SLOW QUASIREGULAR MAPPINGS AND UNIVERSAL COVERINGS

PEKKA PANKKA

Abstract. We define slow quasiregular mappings and study cohomology and universal coverings of closed manifolds receiving slow quasiregular mappings. We show that closed manifolds receiving a slow quasiregular mapping from a punctured ball have the de Rham cohomology type of either $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{S}^1$. We also show that, in the case of manifolds of the cohomology type of $\mathbb{S}^{n-1} \times \mathbb{S}^1$, the universal covering of the manifold has exactly two ends and the lift of the slow mapping into the universal covering has a removable singularity at the point of punctuation. We also obtain exact growth bounds and a global homeomorphism type theorem for slow quasiregular mappings into the manifolds of the cohomology type $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

1. Introduction

In the theory of quasiregular mappings a certain class of spaces, *quasiregularly elliptic manifolds*, have an important role. A continuous mapping $f \colon M \to N$ between connected and oriented Riemannian n-manifolds is called K-*quasiregular*, $K \geq 1$, if f is in the Sobolev class $W^{1,n}_{\mathrm{loc}}(M,N)$ and satisfies an inequality

$$\|Tf\|^n \leq K J_f \quad \text{a.e.} ,$$

where $\|Tf\|$ is the norm of the tangent mapping Tf of f and J_f is the Jacobian determinant of f. A connected and oriented Riemannian n-manifold N is K-*quasiregularly elliptic* if there exists a non-constant K-quasiregular mapping from $\mathbb{R}^n$ into N. A manifold is *quasiregularly elliptic* if it is K-quasiregularly elliptic for some $K \geq 1$.

It is known, by the geometric version of Rickman's Picard theorem [10, 3.1], that an open quasiregularly elliptic manifold has a bounded number of ends. It is also known by a theorem of Bonk and Heinonen [1, Theorem 1.1] that a closed quasiregularly elliptic manifold has a bounded de Rham cohomology. Both results are quantitative in the sense, that bounds depend only on the dimension and the constant K.

Date: 2nd June 2006.

2000 *Mathematics Subject Classification.* 30C65 (53C21 58A12).

Key words and phrases. quasiregular mappings, quasiregularly elliptic manifolds, p-harmonic forms.

The author was partly supported by foundation Vilho, Yrjö ja Kalle Väisälän rahasto and the Academy of Finland, project 53292.

Also local versions of these theorems have been under investigation. In [8] we proved, with Holopainen, a big Picard type version of Rickman's Picard theorem, and in [18] a local version of the theorem of Bonk and Heinonen. For the statements of these local versions, let us give some definitions.

Let N be a manifold and C a compact subset. We say that a component V of $M \setminus C$ is *an end of N with respect to C*, if $\overline{V}$ is non-compact in N. Furthermore, we say that C *separates q ends*, if N has q ends with respect to C. Finally, we say that N *has at least q ends*, if there exists a compact set C, which separates at least q ends. A mapping $f\colon B^n \setminus \{0\} \to N$ has *a removable singularity (at origin) in V* if for every compact set $C \subset N$ there exists $r \in (0,1)$ such that $f(B^n(r) \setminus \{0\}) \subset V \setminus C$. Furthermore, we say that f has *a removable singularity (at the origin)* if either f has a limit or a removable singularity at origin. If f does not have a removable singularity at origin, we say that f has *an essential singularity (at origin).*

Theorem 1 ([8, Theorem 1.3])**.** *Let N be a connected and oriented Riemannian n-manifold. For every $K \geq 1$ there exists $q = q(K,n)$ such that every K-quasiregular mapping $f\colon B^n \setminus \{0\} \to N$ has a removable singularity at the origin if N has at least q ends.*

A local version of the Bonk-Heinonen theorem reads as follows. Here, and in this article, $H^*(M)$ denotes the de Rham cohomology ring of manifold M. The ℓ-th de Rham cohomology group of M we denote by $H^\ell(M)$.

Theorem 2 ([18, Theorem 2])**.** *Let $n \geq 2$ and $K \geq 1$. There exists a constant $C = C(n,K) > 0$ such that if N is a closed, connected, and oriented Riemannian n-manifold with $\dim H^*(N) \geq C$ and $f\colon B^n \setminus \{0\} \to N$ is a K-quasiregular mapping, then the limit $\lim_{x\to 0} f(x)$ exists.*

Theorem 1 yields the geometric version of Rickman's Picard theorem by a simple compactness argument and Theorem 2 implies the theorem of Bonk and Heinonen. However, in order to obtain the theorem of Bonk and Heinonen from Theorem 2, we need an additional growth result from [18]. The same growth result was also used in the proof of Theorem 2 in [18]. Arguments of this type are not needed in the proof of the original Bonk-Heinonen theorem. Let us state this growth result.

Theorem 3 ([18, Theorem 14])**.** *Let N be a closed, connected, oriented Riemannian n-manifold such that $H^\ell(N) \neq 0$ for some $\ell \in \{2,\ldots,n-2\}$, and let $f\colon \mathbb{R}^n \setminus \bar{B}^n \to N$ be a K-quasiregular mapping having an essential singularity at infinity. Then there exists $\alpha > 0$ depending on*

n and K such that

$$\liminf_{r\to\infty} \frac{1}{r^\alpha} \int_{B^n(r)\setminus \bar{B}^n(2)} J_f > 0.$$

This result partly corresponds to an another result of Bonk and Heinonen in [1].

Theorem 4 ([1, Theorem 1.11])**.** *Let $n \geq 2$ and $f\colon \mathbb{R}^n \to N$ be a non-constant K-quasiregular mapping into a closed, connected and oriented Riemannian n-manifold N. If the ℓ-th cohomology group $H^\ell(N)$ of N is nontrivial for some $\ell = 1, \dots, n-1$, then there exists a positive constant $\alpha = \alpha(n, K) > 0$ such that*

$$\liminf_{r\to\infty} \frac{1}{r^\alpha} \int_{B^n(r)} J_f > 0.$$

In Theorem 3 we could replace the assumption on an essential singularity by an assumption that

$$\int_{B^n(r)\setminus \bar{B}^n(a)} J_f \to \infty \quad \text{as } r \to \infty \tag{1}$$

for any $a > 1$. Also non-constant entire quasiregular mappings in Theorem 4 satisfy (1). Having these observations and Theorems 3 and 4 as our motivation, we define *slow quasiregular mappings* as follows.

Let N be a connected and oriented Riemannian n-manifold and $a_0 > 0$. We say that a quasiregular mapping $f\colon \mathbb{R}^n \setminus \bar{B}^n(a_0) \to N$ is *slow* if f satisfies (1) and

$$\frac{1}{r^\alpha} \int_{B^n(r)\setminus \bar{B}^n(a)} J_f \to 0 \quad \text{as } r \to 0 \tag{2}$$

for every $\alpha > 0$ and any $a > a_0$. We say that an entire quasiregular mapping $f\colon \mathbb{R}^n \to N$ is slow if $f|(\mathbb{R}^n \setminus \bar{B}^n)$ is slow. Furthermore, we say that a quasiregular mapping $f\colon B^n \setminus \{0\} \to N$ is slow, if $f \circ \sigma$ is slow for a Möbius mapping σ such that $\sigma(\mathbb{R}^n \setminus \bar{B}^n) = B^n \setminus \{0\}$.

If N is a closed manifold, conditions (1) and (2) can equivalently be formulated using the averaged counting function $A(\cdot; f)$, since

$$A(\Omega; f) = \frac{1}{|N|} \int_\Omega J_f,$$

where Ω is a relatively compact open set contained in the domain of definition of f, and $|N|$ is the volume of N. For a detailed discussion on the averaged counting function of a quasiregular mapping, see e.g. [17] or [19].

One of the elementary properties of slow quasiregular mappings is the stability under composition with BLD-mappings, that is, mappings of bounded length distortion. Whereas a BLD-mapping itself can not be a slow mapping, the composition of a slow mapping with a BLD-mapping is. This observation yields that the class of slow mappings does not depend on the particular choice of the Riemannian metric of the target manifold but only on the bilipschitz equivalence class of the metric. However, simple examples, based e.g. on the inversion in $\bar{\mathbb{R}}^n$, show that the class of slow mappings is not invariant under conformal changes of the Riemannian metric of the target manifold. We use the observation on the composition of a slow mapping with a BLD-mapping frequently in this article especially when liftings of slow mappings into covering spaces are considered. Since the covering map is BLD, the lifting of a slow mapping is also slow. Naturally, here we could also use the fact that the covering map is a local isometry. For detailed discussion on mappings of bounded length distortion, see e.g. [16] and [5].

Having the definition of a slow mapping at our disposal, Theorem 4 can be reformulated as follows.

Theorem 5 ([1, Theorem 1.11])**.** *Let $f\colon \mathbb{R}^n \to N$ be a slow K-quasiregular mapping with N a closed, connected and oriented Riemannian n-manifold. Then $H^*(N) = H^*(\mathbb{S}^n)$.*

In this article we first extend Theorem 3 to correspond to a counterpart of Theorem 5.

Theorem 6. *Let N be a closed, connected, and oriented Riemannian n-manifold receiving a slow quasiregular mapping from $B^n \setminus \{0\}$. Then $H^*(N) = H^*(\mathbb{S}^n)$ or $H^*(N) = H^*(\mathbb{S}^{n-1} \times \mathbb{S}^1)$.*

The proof of this theorem is a refinement of the argument in [18]. Contrary to the proof in [18], we divide this proof into two parts such that the essential growth result for weakly exact $\mathcal{A}$-harmonic forms is discussed separately.

In the case of manifolds of the cohomology type of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ the existence of non-trivial cohomology classes gives us a tool to study the universal covering of the manifold. Especially ends of the universal covering and the behavior of liftings of slow mappings into the universal covering are of our interest. Let us first state a general result on universal coverings and quasiregular mappings. The connection of the fundamental group of a closed manifold to the isometric deck transformations of the universal covering yields the following theorem without additional slowness assumptions on the mapping.

Theorem 7. *Let N be a closed, connected, and oriented Riemannian n-manifold, $\tilde{N}$ the universal covering of N, and $f\colon B^n \setminus \{0\} \to N$ a quasiregular mapping. If $\tilde{N}$ has more than two ends, then f and every lift $\tilde{f}$ of f into $\tilde{N}$ has a limit at the origin.*

In Section 2, we do not prove Theorem 7 only for universal coverings, but extend the theorem to contain also *infinitely branching manifolds.* The proof is based on a topological observation and Theorem 1.

For slow mappings and manifolds of the cohomology type of $\mathbb{S}^{n-1}\times\mathbb{S}^1$, we can improve the result of Theorem 7.

Theorem 8. *Let N be a closed, connected, and oriented Riemannian n-manifold such that $H^*(N) = H^*(\mathbb{S}^{n-1}\times\mathbb{S}^1)$, and let $f\colon B^n\setminus\{0\} \to N$ be a slow quasiregular mapping. Then every lift $\tilde f$ of f into the universal covering $\tilde N$ of N has a removable singularity at the origin and $\tilde N$ has exactly two ends. More precisely, there exists an n-harmonic function u on $\tilde N$ such that u has compact level sets and $\tilde N$ has two ends with respect to any level set of u.*

We also obtain the following corollary.

Corollary 9. *Let N be a closed, connected, and oriented Riemannian n-manifold such that $H^*(N) = H^*(S^{n-1}\times S^1)$, and let $f\colon B^n\setminus\{0\} \to N$ be a slow quasiregular mapping. Then there exists a finite normal subgroup H of $\pi_1(N)$ such that $\pi_1(N)/H$ is isomorphic to $\mathbb{Z}$.*

Although Theorem 8 describes the behavior of lifted mappings only in a qualitative manner, we obtain also quantitive information. Namely, having Theorem 8 at our disposal, we have that slow mappings into closed manifolds of the cohomology type of $\mathbb{S}^{n-1}\times\mathbb{S}^1$ have a logarithmic growth in the following sense.

Theorem 10. *Let N be a closed, connected, and oriented Riemannian n-manifold, $f\colon \mathbb{R}^n\setminus\bar B^n \to N$ a slow K-quasiregular mapping, and $\tilde f\colon \mathbb{R}^n\setminus\bar B^n\to\tilde N$ a lifting of f to the universal covering $\tilde N$ of N. Then there exist constants $C_m>0$ and $C_M>0$ depending only on n, K, N, and the multiplicity of $\tilde f$ such that for any fixed $a\in(1,\infty)$ we have*

$$\liminf_{r\to\infty}\frac{1}{\log(r)}\int_{B^n(r)\setminus\bar B^n(a)} J_f \ge C_m \tag{3}$$

and

$$\limsup_{r\to\infty}\frac{1}{\log(r)}\int_{B^n(r)\setminus\bar B^n(a)} J_f \le C_M. \tag{4}$$

Theorem 8 also yields a local Zorich type theorem for slow quasiregular mappings.

Theorem 11. *Let N be a closed, connected, and oriented Riemannian n-manifold such that $H^*(N) = H^*(\mathbb{S}^{n-1}\times\mathbb{S}^1)$, and let $f\colon B^n\setminus\{0\}\to N$ be a slow K-quasiregular mapping. If f is a local homeomorphism, then there exists a neighborhood W of origin such that every lifting $\tilde f$ of f into the universal cover $\tilde N$ of N is an embedding on $W\setminus\{0\}$.*

This article is organized as follows. In Section 2 we prove Theorem 7 for infinitely branching manifolds. In Section 3 we discuss the connection of quasiregular mappings to $\mathcal{A}$-harmonic forms and to the de Rham cohomology. In Sections 4 and 5 we prove Theorems 6 and

8, respectively, using growth estimates for $\mathcal{A}$-harmonic functions and forms. Section 6 contains the proof of Theorem 10 and Section 7 the proof of Theorem 11. Finally in Section 8, we give examples of slow quasiregular mappings into $\mathbb{S}^n$ and $\mathbb{S}^{n-1} \times \mathbb{S}^1$. Examples show that a theorem of the type of Theorem 10 is not possible for slow quasiregular mappings into spaces of the cohomology type of $\mathbb{S}^n$.

The notation in this article is standard. Given a Riemannian manifold M we denote by $B(x,r)$ the open ball of radius $r > 0$ centered at $x \in M$. An open ball in $\mathbb{R}^n$ with radius r and centered at origin we denote by $B^n(r)$. The unit ball of $\mathbb{R}^n$ is denoted by B^n. The closed balls are denoted by $\bar{B}(x,r)$, $\bar{B}^n(r)$, and $\bar{B}^n$, respectively. For any measurable set A on a Riemannian manifold M, we denote by $|A|$ the measure of A in the Riemannian measure of M. For every path γ on M, we denote by $|\gamma|$ the image of γ.

Acknowledgments. This study was initiated during author's visit to the University of Michigan, Ann Arbor. The author wishes to thank the Department of Mathematics for the hospitality. We would also like to thank Mario Bonk, David Drasin, Juha Heinonen, Ilkka Holopainen, Kirsi Peltonen, and Seppo Rickman for inspiring discussions on the topics of this article.

2. Infinitely branching manifolds and a big Picard type theorem

In this section we prove Theorem 8 for *infinitely branching manifolds*. For the statement, let us give some definitions.

Let M be a manifold. We say that a compact set $C_1 \subset M$ *splits the end V of M separated by a compact set $C_0 \subset M$* if V contains at least two ends of M with respect to C_1. We say that a manifold M is *infinitely branching* if every end of M splits.

Theorem 12. *Let M be an oriented and connected Riemannian n-manifold and $f\colon B^n \setminus \{0\} \to M$ a quasiregular mappings. If M is infinitely branching, then f has a limit at origin.*

The proof of Theorem 12 is based on the following topological lemma and Theorem 1.

Lemma 13. *Let M be a Riemannian n-manifold and $f\colon B^n \setminus \{0\} \to M$ an open and discrete mapping with a removable singularity at origin and W a neighborhood of origin compactly contained in B^n. Then f has a finite multiplicity in $W \setminus \{0\}$. Moreover, if f has a limit at origin, then*

$$\partial f(W \setminus \{0\}) \subset f(\partial W) \cup \{\lim_{x\to 0} f(x)\}, \tag{5}$$

and if f does not have a limit at origin, then

$$\partial f(W \setminus \{0\}) \subset f(\partial W) \tag{6}$$

and $f(W \setminus \{0\})$ is an end of M with respect to $\partial f(W \setminus \{0\})$. Furthermore, the end $f(W \setminus \{0\})$ does not split.

Proof. We prove (5) and (6) simultaneously. Let $y \in \partial f(W \setminus \{0\})$ and let (x_i) be a sequence in $W \setminus \{0\}$ such that $f(x_i) \to y$. If the set $\{x_i\}$ is relatively compact in $W \setminus \{0\}$, then there exists a subsequence (x_{i_j}) and $x \in W \setminus \{0\}$ such that $x_{i_j} \to x$ and $f(x) = y$. This is a contradiction, since f is open. Thus $\{x_i\}$ is not relatively compact in $W \setminus \{0\}$. Hence there exists a subsequence (x_{i_j}) of (x_i) such that either $|x_{i_j}| \to r$ or $x_{i_j} \to 0$. In the former case, we may assume that $x_{i_j} \to x$ for some $x \in \partial W$. Thus either $y \in f(\partial W)$ or $y = \lim_{x\to 0} f(x)$, since f has a removable singularity at origin. This proves (5) and (6).

Suppose that f does not have a limit at origin. Since f has a removable singularity at origin by Theorem 1, $f(W \setminus \{0\})$ is not relatively compact in M. By (6), $\partial f(W \setminus \{0\})$ is compact. Thus $f(W \setminus \{0\})$ is an end of M with respect to $\partial f(W \setminus \{0\})$. To show that $f(W \setminus \{0\})$ does not split, let us assume towards contradiction that there exists a compact set E in M such that E splits $f(W \setminus \{0\})$. Let $V_1, \ldots, V_k$ be the ends of M with respect to E contained in $f(W \setminus \{0\})$. Since f has a removable singularity at origin, there exists $r > 0$ such that $f(B^n(r) \setminus \{0\})) \subset V_i$ for some i. We may assume that $f(B^n(r) \setminus \{0\})) \subset V_1$. Thus $V_2 \cup \cdots \cup V_k \subset f(\overline{W} \setminus B^n(r))$. This is a contradiction, since $f(\overline{W} \setminus B^n(r))$ is compact. Thus $f(W \setminus \{0\})$ does not split.

Let us now show that f has a finite multiplicity in W. We employ here the local topological degree of the mapping f, for details see [19, I.4]. Suppose first that f has a limit at origin. Then f can be continued to the origin and f is quasiregular in B^n. Thus f has finite multiplicity in $\overline{W}$, see e.g. [19, I.4.10(3)].

Suppose now that f does not have a limit at origin. Let Ω be the component of $fW \setminus f(\partial W)$ that is not relatively compact. Fix $y_0 \in \Omega$ and let G be a domain compactly contained in Ω. Let D be a component of $f^{-1}G \cap W$. Since $\overline{G} \subset fW \setminus f(\partial W)$, D is compactly contained in W. Furthermore, since f has a removable singularity at origin, $0 \notin \overline{D}$ and D is compactly contained in $W \setminus \{0\}$. Thus D is a normal domain of f and $fD = G$, see e.g. [19, I.4.7]. Since the set $f^{-1}(y_0)$ is finite, $f^{-1}G$ is a finite union of normal domains of f. Thus, by [19, I.4.10],

$$\begin{aligned} \text{card } (f^{-1}(y) \cap f^{-1}\Omega) &\leq \mu(y, f, f^{-1}G) = \mu(y_0, f, f^{-1}G) \\ &= \sum_{x \in f^{-1}(y_0) \cap f^{-1}\Omega} i(x, f) < \infty \end{aligned}$$

for every $y \in G$. Since for every $y \in \Omega$ we find a domain containing y and y_0 which is compactly contained in Ω, the claim follows from compactness of $\overline{W} \setminus f^{-1}\Omega$. □

Proof of Theorem 12. Since M is infinitely branching, M has infinitely many ends. Hence, by Theorem 1, f has a removable singularity at origin. Suppose f does not have a limit at origin. Then, by Lemma 13, $f(B^n(1/2)\setminus\{0\})$ is an end of M with respect to $\partial f(B^n(1/2)\setminus\{0\})$. Since M is infinitely branching, $f(B^n(1/2)\setminus\{0\})$ splits. This contradicts Lemma 13. Thus f has a limit at origin. □

To obtain Theorem 7 from Theorem 12, it is enough to show that a universal covering with more than two ends is infinitely branching. Although this is well known, we give, for the reader's convenience, a simple proof based on the Riemannian geometry of covering spaces. The results on Riemannian geometry used in the proof are standard, see e.g [2].

Lemma 14. *Let N be a closed Riemannian manifold and $\tilde{N}$ the universal covering of N. Then either $\tilde{N}$ has at most two ends or $\tilde{N}$ is infinitely branching.*

Proof. Suppose $\tilde{N}$ has at least three ends and let C_1 be a compact set separating at least three ends. Let $V_1,\ldots,V_m$, where $3 \le m < \infty$, be the ends of $\tilde{N}$ with respect to C_1.

Let $\varphi\colon \tilde{N} \to N$ be a Riemannian covering map. Since N and C_1 are compact, we may fix $x \in C_1$ and $R > 0$ such that $C_1 \subset \bar{B}(x,R)$ and $R > \operatorname{diam} N$. Since $\tilde{N}$ is a covering space of a compact manifold, it is complete. Hence, by the Hopf-Rinow theorem, closed balls of $\tilde{N}$ are compact and ends V_i are not bounded. Thus $V_i \setminus \bar{B}(x,6R) \neq \emptyset$ for every i and we may, for every $i \le m$, fix $x_i \in V_i \setminus B(x,6R)$ such that $x_i \in \varphi^{-1}(\varphi(x))$.

For every i, we fix a path $\tilde{\gamma}_i$ from x to x_i and a deck transformation $\psi_i\colon \tilde{N} \to \tilde{N}$ corresponding to the loop $\gamma_i := \varphi\circ\tilde{\gamma}_i$, that is, for every $x \in \tilde{N}$ and $\sigma\colon [0,1] \to \tilde{N}$ such that $\sigma(0) = x$ and $\sigma(1) = \psi_i(x)$, $\psi_i\circ\sigma$ is (freely) homotopic to γ_i. Since ψ_i is an isometry, $\psi_i(B(x,2R)) = B(x_i,2R)$ and $\tilde{N}\setminus\psi_i(C_1) = \psi_i(\tilde{N}\setminus C_1)$. Thus $\tilde{N}$ has m ends, say $V_{i1},\ldots,V_{im}$, with respect to $\psi_i(C_1)$ for every i. We may assume that $C_1 \subset V_{i1}$ for every i. Since $V_{ik} \subset V_i$ for every $1 \le i \le m$ and $2 \le k \le m$, we have that every end V_i splits into $m-1$ ends $V_{i2},\ldots,V_{im}$. The claim now follows by induction. □

Remark 15. Lemma 14 reveals that also another line of argument is available for the proof of Theorem 7. Indeed, using the argument of Lemma 14, we have that the fundamental group of N has an exponential growth. Thus the covering space is n-hyperbolic, see [22, Chapter X]. Thus $\tilde{f}$ has a limit at the origin, see e.g. [8, Lemma 3.1]. This argument, however, is not available in the more general case of Theorem 12.

Remark 16. It is easy to construct infinitely branching manifolds which are not universal covering spaces of closed manifolds. Indeed, given any locally finite tree with edges whose length is bounded from below, we may construct a complete Riemannian n-manifold which is roughly isometric to the tree and has finite volume. If the tree is infinitely branching, so is the manifold. Manifolds constructed this way are not n-hyperbolic but n-parabolic. For details on conformal types of Riemannian manifolds, see e.g. [3], [7], and [9].

3. $\mathcal{A}$-HARMONIC FORMS, QUASIREGULAR MAPPINGS, AND DE RHAM COHOMOLOGY

Our discussion on $\mathcal{A}$-harmonic forms in this section is brief. For detailed discussions, see e.g. [11], [13], [14], [15], and [20]. For the connection of $\mathcal{A}$-harmonic forms to quasiregular mappings, see e.g. [1] and [14].

Let $p \in (1,\infty)$. We denote by $W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell M)$ the partial Sobolev space of ℓ-forms. A form $\omega \in L^p_{\mathrm{loc}}(\bigwedge^\ell M)$ is in the space $W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell M)$ if the distributional exterior derivative $d\omega$ exists and $d\omega \in L^p_{\mathrm{loc}}(\bigwedge^{\ell+1} M)$. The global space $W^{d,p}(\bigwedge^\ell M)$ is defined similarly. We say that a form $\omega \in W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell M)$ is *weakly closed* if $d\omega = 0$ and *weakly exact* if $\omega = d\tau$ for some $\tau \in W^{d,p}_{\mathrm{loc}}(\bigwedge^{\ell-1} M)$.

Given $\ell \in \{1,\dots,n-1\}$ let $\mathcal{A}\colon \bigwedge^\ell T^*M \to \bigwedge^\ell T^*M$ be a measurable bundle map such that there exist positive constants a and b satisfying

$$\begin{aligned} \langle \mathcal{A}(\xi)-\mathcal{A}(\zeta), \xi-\zeta\rangle &\geq a(|\xi|+|\zeta|)^{p-2}|\xi-\zeta|^2, && (7)\\ |\mathcal{A}(\xi)-\mathcal{A}(\zeta)| &\leq b(|\xi|+|\zeta|)^{p-2}|\xi-\zeta|, \text{ and} && (8)\\ \mathcal{A}(t\xi) &= t|t|^{p-2}\mathcal{A}(\xi) && (9) \end{aligned}$$

for all $\xi,\zeta \in \bigwedge^\ell T^*_xM$, $t\in\mathbb{R}$, and for almost every $x \in M$. We also assume that $x \mapsto \mathcal{A}_x(\omega)$ is a measurable ℓ-form for every measurable ℓ-form $\omega\colon M \to \bigwedge^\ell T^*M$.

We say that an ℓ-form ξ is *$\mathcal{A}$-harmonic (of type p) on M* if ξ is a weakly closed continuous form in $W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell M)$ and satisfies an equation

$$d^*(\mathcal{A}(\xi)) = 0 \tag{10}$$

weakly, that is,

$$\int_M \langle \mathcal{A}(\xi), d\varphi\rangle = 0$$

for all $\varphi \in C^\infty_0(\bigwedge^{\ell-1} M)$. In the special case $\mathcal{A}(\zeta) = |\zeta|^{p-2}\zeta$, we say that an $\mathcal{A}$-harmonic form ξ is *p-harmonic.*

A continuous function u on M is called *$\mathcal{A}$-harmonic (of type p)* if du is an $\mathcal{A}$-harmonic 1-form. If du is p-harmonic, also u is called *p-harmonic.*

Let $f\colon M \to N$ be a quasiregular mapping between Riemannian manifolds M and N. Since f is almost everywhere differentiable, we may define the pull-back $f^*\omega$ of $\omega \in L^{n/\ell}_{\text{loc}}(\bigwedge^\ell N)$ by

$$(f^*\omega)_x = (T_x f)^* \omega_{f(x)}.$$

The quasiregularity of f yields that $f^*\omega \in L^{n/\ell}_{\text{loc}}(\bigwedge^\ell M)$ and $d(f^*\omega) = f^*(d\omega)$ if $\omega \in W^{d,n/\ell}_{\text{loc}}(\bigwedge^\ell N)$. Thus $f^*\omega \in W^{d,n/\ell}_{\text{loc}}(\bigwedge^\ell M)$ for $\omega \in W^{d,n/\ell}_{\text{loc}}(\bigwedge^\ell N)$. Furthermore, if ω is an (n/ℓ)-harmonic ℓ-form on N, then $f^*\xi$ is $\mathcal{A}$-harmonic, where

$$\mathcal{A}(\omega) = \langle G^*\omega, \omega\rangle^{\frac{(n/\ell)-2}{2}} G^*\omega$$

and

$$G_x = J(x,f)^{2/n}(T_x f)^{-1}((T_x f)^{-1})^t \quad \text{a.e. .}$$

Since the pull-back of a smooth form in a quasiregular mapping may not be smooth, the formula $[\omega] \mapsto [f^*\omega]$ does not induce a homomorphism between de Rham cohomologies of N and M. Since the exterior derivation commutes with the pull-back f^* induced by f, it is natural to consider cohomology groups of forms with Sobolev coefficients, that is, for $p \in (1,\infty)$ we set

$$H^{\ell,p}(M) = \frac{\operatorname{Ker}(d\colon W^{d,p}_{\text{loc}}(\bigwedge^\ell M) \to W^{d,p}_{\text{loc}}(\bigwedge^{\ell+1} M))}{\operatorname{Im}(d\colon W^{d,p}_{\text{loc}}(\bigwedge^{\ell-1} M) \to W^{d,p}_{\text{loc}}(\bigwedge^{\ell} M))}.$$

By the discussion above, f induces for every $\ell \in \{1,\dots,n\}$ a mapping $f^*\colon H^{\ell,n/\ell}(N) \to H^{\ell,n/\ell}(M)$ by $[\xi] \mapsto [f^*\xi]$. Since for all Riemannian manifolds M and all $p \in (1,\infty)$ $H^{*,p}(M)$ is naturally isomorphic to $H^*(M)$, we have homomorphisms $f^*\colon H^\ell(N) \to H^\ell(M)$ for $\ell \in \{1,\dots,n\}$. For $\ell = 0$ the homomorphism $f^*\colon H^{0,n}(N) \to H^{0,n}(M)$, $[v] \mapsto [v \circ f]$, gives the desired mapping. Induced homomorphisms have the usual properties, that is, $\mathrm{id}^* = \mathrm{id}$ and $(f \circ h)^* = h^* \circ f^*$ whenever the composition of quasiregular mappings f and h is defined.

Cohomologies $H^{*,p}(M)$ and $H^*(M)$ are isomorphic by the de Rham theorem for $H^{*,p}(M)$. For the reader's convenience we give an outline of the proof. We follow here [23, Chapter 5].

Let us first construct sheaves $\mathcal{W}^\ell(M)$ for $\ell \geq 0$. Let $x \in M$. We say that forms ξ and η in $W^{d,p}_{\text{loc}}(\bigwedge^\ell M)$ are equivalent at x if there exists a neighborhood U of x such that $\xi = \eta$ almost everywhere in U. Clearly this is an equivalence relation in $W^{d,p}_{\text{loc}}(\bigwedge^\ell M)$. We denote the set of equivalence classes by $\mathcal{W}^{\ell,p}_x(M)$. We let $\mathcal{W}^{\ell,p}(M) = \bigcup_{x\in M} \mathcal{W}^{\ell,p}_x(M)$ and $\pi\colon \mathcal{W}^{\ell,p}(M) \to M$ be the natural projection. We endow $\mathcal{W}^{\ell,p}(M)$ with the topology whose basis is given by the sets

$$\bigcup_{x\in U} \overline{\xi}^x$$

where $\overline{\xi}^x$ is the equivalence class of $\xi \in W^{d,p}_{\text{loc}}(\bigwedge^\ell M)$ at x and U is an open set in M. As in the smooth case, this gives $\mathcal{W}^{\ell,p}(M)$ a topology which makes it a sheaf of real vector spaces.

Since the exterior derivative d induces an operator between germs, we have a sequence of sheaves

$$0 \xrightarrow{\iota} \mathcal{R} \to \mathcal{W}^{0,p}(M) \xrightarrow{d} \mathcal{W}^{1,p}(M) \xrightarrow{d} \cdots, \tag{11}$$

where $\mathcal{R}$ is the constant sheaf $M \times \mathbb{R}$ and ι is the natural injection. We are now left to show that the sequence (11) is a fine (torsionless) resolution of $\mathcal{R}$. By the Poincaré lemma of Iwaniec and Lutoborski [12], the sequence is exact. Thus it is a resolution of $\mathcal{R}$. Since every smooth partition of unity on M induces a partition of unity for $\mathcal{W}^{\ell,p}(M)$, sheaves $\mathcal{W}^{\ell,p}(M)$ are fine. Thus the sequence (11) induces a cohomology theory with coefficients in sheaves of $\mathbb{R}$-modules over M, that is, for every sheaf $\mathcal{T}$ of real vector spaces over M we have

$$H^\ell(M,\mathcal{T}) := H^\ell(\Gamma(\mathcal{W}^{\ell,p}(M) \otimes \mathcal{T})),$$

where $\Gamma(\mathcal{W}^\ell(M) \otimes \mathcal{T})$ is the module of global sections of $\mathcal{W}^{\ell,p}(M) \otimes \mathcal{T}$.

Since all cohomology theories on M, with coefficients in sheaves of $\mathbb{R}$-modules over M, are uniquely isomorphic ([23, Theorem 5.23]), we have that $H^\ell(M,\mathcal{R})$ is isomorphic to the ℓ-th singular cohomology of M with real coefficients and to the ℓ-th de Rham cohomology of M, see e.g. [23, 5.28, 5.30] for details. It is now sufficient to show that $H^\ell(M,\mathcal{R})$ and $H^{\ell,p}(M)$ are isomorphic. Since we may apply the same argument as in the smooth case, we refer to [23, 5.30].

4. Proof of Theorem 6

Theorem 6 is based on the following estimate on the growth of the p-energy of exact $\mathcal{A}$-harmonic forms.

Theorem 17. *Let $n \geq 3$ and η a weakly exact $\mathcal{A}$-harmonic ℓ-form, $\ell \in \{2,\ldots,n-1\}$, on $\mathbb{R}^n \setminus \bar{B}^n$ such that*

$$\int_{\mathbb{R}^n \setminus \bar{B}^n(2)} |\eta|^{n/\ell} = \infty. \tag{12}$$

Then there exists $\gamma = \gamma(n,a,b) > 0$ such that

$$\liminf_{r\to\infty} \frac{1}{r^\gamma} \int_{B^n(r)\setminus \bar{B}^n(2)} |\eta|^{n/\ell} > 0. \tag{13}$$

Here a and b are as in (7) *and* (8)

The proof of Theorem 17 is based on following three lemmata from [18]. We set $D_R = B^n(R) \setminus \bar{B}^n(2)$, $\Omega_R = B^n(2R) \setminus \bar{B}^n(R)$, and $\Omega'_R = B^n(4R) \setminus \bar{B}^n(R/2)$ for $R > 2$.

Lemma 18 ([18, Lemma 16])**.** *Let $1 \leq \ell \leq n$, $p \in (1,\infty)$, $R > 4$, and let ω be a form in $W^{d,p}(\bigwedge^{\ell-1} \bar{B}^n(2R) \setminus B^n)$ such that $d\omega$ is $\mathcal{A}$-harmonic (of type p) in $B^n(2R) \setminus \bar{B}^n$. Then*

$$\|d\omega|D_R\|_p^p \leq C_1 \|d\omega|D_2\|_p^{p-1} \|\omega|D_2\|_p + \frac{C_1}{R} \|d\omega|\Omega_R\|_p^{p-1} \|\omega|\Omega_R\|_p,$$

where $C_1 = 2b/a$, *and constants* a *and* b *are as in* (7) *and* (8).

Lemma 19 ([18, Lemma 17]). *Let* $2 \le \ell \le n-1$, $p \in (1,\infty)$, $R > 8$, *and let* $\tau \in W^{d,p}(\bigwedge^{\ell-1} \bar{B}^n(4R) \setminus B^n)$ *be such that* $\|d\tau|\Omega_R\|_p > 0$. *Then there exists* $\omega \in W^{d,p}(\bigwedge^{\ell-1} \bar{B}^n(4R) \setminus B^n)$ *such that* $d\omega = d\tau$, $\omega|D_2 = \tau|D_2$, *and*

$$\|\omega|\Omega_R\|_p \le 2\|\omega|\Omega_R + d\beta|\Omega_R\|_p$$

for every $\beta \in W^{d,p}(\bigwedge^{\ell-2} \overline{\Omega'_R})$.

Lemma 20 ([18, Lemma 18]). *Let* $2 \le \ell \le n-1$. *Then for every* $R > 0$ *and* $\omega \in W^{d,n/\ell}(\bigwedge^{\ell-1} \overline{\Omega'_R})$ *there exists a closed form* $\omega_0 \in W^{d,n/\ell}(\bigwedge^{\ell-1} \overline{\Omega'_R})$ *such that*

$$\|\omega - \omega_0\|_{n/\ell} \le C(R/2)\|d\omega\|_{n/\ell}, \tag{14}$$

where $C = C(n/\ell) > 0$.

Proof of Theorem 17. Set $p = n/\ell$. Since η is weakly exact, there exists a form τ_0 in $W^{d,p}_{\mathrm{loc}}(\bigwedge^{\ell-1} \mathbb{R}^n \setminus \bar{B}^n)$ such that $\eta = d\tau_0$. By (12), we may fix $R_0 > 2$ such that

$$\|\eta|B^n(R_0) \setminus \bar{B}^n(4)\|_p^p \ge 2C_1\|\eta|D_4\|_p^{p-1}\|\tau_0|D_4\|_p.$$

Let $R > R_0$. Let $\omega \in W^{d,p}(\bigwedge^{\ell-1} B^n(R) \setminus B^n)$ as in Lemma 19 such that $\omega = \tau_0$ on D_4. By Lemma 18 and choices above,

$$\|\eta|D_R\|_p^p \le 2\frac{C_1}{R}\|\eta|\Omega_R\|_p^{p-1}\|\omega|\Omega_R\|_p.$$

By Lemma 20, there exists a closed form $\omega_0 \in W^{1,p}(\bigwedge^{\ell-1} \overline{\Omega'_R})$ such that

$$\|\omega|\Omega'_R - \omega_0\|_p \le C(R/2)\|\eta|\Omega'_R\|_p.$$

Since ω_0 is exact, we have, by Lemma 19,

$$\|\eta|D_R\|_p^p \le 2\frac{C_1}{R}\|\eta|\Omega_R\|_p^{p-1}\|\omega|\Omega_R\|_p \le CC_1\|\eta|\Omega'_R\|_p^p.$$

Hence

$$\int_{B^n(4R)\setminus\bar{B}^n(2)} |\eta|^p \ge (1 + 1/(CC_1)) \int_{B^n(R/2)\setminus\bar{B}^n(2)} |\eta|^p.$$

Thus (13) holds for $\gamma = \log_8(1 + 1/(CC_1))$. □

Having Theorem 17 at our disposal, Theorem 6 follows from a local version of the value distribution result of Mattila and Rickman, see [17, 5.11] and [18, Theorem 6].

Proof of Theorem 6. For $n = 2$ the result follows from the Measurable Riemann Mapping Theorem and uniformization, and does not require slowness, see e.g. [18, Theorem 3] for details.

Suppose now that $n \ge 3$. We replace f with $f \circ \sigma$ where σ is an orientation preserving Möbius mapping such that $\sigma(\mathbb{R}^n \setminus \bar{B}^n) = B^n \setminus \{0\}$. Suppose $\mathrm{Ker} f^* \ne 0$ for some $\ell \in \{2, \ldots, n-1\}$. Since every cohomology class weakly contains an (n/ℓ)-harmonic form by [20, Section 7], we may fix an (n/ℓ)-harmonic ℓ-form on N such that $f^*\xi$ is weakly exact and

$$\int_N |\xi|^{n/\ell} = 1.$$

By [18, Theorem 6], there exists a set $E \subset (1,\infty)$ of finite logarithmic measure such that

$$\lim_{\substack{r\to\infty\\ r\notin E}} \frac{\int_{B^n(r)\setminus \bar{B}^n(2)} |f^*\xi|^{n/\ell}}{\int_{B^n(r)\setminus \bar{B}^n(2)} J_f} = 1.$$

Hence Theorem 17 contradicts the slowness of f and $\mathrm{Ker} f^* = 0$ for $\ell \in \{2,\dots,n-1\}$. Thus either $H^*(N) = H^*(\mathbb{S}^n)$ or $H^*(N) = H^*(\mathbb{S}^{n-1}\times\mathbb{S}^1)$ by Poincaré duality. $\square$

5. Proof of Theorem 8

In this section we assume that N is a closed, connected, and oriented Riemannian n-manifold with $\dim H^1(N) = 1$. Let us fix some notation for this section.

Let $\tilde{N}$ be the universal covering of N and $\varphi\colon \tilde{N} \to N$ a Riemannian covering map. Let us also fix a loop $\gamma_0\colon [0,1] \to N$ and $\Xi \in H^1(N)$ as follows. Let Ξ be a non-trivial cohomology class in $H^1(N)$. Since the integration of smooth 1-forms over 1-chains induces an isomorphism $H^1(N) \to \mathrm{Hom}(H_1(N),\mathbb{R})$, we fix a homology class $c \in H_1(N)$ such that

$$I := \int_c \Xi = \min_{c'} \int_{c'} \Xi > 0, \tag{15}$$

where the minimum is taken over all such $c' \in H_1(N)$ that the integral is positive, see e.g. [4, Section 15.c] for details.

By Hurewicz's theorem, we may represent c by a loop $\gamma_0\colon [0,1] \to N$. Furthermore, we may assume that γ_0 is smooth. For every smooth 1-form ω in Ξ we have

$$\int_{\gamma_0} \omega = \int_c \Xi\,. \tag{16}$$

We set $\gamma\colon \mathbb{R} \to N$ to be the periodic extension of γ_0, i.e. $\gamma(t+k) = \gamma_0(t)$ for all $k \in \mathbb{Z}$ and $t \in [0,1]$.

Let ξ be the n-harmonic 1-form weakly contained in Ξ, i.e. for every smooth form ω representing Ξ there exists $w \in W^{1,n}(N)$ such that $\xi - \omega = dw$. By [21], ξ is locally Hölder continuous. Hence we may integrate ξ over γ_0 and (16) holds also for ξ. Indeed, let $\omega \in \Xi$ and $w \in W^{1,n}(N)$ be such that $\xi = \omega + dw$. Since dw is continuous, $w \in C^1(N)$ and

$$\int_{\gamma_0} \xi = \int_{\gamma_0} \omega + \int_{\gamma_0} dw = \int_c \Xi\,. \tag{17}$$

Since $H^1(\tilde N) = 0$ and $\varphi^*\xi$ is continuous, we may fix $u \in C^1(\tilde N)$ such that $du = \varphi^*\xi$. Since $\varphi^*\xi$ is an n-harmonic 1-form, u is an n-harmonic function. For every total lift $\tilde\gamma$ of γ, we have, by change of variables,

$$\int_{\tilde\gamma|[k,k+1]} \varphi^*\xi = \int_{\varphi\circ\tilde\gamma|[k,k+1]} \xi = \int_{\gamma|[k,k+1]} \xi = \int_c \Xi$$

for every $k \in \mathbb{Z}$, since φ is a local isometry. Thus

$$u(\tilde\gamma(k)) - u(\tilde\gamma(0)) = k \int_c \Xi \,. \tag{18}$$

for every total lift $\tilde\gamma$ of γ and every $k \in \mathbb{Z}$.

The proof of Theorem 8 is based on a study of the pull-back $u\circ f$ of u. The following result, based on Harnack's inequality and a Phragmén-Lindelöf type theorem for $\mathcal{A}$-harmonic functions, is essential in the proof.

Proposition 21. *Let v be an $\mathcal{A}$-harmonic function on $\mathbb{R}^n \setminus \bar B^n$ such that $\limsup_{|x|\to\infty} v(x) = \infty$. Then either*

$$\lim_{|x|\to\infty} v(x) = \infty \quad \text{or} \quad \liminf_{|x|\to\infty} v(x) = -\infty.$$

If $\liminf_{|x|\to\infty} v(x) = -\infty$, then there exists $\gamma = \gamma(n,a,b) > 0$, where a and b are as in (7) *and* (8)*, such that*

$$\liminf_{r\to\infty} \frac{1}{r^\gamma} \int_{B^n(r)\setminus \bar B^n(2)} |\nabla v|^n = \infty. \tag{19}$$

Proof. Suppose that $\liminf_{|x|\to\infty} v(x) > -\infty$. We may assume that v is positive on $\mathbb{R}^n \setminus \bar B^n(2)$, since there exists $c > 0$ such that $v + c$ is positive on $\mathbb{R}^n \setminus \bar B^n(2)$. We show that $\liminf_{|x|\to\infty} v(x) = \infty$.

Let k_0 be such that for every $R > 10$, there exists k_0 balls $B_i = B(x_i, R/4)$ covering $S^{n-1}(R)$ such that $x_i \in S^{n-1}(R)$ for every i.

Let $R > 10$ and $B_1, \ldots, B_{k_0}$ balls as above. By Harnack's inequality, there exists $\theta = \theta(n,a,b) \geq 1$ such that $\max_{B_i} v \leq \theta \min_{B_i} v$ for every i. Thus, by a standard chain argument,

$$\max_{S^{n-1}(R)} v \leq \theta^{k_0} \min_{S^{n-1}(R)} v.$$

Hence $\limsup_{|x|\to\infty} v = \infty$ yields $\liminf_{|x|\to\infty} v = \infty$.

Let us now assume that $\liminf_{|x|\to\infty} v = -\infty$. We show that (19) holds. Since v is bounded on $S^{n-1}(2)$, we may assume, by adding a constant if necessary, that $\min_{S^{n-1}(2)} v < 0$. Fix $R_0 > 2$. By the Maximum Principle, there exists a closed connected set Γ_0 such that $v|\Gamma_0 < 0$ and $\Gamma_0 \cap S^{n-1}(R) \neq \emptyset$ for every $R \geq R_0$.

Since for $r \geq R_0$

$$\mathsf{cap}_n(\bar B^n(r) \cap \Gamma, B^n(2r)) \geq c,$$

where $c > 0$ depends only on n, we have, by a Phragmén-Lindelöf type theorem [19, VII.6.7], that

$$\liminf_{r\to\infty} \frac{\max_{S^{n-1}(r)} v}{r^\beta} > 0,$$

where β depends only on n and constants a and b of $\mathcal{A}$. Here we used the assumption $\limsup_{|x|\to\infty} v(x) = \infty$. Let $c' > 0$ and $R_1 > R_0$ be such that $\max_{S^{n-1}(r)} v \ge c' r^\beta$ for $r \ge R_1$.

Let $R > R_1$ and $A = B^n(2R) \setminus \bar{B}^n(R)$. By the Maximum Principle, there exists a continuum Γ_1 connecting $S^{n-1}(R)$ to $S^{n-1}(2R)$ in $\overline{A}$ such that $v|\Gamma_1 \ge c' R^\beta$. By a standard capacity estimate,

$$\begin{aligned} \int_{B^n(2R)\setminus \bar{B}^n(2)} |\nabla v|^n \,\mathrm{d}x &\ge \int_A |\nabla v|^n \ge (c'R^\beta)^n \int_A \left| \frac{\nabla v}{c'R^\beta}\right|^n \,\mathrm{d}x \\ &\ge c'C'R^{n\beta} \mathsf{M}_n(\Delta(\Gamma_1, \Gamma_0 \cap A; A)) \ge c'CR^{n\beta}, \end{aligned}$$

where C and C' depend only on n. Thus (19) holds for $\gamma = n\beta$. □

For the proof of Theorem 8, let us also fix a deck transformation h on $\tilde{N}$ corresponding to γ_0, i.e. given $x \in \tilde{N}$ and $\sigma\colon [0,1] \to \tilde{N}$ such that $\sigma(0) = x$ and $\sigma(1) = h(x)$ then $\varphi \circ \sigma$ is (freely) homotopic to γ_0. Then for every $t \in \mathbb{R}$ and $x \in \tilde{N}$ we have $h(\tilde{\gamma}(t)) = \tilde{\gamma}(t+1)$ and $u(h(x)) = u(x) + I$, where I is the integral in (15) and $\tilde{\gamma}$ is any total lift of γ. The deck transformation h has also the following covering property.

Lemma 22. *Suppose that V is an end of $\tilde{N}$ such that u is bounded from below in V and there exists a total lift $\tilde{\gamma}$ of γ such that $|\tilde{\gamma}|[t_0,\infty)| \subset V$ for some $t_0 \in \mathbb{R}$, then*

$$\tilde{N} = \bigcup_{k\in\mathbb{N}} h^{-k} V \tag{20}$$

Proof. Let $R > 0$. We show that $B(\tilde{\gamma}(t_0), R)$ is contained in $h^{-k}(V)$ for some $k \in \mathbb{N}$. Set $x_0 = \tilde{\gamma}(t_0)$. Since $\tilde{N}$ is complete, $\bar{B}(x_0,R)$ is compact. Thus $V \cup B(x_0,R)$ is an end of $\tilde{N}$ and u is bounded from below in $V \cup B(x_0,R)$. We fix $k \in \mathbb{N}$ such that $kI > \max_{\partial V} u - \inf_{V\cap B(x_0,R)} u$. Then

$$\begin{aligned} u(x) &= u(h^k(x)) - kI < u(h^k(x)) - \max_{\partial V} u + \inf_{V\cup B(x_0,R)} u \\ &\le \inf_{V\cup B(x_0,R)} u \end{aligned}$$

for every $x \in h^{-k}(\partial V)$. Thus $h^{-k}(\partial V)$ does not intersect $V \cup B(x_0,R)$. Since $h^{-k}(\tilde{\gamma}(t_0+k)) = \tilde{\gamma}(t_0)$, $h^{-k}V$ intersects V. Hence, by connectedness of $V \cup B(x_0,R)$, $V \cup B(x_0,R) \subset h^{-k}V$. □

Proof of Theorem 8. We replace f with $f \circ \sigma$, where σ is a sense-preserving Möbius mapping such that $\sigma(\mathbb{R}^n \setminus \bar{B}^n) = B^n \setminus \{0\}$. We may assume that f is continuous up to S^{n-1}, since we may replace f with the map $x \mapsto f(x/2)$ if this is not the case.

Let us first show that a lift $\tilde{f}$ of f has a removable singularity at infinity. Suppose towards contradiction that $\tilde{f}$ has an essential singularity at infinity. Let $v = u \circ \tilde{f}$. Then v is an $\mathcal{A}$-harmonic function on $\mathbb{R}^n \setminus \bar{B}^n$. Since $\tilde{f}$ has an essential singularity at infinity, for every $k \geq 1$ the set $\tilde{f}(\mathbb{R}^n \setminus \bar{B}^n(k))$ covers whole $\tilde{N}$ except a possible set of zero capacity. Thus we may fix a sequence (x_k) in $\mathbb{R}^n \setminus \bar{B}^n$ such that $x_k \in \mathbb{R}^n \setminus \bar{B}^n(k)$ and $v(x_k) = 0$ for every k. Similarly, we may fix a sequence (y_k) in $\mathbb{R}^n \setminus \bar{B}^n$ such that $y_k \in \mathbb{R}^n \setminus \bar{B}^n(k)$ and $v(y_k) \to \infty$ as $k \to \infty$. Since v is $\mathcal{A}$-harmonic, there exists, by Proposition 21, $\gamma > 0$ such that

$$\liminf_{R\to\infty} \frac{1}{R^\gamma} \int_{B^n(R)\setminus \bar{B}^n(2)} |\nabla v|^n = \infty.$$

On the other hand,

$$f^*(\xi) = \tilde{f}^* \varphi^* \xi = \tilde{f}^* du = d(u \circ \tilde{f}) = dv.$$

Thus, by change of variables,

$$\frac{1}{R^\gamma} \int_{B^n(R)\setminus \bar{B}^n(2)} |dv|^n \;\leq\; K_O(f) \|\xi\|_\infty^n \frac{1}{R^\gamma} \int_{B^n(R)\setminus \bar{B}^n(2)} J_f \to 0$$

as $R \to \infty$, since f is slow. This is a contradiction and $\tilde{f}$ has a removable singularity at infinity.

Let us now show that $\tilde{N}$ has exactly two ends. Since f is slow, $\tilde{f}$ does not have a limit in $\tilde{N}$ at infinity. By Lemma 13, $\tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$ is an end of $\tilde{N}$ with respect to $\partial \tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$ and, by Theorem 7, $\tilde{N}$ has at most two ends. Suppose towards contradiction that $\tilde{N}$ has only one end. Then $\tilde{N} \setminus \tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$ is compact. Since u is unbounded from above and below in $\tilde{N}$, u is unbounded from above and below in $\tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$. Thus

$$-\infty = \liminf_{|x|\to\infty} v(x) < \limsup_{|x|\to\infty} v(x) = \infty,$$

where $v = u \circ \tilde{f}$ as above. Following the reasoning above, this is a contradiction with the assumption that f is slow and Proposition 21. Therefore $\tilde{N}$ has exactly two ends.

Finally, we show that u has compact level sets. As the first step, we show that u is unbounded in $\tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$. Since $\tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$ is an end of $\tilde{N}$, there exists a sequence (x_k) in $\tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$ such that $\mathrm{dist}\left(x_k, \tilde{f}(S^{n-1})\right) \geq k\ell(\gamma_0)$ and $\varphi(x_k) = \gamma(0)$. For all $k \geq 1$, we denote by $\tilde{\gamma}_k$ the total lift of $\gamma|[0,\infty)$ with $\tilde{\gamma}(0) = x_k$. If $\tilde{\gamma}_k$ is contained in $\tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$ for some k, u is unbounded from above, since

$$u(\tilde{\gamma}_k(m)) = u(x_k) + mI \to \infty$$

as $m \to \infty$. On the other hand, if $\tilde{\gamma}_k$ is not contained in $\tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$ for any k, then

$$u(x_k) \le \max_{x \in \tilde{f}(S^{n-1})} u(x) - kI \to -\infty$$

as $k \to \infty$ and u is unbounded from below. Since u is unbounded in $\tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$, we may assume u to be unbounded from above by replacing ξ with $-\xi$ and by replacing γ_0 with $t \mapsto \gamma_0(1-t)$ if necessary. Thus, by Proposition 21 and reasoning above, $\lim_{|x|\to\infty} v(x) = \infty$.

For the second step, we fix $c \in \mathbb{R}$ such that $c > \max_{f(\mathbb{S}^{n-1})} u$. We show that $u^{-1}(c)$ is compact and separates two ends of $\tilde{N}$. Let $E = u^{-1}(c) \cap \tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$. Since $\tilde{f}$ has a removable singularity at infinity and $\lim_{|x|\to\infty} v = \infty$, E is compact. Since u is unbounded from above and below in $\tilde{N}$ and bounded from below in $\tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$, E separates two ends of $\tilde{N}$. It is now sufficient to show that $u^{-1}(c) \subset \tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$.

Let V and W be the ends of $\tilde{N}$ with respect to E such that $V \subset \tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$. We fix a total lift $\tilde{\gamma}$ of γ and $t_0 \in \mathbb{R}$ such that $\tilde{\gamma}(t_0) \in V$ and

$$u(\tilde{\gamma}(t_0)) > c - \min_{t\in[0,1]} \int_{\gamma_0|[0,t]} \xi.$$

Since

$$\begin{aligned} u(\tilde{\gamma}(t_0+k+t)) &= u(\tilde{\gamma}(t_0+t)) + kI \\ &\ge u(\tilde{\gamma}(t_0)) + kI + \int_{\gamma|[t_0,t_0+t]} \xi > c + kI \end{aligned}$$

for every $k \in \mathbb{Z}$ and $t \in [0,1]$, we have that $|\tilde{\gamma}|[t_0,\infty)| \subset V$.

Since the sets $h^{-k}(V)$, $k \in \mathbb{N}$, cover $\tilde{N}$ by Lemma 22, it suffices to show that $h^{-k}(V) \cap u^{-1}(c) \subset \tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$ whenever $\tilde{f}(\mathbb{R}^n \setminus \bar{B}^n) \subset h^{-k}V$. Let $k \in \mathbb{N}$ be such that $\tilde{f}(\mathbb{R}^n \setminus \bar{B}^n) \subset h^{-k}V$. Since

$$u|\partial h^{-k}V = u|h^{-k}(E) \le c - kI < c$$

and $u|\tilde{f}(S^{n-1}) < c$, we have that $u < c$ on $\partial(h^{-k}V \setminus \tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$. Thus, by the Maximum Principle, $u < c$ on $h^{-k}V \setminus \tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$. Hence $h^{-k}V \cap u^{-1}(c) \subset \tilde{f}(\mathbb{R}^n \setminus \bar{B}^n)$.

As the third step, we note that, since $u^{-1}(c)$ is compact for $c > \max_{\tilde{f}(S^{n-1})} u$ and sets $u^{-1}(c-kI)$ are homeomorphic to $u^{-1}(c)$ for all $k \in \mathbb{Z}$ and all $c \in \mathbb{R}$, all level sets of u are compact. □

Proof of Corollary 9. Let $F\colon \pi_1(N) \to \mathbb{R}$ be a homomorphism

$$\bar{\sigma} \mapsto \int_\sigma \xi,$$

where $\bar{\sigma}$ is the homotopy class of a loop σ in N. Let $H = \mathrm{Ker}F$. Then H is a normal subgroup of $\pi_1(N)$ and $\pi_1(N)/H$ is isomorphic to the image of F. Since the image of F is generated by one element, the claim follows, if we show that H is finite.

Fix $x \in \tilde{N}$ and $y = \varphi(x)$. For every homotopy class $\bar{\sigma}$ in H, we fix a representative $\sigma\colon [0,1] \to N$ starting from y and a lift $\tilde{\sigma}$ of σ starting from x. Since $u(\tilde{\sigma}(1)) = u(\tilde{\sigma}(0))$, we have that H is in one-to-one correspondence with $u^{-1}(u(x)) \cap \varphi^{-1}(y)$. Since $u^{-1}(u(x))$ is compact and $\varphi^{-1}(y)$ is discrete, H is finite. $\square$

6. Slow mappings into spaces of the cohomology type of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ have logarithmic growth

In this section we prove Theorem 10. In the proof we use notation fixed in Section 5, that is, $\tilde{N}$ is the universal covering of N, $\varphi\colon \tilde{N} \to N$ and $h\colon \tilde{N} \to \tilde{N}$ are the Riemannian covering map and the fixed deck transformation, respectively, $\Xi \in H^1(N)$ the fixed cohomology class, ξ the n-harmonic form in Ξ, u an n-harmonic function on $\tilde{N}$ such that $du = \varphi^*\xi$, and I the integral in (15). The proof of Theorem 10 is based on following observations.

Lemma 23. *For almost every $c \in \mathbb{R}$ and every $k \in \mathbb{Z}$ we have*

$$|u^{-1}(c, c+kI)| = C_N k$$

where $C_N > 0$ is a constant depending only on N.

Proof. Since h is an isometry with the property $u \circ h = u + I$, sets $u^{-1}(c+iI, c+(i+1)I)$ have the same volume for every $i \in \mathbb{Z}$ and every $c \in \mathbb{R}$. For almost every $c \in \mathbb{R}$ we have $|u^{-1}(c)| = 0$. Let $c \in \mathbb{R}$ be such that $|u^{-1}(c)| = 0$. Since $|u^{-1}(c+iI)| = 0$ for every $i \in \mathbb{Z}$,

$$|u^{-1}(c, c+kI)| = k|u^{-1}(c, c+I)| \tag{21}$$

for every $k \in \mathbb{Z}$. Since φ is a local isometry, $|\varphi(u^{-1}(c))| = 0$. For every $y \in N$ we also have

$$\text{card } (\varphi^{-1}(y) \cap u^{-1}[c, c+I)) = \text{card } H,$$

where H is as in Corollary 9. Thus

$$(\text{card } H)|N| = \int_{u^{-1}(c,c+I)} J_\varphi = |u^{-1}(c, c+I)|. \tag{22}$$

$\square$

The following lemma is a simplified version of [6, 3.8] adapted to our setting.

Lemma 24. *Let $c \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then*

$$\mathsf{M}_n(\Delta(u^{-1}(c), u^{-1}(c+kI))) = C'_N k^{1-n}, \tag{23}$$

where $C'_N > 0$ is a constant depending only on N.

Proof. Since u is n-harmonic,

$$\int_{u^{-1}(c,c+kI)} \left| \nabla \frac{u-c}{(c+kI)-c} \right|^n = \inf \int_{u^{-1}(c,c+kI)} |\nabla w|^n,$$

where the infimum is taken over all functions $w \in W_0^{1,n}(u^{-1}(c,\infty))$ such that $w \geq 1$ on $u^{-1}(c+kI)$. Since the level sets $u^{-1}(c)$ and $u^{-1}(c+kI)$ separate $\tilde{N}$,

$$\begin{aligned}\mathsf{M}_n(\Delta(u^{-1}(c), u^{-1}(c+kI))) &= \int_{u^{-1}(c,c+kI)} \left| \nabla \frac{u-c}{(c+kI)-c} \right|^n \\ &= \frac{1}{(kI)^n} \int_{u^{-1}(c,c+kI)} |du|^n.\end{aligned}$$

Since $du = \varphi^*\xi$ and φ is a local isometry, $|du|^n = |\xi|^n \circ \varphi$. Thus

$$\int_{u^{-1}(c,c+kI)} |du|^n = \int_{u^{-1}(c,c+kI)} |\varphi^*\xi|^n = k \int_N |\xi|^n.$$

Since the constant

$$C'_N = \frac{1}{I^n} \int_N |\xi|^n$$

does not depend on Ξ, (23) follows. □

Proof of Theorem 10. By monotonicity, it is sufficient to show that there exist positive constants $C_m > 0$ and $C_M > 0$ depending only on n, K, N, and the multiplicity of $\tilde{f}$ such that

$$C_m j + C'_m \leq \int_{B^n(2^j a)\setminus B^n(a)} J_{\tilde{f}} \leq C_M j + C'_M$$

for $j \in \mathbb{Z}$ large enough and given $a \in (0,1)$, where C'_m and C'_M are constants independent of j.

Let u be as in Section 5 and $v = u \circ \tilde{f}$. Let $m(r) = \min_{S^{n-1}(r)} v$ for all $r \in (0,1)$. By Proposition 21, we may assume that $\lim_{|x|\to 0} v = \infty$. Thus we may fix $a \in (2,\infty)$ such that $m(a) > \max_{S^{n-1}(2)} v$ and $m(a/2) > 0$. Since v is $\mathcal{A}$-harmonic, we may, as in the proof of Proposition 21, fix $\theta' = \theta'(n,K) \geq 1$ such that $\max_{S^{n-1}(r)} v \leq \theta' \min_{S^{n-1}(r)} v$ for every $r \geq a$.

Let $R \in (a,\infty)$. By the Maximum Principle,

$$u^{-1}\left(\theta' m(a), m(R)\right) \subset \tilde{f}(B^n(R) \setminus \bar{B}^n(a)) \subset u^{-1}\left(m(a), \theta' m(R)\right).$$

Thus

$$\left|u^{-1}(\theta' m(a), m(R))\right| \leq \left|\tilde{f}(B^n(R) \setminus \bar{B}^n(a))\right| \leq \left|u^{-1}(m(a), \theta' m(R))\right|.$$

By Lemma 23,

$$\begin{aligned}\left|u^{-1}(m(a), \theta' m(R))\right| &\leq C_N \left(\frac{\theta' m(R) - m(a)}{I} + 1 \right) \\ &= \tilde{C}_M m(R) + \tilde{C}'_M,\end{aligned}$$

where C_N is a constant depending only on N and $\tilde{C}_M > 0$ depends only on n, K, and N. Similarly,

$$\begin{aligned} \left|u^{-1}(\theta' m(a), m(R))\right| &= \left|u^{-1}(m(a), m(R))\right| - \left|u^{-1}(m(a), \theta' m(a))\right| \\ &\geq C_N \left(\frac{m(R) - m(a)}{I} - 1\right) - C \\ &= \tilde{C}_m m(R) + \tilde{C}'_m, \end{aligned}$$

where C_N is a constant depending on N and $\tilde{C}_m > 0$ depends only on n, K, and N. Thus

$$\tilde{C}_m m(R) + \tilde{C}'_m \leq \left|\tilde{f}(B^n(R) \setminus B^n(a)\right| \leq \tilde{C}_M m(R) + \tilde{C}'_M \tag{24}$$

for $R \in (a, \infty)$, where $\tilde{C}_m > 0$ and $\tilde{C}_M > 0$ depend only on n, K, and N, and constants $\tilde{C}'_m$ and $\tilde{C}'_M$ are independent of R. On the other hand, by Theorem 8 and Lemma 13, $\tilde{f}$ has finite multiplicity, say μ. Thus

$$\left|\tilde{f}(B^n(2^j a) \setminus \bar{B}^n(a)\right| \leq \int_{B^n(2^j a)\setminus \bar{B}^n(a)} J_{\tilde{f}} \leq \mu \left|\tilde{f}(B^n(2^j a) \setminus \bar{B}^n(a)\right|. \tag{25}$$

By (24), it is sufficient to show that $m(2^j a) \sim j$ for large j. Set

$$\Gamma_{r,R} = \Delta(S^{n-1}(r), S^{n-1}(R); \mathbb{R}^n \setminus \bar{B}^n)$$

and

$$\tilde{\Gamma}_{c,d} = \Delta(u^{-1}(c), u^{-1}(d); \tilde{N})$$

for $R > r > 1$ and $d > c$.

Since sets $u^{-1}(m(a))$, $u^{-1}(\theta' m(a))$, $u^{-1}(m(R))$ and $u^{-1}(\theta' m(R))$ separate $\tilde{N}$ into two ends, every path in $\tilde{f}\Gamma_{a,R}$ has a subpath in $\tilde{\Gamma}_{\theta' m(a), m(R)}$. Thus, by the K_0-inequality,

$$\mathsf{M}_n(\tilde{\Gamma}_{\theta' m(a), m(R)}) \geq \mathsf{M}_n(\tilde{f}\Gamma_{a,R}) \geq \frac{\mathsf{M}_n(\Gamma_{a,R})}{K^{n-1}\mu}. \tag{26}$$

Let us now show that every path in $\tilde{\Gamma}_{m(a),\theta' m(R)}$ has a subpath in $\tilde{f}\Gamma_{a,R}$. Let $\tilde{\alpha}\colon [0,1] \to \tilde{N}$ be a path in $\tilde{\Gamma}_{m(a),\theta' m(R)}$. We may assume that $\tilde{\alpha}$ is contained in $u^{-1}[m(a), \theta' m(R)]$. Let $\hat{\alpha}$ be a maximal lift of $\tilde{\alpha}$ in $\tilde{f}$ such that $\hat{\alpha}(0) \in S^{n-1}(a)$. Since $v \circ \hat{\alpha} = u \circ \tilde{\alpha}$, when defined, and

$$\max_{S^{n-1}(2)} v < m(a) \leq u(\tilde{\alpha}(t)) \leq \theta' m(R),$$

for all $t \in [0,1]$, we have that $\hat{\alpha}$ is a total lift of $\tilde{\alpha}$. Since $\hat{\alpha}(1) \geq \max_{S^{n-1}(R)} v$, we have by the Maximum Principle that $\hat{\alpha}(1) \notin B^n(R)$. Hence $\hat{\alpha}$ has a subpath $\hat{\beta}$ in $\Gamma_{a,R}$. Thus $\tilde{f} \circ \hat{\beta}$ is a subpath of $\tilde{\alpha}$ contained in $\tilde{f}\Gamma_{a,R}$. Hence $\tilde{\Gamma}_{m(a),\theta' m(R)}$ is minorized by $\tilde{f}\Gamma_{a,R}$ and Poletsky's inequality yields

$$\mathsf{M}_n(\tilde{\Gamma}_{m(a),\theta' m(R)}) \leq \mathsf{M}_n(\tilde{f}\Gamma_{a,R}) \leq K^{n-1}\mathsf{M}_n(\Gamma_{a,R}). \tag{27}$$

Combining (26) and (27) with the modulus of the spherical ring, we have that

$$\frac{1}{C}\left(\log\frac{R}{a}\right)^{1-n} \le \mathsf{M}_n(\tilde{\Gamma}_{m(a),\theta' m(R)}) \le C\left(\log\frac{R}{a}\right)^{1-n} \tag{28}$$

where C depends only on n, K, and μ. Thus, by (28) and Lemma 23, we have

$$C'_m \log R + C''_m \le m(R) \le C'_M \log R + C''_M \tag{29}$$

where C'_m and C'_M depend only on n, K, N, and multiplicity μ of $\tilde{f}$, and constants C''_m and C''_M are independent of R. □

7. A local Zorich type theorem for slow quasiregular mappings

The proof of Theorem 11 is based on following observations on the universal covering $\tilde{N}$ of N.

Let $\varphi\colon \tilde{N} \to N$ be a Riemannian covering map. Let also Ξ be a cohomology class in $H^1(N)$ and γ_0 a loop in N as in Section 5. As before, let $h\colon \tilde{N} \to \tilde{N}$ be a deck transformation of $\tilde{N}$ with respect to φ corresponding to γ_0.

Let ζ be the harmonic 1-form in Ξ, that is, ζ satisfies Laplace equation $\Delta\zeta = 0$. We fix a harmonic function w on $\tilde{N}$ such that $dw = \varphi^*\zeta$. Since w is smooth, we may assume, by Sard's theorem, that $w^{-1}(0)$ is a smooth submanifold of $\tilde{N}$. By our choices, $w(h(x)) = w(x) + I$, where I is the integral in (15).

Let u be an n-harmonic function on $\tilde{N}$ as in Section 5. Since the function $w - u$ is bounded on $\tilde{N}$ and u has compact level sets, also w has compact level sets.

Let $\Omega = w^{-1}(0, I)$, $\Omega_i = h^i\Omega$, and

$$\Omega_{ij} = \Omega_i \cup \bigcup_{k=i+1}^{j-2} \overline{\Omega_k} \cup \Omega_{j-1}$$

for every i and j in $\mathbb{Z}$ such that $i < j$. Denote also

$$\Omega_{i+} = \Omega_i \cup \bigcup_{k=i}^{\infty} \overline{\Omega_k}$$

for every i. By the properties of h, we have that $\Omega_i = w^{-1}(iI, (i+1)I)$, $\Omega_{ij} = w^{-1}(iI, jI)$, and $\Omega_{i+} = w^{-1}(iI, \infty)$.

The proof of Theorem 11 is based on following two topological observations.

Lemma 25. *There exists $k \ge 0$ such that Ω_{i+} is simply connected in $\Omega_{(i-k)+}$ for every i, that is, for every i and every loop α in Ω_{i+} there exists homotopy contracting α in $\Omega_{(i-k)+}$.*

Proof. Let us first fix $k \geq 0$. Since $\overline{\Omega_{01}} = w^{-1}[0, 2I]$ is a compact manifold with boundary, the fundamental group of $\overline{\Omega_{01}}$ is finitely generated. Thus we may fix loops $\{\alpha_0, \ldots, \alpha_d\}$ generating $\pi_1(\overline{\Omega_{01}})$. Since $\tilde{N}$ is simply connected, there exists $k \geq 0$ such that loops α_i are contractible in $\Omega_{-k,k+1}$. Hence all loops in Ω_{01} are contractible in $\Omega_{-k,k+1}$.

To show that Ω_{i+} is simply connected in $\Omega_{(i-k)+}$ for every i, it is sufficient to show that Ω_{ij} is simply connected in $\Omega_{i-k,j+k}$ for every $j > i$. We do this by induction. Since $\Omega_{i,i+1} = h^i\Omega_{01}$, the claim holds for $j = i + 1$.

Suppose that the claim holds for $j \geq i + 1$ and let $\beta\colon [0, 1] \to \tilde{N}$ be a loop in $\Omega_{i,j+1}$. We may assume that β is not contained in $\Omega_{j,j+1}$. We have, by compactness, a finite number of maximal essentially disjoint intervals $[a_\ell, b_\ell]$ such that $\beta([a_\ell, b_\ell]) \subset \overline{\Omega_{j,j+1}}$, $\beta([a_\ell, b_\ell]) \cap \overline{\Omega_{j+1}} \neq \emptyset$, and that β is contained in $\Omega_{i,j}$ outside these intervals. For every ℓ we choose a path $\beta_\ell\colon [a_\ell, b_\ell] \to \tilde{N}$ in $\Omega_{i,j}$ with the end points $\beta_\ell(a_\ell) = \beta(a_\ell)$ and $\beta_\ell(b_\ell) = \beta(b_\ell)$. Since $\beta|[a_\ell, b_\ell]$ is homotopic to β_ℓ in $\Omega_{j-k,(j+1)+k}$, β is homotopic to a path β' in $\Omega_{i-k,(j+1)+k}$, where $\beta'|[a_\ell, b_\ell] = \alpha_\ell$ for every ℓ, and β' coincides with β outside intervals $[a_\ell, b_\ell]$. By the induction assumption, β' is contractible in $\Omega_{i-k,j+k}$. Thus β is contractible in $\Omega_{i-k,(j+1)+k}$. The claim now follows. $\square$

Lemma 26. *Let M be an n-manifold and let $f\colon B^n \setminus \{0\} \to M$ be a local homeomorphism with a removable singularity, which is not a limit, at origin. Then there exists a neighborhood W of origin such that f is a covering map in $W \setminus \{0\}$.*

Proof. Let $R \in (0, 1)$. By Lemma 13, $f(B^n(R) \setminus \{0\})$ is an end of M with respect to $\partial f(B^n(R) \setminus \{0\})$. Let Ω be the end of M with respect to $f(S^{n-1}(R))$ contained in $f(B^n(R) \setminus \{0\})$. Let W' be the component of $f^{-1}\Omega$ contained in $B^n(R) \setminus \{0\}$. Since f has a removable singularity at origin, f is a proper map in W' and $W' \cup \{0\}$ is a neighborhood of origin. Hence f is a covering map in $W \setminus \{0\}$ for $W = W' \cup \{0\}$. $\square$

Proof of Theorem 11. Suppose that f is a slow quasiregular local homeomorphism. Then also $\tilde{f}$ is a local homeomorphism. By Theorem 8, $\tilde{f}$ has a removable singularity at the origin. Hence, by Lemma 26, $\tilde{f}$ is a covering map in a neighborhood W of origin. Since $\tilde{f}(W \setminus \{0\})$ is an end of $\tilde{N}$, there exists i such that either $\Omega_{i+} \subset \tilde{f}(BW \setminus \{0\})$ or $\tilde{N} \setminus \Omega_{i+} \subset \tilde{f}(W \setminus \{0\})$. We may assume that $\Omega_{i+} \subset \tilde{f}(W \setminus \{0\})$ for some i. Let $k \geq 0$ be as in Lemma 25. Since $\tilde{f}$ has the homotopy lifting property in $\tilde{f}(W \setminus \{0\})$ and $\Omega_{(i+k)+}$ is simply connected in Ω_{i+}, $\tilde{f}$ is one-to-one in $\tilde{f}^{-1}(\Omega_{(i+k)+})$. By the unique path lifting property and connectedness of $\tilde{f}(W \setminus \{0\})$, $\tilde{f}$ is one-to-one and hence an embedding in $W \setminus \{0\}$. $\square$

8. Examples

In this section we show that Theorem 6 is sharp by constructing examples of slow quasiregular mappings from $B^n \setminus \{0\}$ into $\mathbb{S}^n$ and $\mathbb{S}^{n-1} \times \mathbb{S}^1$. Let us first consider the case of $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

Example 27. A slow quasiregular mapping from $B^n \setminus \{0\}$ into $\mathbb{S}^{n-1} \times \mathbb{S}^1$ is constructed in [18]. For the reader's convenience, we sketch the short construction. Let $\psi\colon \bar{B}^n(2) \setminus B^n \to \mathbb{S}^{n-1} \times \mathbb{S}^1$, $x \mapsto (x/|x|, e^{i2\pi|x|})$. Extend ψ into $\mathbb{R}^n \setminus \bar{B}^n$ periodically, i.e. $\psi(2^k x) = \psi(x)$ for every $k \geq 0$ and $x \in \bar{B}^n(2) \setminus B^n$. Let us now fix a Möbius mapping σ such that σ is sense-preserving and $\sigma(B^n \setminus \{0\}) = \mathbb{R}^n \setminus \bar{B}^n$. Then $f = \psi \circ \sigma$ is a slow quasiregular mapping from $B^n \setminus \{0\}$ into $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

Example 28. For an example of a slow quasiregular mapping from $B^n \setminus \{0\}$ into $\mathbb{S}^n$, it is sufficient to construct an entire slow quasiregular mapping from $\mathbb{R}^n$ into $\mathbb{S}^n$. We thank Juha Heinonen and Seppo Rickman for bringing examples of this type to our attention.

First step is to construct a quasiregular mapping $h\colon B^n \to \mathbb{S}^n$ such that h is finite-to-one and onto mapping such that h is the identity on $\partial B^n = \mathbb{S}^{n-1} \subset \mathbb{S}^n$. Since we may identify B^n with the lower hemisphere $\mathbb{S}^n_-$ of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, the restriction to $\mathbb{S}^n_-$ of the mapping $\mathbb{R}^{n-1} \times \mathbb{C} \to \mathbb{R}^{n-1} \times \mathbb{C}$, $(x, z) \mapsto (x, z^3)$, satisfies our needs.

Let (y_k) be a sequence in $\mathbb{R}^n$ such that distance of the points in the sequence is at least 3. We identify $\mathbb{R}^n$ with $\mathbb{S}^n \setminus \{e_{n+1}\}$ and let $f\colon \mathbb{R}^n \to \mathbb{S}^n$ be the identity outside open balls $B^n(y_k, 1)$ and $f(x) = h(x - y_k) + y_k$ in $B^n(y_k, 1)$ for every k. Since, for every k, f is BLD in $B^n(y_k, 2)$ with a constant independent of k, f is quasiregular. Furthermore,

$$\int_{B^n(r)} J_f \sim \text{card } \{k\colon |y_k| \leq r\}$$

for large r. Thus f satisfies (1) and (2) when we choose the sequence properly. This example also shows that a result like Theorem 10 is not possible when we consider slow mappings into manifolds of the cohomology type of $\mathbb{S}^n$.

References

[1] M. Bonk and J. Heinonen. Quasiregular mappings and cohomology. *Acta Math.*, 186(2):219–238, 2001.

[2] I. Chavel. *Riemannian geometry—a modern introduction*, volume 108 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1993.

[3] T. Coulhon, I. Holopainen, and L. Saloff-Coste. Harnack inequality and hyperbolicity for subelliptic p-Laplacians with applications to Picard type theorems. *Geom. Funct. Anal.*, 11(6):1139–1191, 2001.

[4] W. Fulton. *Algebraic topology*, volume 153 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[5] J. Heinonen and S. Rickman. Geometric branched covers between generalized manifolds. *Duke Math. J.*, 113(3):465–529, 2002.

[6] I. Holopainen. Nonlinear potential theory and quasiregular mappings on Riemannian manifolds. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (74):45, 1990.

[7] I. Holopainen. Quasiregular mappings and the p-Laplace operator. In *Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)*, volume 338 of *Contemp. Math.*, pages 219–239. Amer. Math. Soc., Providence, RI, 2003.

[8] I. Holopainen and P. Pankka. A big Picard theorem for quasiregular mappings into manifolds with many ends. *Proc. Amer. Math. Soc.*, 133(4):1143–1150 (electronic), 2005.

[9] I. Holopainen and S. Rickman. Classification of Riemannian manifolds in nonlinear potential theory. *Potential Anal.*, 2(1):37–66, 1993.

[10] I. Holopainen and S. Rickman. Ricci curvature, Harnack functions, and Picard type theorems for quasiregular mappings. In *Analysis and topology*, pages 315–326. World Sci. Publishing, River Edge, NJ, 1998.

[11] T. Iwaniec. p-harmonic tensors and quasiregular mappings. *Ann. of Math. (2)*, 136(3):589–624, 1992.

[12] T. Iwaniec and A. Lutoborski. Integral estimates for null Lagrangians. *Arch. Rational Mech. Anal.*, 125(1):25–79, 1993.

[13] T. Iwaniec and G. Martin. Quasiregular mappings in even dimensions. *Acta Math.*, 170(1):29–81, 1993.

[14] T. Iwaniec and G. Martin. *Geometric function theory and non-linear analysis.* Oxford Mathematical Monograph. The Clarendon Press Oxford University Press, New York, 2001.

[15] T. Iwaniec and C. Scott and B. Stroffolini. Nonlinear Hodge theory on manifolds with boundary. *Ann. Mat. Pura Appl. (4)*, 177:37–115, 1999.

[16] O. Martio and J. Väisälä. Elliptic equations and maps of bounded length distortion. *Math. Ann.*, 282(3):423–443, 1988.

[17] P. Mattila and S. Rickman. Averages of the counting function of a quasiregular mapping. *Acta Math.*, 143(3-4):273–305, 1979.

[18] P. Pankka. Quasiregular mappings from a punctured ball into compact manifolds. *Conform. Geom. Dyn.*, 10:41–62 (electronic), 2006.

[19] S. Rickman. *Quasiregular mappings*, volume 26 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1993.

[20] C. Scott. L^p theory of differential forms on manifolds. *Trans. Amer. Math. Soc.*, 347(6):2075–2096, 1995.

[21] N. Ural'tseva. Degenerate quasilinear elliptic systems. *Zap. Naučn. Sem. Leningrad. Odetl. Mat. Inst. Steklov*, 7:184–222, 1968.

[22] N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon. *Analysis and geometry on groups*, volume 100 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 1992.

[23] F. W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1983.

Department of Mathematics and Statistics,
P.O. Box 68, FIN-00014 University of Helsinki, Finland.
E-mail: pekka.pankka@helsinki.fi