions f on $\mathbb{D}$ which satisfy the condition

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty.$$

It is well known that $\mathcal{B}^\alpha$ is a Banach space under the norm

$$\|f\|_\alpha = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)|.$$

(See the expository article [25] by Zhu for more information on these spaces.) Here α could be any positive index, but we will be mainly interested in the range $0 < \alpha \le 1$. Note that $\mathcal{B} = \mathcal{B}^1$ is just the standard Bloch space. For $0 < \alpha < 1$ it was proved by Hardy and Littlewood that a function f belongs to $\mathcal{B}^\alpha$ if and only if it is analytic in $\mathbb{D}$ and satisfies a Lipschitz condition of order $1-\alpha$, that is,

$$\sup_{z,w\in\mathbb{D}} \frac{|f(z)-f(w)|}{|z-w|^{1-\alpha}} < \infty$$

(see [4, Theorem 5.1]). In fact, the two suprema above are comparable to each other. Moreover, one should note that every Lipschitz function in $\mathbb{D}$ is boundary-regular in the sense that it extends continuously to the closed unit disc.

Before explaining our main results we need to fix some notation. For $z, w \in \mathbb{D}$, the pseudo-hyperbolic distance is defined by $\rho(z,w) = |z-w|/|1-\overline{w}z|$. The hyperbolic distance between z and w is then

$$\inf_\gamma \int_\gamma \frac{|d\zeta|}{1-|\zeta|^2} = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)},$$

where the infimum is taken over all rectifiable paths joining z and w in $\mathbb{D}$. When ϕ is an analytic self-map of $\mathbb{D}$, we will use the short-hand notation

$$\mathcal{D}^\alpha\phi(z) = \left(\frac{1-|z|^2}{1-|\phi(z)|^2}\right)^\alpha \phi'(z).$$

In the Bloch case $\alpha = 1$ we just write $\mathcal{D}\phi$ for $\mathcal{D}^1\phi$. It should be noted that $\mathcal{D}\phi$ is the hyperbolic derivative of ϕ in the sense that

$$|\mathcal{D}\phi(z)| = \lim_{w\to z} \frac{\rho(\phi(z),\phi(w))}{\rho(z,w)}.$$

More generally one can regard $\mathcal{D}^\alpha\phi$ as a derivative relative to a metric induced by the arc length element $(1-|\zeta|^2)^{-\alpha}|d\zeta|$ (see [25, §4]).

The importance of $\mathcal{D}^\alpha$-derivatives to the study of composition operators on $\mathcal{B}^\alpha$ stems from the identity

$$(1-|z|^2)^\alpha |(C_\phi f)'(z)| = |\mathcal{D}^\alpha\phi(z)| \cdot (1-|\phi(z)|^2)^\alpha |f'(\phi(z))|, \tag{1.1}$$

which basically shows that the condition $\|\mathcal{D}^\alpha\phi\|_\infty < \infty$ is sufficient for C_ϕ to be bounded on $\mathcal{B}^\alpha$. For $\alpha = 1$ this is always true: the classical Schwarz–Pick inequality actually says that $\|\mathcal{D}\phi\|_\infty \le 1$. For $0 < \alpha < 1$ this is not the case, and Madigan [13] observed that the condition is also necessary for the boundedness of C_ϕ on $\mathcal{B}^\alpha$ (see also [2, Theorem 4.9]).

We are now ready to state our main results. Here we consider two analytic maps $\phi, \psi : \mathbb{D} \to \mathbb{D}$ and we let $T = C_\phi - C_\psi$. We also agree to write $\rho(z) = \rho(\phi(z), \psi(z))$ for the pseudo-hyperbolic distance between $\phi(z)$ and $\psi(z)$. Our first theorem deals with the standard Bloch case, characterizing the compactness and weak compactness of T.

Theorem 1.1. *T is (weakly) compact on $\mathcal{B}$ if and only if*

$$\mathcal{D}\phi(z)\rho(z) \to 0 \quad \textit{as } |\phi(z)| \to 1, \tag{B1}$$

$$\mathcal{D}\psi(z)\rho(z) \to 0 \quad \textit{as } |\psi(z)| \to 1. \tag{B2}$$

Recently Hosokawa and Ohno [8] characterized the compactness of T on $\mathcal{B}$ by requiring (B1) and (B2) plus an additional condition which essentially says that

$$\mathcal{D}\phi(z) - \mathcal{D}\psi(z) \to 0 \quad \text{as } |\phi(z)| \wedge |\psi(z)| \to 1.$$

(We use $\wedge$ to refer to the minimum of two real numbers, and $\vee$ to the maximum.) Our contribution is to show that this third condition is actually implied by (B1) and (B2), so it can be dispensed with.

As an immediate corollary to Theorem 1.1 we obtain a very simple sufficient condition for the compactness of T.

Corollary 1.2. *If $\rho(z) \to 0$ as $|\phi(z)| \vee |\psi(z)| \to 1$, then T is compact on $\mathcal{B}$.*

To understand the conditions of Theorem 1.1 and Corollary 1.2, one should recall that Madigan and Matheson [14] showed that a single composition operator C_ϕ is (weakly) compact on $\mathcal{B}$ if and only if $\mathcal{D}\phi(z) \to 0$ as $|\phi(z)| \to 1$. This is just a natural "little-oh" variant of the Schwarz–Pick inequality. On the other hand, the condition of Corollary 1.2 is known to guarantee the compactness of T on various spaces, such as (weighted) Bergman and Diriclet spaces and Hardy spaces (see [16], [9]). In fact, it was shown by MacCluer, Ohno and Zhao [11] to characterize the compactness of T on the space H^∞ of bounded analytic functions.

Our second theorem is concerned with the Lipschitz case $0 < \alpha < 1$. As mentioned above, we have to assume that the $\mathcal{D}^\alpha$-derivatives of the symbols are bounded so as to guarantee the boundedness of the induced operators.

Theorem 1.3. *Let $0 < \alpha < 1$ and assume that $\|\mathcal{D}^\alpha \phi\|_\infty < \infty$ and $\|\mathcal{D}^\alpha \psi\|_\infty < \infty$. Then T is (weakly) compact on $\mathcal{B}^\alpha$ if and only if*

$$\mathcal{D}^\alpha\phi(z)\rho(z) \to 0 \quad \textit{as } |\phi(z)| \to 1, \tag{L1}$$

$$\mathcal{D}^\alpha\psi(z)\rho(z) \to 0 \quad \textit{as } |\psi(z)| \to 1, \tag{L2}$$

$$\mathcal{D}^\alpha\phi(z) - \mathcal{D}^\alpha\psi(z) \to 0 \quad \textit{as } |\phi(z)| \wedge |\psi(z)| \to 1. \tag{L3}$$

Conditions (L1) and (L2) are obvious analogues of those in Theorem 1.1. In the present case, however, one has to impose the additional condition (L3) to guarantee the (weak) compactness of T. In fact, we will later construct symbols ϕ and ψ, both satisfying Madigan's boundedness condition, such that $\rho(z) \to 0$ as $|\phi(z)| \vee |\psi(z)| \to 1$ but (L3) fails and consequently T is non-compact on $\mathcal{B}^\alpha$.

We will approach Theorems 1.1 and 1.3 in a unified way. In fact, in Section 2 we will consider a very general setup where we have the difference of two weighted composition operators acting between two weighted H^∞-type spaces. The proof of Theorem 1.1 occupies Section 3. In Section 4 we will consider Theorem 1.3, especially addressing the necessity of condition (L3).

In Section 5, which is largely independent of the previous sections, we briefly revisit the theory of a single composition operator on the Lipschitz spaces. We will answer a question of Cowen and MacCluer on the boundedness of such an operator. In addition, we will explore the function-theoretic relationship between various compactness criteria given in the literature.

2. Differences of weighted composition operators on weighted spaces

In this section we will examine the following general setup. Given analytic functions $\phi : \mathbb{D} \to \mathbb{D}$ and $u : \mathbb{D} \to \mathbb{C}$ we define the *weighted composition operator*

$$W_{\phi,u} : H(\mathbb{D}) \to H(\mathbb{D}), \quad f \mapsto u(f \circ \phi).$$

We also define the weighted function spaces

$$H^\infty_\alpha = \Big\{ f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f(z)| < \infty \Big\}$$

for $0 < \alpha < \infty$. These are Banach spaces under the norm determined by the above supremum, which we will denote by $\|f\|_{H^\infty_\alpha}$. Montes-Rodríguez [15] and Contreras and Hernández-Díaz [1] have studied $W_{\phi,u}$ as an operator acting between this type of weighted spaces (with even more general weights). In particular, they have shown that $W_{\phi,u}$ is a bounded operator from H^∞_α to H^∞_β if and only if the pair (ϕ, u) satisfies

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |u(z)|}{(1-|\phi(z)|^2)^\alpha} < \infty. \tag{2.1}$$

They also characterized the (weak) compactness of $W_{\phi,u}$ by the corresponding "little-oh" condition as $|\phi(z)| \to 1$.

One should note that the differentiation map $f \mapsto f'$ is a linear isometry from $\mathcal{B}^\alpha$ onto H^∞_α, provided that in $\mathcal{B}^\alpha$ we identify functions differing by a constant. Hence the unweighted composition operator C_ϕ acting between Bloch-type spaces modulo constants is similar to the weighted composition operator $W_{\phi,\phi'}$ acting between the corresponding weighted H^∞-spaces. Since the identification of functions differing by a constant does not affect the boundedness or compactness properties of the operator (see e.g. [1, §6]), the above-mentioned general results yield conditions for the boundedness and (weak) compactness of C_ϕ as an operator from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$. These conditions have also been derived in [24]. In particular, if $0 < \alpha = \beta < 1$, Madigan's boundedness condition $\|\mathcal{D}^\alpha \phi\|_\infty < \infty$ is obtained.

In the present section our goal is to investigate the compactness of the difference of two weighted composition operators on weighted spaces of the above type. To this end we introduce analytic maps $\phi, \psi : \mathbb{D} \to \mathbb{D}$ and $u, v : \mathbb{D} \to \mathbb{C}$ and look at the operator

$$T = W_{\phi,u} - W_{\psi,v}.$$

Our general result is the following. Recall that we use $\rho(z)$ to denote the pseudo-hyperbolic distance between $\phi(z)$ and $\psi(z)$.

Theorem 2.1. *Let α and β be positive real numbers, and assume that the pairs (ϕ, u) and (ψ, v) both satisfy (2.1). Then the operator T is (weakly) compact from H^∞_α to H^∞_β if and only if*

$$\frac{(1-|z|^2)^\beta u(z)}{(1-|\phi(z)|^2)^\alpha} \rho(z) \to 0 \quad \text{as } |\phi(z)| \to 1, \tag{2.2}$$

$$\frac{(1-|z|^2)^\beta v(z)}{(1-|\psi(z)|^2)^\alpha} \rho(z) \to 0 \quad \text{as } |\psi(z)| \to 1, \tag{2.3}$$

$$\frac{(1-|z|^2)^\beta u(z)}{(1-|\phi(z)|^2)^\alpha} - \frac{(1-|z|^2)^\beta v(z)}{(1-|\psi(z)|^2)^\alpha} \to 0 \quad \text{as } |\phi(z)| \wedge |\psi(z)| \to 1. \tag{2.4}$$

To prepare for the proof of this theorem we have to recall some notions related to weak compactness. A Banach space X is said to have the *Dunford–Pettis property* if $x_n^*(x_n) \to 0$ whenever $x_n \to 0$ weakly in X and $x_n^* \to 0$ weakly in the dual space X^*. Equivalently, this means that every weakly compact linear operator from X into some Banach space is completely continuous, i.e. maps weakly null sequences into norm-null sequences. A well-known example of a space with this property is c_0, the space of null sequences of scalars endowed with the supremum norm. For a survey of the Dunford–Pettis property we refer to [3].

The auxiliary functions provided by the next lemma will be used to construct appropriate weakly convergent test function sequences. Instead of this quite elementary lemma, one could utilize more refined results on interpolating functions here (see e.g. [5, VII.2]).

Lemma 2.2. *Let (a_n) be a sequence in $\mathbb{D}$ such that $a_n \to 1$. Then there exist numbers $0 < \epsilon_n < 1$ and $0 < \delta_n < \delta_n' < \pi$ and functions $Q_n \in H^\infty$ such that $\epsilon_n \to 0$, $\delta_n' \to 0$, $\|Q_n\|_\infty \le 1$, $|Q_n(a_n)| \ge 1/2$ and $|Q_n(e^{it})| \le \epsilon_n$ when $|t| \le \delta_n$ or $\delta_n' \le |t| \le \pi$.*

Proof. The functions Q_n can be realized as outer functions satisfying

$$\log|Q_n(z)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\log q_n(t)\,dt,$$

where $q_n(t) = 1$ for $\delta_n < |t| < \delta_n'$ and $q_n(t) = \epsilon_n$ otherwise. We leave it to the reader to check that the numbers ϵ_n, δ_n and δ_n' can be chosen in such a way that the requirements of the lemma are fulfilled. □

One more lemma will be needed. It is certainly known to specialists but we sketch the proof for completeness. Here and throughout the paper we will use the abbreviated notation $A \lesssim B$ to mean $A \le CB$ for some inessential constant $C > 0$ depending possibly on α, and $A \sim B$ if $A \lesssim B \lesssim A$.

Lemma 2.3. *For $f \in H_\alpha^\infty$ and $z, w \in \mathbb{D}$,*

$$\big|(1-|z|^2)^\alpha f(z) - (1-|w|^2)^\alpha f(w)\big| \lesssim \|f\|_{H_\alpha^\infty}\rho(z,w).$$

Proof. Assume $\|f\|_{H_\alpha^\infty} \le 1$. Then $|f(\zeta)| \le (1-|\zeta|^2)^{-\alpha}$ and $|f'(\zeta)| \lesssim (1-|\zeta|^2)^{-\alpha-1}$ for $\zeta \in \mathbb{D}$ (see e.g. [4, Theorem 5.5]). Write $h(\zeta) = (1-|\zeta|^2)^\alpha f(\zeta)$. It is straightforward to check that $|\nabla(1-|\zeta|^2)^\alpha| \lesssim (1-|\zeta|^2)^{\alpha-1}$. Therefore, by the product rule of differentiation,

$$\begin{aligned}|\nabla h(\zeta)| &\lesssim (1-|\zeta|^2)^{\alpha-1}(1-|\zeta|^2)^{-\alpha} + (1-|\zeta|^2)^{\alpha}(1-|\zeta|^2)^{-\alpha-1}\\ &\lesssim (1-|\zeta|^2)^{-1}.\end{aligned}$$

Since $(1-|\zeta|^2)^{-1}|d\zeta|$ is the element of arc length in the hyperbolic metric, we have established the assertion of the lemma with the hyperbolic distance in place of the pseudo-hyperbolic one; that is,

$$|h(z) - h(w)| \lesssim \log\frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

To finish the proof, we consider two cases. If $\rho(z,w) < 1/2$, routine estimates show that the logarithm here is less than $3\rho(z,w)$. If $\rho(z,w) \ge 1/2$, we just observe that since $|h|$ is bounded by 1, we trivially have $|h(z)-h(w)| \le 2 \le 4\rho(z,w)$. □

Proof of Theorem 2.1. Necessity. Assume that T is weakly compact. We first prove condition (2.2). Let (z_n) be a sequence of points in $\mathbb{D}$ such that $|\phi(z_n)| \to 1$. By passing to a subsequence and applying a rotation argument we may assume that $\phi(z_n) \to 1$. Let (Q_n) be the sequence of functions provided by Lemma 2.2 with respect to the points $(\phi(z_n))$. By passing to a further subsequence we may assume that the quantities of the lemma satisfy $\epsilon_n \leq 2^{-n}$ and $\delta'_{n+1} \leq \delta_n$ for all n.

Now define

$$f_n(z) = \frac{Q_n(z)}{(1 - \overline{\phi(z_n)}z)^\alpha} \cdot \frac{z - \psi(z_n)}{1 - \overline{\psi(z_n)}z}.$$

Then $|f_n(z)| \leq |Q_n(z)|/(1-|z|)^\alpha$. Since the sets $\{e^{it} : |Q_n(e^{it})| > \epsilon_n\}$ are pairwise disjoint and $\sum_n \epsilon_n \leq 1$, it is easy to see that the mapping $(\xi_n) \mapsto \sum_n \xi_n f_n$ takes the sequence space c_0 continuously into H^∞_α. Therefore $f_n \to 0$ weakly in H^∞_α, and since T was assumed weakly compact, the Dunford–Pettis property of c_0 yields that $\|Tf_n\|_{H^\infty_\beta} \to 0$. However, by the definition of f_n and the fact that $|Q_n(\phi(z_n))| \geq 1/2$ we have

$$\|Tf_n\|_{H^\infty_\beta} \geq (1-|z_n|^2)^\beta |Tf_n(z_n)| \geq \frac{1}{2}\frac{(1-|z_n|^2)^\beta |u(z_n)|}{(1-|\phi(z_n)|^2)^\alpha}\rho(z_n),$$

so the right-hand side here must also converge to zero. This proves (2.2), and (2.3) is analogous.

For the proof of (2.4) we begin with any sequence (z_n) for which $|\phi(z_n)| \to 1$ and $|\psi(z_n)| \to 1$. Again we may assume $\phi(z_n) \to 1$ and in view of (2.2) and (2.3) also that $\rho(z_n) \to 0$. We then proceed as above, choosing functions Q_n corresponding to the sequence $(\phi(z_n))$ by Lemma 2.2, passing to a subsequence, and defining test functions

$$g_n(z) = \frac{Q_n(z)}{(1 - \overline{\phi(z_n)}z)^\alpha}.$$

As previously, we deduce that $\|Tg_n\|_{H^\infty_\beta} \to 0$. Now we have the estimate

$$\begin{aligned}\|Tg_n\|_{H^\infty_\beta} &\geq (1-|z_n|^2)^\beta |Tg_n(z_n)| \\ &= \left| \frac{(1-|z_n|^2)^\beta u(z_n) Q_n(\phi(z_n))}{(1-|\phi(z_n)|^2)^\alpha} - \frac{(1-|z_n|^2)^\beta v(z_n) Q_n(\psi(z_n))}{(1-\overline{\phi(z_n)}\psi(z_n))^\alpha} \right|.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\left| \frac{(1-|z_n|^2)^\beta v(z_n) Q_n(\phi(z_n))}{(1-|\psi(z_n)|^2)^\alpha} - \frac{(1-|z_n|^2)^\beta v(z_n) Q_n(\psi(z_n))}{(1-\overline{\phi(z_n)}\psi(z_n))^\alpha} \right| \\ &\quad = \frac{(1-|z_n|^2)^\beta |v(z_n)|}{(1-|\psi(z_n)|^2)^\alpha} \big| (1-|\phi(z_n)|^2)^\alpha g_n(\phi(z_n)) - (1-|\psi(z_n)|^2)^\alpha g_n(\psi(z_n)) \big|,\end{aligned}$$

where the first factor stays bounded because $W_{\psi,v}$ is a bounded operator and the second factor converges to zero by Lemma 2.3. Putting these observations together we conclude that the difference in (2.4) tends to zero along the sequence (z_n). This completes the proof of the necessity part.

Sufficiency. We assume conditions (2.2)–(2.4) and prove that T is compact. As usual, let (f_n) be a sequence in H^∞_α such that $\|f_n\|_{H^\infty_\alpha} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. We have to show that $\|Tf_n\|_{H^\infty_\beta} \to 0$.

To prepare, we let $\epsilon > 0$ and use (2.2)–(2.4) to find $r \in (0, 1)$ large enough such that

$$\frac{(1-|z|^2)^\beta |u(z)|}{(1-|\phi(z)|^2)^\alpha}\rho(z) \le \epsilon \quad \text{when } |\phi(z)| > r, \tag{2.5}$$

$$\frac{(1-|z|^2)^\beta |v(z)|}{(1-|\psi(z)|^2)^\alpha}\rho(z) \le \epsilon \quad \text{when } |\psi(z)| > r, \tag{2.6}$$

$$\left|\frac{(1-|z|^2)^\beta u(z)}{(1-|\phi(z)|^2)^\alpha} - \frac{(1-|z|^2)^\beta v(z)}{(1-|\psi(z)|^2)^\alpha}\right| \le \epsilon \quad \text{when } |\phi(z)| > r,\ |\psi(z)| > r. \tag{2.7}$$

We divide the argument into a few cases. First of all, it is clear that for points z with $|\phi(z)| \le r$ and $|\psi(z)| \le r$, the quantity

$$(1-|z|^2)^\beta |Tf_n(z)| = (1-|z|^2)^\beta |f_n(\phi(z))u(z) - f_n(\psi(z))v(z)|$$

converges to zero uniformly. Then suppose $|\psi(z)| > r$. We write $(1-|z|^2)^\beta |Tf_n(z)| = |A_n(z) + B_n(z)|$, where

$$A_n(z) = \left[\frac{(1-|z|^2)^\beta u(z)}{(1-|\phi(z)|^2)^\alpha} - \frac{(1-|z|^2)^\beta v(z)}{(1-|\psi(z)|^2)^\alpha}\right](1-|\phi(z)|^2)^\alpha f_n(\phi(z)),$$

$$B_n(z) = \frac{(1-|z|^2)^\beta v(z)}{(1-|\psi(z)|^2)^\alpha}\big[(1-|\phi(z)|^2)^\alpha f_n(\phi(z)) - (1-|\psi(z)|^2)^\alpha f_n(\psi(z))\big].$$

Here $|B_n(z)| \lesssim \epsilon$ by Lemma 2.3 and inequality (2.6). As regards $A_n(z)$, we observe that in the set where $|\phi(z)| \le r$ clearly $A_n(z) \to 0$ uniformly. On the other hand, if $|\phi(z)| > r$, then (2.7) implies $|A_n(z)| \le \epsilon$. Hence

$$\limsup_{n\to\infty} \sup\big\{(1-|z|^2)^\beta |Tf_n(z)| : |\psi(z)| > r\big\} \lesssim \epsilon.$$

Finally note that by symmetry considerations the same result also holds in the set where $|\phi(z)| > r$. Since ϵ was arbitary, we conclude that $(1-|z|^2)^\beta |Tf_n(z)| \to 0$ uniformly for $z \in \mathbb{D}$, and the proof of the sufficiency part is complete. ☐

3. The Bloch case

In this section we consider the difference operator $T = C_\phi - C_\psi$ as acting on the classical Bloch space $\mathcal{B}$. Note that ϕ and ψ can be any analytic self-maps of $\mathbb{D}$ because it follows from the Schwarz–Pick lemma that every composition operator is bounded on $\mathcal{B}$.

The following result is a corollary to Theorem 2.1 and the similarity argument explained before the statement of the theorem in Section 2. It was obtained earlier by Hosokawa and Ohno [8] (in a slightly different formulation).

Theorem 3.1 (Hosokawa–Ohno)**.** *T is (weakly) compact on $\mathcal{B}$ if and only if*

$$\mathcal{D}\phi(z)\rho(z) \to 0 \quad \textit{as } |\phi(z)| \to 1, \tag{B1}$$

$$\mathcal{D}\psi(z)\rho(z) \to 0 \quad \textit{as } |\psi(z)| \to 1, \tag{B2}$$

$$\mathcal{D}\phi(z) - \mathcal{D}\psi(z) \to 0 \quad \textit{as } |\phi(z)| \wedge |\psi(z)| \to 1. \tag{B3}$$

It turns out, however, that condition (B3) is implied by (B1) and (B2). Thus it can be dispensed with and we obtain Theorem 1.1, which we restate here.

Theorem 3.2. *T is (weakly) compact on $\mathcal{B}$ if and only if* (B1) *and* (B2) *hold.*

The proof of our result is based on a pair of rather elementary lemmas concerning continuity properties of hyperbolic derivatives. The first lemma is a special case of Theorem 6 in [6] (see also Remark 4.7 at the end of Section 4).

Lemma 3.3. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic map. Then $|\mathcal{D}\phi(z) - \mathcal{D}\phi(w)| \lesssim \rho(z, w)$ for all $z, w \in \mathbb{D}$.*

Lemma 3.4. *Let $\phi, \psi : \mathbb{D} \to \mathbb{D}$ be analytic maps. Then*

$$|\mathcal{D}\phi(z) - \mathcal{D}\psi(z)| \lesssim \frac{1}{r} \sup\{\rho(w) : \rho(z, w) \le r\}$$

for all $0 < r < 1$ and $z \in \mathbb{D}$.

Proof. Let $\sigma_w(z) = (w - z)/(1 - \overline{w}z)$, so that σ_w is the conformal automorphism of $\mathbb{D}$ that interchanges 0 and w. We begin by establishing the general inequality

$$|\sigma_w(z) - \sigma_{w'}(z')| \lesssim \rho(z, z') + \rho(w, w'), \tag{3.1}$$

which holds for all points $z, z', w, w' \in \mathbb{D}$ uniformly. To verify this, first note that $|\sigma_w(z) - \sigma_w(z')| \lesssim \rho(z, z')$ by the conformal invariance of the pseudo-hyperbolic distance. In addition, since $\partial_w\sigma_w(z) = 1/(1 - \overline{w}z)$ and $\partial_{\overline{w}}\sigma_w(z) = \sigma_w(z) \cdot z/(1 - \overline{w}z)$ are both less than $1/(1 - |w|)$ in modulus, we may argue as in the proof of Lemma 2.3 to get $|\sigma_w(z) - \sigma_{w'}(z)| \lesssim \rho(w, w')$. These observations, along with an application of the triangle inequality, yield (3.1).

To proceed to the actual proof, we note that for $z \in \mathbb{D}$ the derivative of $\sigma_{\phi(z)} \circ \phi \circ \sigma_z$ at the origin equals $\mathcal{D}\phi(z)$. Therefore, if $0 < r < 1$ is given, the Cauchy integral formula for derivatives yields the representation

$$\mathcal{D}\phi(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{(\sigma_{\phi(z)} \circ \phi \circ \sigma_z)(\zeta)}{\zeta^2}\, d\zeta.$$

An analogous formula holds for $\mathcal{D}\psi(z)$. Now we can apply (3.1) to get the estimate

$$|(\sigma_{\phi(z)} \circ \phi \circ \sigma_z)(\zeta) - (\sigma_{\psi(z)} \circ \psi \circ \sigma_z)(\zeta)| \lesssim \rho(\sigma_z(\zeta)) + \rho(z).$$

As ζ traverses the set $|\zeta| = r$, the point $w = \sigma_z(\zeta)$ runs through the pseudo-hyperbolic circle $\rho(z, w) = r$. Thus, denoting the supremum in the statement of the lemma by S, we arrive at the estimate

$$|\mathcal{D}\phi(z) - \mathcal{D}\psi(z)| \lesssim \frac{1}{2\pi} \int_{|\zeta|=r} \frac{S}{r^2}\, |d\zeta| = \frac{S}{r},$$

and the proof is complete. □

Proof of Theorem 3.2. We assume that conditions (B1) and (B2) of Theorem 3.1 hold, and we will prove that then (B3) is necessarily satisfied. Suppose (z_n) is a sequence in $\mathbb{D}$ for which $|\phi(z_n)| \to 1$ and $|\psi(z_n)| \to 1$. We wish to show that $\mathcal{D}\phi(z_n) - \mathcal{D}\psi(z_n) \to 0$.

Let $\epsilon > 0$. By Lemma 3.3 there exists $r \in (0, 1)$ such that

$$|\mathcal{D}\phi(z_n) - \mathcal{D}\psi(z_n)| \le |\mathcal{D}\phi(w) - \mathcal{D}\psi(w)| + \epsilon$$

whenever $\rho(z_n, w) \le r$. On the other hand, by Lemma 3.4 we can extract points $w_n \in \mathbb{D}$ with $\rho(z_n, w_n) \le r$ such that $|\mathcal{D}\phi(z_n) - \mathcal{D}\psi(z_n)| \lesssim \rho(w_n)$. On multiplying these two inequalities together we obtain

$$|\mathcal{D}\phi(z_n) - \mathcal{D}\psi(z_n)|^2 \lesssim |\mathcal{D}\phi(w_n) - \mathcal{D}\psi(w_n)|\rho(w_n) + \epsilon \tag{3.2}$$

for all n. Since $\rho(\phi(z_n), \phi(w_n)) \leq \rho(z_n, w_n) \leq r$, we necessarily have $|\phi(w_n)| \to 1$. Similarly $|\psi(w_n)| \to 1$. Hence conditions (B1) and (B2) imply that the first term on the right-hand side of (3.2) tends to zero. Therefore

$$\limsup_{n\to\infty} |\mathcal{D}\phi(z_n) - \mathcal{D}\psi(z_n)|^2 \lesssim \epsilon.$$

Since ϵ was arbitrary, the limit superior here must be zero. □

4. The Lipschitz case

When applied to differences of composition operators on the Lipschitz spaces $\mathcal{B}^\alpha$, where $0 < \alpha < 1$, Theorem 2.1 yields Theorem 1.3, which we restate here.

Theorem 4.1. *Let $0 < \alpha < 1$ and assume that $\phi, \psi : \mathbb{D} \to \mathbb{D}$ are analytic maps with $\|\mathcal{D}^\alpha \phi\|_\infty < \infty$ and $\|\mathcal{D}^\alpha \psi\|_\infty < \infty$. Then $T = C_\phi - C_\psi$ is (weakly) compact on $\mathcal{B}^\alpha$ if and only if*

$$\text{(L1)} \qquad \mathcal{D}^\alpha\phi(z)\rho(z) \to 0 \quad \text{as } |\phi(z)| \to 1,$$

$$\text{(L2)} \qquad \mathcal{D}^\alpha\psi(z)\rho(z) \to 0 \quad \text{as } |\psi(z)| \to 1,$$

$$\text{(L3)} \qquad \mathcal{D}^\alpha\phi(z) - \mathcal{D}^\alpha\psi(z) \to 0 \quad \text{as } |\phi(z)| \wedge |\psi(z)| \to 1.$$

We first point out some implications of the theorem. Let us recall here that all functions in $\mathcal{B}^\alpha$, hence ϕ and ψ, extend continuously to the closed disc $\overline{\mathbb{D}}$. Assume for the moment that $\zeta \in \partial\mathbb{D}$ is a point for which $|\phi(\zeta)| = 1$. Then it is known that ϕ has a finite angular derivative at ζ, say $\phi'(\zeta) = \delta$, and therefore $\mathcal{D}^\alpha\phi(z) \to \delta/|\delta|^\alpha$ as $z \to \zeta$ non-tangentially. (For these facts, see Section 5.) So, if (L1) holds, we actually have $\rho(z) \to 0$ as $z \to \zeta$ non-tangentially. In particular, then $\psi(\zeta) = \phi(\zeta)$, and with the aid of (L3) we further obtain $\psi'(\zeta) = \phi'(\zeta)$. Thus a necessary condition for the (weak) compactness of T on $\mathcal{B}^\alpha$ is that the symbols ϕ and ψ have the same unimodular boundary values and that their angular derivatives at those boundary points coincide. This condition is known to be necessary for the compactness of T on many other spaces as well, including the Hardy space H^2 (see [10] or [2, Theorem 9.16]).

The above reasoning leads to an interesting question, which we have been unable to answer.

Question 4.2. Let $0 < \alpha < 1$ and suppose $\|\mathcal{D}^\alpha\phi\|_\infty < \infty$ and $\|\mathcal{D}^\alpha\psi\|_\infty < \infty$. If T is compact on $\mathcal{B}^\alpha$, does it follow that $\rho(z) \to 0$ as $|\phi(z)| \vee |\psi(z)| \to 1$?

We observed above that a non-tangential version of this holds true. If the answer to the general question were positive, then the (weak) compactness of T would be characterized by the stated condition together with condition (L3), so Theorem 4.1 could be simplified considerably. Let us recall here that the answer is positive in the larger space H^∞ of bounded analytic functions [11].

We proceed to give a simple family of examples to illustrate the application of Theorem 4.1. It will be convenient to employ $\phi(z) = (1 + z)/2$ as a kind of reference map from which we are to build other maps with various properties. We will make repeated use of the identity

$$\text{(4.1)} \qquad 1 - |\phi(z)|^2 = \frac{1}{2}(1 - |z|^2) + \frac{1}{4}|z - 1|^2,$$

which can be verified by a direct calculation.

Example 4.3. Let ϕ be as above and put

$$\psi = \phi + \lambda, \quad \text{where} \quad \lambda(z) = c_p(z - 1)^p,\ p \geq 2.$$

Here $c_p > 0$ is chosen small enough in order that $\overline{\psi(\mathbb{D})} \subset \mathbb{D} \cup \{1\}$. For instance, if $c_p = 2^{-p-2}$, then $|\lambda(z)| \leq |z-1|^2/16$ and hence $1 - |\psi| \geq 1 - |\phi| - |\lambda| \geq (1 - |\phi|)/2$. Also note that since ϕ' and λ' are bounded in $\mathbb{D}$, both ϕ and ψ induce bounded composition operators on $\mathcal{B}^\alpha$.

Case $p > 2$. We can estimate

$$\rho(z) \leq \frac{|\lambda(z)|}{1 - |\phi(z)|} \lesssim \frac{|z-1|^p}{|z-1|^2} = |z-1|^{p-2},$$

so obviously $\rho(z) \to 0$ as $z \to 1$. Thus (L1) and (L2) are satisfied. To address (L3) we observe that, by (4.1) and the definition of λ, the ratio of $1 - |\phi(z)|$ and $1 - |\psi(z)|$ tends to 1 as $z \to 1$. Since $\lambda'(z) = p(z-1)^{p-1} \to 0$, it follows rather easily that (L3) is satisfied too. So $T = C_\phi - C_\psi$ is compact in this case.

Case $p = 2$. Now it is easily seen that $|1 - \phi(z)\overline{\psi(z)}| \sim 1 - |z|^2 + |z-1|^2$ and hence

$$\rho(z) \sim \frac{|z-1|^2}{1 - |z|^2 + |z-1|^2} = \left(1 + \frac{1 - |z|^2}{|z-1|^2}\right)^{-1}.$$

On the other hand,

$$|\mathcal{D}^\alpha \phi(z)| \sim \frac{(1-|z|^2)^\alpha}{(1 - |z|^2 + |z-1|^2)^\alpha} = \left(1 + \frac{|z-1|^2}{1 - |z|^2}\right)^{-\alpha}.$$

As a consequence we see that the limiting behaviour of $\rho(z)$ and $|\mathcal{D}^\alpha \phi(z)|$ as z tends to 1 depends strongly on the path of approach taken via the expression $(1-|z|^2)/|z-1|^2$. Indeed, if $(1-|z|^2)/|z-1|^2$ tends to zero or infinity (e.g. if $z \to 1$ non-tangentially), then one of these quantities converges to zero and the other is ~ 1, so $\mathcal{D}^\alpha\phi(z)\rho(z) \to 0$ in this case. But if $(1-|z|^2)/|z-1|^2$ tends to a positive constant (e.g. if $z \to 1$ along a circle that touches $\partial\mathbb{D}$ at 1), then $|\mathcal{D}^\alpha\phi(z)|\rho(z) \sim 1$. Therefore (L1) fails and T is non-compact.

The preceding examples leave open the natural question whether condition (L3) could be dispensed with in Theorem 4.1, as it was possible to do in the Bloch case in Section 3. We conclude the present section by giving a negative answer to this question. We will again start from the map $\phi(z) = (1+z)/2$, but the procedure used to construct the other map ψ will be somewhat complicated and requires careful analysis of the growth properties of $\mathcal{D}^\alpha$-derivatives.

Theorem 4.4. *Let ϕ be as above and $0 < \alpha < 1$. There is a map ψ, analytic on $\mathbb{D}$, with the following properties: (i) $\overline{\psi(\mathbb{D})} \subset \mathbb{D} \cup \{1\}$ and $\psi(1) = 1$, (ii) $\|\mathcal{D}^\alpha\psi\|_\infty < \infty$, (iii) $\rho(z) \to 0$ as $z \to 1$, and (iv) $\mathcal{D}^\alpha\phi(z) - \mathcal{D}^\alpha\psi(z) \not\to 0$ as $z \to 1$.*

In particular, conditions (L1) *and* (L2) *of Theorem 4.1 are satisfied but* (L3) *fails.*

We will make use of auxiliary functions κ_a and λ_a defined on $\mathbb{D}$ by

$$\kappa_a(z) = \frac{1 - |a|}{1 - \overline{a}z},$$
$$\lambda_a(z) = (z-1)^3 \kappa_a(z),$$

and depending on a parameter $a \in \mathbb{D}$. Note that $|\kappa_a(z)| \leq 1$ and so $|\lambda_a(z)| \leq |z-1|^3$ for all $z \in \mathbb{D}$. In addition,

$$\kappa_a'(z) = \frac{\overline{a}(1-|a|)}{(1-\overline{a}z)^2},$$
$$\lambda_a'(z) = (z-1)^3\kappa_a'(z) + 3(z-1)^2\kappa_a(z).$$

The intuitive idea behind our construction can be described as follows. We will employ the functions λ_a as elementary perturbations to the map ϕ. Adding λ_a to ϕ does not essentially alter the behaviour of the map near 1 in the hyperbolic scale. However, by a judicious choice of a we can influence the $\mathcal{D}^\alpha$-derivative of the resulting map just in the right way (Lemma 4.5). We will also observe (Lemma 4.6) that each perturbation λ_a is "local" in the sense that when a is close to the boundary of $\mathbb{D}$, the support of λ_a in the disc is essentially concentrated around the radius through point a. Thus it makes sense to define $\psi = \phi + c\sum_k \lambda_{a_k}$ where $c > 0$ is a small constant and (a_k) is a certain sequence of points in $\mathbb{D}$ converging to 1.

Lemma 4.5. *There are constants $c_1, c_2 > 0$ and $q \in (\frac{1}{2}, 1)$ depending only on α such that if $q < |a| < 1$ and*

$$1 - |a| = |a-1|^{\frac{3-2\alpha}{1-\alpha}}, \tag{4.2}$$

then, for all $z \in \mathbb{D}$,

$$\left(\frac{1-|z|^2}{1-|\phi(z)|^2}\right)^\alpha |\lambda_a'(z)| \le c_1 \tag{4.3}$$

and

$$\left(\frac{1-|a|^2}{1-|\phi(a)|^2}\right)^\alpha |\lambda_a'(a)| \ge c_2. \tag{4.4}$$

Proof. Let us write

$$A_a(z) = \left(\frac{1-|z|^2}{1-|\phi(z)|^2}\right)^\alpha (z-1)^3\kappa_a'(z),$$
$$B_a(z) = \left(\frac{1-|z|^2}{1-|\phi(z)|^2}\right)^\alpha (z-1)^2\kappa_a(z),$$

so that the expression on the left-hand side of (4.3) equals $|A_a(z) + 3B_a(z)|$.

We first prove (4.3) by showing that both A_a and B_a are uniformly bounded by a constant independent of a. Since $1 - |\phi(z)|^2 \gtrsim |z-1|^2$ and $|\kappa_a'(z)| \le 1/|1-\overline{a}z|$, we get

$$|A_a(z)| \lesssim \frac{(1-|z|^2)^\alpha |z-1|^3}{|z-1|^{2\alpha}|1-\overline{a}z|} \lesssim \frac{|z-1|^{3-2\alpha}}{|1-\overline{a}z|^{1-\alpha}}. \tag{4.5}$$

If $|z-1| \le 2|a-1|$, this is $\lesssim |a-1|^{3-2\alpha}/(1-|a|)^{1-\alpha}$, which is a constant by (4.2). If $|z-1| > 2|a-1|$, then $|1-\overline{a}z| \ge |z-1| - |a-1| \gtrsim |z-1|$, so clearly $A_a(z)$ is uniformly bounded also in this case. With regard to B_a, we use the simple estimates $1-|\phi(z)|^2 \gtrsim 1-|z|^2$ and $|\kappa_a(z)| \le 1$ to get $|B_a(z)| \lesssim |z-1|^2 \lesssim 1$. This completes the proof of the upper estimate (4.3).

To establish the lower estimate (4.4) we first observe that (4.1) and (4.2) imply $1-|\phi(a)|^2 \lesssim |a-1|^2$. In addition, $|\kappa_a'(a)| \gtrsim 1/(1-|a|)$. Hence

$$|A_a(a)| \gtrsim \frac{(1-|a|)^\alpha |a-1|^3}{|a-1|^{2\alpha}(1-|a|)} = \frac{|a-1|^{3-2\alpha}}{(1-|a|)^{1-\alpha}} = 1,$$

again by (4.2). Since $|B_a(a)| \lesssim |a-1|^2$, which tends to zero as $a \to 1$ (equivalently $|a| \to 1$), we conclude that (4.4) holds when $|a|$ is sufficiently close to 1. ☐

Lemma 4.6. *Assume that $a \in \mathbb{D}$ satisfies (4.2). For every $\epsilon > 0$, there exists $\delta > 0$ such that if $\theta = \arg a \in (0, \delta)$, then $|\kappa_a(z)| \le \epsilon$ and $|\kappa_a'(z)| \le \epsilon$ whenever $z \in \mathbb{D}$ such that $\arg z \in [0, 2\pi] \setminus (\frac{1}{2}\theta, \frac{3}{2}\theta)$.*

Proof. Write $z = re^{it}$ so that $t = \arg z$. In view of the expressions given for κ_a and κ_a' after the statement of Theorem 4.4, it suffices to show that the quotient

$$\frac{1-|a|}{|1-\overline{a}re^{it}|^2} = \frac{|a-1|^{\frac{3-2\alpha}{1-\alpha}}}{|1-\overline{a}re^{it}|^2} \tag{4.6}$$

can be made arbitrarily small for θ and $z = re^{it}$ as specified in the lemma. In the sequel we may actually assume $r \geq 1/2$; otherwise we would have $|1-\overline{a}re^{it}|^2 \geq 1/4$ for all a and t, yielding the claim immediately as $\theta \to 0+$ (or equivalently $a \to 1$).

Let us first consider the denominator (4.6). We have

$$|1-\overline{a}re^{it}|^2 = 1 + |a|^2r^2 - 2|a|r\cos(t-\theta).$$

Here $\cos(t-\theta)$ is at its maximum when $|t-\theta|$ is the smallest possible, i.e. equals $\theta/2$. Moreover, we have the elementary estimate $\cos(\theta/2) \leq 1 - c\theta^2$ for some $c > 0$. Thus

$$|1-\overline{a}re^{it}|^2 \geq (1-|a|r)^2 + 2|a|rc\theta^2 \geq (1-|a|)^2 + c|a|\theta^2.$$

The numerator of (4.6) can be estimated in the same way. Since $\cos\theta \geq 1 - \theta^2/2$, we have

$$|a-1|^2 = 1 + |a|^2 - 2|a|\cos\theta \leq (1-|a|)^2 + |a|\theta^2.$$

These estimates combine to show that $|a-1|^2 \lesssim |1-\overline{a}re^{it}|^2$. Since the exponent $(3-2\alpha)/(1-\alpha)$ in the numerator of (4.6) is greater than 2, it follows that the whole quotient converges to zero as $\theta \to 0+$ (or $a \to 1$), the convergence being uniform in r and t. This completes the proof. ☐

Proof of Theorem 4.4. To begin with, we employ Lemma 4.6 inductively to find a sequence (a_k) in $\mathbb{D}$, approaching point 1 along the curve (4.2), such that if $\theta_k = \arg a_k$, then $0 < \theta_{k+1} \leq \theta_k/3$ for every k and

$$|\kappa_{a_k}(z)| \leq 2^{-k}, \qquad |\kappa_{a_k}'(z)| \leq 2^{-k} \qquad \text{if } \arg z \in [0, 2\pi] \setminus (\tfrac{1}{2}\theta_k, \tfrac{3}{2}\theta_k). \tag{4.7}$$

Since the intervals $(\frac{1}{2}\theta_k, \frac{3}{2}\theta_k)$ are disjoint, inequalities (4.7) are certainly satisfied at every point z of $\mathbb{D}$ for all indices k with the possible exception of one k (depending on z). For this exceptional k we nevertheless have the trivial bound $|\kappa_{a_k}(z)| \leq 1$.

We may clearly assume that $|a_1|$, and hence each $|a_k|$, is greater than the number q of Lemma 4.5. Let

$$\lambda(z) = \frac{1}{64}\sum_{k=1}^{\infty} \lambda_{a_k}(z) = \frac{(z-1)^3}{64}\sum_{k=1}^{\infty} \kappa_{a_k}(z).$$

By the remarks above we see that λ is a well-defined analytic function in $\mathbb{D}$ (with continuous extension to $\overline{\mathbb{D}}$) and

$$|\lambda(z)| \leq \frac{1}{32}|z-1|^3 \leq \frac{1}{16}|z-1|^2.$$

Put $\psi = \phi + \lambda$. Since $1 - |\phi(z)| \geq |z-1|^2/8$, we have $1 - |\psi(z)| \geq (1-|\phi(z)|)/2$, so ψ is an analytic function satisfying requirement (i) of the theorem. Moreover, we may estimate

$$\rho(z) \leq \frac{|\lambda(z)|}{1-|\phi(z)|-|\lambda(z)|} \leq \frac{|z-1|^3/32}{|z-1|^2/16} = \frac{1}{2}|z-1|,$$

from which (iii) obtains.

It remains to verify (ii) and (iv). The preceding observations imply that $1-|\phi(z)|$ is comparable to $1-|\psi(z)|$ and their ratio tends to one as $z \to 1$. Therefore it is enough to show that the expression

$$\left(\frac{1-|z|}{1-|\phi(z)|}\right)^{\alpha} |\lambda'(z)|$$

stays bounded in $\mathbb{D}$ and does not converge to zero as $z \to 1$. To accomplish this we observe that by the definition of λ_{a_k} and inequalities (4.7) we have $|\lambda'_{a_k}(z)| \leq 2^{-k} \cdot 5|z-1|^2$ for all $z \in \mathbb{D}$ and all except at most one k. The first claim follows from this and the first part of Lemma 4.5 (applied to the exceptional λ_{a_k}). To verify the second claim we apply the second part of Lemma 4.5 to conclude that the above expression does not converge to zero as we approach point 1 along the sequence (a_k). □

Remark 4.7. The argument presented in Section 3 to get rid of condition (B3) in Theorem 3.1 fails in the context of Lipschitz spaces because there is no counterpart of Lemma 3.4 for $\mathcal{D}^\alpha$-derivatives in general. Lemma 3.3, in turn, could be carried over to the Lipschitz case; namely, for any α one has

$$|\mathcal{D}^\alpha \phi(z) - \mathcal{D}^\alpha \phi(w)| \lesssim \|\mathcal{D}^\alpha \phi\|_\infty \rho(z, w).$$

This can be deduced from the generalized Schwarz–Pick estimates obtained in [12]. In fact, Theorem 3 (and its proof) in [12] shows that

$$\frac{(1-|z|^2)^{\alpha+1}}{(1-|\phi(z)|^2)^{\alpha}} |\phi''(z)| \lesssim \|\mathcal{D}^\alpha \phi\|_\infty,$$

and by applying the product rule of differentiation (cf. the proof of Lemma 2.3) we get $|\nabla \mathcal{D}^\alpha \phi(z)| \lesssim \|\mathcal{D}^\alpha \phi\|_\infty/(1-|z|^2)$, which yields the desired estimate.

5. A single composition operator revisited

In this last section we briefly revisit the boundedness and compactness problems for a single composition operator on the Lipschitz spaces. We assume throughout this section that ϕ is an analytic self-map of the unit disc and $0 < \alpha < 1$.

An early contribution to the study of composition operators on analytic Lipschitz spaces was due to Roan [19], who sought after conditions for the boundedness and compactness of such operators. In his Corollary 1 the following result on boundedness is given:

- C_ϕ is bounded on $\mathcal{B}^\alpha$ if and only if $\phi \in \mathcal{B}^\alpha$ and there exist $M < \infty$ and $r < 1$ such that $|\phi'(z)| \leq M$ whenever $|\phi(z)| \geq r$.

Unfortunately, as noticed by Cowen and MacCluer [2, page 196], there appears to be an error in Roan's proof for the necessity of his condition. Thus Cowen and MacCluer mention it as an open question whether the result still holds. As a by-product of the work done in Section 4 we can give a negative answer to their question: *There are functions that fail Roan's condition but nonetheless induce a bounded composition operator on $\mathcal{B}^\alpha$.*

Example 5.1. Let $\psi = \phi + c\sum_k \lambda_{a_k}$ be the function constructed in the proof of Theorem 4.4. Then $a_k \to 1$ and $\psi(a_k) \to 1$ as $k \to 1$. Consider the derivative of ψ at a_k. By the second part of Lemma 4.5 plus equations (4.1) and (4.2) we have

$$|\lambda'_{a_k}(a_k)| \gtrsim \left(\frac{|a_k-1|^2}{1-|a_k|^2}\right)^{\alpha} \gtrsim |a_k - 1|^{-\alpha/(1-\alpha)},$$

so $|\lambda'_{a_k}(a_k)| \to \infty$ as $k \to \infty$. Arguing as at the end of the proof of Theorem 4.4 we now see that $|\psi'(a_k)| \to \infty$.

We conclude by making a few remarks on compactness. Let us make here the standing assumption that $\|\mathcal{D}^\alpha \phi\|_\infty < \infty$, so C_ϕ is always bounded on $\mathcal{B}^\alpha$. In the existing literature there are (at least) three different characterizations for the compactness of C_ϕ. First of all, in the work cited above, Roan stated the following:

- C_ϕ is compact on $\mathcal{B}^\alpha$ if and only if for every $\epsilon > 0$ there exists $r < 1$ such that $|\phi'(z)| \leq \epsilon$ whenever $|\phi(z)| \geq r$.

Later Shapiro [22] investigated the compactness problem in a general setting of boundary-regular "small" function spaces with conformal invariance. By a spectral-theoretic argument he obtained the surprising result that a necessary condition for the compactness of C_ϕ on such spaces is $\|\phi\|_\infty < 1$. In the Lipschitz case it follows almost trivially that his condition is also sufficient, thus yielding a complete characterization of compactness as follows:

- C_ϕ is compact on $\mathcal{B}^\alpha$ if and only if $\|\phi\|_\infty < 1$.

Finally, it is certainly possible to characterize the compactness of C_ϕ by an appropriate "little-oh" version of Madigan's [13] boundedness condition. That is:

- C_ϕ is (weakly) compact on $\mathcal{B}^\alpha$ if and only if $\mathcal{D}^\alpha \phi(z) \to 0$ as $|\phi(z)| \to 1$.

This result, although not explicitly stated by Madigan, follows by fairly standard arguments from the identity (1.1) and lends itself to many generalizations (see [1], [15], [24]). Of course, it could also be deduced from our Theorem 1.3 by taking $\psi \equiv 0$.

A natural question now arises: can one demonstrate the equivalence of these three compactness conditions by function-theoretic arguments, without invoking operator theory? Obviously, if Shapiro's condition holds, then the other two become trivial. Also, assuming the finiteness of $\|\mathcal{D}^\alpha \phi\|_\infty$ (or only that $\phi \in \mathcal{B}^\alpha$), a simple reasoning shows that Roan's condition implies the $\mathcal{D}^\alpha$-condition. However, there appears to be no function-theoretic argument to infer Shapiro's condition from the $\mathcal{D}^\alpha$-condition.

Our aim is to give such an argument. The key to it is the notion of angular derivatives and the following proposition. It should be noted that the proposition is already known (see [2, Corollary 4.10]), but the existing proof depends on the above-mentioned result of Shapiro. In what follows we will give a simple function-theoretic proof.

Proposition 5.2. *Let $0 < \alpha < 1$ and suppose C_ϕ is a bounded operator on $\mathcal{B}^\alpha$, that is, $\|\mathcal{D}^\alpha \phi\|_\infty < \infty$. Then ϕ has a finite angular derivative at every $\zeta \in \partial\mathbb{D}$ with $|\phi(\zeta)| = 1$.*

Let us recall that an analytic map $\phi : \mathbb{D} \to \mathbb{D}$ is said to have a *finite angular derivative* at $\zeta \in \partial\mathbb{D}$ if there exists a point $\omega \in \partial\mathbb{D}$ such that the difference quotient $(\phi(z) - \omega)/(z - \zeta)$ tends to a finite limit as $z \to \zeta$ non-tangentially. The limit is denoted by $\phi'(\zeta)$ and called the *angular derivative* of ϕ at ζ. Clearly then $\phi(\zeta) = \omega$ as a non-tangential limit.

The main result about angular derivatives is the following classical theorem. See, for example, [2, Theorem 2.44].

Theorem 5.3 (Julia–Carathéodory)**.** *For $\zeta \in \partial\mathbb{D}$ the following are equivalent:*

1. *ϕ has a finite angular derivative at ζ.*
2. *ϕ has a non-tangential limit of modulus 1 at ζ, and ϕ' has a finite non-tangential limit at ζ.*
3. *The quantity $d(\zeta) = \liminf_{z \to \zeta}(1 - |\phi(z)|)/(1 - |z|)$ is finite.*

Furthermore, under these conditions the non-tangential limit of ϕ' at ζ, the angular derivative $\phi'(\zeta)$ and the number $d(\zeta)\phi(\zeta)\overline{\zeta}$ all agree and the limit inferior in (3) *is a non-tangential limit.*

Proof of Proposition 5.2. Assume $\phi(1) = 1$. We then claim that ϕ has a finite angular derivative at 1. Let us define, for $0 < r < 1$,

$$h(r) = \left(\frac{1-r}{1-\phi(r)}\right)^{\alpha} \phi'(r), \qquad u(r) = \frac{1-\phi(r)}{1-r}.$$

The hypothesis of the proposition implies that h is a bounded function, and in view of the Julia–Carathéodory theorem the claim will follow if we show that u is also bounded.

Note that $\phi'(r) = -(1-r)u'(r) + u(r)$ and so

$$h(r) = u(r)^{-\alpha}[-(1-r)u'(r) + u(r)] = -(1-r)u(r)^{-\alpha}u'(r) + u(r)^{1-\alpha}.$$

If we write $v(r) = u(r)^{1-\alpha}$, then this is equivalent to

$$-\frac{1}{1-\alpha}(1-r)v'(r) + v(r) = h(r).$$

The general solution of this differential equation is

$$v(r) = -\frac{1-\alpha}{(1-r)^{1-\alpha}} \int_1^r \frac{h(s)}{(1-s)^{\alpha}}\, ds + \frac{C}{(1-r)^{1-\alpha}}.$$

Since h is bounded, the first term here is a bounded function of r. Moreover, the definition of v implies that $v(r)$ is of the order $o(1/(1-r)^{1-\alpha})$ as $r \to 1-$, so we must have $C = 0$. Hence v and u are bounded. $\square$

As a corollary we obtain the desired result that the "little-oh" condition for the $\mathcal{D}^{\alpha}$-derivative actually trivializes to Shapiro's compactness condition.

Corollary 5.4. *Let $0 < \alpha < 1$ and suppose $\phi \in \mathcal{B}^{\alpha}$ such that $\mathcal{D}^{\alpha}\phi(z) \to 0$ as $|\phi(z)| \to 1$. Then $\|\phi\|_{\infty} < 1$.*

Proof. Assume to the contrary that $|\phi(\zeta)| = 1$ for some $\zeta \in \partial\mathbb{D}$. By Proposition 5.2 ϕ has a finite angular derivative, say δ, at ζ. But by the Julia–Carathéodory theorem $(1-|\phi(z)|)/(1-|z|) \to |\delta|$ and $\phi'(z) \to \delta$ as $z \to \zeta$ non-tangentially. Hence $|\mathcal{D}^{\alpha}\phi(z)| \to |\delta|^{1-\alpha}$, which is a contradiction. $\square$

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