ASYMPTOTICAL BEHAVIOUR OF A CLASS OF SEMILINEAR DIFFUSION EQUATIONS

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Abstract. We improve the existing results of the long time asymptotical behaviour of some basic semilinear diffusion equations (with space variable in the whole space $\mathbb{R}$). Our method is elementary: it is based on explicit calculations, weighted sup–norm estimates and a fixed point argument.

1. Main result.

We consider in this paper the classical semilinear diffusion equation

$$\partial_t u \;=\; \partial_x^2 u + \mathcal{N}(u) \tag{1}$$

and the associated Cauchy problem with small, integrable initial data. Here $\mathcal{N}$ is a nonlinear function $\mathbb{D} := \{z \in \mathbb{C} \mid |z| \le 1\} \to \mathbb{C}$ to be specified later, or more generally, a nonlinear operator. The unknown function u is defined on $\mathbb{R} \times [0, \infty[=: \mathbb{R} \times \mathbb{R}^+$ and may be complex valued, and we denote the variables by $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ and $\partial_x := \partial/\partial x$ and $\partial_t := \partial/\partial t$.

In [1] the authors proved, among other things, that for $\mathcal{N}(u) := u^p$, $p = 4, 5, \dots$ the solution u has the form $u = e^{t\partial_x^2} f + v$ for some integrable $f : \mathbb{R} \to \mathbb{C}$, where

$$\|v(\cdot, t)\|_\infty := \sup_{x\in\mathbb{R}} |v(x, t)| \le \frac{C_\varepsilon}{t^{1-\varepsilon}}, \tag{2}$$

$\varepsilon > 0$ arbitrary. In other words, the solution to the nonlinear Cauchy problem behaves asymptotically like the solution of the corresponding linear problem (of order $1/\sqrt{t}$ in the sup–norm for large t), with a correction of order $1/t$ in the sup–norm.

Similar results were obtained recently by Zhao, [7], Theorem 1.10, for $\mathcal{N}(u) = O(1)|u|^p$, for non–integer p, but only in the case $p > 5$.

In this paper we prove the result e.g. for $\mathcal{N}(u) := \alpha u^p |u|^q$, for all constants $\alpha \in \mathbb{R}$, $p \in \mathbb{N}$, $q \ge 0$ with $p + q \ge 4$, which is a considerable improvement. Moreover, our method of proof is quite elementary: it is based on explicit calculations and weighted norm estimates in suitable Banach spaces. Also the representation of the leading term is most concrete in our work: it is explicitely the Gaussian function $\varphi(x, t) := (t + 1)^{-1/2} e^{-\frac{1}{4}x^2/(t+1)}$.

On the other hand the renormalization group method used in [1] seems to provide a shorter proof (for the more restricted case).

The nonlinear term can actually be more general in [1] and [7], and also in our paper. We refer to the cited papers and Section 3 for details.

We refer to [2], [5], [6] and [8] for the literature concerning (1). We just recall that a time–global solution is known to exist for small initial data, for $\mathcal{N}(u) := |u|^p$, if $p > 3$. If $p \le 3$, the solution blows up in finite time for general small data. See the classical paper [3], and also [4].

We now formulate the main result. The proof is postponed to Section 4. Assume that the bounded, measurable function $f : \mathbb{R} \to \mathbb{C}$ satisfies

$$\sup_{x\in\mathbb{R}} |f(x)|(1 + |x|)^{m+2+\rho} \le C, \tag{3}$$

where $m > 2$ and $\rho > 0$ can be fixed arbitrarily according to the wishes of the reader, and the small enough $C = C_{\mathcal{N}} > 0$ is fixed later. Let the function $\mathcal{N} : \mathbb{D} \to \mathbb{C}$ (or more generally, the operator $\mathcal{N}$) be as in Section 2. (For example, $\mathcal{N}(u) := \alpha u^p|u|^q$, where $\alpha \in \mathbb{R}$, $p \in \mathbb{N}$, $q \ge 0$ with $p + q \ge 4$; for this and other examples, see Section 3.)

Theorem 1. *The Cauchy problem*

$$\begin{aligned} \partial_t u &= \partial_x^2 u + \mathcal{N}(u) \qquad on \ \ \mathbb{R}\times\mathbb{R}^+ && (4)\\ u(x,0) &= f(x) \qquad for\ all \ \ x \in \mathbb{R}, && (5)\end{aligned}$$

has a unique classical solution $u : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{C}$*, which satisfies for some* $A \in \mathbb{C}$

$$u = \frac{A}{\sqrt{t+1}} e^{-\frac14 x^2/(t+1)} + v \quad \text{with} \ \ \|v(\cdot,t)\|_\infty \le \frac{C}{t+1}. \tag{6}$$

Notations. We denote by $C_w(\mathbb{R}\times\mathbb{R}^+)$ the Banach space of continuous functions $u : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{C}$ with a finite norm

$$|||u||| := \sup_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} \Big(1 + \frac{|x|}{\sqrt{t+1}}\Big)^m |u(x,t)|, \tag{7}$$

where $m > 2$ is fixed as above. We also denote $C_W(\mathbb{R}\times\mathbb{R}^+)$ the subspace of $C_w(\mathbb{R}\times\mathbb{R}^+)$ consisting of functions with finite norm

$$\|v\| := \sup_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} \Big(1 + \frac{|x|}{\sqrt{t+1}}\Big)^m (t+1)|v(x,t)|, \tag{8}$$

Given a fixed $t \in \mathbb{R}^+$, we also denote (with some abuse of notation) $\|u(\cdot,t)\|_t := \|u\|_t := \sup_{x\in\mathbb{R}}(1 + |x|/\sqrt{t+1})^m|u(x,t)|$.

We shall assume $|||u||| \le 1$ in the following, hence the expression $\mathcal{N}(u)$ in (4) is well defined.

We define $\varphi(x,t) := (t+1)^{-1/2}e^{-\frac14 x^2/(t+1)}$. By C, c, C' (respectively, c_j) etc. we denote strictly positive constants (resp., constant depending on j) which may vary from place to place but not in the same inequality.

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2. The nonlinear term.

We now present the assumption on $\mathcal{N}$ in Theorem 1. We first give a general form, and then consider some concrete examples in the next section.

So, in Theorem 1 we assume that the nonlinear operator $\mathcal{N} : C_w(\mathbb{R}\times\mathbb{R}^+) \cap \{|||u||| \le 1\} \to C_w(\mathbb{R}\times\mathbb{R}^+)$ satisfies the following: For some $\alpha \in \mathbb{C}$, for some $p \in \{0, 1, 2, \ldots\} =: \mathbb{N}$ and $q \ge 0$ with $p + q \ge 4$, for some $0 < C \le 1/2$, for every $A \in \mathbb{C}$, $|A| < C$, for every function $v \in C_W(\mathbb{R}\times\mathbb{R}^+)$ with $\|v\| \le C$, we have

$$\mathcal{N}(A\varphi + v) = \alpha A^p|A|^q\varphi^{p+q} + \mathcal{M}_{A,v}. \tag{9}$$

Here $\mathcal{M}_{A,v} \in C_W(\mathbb{R} \times \mathbb{R}^+)$, and we assume it satisfies the Lipschitz type condition

$$\begin{aligned} &\sup_{x\in\mathbb{R}} \Big(1 + \frac{|x|}{\sqrt{t+1}}\Big)^{2m} |\mathcal{M}_{A_1,v_1}(x,t) - \mathcal{M}_{A_2,v_2}(x,t)| \\ \le\ & \frac{C}{(t+1)^{5/2}}(|A_1 - A_2| + \|v_1 - v_2\|)(|A_1| + |A_2| + \|v_1\| + \|v_2\|) \\ =:\ & \frac{C}{(t+1)^{5/2}} L(A_1, A_2, v_1, v_2) \qquad \text{for all } t, \end{aligned} \tag{10}$$

for every A_j and v_j, $j = 1, 2$ (as A and v above).

So the idea is that the term $\alpha A^p |A|^q \varphi^{p+q}$ contains the largest stuff for large t; for example in the case $p + q = 4$ it is of order $(t + 1)^{-2}$. The point is to obtain the bound $(t + 1)^{-5/2}$ for large t for the remainder term, see (10).

More generally, $\mathcal{N}$ may also be a finite sum of such operators, i.e.

$$\mathcal{N}(A\varphi + v) = \sum_{k=1}^{n} \alpha_k A^{p_k} |A|^{q_k} \varphi^{p_k+q_k} + \mathcal{M}_{A,v}, \tag{11}$$

with α_k and (p_k, q_k) as α and (p, q) above, and $\mathcal{M}_{A,v}$ as in (10); we thus have

Lemma 1. *The acceptable nonlinear terms form a linear space.*

3. Examples of the nonlinear term.

Example 1. *The function $\mathcal{N}(u) := u^p$, where $p \in \mathbb{N}$, $p \ge 4$, satisfies the requirements of Section 2.*

For example, if $p = 4$, we have the representation (9) with $q = 0$ and $\alpha = 1$, since

$$(A\varphi + v)^4 = (A\varphi)^4 + \sum_{j=1}^{4} c_j v^j (A\varphi)^{4-j} =: (A\varphi)^4 + \mathcal{M}_{A,v}. \tag{12}$$

Given (A_1, v_1) and (A_2, v_2), let us for example show that the term $c_1 v_1 (A_1\varphi)^3 - c_1 v_2 (A_2\varphi)^3$ of $\mathcal{M}_{A_1,v_1} - \mathcal{M}_{A_2,v_2}$ satisfies the bound required in (10). Indeed,

$$\begin{aligned} & |v_1(A_1\varphi)^3 - v_2(A_2\varphi)^3| \\ =\ & |((A_1\varphi)^3 - (A_2\varphi)^3)v_1 + (A_2\varphi)^3(v_1 - v_2)| \\ \le\ & |A_1 - A_2| \max(|A_1|^2, |A_2|^2)\varphi^3 |v_1| + |A_2|^3 \varphi^3 |v_1 - v_2|, \end{aligned}$$

and here $|\varphi(x,t)| \le C(t+1)^{-1/2}(1 + |x|/\sqrt{t+1})^{-m}$ and $|v_j(x,t)| \le C(t+1)^{-1}(1 + |x|/\sqrt{t+1})^{-m}\|v_j\|$ and $|v_1(x,t) - v_2(x,t)| \le C(t+1)^{-1}(1 + |x|/\sqrt{t+1})^{-m}\|v_1 - v_2\|$ imply the requirement of (10).

The following important example shows that $\mathcal{N}$ does not need to be an analytic function of u, as in [1].

Example 2. *The function $\mathcal{N}(u) := u^p|u|^q$, where $p + q \ge 4$, $p \in \mathbb{N}$, also satisfies the assumptions of Section 2.*

Proof . Let A and v be given. We first derive two representations for $\mathcal{M}_{A,v}$. If (x, t) is such that

$$|A|\varphi(x,t) \ge 10|v(x,t)|, \tag{13}$$

we write (omitting the variables for notational simplicity)

$$u^p|u|^q = (A\varphi + v)^p |A\varphi + v|^q =: \Big((A\varphi)^p + \mathcal{M}'\Big)|A\varphi + v|^q. \tag{14}$$

Moreover, using the Taylor series of $(1+z)^{q/2}$ for small z,

$$\begin{aligned} |A\varphi + v|^q &= (|A|\varphi)^q \Big| 1 + \frac{v}{A\varphi}\Big|^q \\ &= (|A|\varphi)^q \Big(1 + 2\mathrm{Re}v/(|A|\varphi) + (\mathrm{Re}v/|A|\varphi)^2 + (\mathrm{Im}v/(|A|\varphi))^2\Big)^{q/2} \\ &= (|A|\varphi)^q + (|A|\varphi)^q \sum_{n=1}^{\infty} c_n \mathcal{T}_{n,A,v}, \qquad (15) \end{aligned}$$

where $|c_n| \leq 1$ for all n and

$$\mathcal{T}_{n,A,v} := \Big(2\frac{\mathrm{Re}v}{|A|\varphi} + \Big(\frac{\mathrm{Re}v}{|A|\varphi}\Big)^2 + \Big(\frac{\mathrm{Im}v}{|A|\varphi}\Big)^2\Big)^n. \qquad (16)$$

Hence, (14) equals

$$\begin{aligned} & A^p|A|^q\varphi^{p+q} + \sum_{n=1}^{\infty} c_n A^p|A|^q\varphi^{p+q}\mathcal{T}_{n,A,v} + \mathcal{M}'|A|^q\varphi^q + \mathcal{M}'|A|^q\varphi^q \sum_{n=1}^{\infty} c_n \mathcal{T}_{n,A,v} \\ &=: A^p|A|^q\varphi^{p+q} + \mathcal{M}_{A,v}. \qquad (17) \end{aligned}$$

On the other hand, if (x,t) satisfies $0 < |A|\varphi(x,t) \leq 10|v(x,t)|$, we directly write $\mathcal{N}(A\varphi) = A^p|A|^q\varphi^{p+q} + \mathcal{M}_{A,v}$, where

$$\mathcal{M}_{A,v} := -A^p|A|^q\varphi^{p+q} + (A\varphi + v)^p|A\varphi + v|^q. \qquad (18)$$

In the rest of the proof we are given two pairs (A_1, v_1) and (A_2, v_2), and our aim is to show that for all x and t

$$\begin{aligned} & |\mathcal{M}_{A_1,v_1}(x,t) - \mathcal{M}_{A_2,v_2}(x,t)| =: |\mathcal{M}_1(x,t) - \mathcal{M}_2(x,t)| \\ &\leq \frac{C}{(t+1)^{5/2}}\Big(1 + \frac{|x|}{\sqrt{t+1}}\Big)^{-2m} L(A_1, A_2, v_1, v_2), \qquad (19) \end{aligned}$$

which implies (10); we distinguish between three cases.

1°. Given (x,t) such that both pairs (A_1, v_1) and (A_2, v_2) satisfy (13), we use the representation (17). The proof of (19) is in this case lengthy, but straightforward, and we omit some details. Let us consider only as an example some terms of $|\mathcal{M}_1 - \mathcal{M}_2|$, like

$$\Big| A_1^p|A_1|^q\varphi^{p+q}\mathcal{T}_{n,A_1,v_1} - A_2^p|A_2|^q\varphi^{p+q}\mathcal{T}_{n,A_2,v_2}\Big|. \qquad (20)$$

If $|A_1 - A_2| \geq 10|A_1|$, then $|A_2| \geq 9|A_1|$ by the triangle inequality, and hence also $|A_1 - A_2| \geq |A_2| - |A_1| \geq |A_2| - |A_2|/9 \geq |A_2|/2$. In this case we thus have (see (10))

$$L(A_1, A_2, v_1, v_2) \geq C|A_2|^2. \qquad (21)$$

On the other hand, the first term in (20) has the bound

$$\begin{aligned} & C|A_1|^{p+q}\varphi^{p+q}(x,t)\Big(\frac{|v_1(x,t)|}{|A_1|\varphi(x,t)}\Big)^n \\ &= C|A_1|^{p+q-1}\varphi^{p+q-1}(x,t)|v_1(x,t)|\Big(\frac{|v_1(x,t)|}{|A_1|\varphi(x,t)}\Big)^{n-1} \\ &\leq C'|A_1|^{p+q-1}\varphi^{p+q-1}(x,t)|v_1(x,t)| \\ &\leq \frac{C''|A_1|^2}{(t+1)^{5/2}}\Big(1 + \frac{|x|}{\sqrt{t+1}}\Big)^{-2m} \leq \frac{C''|A_2|^2}{(t+1)^{5/2}}\Big(1 + \frac{|x|}{\sqrt{t+1}}\Big)^{-2m}, \qquad (22) \end{aligned}$$

and the second term of (20) satisfies the same bound. This, in view of (21), implies (19). In the same way one treats the case $|A_1 - A_2| \geq 10|A_2|$. So we are left with the case $|A_1 - A_2| \leq 10 \min(|A_1|, |A_2|)$, which implies

$$C^{-1}|A_1| \leq |A_2| \leq C|A_1|. \tag{23}$$

Then

$$\begin{aligned} & \Big| A_1^p |A_1|^q \varphi^{p+q} \mathcal{T}_{n,A_1,v_1} - A_2^p |A_2|^q \varphi^{p+q} \mathcal{T}_{n,A_2,v_2} \Big| \\ \leq\ & C|A_1 - A_2| \max(|A_1|, |A_2|)^{p-1} (\max |A_1|, |A_2|)^q \varphi^{p+q} \\ \cdot\ & \max(|\mathcal{T}_{n,A_1,v_1}|, |\mathcal{T}_{n,A_2,v_2}|) \\ +\ & C|A_1 - A_2| \max(|A_1|, |A_2|)^p (\max |A_1|, |A_2|)^{q-1} \varphi^{p+q} \\ \cdot\ & \max(|\mathcal{T}_{n,A_1,v_1}|, |\mathcal{T}_{n,A_2,v_2}|) \\ +\ & C \max(|A_1|, |A_2|)^p (\max |A_1|, |A_2|)^q \varphi^{p+q} |\mathcal{T}_{n,A_1,v_1} - \mathcal{T}_{n,A_2,v_2}| \\ \leq\ & C'|A_1 - A_2| |A_1|^{p+q-1} \varphi^{p+q} \max(|\mathcal{T}_{n,A_1,v_1}|, |\mathcal{T}_{n,A_2,v_2}|) \\ +\ & C'|A_1|^{p+q} \varphi^{p+q} |\mathcal{T}_{n,A_1,v_1} - \mathcal{T}_{n,A_2,v_2}| \end{aligned} \tag{24}$$

Consider here for example the second but last line. The factor $|\mathcal{T}_{n,A_1,v_1}(x,t)|$ has the bound $C_n(|v_1(x,t)||A_1|^{-1}\varphi(x,t)^{-1})^n$. Hence,

$$\begin{aligned} & |A_1 \varphi^{p+q}(x,t)| \mathcal{T}_{n,A_1,v_1}(x,t) \\ \leq\ & c_n |\varphi^{p+q-1}(x,t)||v_1(x,t)| \Big(\frac{|v_1(x,t)|}{|A_1|\varphi(x,t)} \Big)^{n-1} \\ \leq\ & c_n' |\varphi^{p+q-1}(x,t)||v_1(x,t)| \\ \leq\ & c_n''(t+1)^{-5/2} \Big(1 + \frac{|x|}{\sqrt{t+1}} \Big)^{-2m}, \end{aligned} \tag{25}$$

applying the definition of the norm for v_1, see (8). Because of (23), the same bound holds for $\mathcal{T}_{n,A_2,v_2}$ replacing $\mathcal{T}_{n,A_1,v_1}$. As a conclusion, this line of (24) satisfies (19).

For the last line of (24) one needs to derive the estimate

$$\begin{aligned} & |\mathcal{T}_{n,A_1,v_1}(x,t) - \mathcal{T}_{n,A_2,v_2}(x,t)| \\ \leq\ & \frac{C|A_1 - A_2|}{|A_1|} \max \Bigg(\Big(\frac{|v_1(x,t)|}{|A_1|\varphi(x,t)} \Big)^n, \Big(\frac{|v_2(x,t)|}{|A_2|\varphi(x,t)} \Big)^n \Bigg) \\ +\ & \frac{C}{t+1} \|v_1 - v_2\| \Big(1 + \frac{|x|}{\sqrt{t+1}} \Big)^{-m} \\ \cdot\ & \max \Bigg(\frac{|v_1(x,t)|^{n-1}}{(|A_1|\varphi(x,t))^n}, \frac{|v_2(x,t)|^{n-1}}{(|A_2|\varphi(x,t))^n} \Bigg) \end{aligned} \tag{26}$$

which is a consequence of the mean value theorem and the definition of the norm. Together with (21) this suffices to imply (19), see also (25).

2°. If none of the pairs (A_j, v_j) satisfies (13) at the point (x,t), we use (18). Consider for example the latter term in (18):

$$\begin{aligned}
&\Big|(A_1\varphi+v_1)^p|A_1\varphi+v_1|^q-(A_2\varphi+v_2)^p|A_2\varphi+v_2|^q\Big| \\
\le\ & |A_1\varphi+v_1-(A_2\varphi+v_2)| \\
\cdot\ & \max((A_1\varphi+v_1)^{p-1}|A_1\varphi+v_1|^q,(A_2\varphi+v_2)^{p-1}|A_2\varphi+v_2|^q) \\
+\ & |A_1\varphi+v_1-(A_2\varphi+v_2)| \\
\cdot\ & \max((A_1\varphi+v_1)^{p}|A_1\varphi+v_1|^{q-1},(A_2\varphi+v_2)^{p}|A_2\varphi+v_2|^{q-1}). \qquad (27)
\end{aligned}$$

Here we use

$$|A_1\varphi+v_1-(A_2\varphi+v_2)| \le CL(A_1,A_2,v_1,v_2), \qquad (28)$$

and (by the negation of (13))

$$|A_j\varphi(x,t)+v_j(x,t)| \le C|v_j(x,t)| \le \frac{C'\|v_j\|}{t+1}\Big(1+\frac{|x|}{\sqrt{t+1}}\Big)^{-m}. \qquad (29)$$

Hence, (19) follows.

3°. Thus suppose for example that (A_1, v_1) satisfies (13), but (A_2, v_2) not. If moreover $|A_1| \le 2|A_2|$, we have the bound $|A_1\varphi(x,t)| \le 2|A_2\varphi(x,t)| \le 2|v_2(x,t)|$. Hence (18) can be used for both (A_1, v_1) and (A_2, v_2) as in 2°, to prove (19).

We are thus left with the final case $|A_1| \ge 2|A_2|$. This implies $|A_1 - A_2| \ge |A_1|/2$. Moreover, by the definition of the norm and (13), $|A_1| \ge C|A_1\varphi| \ge C|v_1|$. This also implies

$$|A_1| + |v_1 - v_2| \ge C(|A_1| + |v_2|). \qquad (30)$$

(If $|v_1 - v_2| \le |v_2|/10$, then $|v_1| \ge |v_2|/2$. As a consequence, the right hand side of (30) is majorized by a constant times $|A_1|$. Hence, (30) must hold.) Summarizing, we get

$$\begin{aligned}
& L(A_1,A_2,v_1,v_2) \ge (|A_1|+|A_2|+|v_1|+|v_2|)(|A_1-A_2|+|v_1-v_2|) \\
\ge\ & C(|A_1|+|A_2|+|v_1|+|v_2|)(A_1+|v_1-v_2|) \\
\ge\ & C'(|A_1|+|A_2|+|v_1|+|v_2|)^2. \qquad (31)
\end{aligned}$$

On the other hand, the method of the proof of 1° implies the estimate

$$\begin{aligned}
& |\mathcal{M}_1(x,t)| \\
\le\ & \frac{C}{(t+1)^{5/2}}\Big(1+\frac{|x|}{\sqrt{t+1}}\Big)^{-2m}(|A_1|+|A_2|+|v_1(x,t)|+|v_2(x,t)|)^2, \qquad (32)
\end{aligned}$$

and 2° implies the same estimate for $|\mathcal{M}_2(x,t)|$. Combining this, (32) and (31) implies (19). □

Example 3.

(i) Functions like

$$\mathcal{N}(u) := \frac{u^p|u|^q}{1+|u|^r} \qquad (33)$$

would also be acceptable, with p and q as in Example 2 and $r \ge 0$. Apply $1/(1-z) = \sum_{n=0}^\infty z^n$ and the method above.

(ii) Also functions like $\mathcal{N}(u) := \sin(u^p|u|^q)$, with p and q as in Example 2, would satisfy the assumptions. Here the sinus could be replaced by any analytic function

g of one complex variable which satisfies, say, $g(0) = 0$ and $g([-1, 1]) \subset \mathbb{R}$. Use the Taylor series to verify this.

(iii) One could add to any of the examples above a nonlinear operator of the form

$$u \mapsto \frac{1}{(t+1)^{3/2}} \mathcal{B}(u, u) \tag{34}$$

where $\mathcal{B} : C_w(\mathbb{R} \times \mathbb{R}^+) \to C_w(\mathbb{R} \times \mathbb{R}^+)$ is, say, any bounded bilinear operator such that

$$\sup_x \Big(1 + \frac{|x|}{\sqrt{t+1}}\Big)^{-2m} |\mathcal{B}(u_1, u_2)(x, t)| \le C\|u_1\|_t \|u_2\|_t \tag{35}$$

for $u_j \in C_w(\mathbb{R} \times \mathbb{R}^+)$. A term like that would satisfy (10), since (writing $u_j := A_j\varphi + v_j$)

$$\begin{aligned}
& \sup_x \Big(1 + \frac{|x|}{\sqrt{t+1}}\Big)^{-2m} \frac{1}{(t+1)^{3/2}} |\mathcal{B}(u_1, u_1)(x, t) - \mathcal{B}(u_2, u_2)(x, t)| \\
\le\ & \sup_x \Big(1 + \frac{|x|}{\sqrt{t+1}}\Big)^{-2m} \frac{1}{(t+1)^{3/2}} |\mathcal{B}(u_1 - u_2, u_1)(x, t) - \mathcal{B}(u_2, u_2 - u_1)(x, t)| \\
\le\ & \frac{C}{(t+1)^{3/2}} \|A_1\varphi - v_1 - A_2\varphi + v_2\|_t \|u_2\|_t + \ldots \\
\le\ & \frac{C}{(t+1)^{3/2}} \Big(|A_1 - A_2| \|\varphi\|_t + \|v_1 - v_2\|_t\Big) \|u_2\|_t + \ldots \\
\le\ & \frac{C'}{(t+1)^{3/2}} \Big(\frac{|A_1 - A_2|}{\sqrt{t+1}} + \frac{\|v_1 - v_2\|}{t+1}\Big) \cdot \frac{1}{\sqrt{t+1}} + \ldots \\
\le\ & \frac{C''}{(t+1)^{5/2}} L(A_1, A_2, v_1, v_2).
\end{aligned} \tag{36}$$

It is easy to find analogous examples for higher order multilinear operators.

4. Proofs.

In this section we prove Theorem 1. A function $u \in C_w(\mathbb{R} \times \mathbb{R}^+)$ is a solution of (4)–(5) if it satisfies the integral equation

$$u(x, t) = e^{t\partial_x^2} f(x) + \int_0^t e^{(t-s)\partial_x^2} \mathcal{N}(u)(\cdot, s) ds. \tag{37}$$

By standard methods one shows that a solution $u \in C_w(\mathbb{R} \times \mathbb{R}^+)$ to (37) is enough many times differentiable to be a classical solution to (4)–(5). We write u as

$$u(x, t) =: A\varphi(x, t) + v(x, t) := \frac{A}{\sqrt{t+1}} e^{-\frac{1}{4}x^2/(t+1)} + v(x, t), \tag{38}$$

where the constant $A \in \mathbb{R}$ is chosen in Lemma 2 and Lemma 3. Our proof is based on the fact that a proper choice of A leads to the cancellation of the terms of order $t^{-1/2}$ in (37), i.e. the sup–norm of v decays as t^{-1} for large t. This is proven in the following lemma, see also (53).

Lemma 2. *Let the initial data f be given as in (3), and let $\mathcal{N}$ be as in (11). Denote*

$$A_0 := \int_{-\infty}^{\infty} f(x)dx\Big(\int_{-\infty}^{\infty} e^{-\frac{1}{4}x^2}dx\Big)^{-1} \qquad \text{and} \qquad g := f - A_0 e^{-\frac{1}{4}x^2}. \tag{39}$$

The function $u = A\varphi + v$ satisfies the integral equation (37) if the pair (A, v) satisfies the equations (with some fixed numbers d_k depending on p_k and q_k only)

$$A = A_0 + \sum_k d_k \alpha_k A^{p_k}|A|^{q_k} + \int_0^{\infty}\int_{-\infty}^{\infty} \mathcal{M}_{A,v}(y,s)dyds, \tag{40}$$

$$v(x,t) = e^{t\partial_x^2} g(x) + \int_0^t e^{(t-s)\partial_x^2}\mathcal{M}_{A,v}(\cdot,s)ds + \mathcal{R}_{A,v}(x,t)$$

$$- \varphi(x,t)\int_0^{\infty}\int_{-\infty}^{\infty}\mathcal{M}_{A,v}(y,s)dyds. \tag{41}$$

Here $\mathcal{M}_{A,v} := \mathcal{M} := \mathcal{N}(u) - \sum_k \alpha_k A^{p_k}|A|^{q_k}\varphi^{b_k} \in C_W(\mathbb{R}\times\mathbb{R}^+)$ with $b_k := p_k + q_k \geq 4$, and the function $\mathcal{R}_{A,v} \in C_W(\mathbb{R}\times\mathbb{R}^+)$ satisfies

$$\|\mathcal{R}_{A_1,v_1} - \mathcal{R}_{A_2,v_2}\| \leq CL(A_1, A_2, v_1, v_2) \tag{42}$$

for arbitrary (A_1, v_1) and (A_2, v_2).

Explanation. We later want to show that (41) has a solution v which belongs to $C_W(\mathbb{R}\times\mathbb{R}^+)$, so it is of order $(t+1)^{-1}$ for large t. (That is essentially the main result of our paper.) On the right hand side (41) the first and third terms are readily of order $(t+1)^{-1}$, but the second term is only of order $(t+1)^{-1/2}$. Fortunately, this large part of the second term is explicit, i.e. exactly the negative of the fourth term in (41). So these terms will cancel out, and v becomes of order $(t+1)^{-1}$.

Proof . We need to show that if (A, v) satisfies (40) and (41), then $A\varphi + v = e^{t\partial_x^2}(A_0 e^{-\frac{1}{4}x^2} + g) + \int_0^t e^{(t-s)\partial_x^2}\mathcal{N}(A\varphi + v)(\cdot, s)ds$. Taking into account (39)–(41) this is reduced to proving

$$\int_0^t e^{(t-s)\partial_x^2}\mathcal{N}(A\varphi + v)(\cdot,s)ds$$

$$= \sum_k d_k\alpha_k A^{p_k}|A|^{q_k}\varphi + \int_0^t e^{(t-s)\partial_x^2}\mathcal{M}(\cdot,s)ds + \mathcal{R}_{A,v}. \tag{43}$$

But we have $\mathcal{N}(u) = \mathcal{M} + \sum_k \alpha_k A^{p_k}|A|^{q_k}\varphi^{b_k}$ with $b_k = p_k + q_k$, hence, it remains to show

$$A^{p_k}|A|^{q_k}\int_0^t e^{(t-s)\partial_x^2}\varphi^{b_k}ds = d_k A^{p_k}|A|^{q_k}\varphi + \mathcal{R}_k \tag{44}$$

for all k, for some $\mathcal{R}_k \in C_W(\mathbb{R}\times\mathbb{R}^+)$ with the property (42) (and with a proper d_k).

We have the general formula

$$\int_{-\infty}^{\infty} e^{-a(x-y)^2-by^2}\,dy = \int_{-\infty}^{\infty} e^{-ax^2+\frac{a^2}{a+b}x^2-(\frac{a}{\sqrt{a+b}}x-\sqrt{a+b}y)^2}\,dy$$

$$= \frac{\sqrt{\pi}}{\sqrt{a+b}}e^{-abx^2/(a+b)}, \tag{45}$$

hence, the expression $e^{(t-s)\partial_x^2}\varphi^{b_k}(\cdot,s)$ equals

$$\frac{1}{\sqrt{4\pi}}\frac{1}{\sqrt{t-s}}\int_{-\infty}^{\infty} e^{-\frac{1}{4}(x-y)^2/(t-s)}\Big(\frac{1}{\sqrt{s+1}}e^{-\frac{1}{4}y^2/(s+1)}\Big)^{b_k}dy$$

$$= \frac{1}{\sqrt{t-s}}\frac{1}{(s+1)^{b_k/2}}\frac{1}{\sqrt{(t-s)^{-1}+b_k(s+1)^{-1}}}e^{-\frac{1}{4}x^2/(t-\gamma(s))}$$

$$= (s+1)^{-b_k/2+1/2}\frac{1}{\sqrt{b_k(t+1)-(b_k-1)(s+1)}}e^{-\frac{1}{4}x^2/(t-\gamma(s))}, \tag{46}$$

where $\gamma(s) = s - s/b_k - 1/b_k$. We can write (46) as

$$(s+1)^{-b_k/2+1/2}\frac{1}{\sqrt{b_k}}\frac{1}{\sqrt{t+1}}\frac{1}{\sqrt{1-\frac{b_k-1}{b_k}\frac{s+1}{t+1}}}e^{-\frac{1}{4}x^2/(t-\gamma(s))}$$

$$= \sum_{n=0}^{\infty} c_n\frac{(s+1)^{-b_k/2+1/2+n}}{(t+1)^{n+1/2}}e^{-\frac{1}{4}x^2/(t-\gamma(s))} \tag{47}$$

with $c_0 = b_k^{-1/2}$. We perform the integration $\int ds$ of (47) by parts and obtain

$$\Big[\sum_{n=0}^{\infty} c_n'\frac{(s+1)^{n+3/2-b_k/2}}{(t+1)^{n+1/2}}e^{-\frac{1}{4}x^2/(t-\gamma(s))}\Big]_{s=0}^{t}$$

$$+ \int_0^t \sum_{n=0}^{\infty} c_n''\frac{(s+1)^{n+3/2-b_k/2}}{(t+1)^{n+1/2}}\frac{x^2}{(t-\gamma(s))^2}e^{-\frac{1}{4}x^2/(t-\gamma(s))}ds. \tag{48}$$

The number d_k is chosen to be c_0'. Except for the term with $n=0$ and $\Big]_{s=0}$, the sum of the other terms of the first line is bounded by $C(t+1)^{-1}e^{-cx^2/(t+1)}$ (notice $b_k \geq 4$). The second line of (48) also satisfies this bound, since we can use the estimate

$$\Big|\frac{x^2}{(t-\gamma(s))^2}e^{-\frac{1}{4}x^2/(t-\gamma(s))}\Big| \leq \frac{C}{t-\gamma(s)}e^{-cx^2/(t-\gamma(s))}$$

and since $\gamma(s) \leq ct$ for a c, $0 < c < 1$. Multiplied by $A^{p_k}|A|^{q_k}$, all of these terms fall into $\mathcal{R}_k$ in (44). (It may happen that $-b_k/2+1/2+n = -1$ for some n, and then the above reasoning cannot be used. But this is only possible if $b_k = 5,7,9\ldots$, which is so large a number that the corresponding term in (47) is easily seen to fall into $\mathcal{R}_k$.)

So we are left with

$$d_k\frac{1}{\sqrt{t+1}}e^{-\frac{1}{4}x^2/(t-\gamma(0))}. \tag{49}$$

But using standard Taylor series developements this can also be written as d_k times $(t+1)^{-1/2}e^{-\frac{1}{4}x^2/(t+1)} = \varphi(x,t)$ plus an expression bounded by $C(t+1)^{-1}e^{-cx^2/(t+1)}$:

we have

$$
\begin{aligned}
&\left|e^{-\frac{1}{4}x^2/(t+1)} - e^{-\frac{1}{4}x^2/(t-\gamma(0))}\right| \\
= \ & e^{-\frac{1}{4}x^2/(t+1)}\left|1 - e^{-\frac{1}{4}(1+\gamma(0))x^2(t+1)^{-1}(t-\gamma(0))^{-1}}\right| \\
\le \ & Ce^{-\frac{1}{4}x^2/(t+1)}\sum_{m=1}^{\infty}\frac{1}{m!}\Big(\frac{C(1+\gamma(0))x^2}{(t+1)(t-\gamma(0))}\Big)^m \\
\le \ & C'e^{-\frac{1}{4}x^2/(t+1)}\sum_{m=1}^{\infty}\frac{1}{m!}\Big(\frac{C'x^2}{(t+1)^2}\Big)^m \\
\le \ & \frac{C''}{t+1}e^{-cx^2/(t+1)}. \qquad \square
\end{aligned} \tag{50}
$$

Our main result naturally follows from

Lemma 3. *The equation system (40)–(41) has a unique solution (A, v), if $\|f\|_0$ is small enough.*

To prove Lemma 3 we shall need

Lemma 4. *We have for every $n > 2$ and $1 > \varepsilon > 0$*

$$
\begin{aligned}
&\int_{-\infty}^{\infty} e^{-\frac{1}{4}(x-y)^2/(t-s)}(1+|y|/\sqrt{s+1})^{-n}dy \\
\le \ & C(n,\varepsilon)\sqrt{s+1}\Big(1+\frac{|x|}{\sqrt{t+1}}\Big)^{-n+1+\varepsilon}.
\end{aligned} \tag{51}
$$

Proof . The integral can be estimated by

$$
\begin{aligned}
&\int_{-\infty}^{\infty} e^{-\frac{1}{4}(x-y)^2/(t-s)}(1+|y|/\sqrt{s+1})^{-n+1+\varepsilon}(1+|y|/\sqrt{s+1})^{-1-\varepsilon}dy \\
\le \ & \sup_{y\in\mathbb{R}} e^{-\frac{1}{4}(x-y)^2/(t-s)}(1+|y|/\sqrt{s+1})^{-n+1+\varepsilon}\cdot\int_{-\infty}^{\infty}(1+|y|/\sqrt{s+1})^{-1-\varepsilon}dy \\
\le \ & C(n,\varepsilon)\Big(1+\frac{|x|}{\sqrt{t+1}}\Big)^{-n+1+\varepsilon}\sqrt{s+1}. \qquad \square
\end{aligned} \tag{52}
$$

Proof of Lemma 3. It is convenient to define

$$
\mathbf{B} := \Big\{ \ (A,v) \ \Big| \ A\in\mathbb{R}\ ,\ |A|\le B_1\ , \ v\in C_W(\mathbb{R}\times\mathbb{R}^+)\ ,\ \|v\|\le B_2\Big\}, \tag{53}
$$

where the small enough positive numbers B_1 and B_2 are fixed later. We want to show that the integral equation (40)–(41) has a unique solution in the complete metric space $\mathbf{B}\subset\mathbb{R}\times C_W(\mathbb{R}\times\mathbb{R}^+)$, by applying the contraction mapping principle.

1°. We first assume that (A_1, v_1) and $(A_2, v_2)\in\mathbf{B}$, and consider the right hand sides of (41) for them, call these $\tilde{v}_1$ and $\tilde{v}_2$. We show that $\tilde{v}_1$ and $\tilde{v}_2$ are elements of $C_W(\mathbb{R}\times\mathbb{R}^+)$ such that

$$
\|\tilde{v}_1 - \tilde{v}_2\| \le CL(A_1, A_2, v_1, v_2). \tag{54}
$$

From (10) we have

$$|\mathcal{M}_1(y,s) - \mathcal{M}_2(y,s)| \le CL(A_1, A_2, v_1, v_2)(s+1)^{-5/2}\Big(1 + \frac{|y|}{\sqrt{s+1}}\Big)^{-2m}. \tag{55}$$

We define

$$\int_y^\infty \mathcal{M}_{A,v}(z,s)dz =: \mathcal{F}_{A,v}(y,s) \qquad \text{for } y \ge 0 \tag{56}$$

and

$$\int_{-\infty}^y \mathcal{M}_{A,v}(z,s)dz =: \mathcal{G}_{A,v}(y,s) \qquad \text{for } y \le 0; \tag{57}$$

hence, from (55) we obtain estimates for $\mathcal{F}_j := \mathcal{F}_{A_j,v_j}$ and $\mathcal{G}_j := \mathcal{G}_{A_j,v_j}$, $j = 1, 2$:

$$\begin{aligned} & |\mathcal{F}_1(y,s) - \mathcal{F}_2(y,s)| \\ \le\ & CL(A_1, A_2, v_1, v_2)(s+1)^{-2}\Big(1 + \frac{|y|}{\sqrt{s+1}}\Big)^{-2m+1} \quad \text{for } y \ge 0 \text{ and} \\ & |\mathcal{G}_1(y,s) - \mathcal{G}_2(y,s)| \\ \le\ & CL(A_1, A_2, v_1, v_2)(s+1)^{-2}\Big(1 + \frac{|y|}{\sqrt{s+1}}\Big)^{-2m+1} \quad \text{for } y \le 0. \end{aligned} \tag{58}$$

We apply integration by parts. In order to obtain a properly behaving integral function of $\mathcal{M}_j(y,\cdot)$ we have to split the y–integration domain to two parts :

$$\begin{aligned} e^{(t-s)\partial_x^2}\mathcal{M}_j(\cdot,s) &= \int_{-\infty}^\infty \frac{1}{\sqrt{t-s}} e^{-\frac14 (x-y)^2/(t-s)} \mathcal{M}_j(y,s)dy \\ &= -\Big[\frac{1}{\sqrt{t-s}} e^{-\frac{(x-y)^2}{4(t-s)}} \mathcal{F}_j(y,s)\Big]_{y=0}^\infty + \int_0^\infty \frac12 \frac{x-y}{(t-s)^{3/2}} e^{-\frac{(x-y)^2}{4(t-s)}} \mathcal{F}_j(y,s)dy \\ &+ \Big[\frac{1}{\sqrt{t-s}} e^{-\frac{(x-y)^2}{4(t-s)}} \mathcal{G}_j(y,s)\Big]_{y=-\infty}^0 - \int_{-\infty}^0 \frac12 \frac{x-y}{(t-s)^{3/2}} e^{-\frac{(x-y)^2}{4(t-s)}} \mathcal{G}_j(y,s)dy. \end{aligned} \tag{59}$$

Here the first term equals $\frac{1}{\sqrt{t-s}} e^{-\frac14 x^2/(t-s)} \mathcal{F}_j(0,s)$. Moreover, we obtain from the inequality $\sqrt{z}e^{-z} \le Ce^{-z/2}$ $(z \ge 0)$

$$\begin{aligned} & \Big|\int_0^\infty \frac{x-y}{(t-s)^{3/2}} e^{-\frac14 (x-y)^2/(t-s)} \Big(\mathcal{F}_1(y,s) - \mathcal{F}_2(y,s)\Big)dy\Big| \\ \le\ & \int_0^\infty \frac{C}{t-s} e^{-\frac18 (x-y)^2/(t-s)} \Big|\mathcal{F}_1(y,s) - \mathcal{F}_2(y,s))\Big|dy \end{aligned} \tag{60}$$

Integrating this $\int_0^{t/2} ds$ we get, using (58), and Lemma 4 (take $\varepsilon \le m-2$) the bound

$$\int_0^{t/2} \frac{CL(A_1,A_2,v_1,v_2)}{(t-s)} \frac{1}{(s+1)^{3/2}} \Big(1+\frac{|x|}{\sqrt{t+1}}\Big)^{-2m+2+\varepsilon} ds$$

$$\le \frac{C'L(A_1,A_2,v_1,v_2)}{t+1} \int_0^{t/2} \frac{1}{(s+1)^{3/2}} \Big(1+\frac{|x|}{\sqrt{t+1}}\Big)^{-m} ds$$

$$\le \frac{C''L(A_1,A_2,v_1,v_2)}{t+1} (1+|x|/\sqrt{t+1})^{-m}. \tag{61}$$

Integrating (60) as $\int_{t/2}^t ds$ is different: we have $t-s \le s+1$, hence, we obtain by (58) the bound

$$\int_{t/2}^{t} \int_0^{\infty} \frac{CL(A_1,A_2,v_1,v_2)}{(t-s)} e^{-\frac18 (x-y)^2/(t-s)} \frac{1}{(s+1)^2} \Big(1+\frac{|y|}{\sqrt{t+1}}\Big)^{-m} dyds$$

$$\le C'L(A_1,A_2,v_1,v_2) \int_{t/2}^{t} \frac{1}{(s+1)^2} \frac{1}{t-s}$$

$$\cdot \Big(\int_{|x-y|\ge|x|/2} e^{-\frac18 (x-y)^2/(t-s)} \Big(1+\frac{|y|}{\sqrt{t+1}}\Big)^{-m} dy$$

$$+ \int_{|x-y|\le|x|/2} e^{-\frac18 (x-y)^2/(t-s)} \Big(1+\frac{|y|}{\sqrt{t+1}}\Big)^{-m} dy\Big) ds$$

$$\le C'L(A_1,A_2,v_1,v_2) \int_{t/2}^{t} \frac{1}{(s+1)^2} \frac{1}{t-s}$$

$$\cdot \Big(e^{-\frac{1}{32} x^2/s} \int_{|x-y|\ge|x|/2} e^{-\frac{1}{16} (x-y)^2/(t-s)} dy$$

$$+ \Big(1+\frac{|x|}{\sqrt{t+1}}\Big)^{-m} \int_{|x-y|\le|x|/2} e^{-\frac18 (x-y)^2/(t-s)} dy\Big) ds$$

$$\le C''L(A_1,A_2,v_1,v_2) \Big(1+\frac{|x|}{\sqrt{t+1}}\Big)^{-m} \cdot \int_{t/2}^{t} \frac{1}{(s+1)^2} \frac{1}{\sqrt{t-s}} ds$$

$$\le \frac{C'''L(A_1,A_2,v_1,v_2)}{t+1} \Big(1+\frac{|x|}{\sqrt{t+1}}\Big)^{-m}. \tag{62}$$

Similar representations and bounds apply to the terms with $\mathcal{G}_j$.

So we are left with

$$\int_0^t \frac{1}{\sqrt{t-s}} e^{-\frac14 (x-y)^2/(t-s)} \mathcal{F}_j(y,s) ds \tag{63}$$

and the similar integral for $\mathcal{G}_j$. These contain some stuff of order $(t+1)^{-1/2}$ only, but that will be cancelled out by the last line of (41). Let us see.

Integrating again by parts yields

$$\begin{aligned}
&\int_0^{t/2} \frac{1}{\sqrt{t-s}} e^{-\frac{1}{4}x^2/(t-s)} \mathcal{F}_j(0,s)ds \\
&= \Big[\frac{1}{\sqrt{t-s}} e^{-\frac{1}{4}x^2/(t-s)} \int_s^\infty \mathcal{F}_j(0,\sigma)d\sigma\Big]_{s=0}^{t/2} \\
&- \int_0^{t/2} \Big(\frac{C_1}{(t-s)^{3/2}} + \frac{C_2 x^2}{(t-s)^{5/2}}\Big) e^{-\frac{1}{4}x^2/(t-s)} \int_s^\infty \mathcal{F}_j(0,\sigma)d\sigma ds \\
&=: \frac{1}{\sqrt{t}} e^{-\frac{1}{4}x^2/t} \int_0^\infty \mathcal{F}_j(0,\sigma)d\sigma + \frac{C}{\sqrt{t}} e^{-\frac{1}{2}x^2/t} \int_{t/2}^\infty \mathcal{F}_j(0,\sigma)d\sigma + \mathcal{Y}_j(x,t),
\end{aligned} \tag{64}$$

From (58) we obtain $|\int_s^\infty (\mathcal{F}_1(0,\sigma) - \mathcal{F}_2(0,\sigma))d\sigma| \le CL(A_1, A_2, v_1, v_2)(s+1)^{-1}$; hence, we get the bound

$$\begin{aligned}
&|\mathcal{Y}_1(x,t) - \mathcal{Y}_2(x,t)| \\
&\le CL(A_1, A_2, v_1, v_2) \int_0^{t/2} \Big(\frac{C_1}{(t-s)^{3/2}} + \frac{C_2 x^2}{(t-s)^{5/2}}\Big) e^{-\frac{1}{4}x^2/(t-s)} (s+1)^{-1/2} ds \\
&\le C' L(A_1, A_2, v_1, v_2) \int_0^{t/2} \frac{1}{(t-s)^{3/2}} \frac{1}{\sqrt{s+1}} e^{-\frac{1}{8}x^2/(t-s)} ds \\
&\le C'' L(A_1, A_2, v_1, v_2) \frac{1}{(t+1)^{3/2}} \int_0^{t/2} \frac{1}{\sqrt{s+1}} e^{-\frac{1}{8}x^2/t} ds \\
&\le C''' L(A_1, A_2, v_1, v_2) \frac{1}{t+1} \Big(1 + \frac{|x|}{\sqrt{t+1}}\Big)^{-m}.
\end{aligned} \tag{65}$$

The second term of the last line of (64) easily leads to a smaller bound because of (58).

Moreover, returning to (63)

$$\Big| \int_{t/2}^{t} \frac{1}{\sqrt{t-s}} e^{-\frac{1}{4}x^2/(t-s)} (\mathcal{F}_1(0,s) - \mathcal{F}_2(0,s)) ds \Big| \tag{66}$$

is easily seen to be at most $CL(A_1, A_2, v_1, v_2)(t+1)^{-1} e^{-cx^2/(t+1)}$, since, in (58), $(s+1)^{-2}$ can be replaced by $C(t+1)^{-2}$.

Hence, the integral $\int_0^t ds$ of (59) is

$$\frac{1}{\sqrt{t}} e^{-\frac{1}{4}x^2/t} \int_0^\infty (\mathcal{F}_j(0,s) + \mathcal{G}_j(0,s)) ds + \mathcal{W}_j(x,t), \tag{67}$$

where $\|\mathcal{W}_1(\cdot,t) - \mathcal{W}_2(\cdot,t)\|_t \le CL(A_1,A_2,v_1,v_2)/(t+1)$. In (67) we can replace $t^{-1/2}e^{-\frac14 x^2/t}$ by $(t+1)^{-1/2}e^{-\frac14 x^2/(t+1)}$, since the difference of these two is bounded by

$$\begin{aligned}
&\Big|\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t+1}}\Big| e^{-\frac14 x^2/t} + \frac{1}{\sqrt{t}}\Big| e^{-\frac14 x^2/t} - e^{-\frac14 x^2/(t+1)}\Big| \\
\le\ & C\frac{1}{t+1}e^{-\frac14 x^2/(t+1)} + \frac{1}{\sqrt{t}}e^{-\frac14 x^2/t}\Big(1 - e^{-\frac14 x^2 t^{-1}(t+1)^{-1}}\Big) \\
\le\ & C'\frac{1}{t+1}e^{-\frac14 x^2/(t+1)} + \frac{1}{\sqrt{t}}e^{-\frac14 x^2/t}\sum_{m=1}^{\infty}\frac{1}{m!}\Big(\frac{x^2}{t(t+1)}\Big)^m \\
\le\ & \frac{C''}{t+1}e^{-\frac14 x^2/(t+1)} + \frac{C''}{t^{3/2}}e^{-cx^2/t}. \qquad (68)
\end{aligned}$$

Thus we have shown that (54) holds.

2°. We prove that if (A_1, v_2) and $(A_2, v_2) \in \mathbf{B}$, then the right hand sides of (40), call them $\tilde{A}_1$ and $\tilde{A}_2$, satisfy

$$|\tilde{A}_1 - \tilde{A}_2| \le CL(A_1,A_2,v_1,v_2). \qquad (69)$$

Clearly, every $|A_1^{p_k}|A_1|^{q_k} - A_2^{p_k}|A_2|^{q_k}\ |$ satisfies this bound. Moreover, by (58)

$$\begin{aligned}
&\Big|\int_0^\infty (\mathcal{F}_1(0,s) - \mathcal{F}_2(0,s))ds\Big| \\
\le\ & CL(A_1,A_2,v_1,v_2)\int_0^\infty (s+1)^{-b_k/2}ds \le C'L(A_1,A_2,v_1,v_2),
\end{aligned}$$

and the same is true for $\int \mathcal{G}$.

3°. We summarize. Taking into account the definition of $\mathbf{B}$, (53), and $L(A_1,A_2,v_1,v_2)$, (10), and choosing the constants B_1 and B_2 small enough in (53), and looking at (54) and (69), we find that the mapping $\Psi : (A,v) \mapsto (\tilde{A},\tilde{v})$ (notation as in the beginning of 1° and 2° above) has the following properties:

(i) The set $\mathbf{B}$ is invariant for Ψ. To see this, take $(A_2,v_2) = (0,0)$ in the proofs above; in addition, take f small enough in (39). Notice that in (41) we have

$$\|e^{t\partial_x^2}g\| \le C\sup_{x\in\mathbb{R}}(1+|x|)^{m+2+\rho}|f(x)|, \qquad (70)$$

since

$$\begin{aligned}
e^{t\partial_x^2}g(x) &= \frac{C}{\sqrt{t+1}}\int_{-\infty}^{\infty} e^{-\frac14(x-y)^2/t}(f(y) - A_0e^{-\frac14 y^2})dy \\
&= -\int_{-\infty}^{\infty}\frac{C'(x-y)}{(t+1)^{3/2}}e^{-\frac14(x-y)^2/t}\int_{-\infty}^{y}(f(z) - A_0e^{-\frac14 z^2})dzdy. \qquad (71)
\end{aligned}$$

Here $f(z) - A_0e^{-\frac14 z^2}$ is a function whose integral over the real line vanishes and which moreover is bounded by $C(1+|z|)^{-m-2-\rho}$. Hence,

$$\Big|\int_{-\infty}^{y}(f(z) - A_0e^{-\frac14 z^2})dz\Big| \le \frac{C}{(1+|y|)^{m+1+\rho}}, \qquad (72)$$

and we can bound (71) by

$$\int_{-\infty}^{\infty} \frac{C}{t+1} e^{-c(x-y)^2/t} \frac{1}{(1+|y|)^{m+1+\rho}} dy \le \frac{C'}{t+1} \frac{1}{(1+|x|/\sqrt{t+1})^m}. \tag{73}$$

So (70) follows.

(ii) Ψ is a strict contraction in $\mathbf{B}$.

So the contraction mapping principle applies to complete the proof of our lemma. □

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