COMPOSITION OPERATORS AND VECTOR-VALUED BMOA

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Abstract. Analytic composition operators $C_\varphi\colon f \mapsto f \circ \varphi$ are studied on certain X-valued versions of BMOA, the space of analytic functions on the unit disk that have bounded mean oscillation on the unit circle, where X is a complex Banach space. It is shown that if X is reflexive and C_φ is compact on the usual scalar-valued BMOA space, then C_φ is weakly compact on the X-valued space $\mathrm{BMOA}_{\mathcal{C}}(X)$ defined in terms of Carleson measures. A related function theoretic characterization is given of the compact composition operators on BMOA.

1. Introduction

Let φ be an analytic self-map of the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. Compactness properties of the composition operators

$$C_\varphi\colon f \mapsto f \circ \varphi$$

have been intensively studied on various Banach spaces of analytic functions on $\mathbb{D}$ (see [CoM] for the basic results related e.g. to the classical Hardy spaces). Recently the question of which composition operators are weakly compact has been studied also in the vector-valued setting where the functions f take values in some complex Banach space X, see e.g. [LST], [BDL], [L], [LT]. In this setting C_φ is usually never compact if X is infinite-dimensional. The purpose of this paper is to continue the study from [L] and [BDL] by considering the weak compactness of C_φ on certain vector-valued BMOA spaces, which are X-valued generalizations of the classical space BMOA of analytic functions on $\mathbb{D}$ that have bounded mean oscillation on the unit circle $\mathbb{T}$.

Compactness and weak compactness of C_φ on the scalar-valued BMOA space have been studied in several recent papers, see e.g. [BCM], [Sm], [MT], [CM], [WX]. In [L] some of these results were extended to the setting of the space $\mathrm{BMOA}(X)$, which is defined as a Möbius invariant version of the vector-valued Hardy space $H^1(X)$. There are also other interesting possibilities of approaching BMOA in the vector-valued setting (see e.g. [Bl], [Bl2], [BP]). One alternative arises by considering the weak vector-valued BMOA space $w\mathrm{BMOA}(X)$, which consists of the analytic functions $f\colon \mathbb{D} \to X$ such

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that $x^* \circ f \in$ BMOA for all $x^* \in X^*$. Some properties of composition operators on a wide class of such weak spaces, including $w\mathrm{BMOA}(X)$, follow from general results of Bonet, Domański and Lindström [BDL].

In this paper we study the weak compactness of composition operators on $\mathrm{BMOA}_{\mathcal{C}}(X)$, a vector-valued version of BMOA defined in terms of Carleson measures, which was considered earlier by Blasco [Bl2] in connection with vector-valued multipliers, see also [BP]. We are partly motivated by the fact that the spaces $\mathrm{BMOA}(X)$, $w\mathrm{BMOA}(X)$ and $\mathrm{BMOA}_{\mathcal{C}}(X)$ are usually different. In fact, it was shown by Blasco [Bl2] that $\mathrm{BMOA}(X)$ and $\mathrm{BMOA}_{\mathcal{C}}(X)$ coincide (and the respective norms are equivalent) only if X is isomorphic to a Hilbert space. We will show that the spaces $\mathrm{BMOA}_{\mathcal{C}}(X)$ and $w\mathrm{BMOA}(X)$ never coincide if X is infinite-dimensional.

Our main result states that if φ induces a compact composition operator on BMOA and X is reflexive, then C_φ is weakly compact on $\mathrm{BMOA}_{\mathcal{C}}(X)$. This result complements the earlier ones from [L] and [BDL]. The proof will be based on a function theoretic condition which characterizes the compact composition operators on the scalar-valued BMOA. The necessity part of this characterization will be established in Section 2. In Section 3 we provide some basic properties of the space $\mathrm{BMOA}_{\mathcal{C}}(X)$ and composition operators. Our main result will be proved in Section 4. As a consequence, we characterize the weakly compact composition operators on $\mathrm{BMOA}_{\mathcal{C}}(X)$ under some restrictions on φ for reflexive Banach spaces X.

2. Compactness of composition operators on BMOA

The space BMOA consists of the analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ which are Poisson integrals of functions that have bounded mean oscillation on $\mathbb{T}$. We recall the following equivalent reformulation of BMOA as a Möbius invariant version of the Hardy space H^2 (see [B]). An analytic function $f\colon \mathbb{D} \to \mathbb{C}$ belongs to BMOA if and only if

$$\|f\|_* = \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^2} < \infty,$$

where $\sigma_a(z) = (a - z)/(1 - \overline{a}z)$ for $a, z \in \mathbb{D}$, and $\|\cdot\|_{H^p}$ denotes the usual norm on the Hardy space H^p ($1 \le p < \infty$) given by $\|f\|_{H^p}^p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}$. The map $f \mapsto \|f\|_*$ is a seminorm. We equip BMOA with the complete norm $\|f\|_{\mathrm{BMOA}} = |f(0)| + \|f\|_*$. Recall that according to the John-Nirenberg theorem [B, p. 15] the map $f \mapsto \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^p}$ defines an equivalent seminorm on BMOA for any $1 \le p < \infty$. We refer to [G, Chapter VI] for further properties of BMOA.

It is well-known known that for every analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$ the operator $C_\varphi\colon f \mapsto f \circ \varphi$ is bounded on BMOA, see [St, Theorem 3], [AFP, Theorem 12]. There also are several (equivalent) characterizations of the compact composition operators on BMOA, see [BCM], [Sm], [WX]. Recall that the Nevanlinna counting function $N(\varphi, \cdot)$ of an analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$ is defined by $N(\varphi, z) = \sum_{w\in\varphi^{-1}(z)} \log(1/|w|)$ for $z \in \mathbb{D} \setminus \{\varphi(0)\}$, where each point in the preimage $\varphi^{-1}(z)$ is counted according to its multiplicity. The following result is due to Smith [Sm, Theorem 1.1]. The operator C_φ is compact on

BMOA if and only if

$$\lim_{r\to 1} \sup_{\{a\in\mathbb{D}\colon |\varphi(a)|>r\}} \sup_{0<|w|<1} |w|^2 N(\sigma_{\varphi(a)}\circ\varphi\circ\sigma_a, w) = 0 \tag{2.1}$$

and

$$\lim_{t\to 1} \sup_{\{a\in\mathbb{D}\colon |\varphi(a)|\le R\}} m(\{\zeta\in\mathbb{T}\colon |(\varphi\circ\sigma_a)(\zeta)|>t\}) = 0, \tag{2.2}$$

for every $R\in(0,1)$, where m is the Lebesgue measure on $\mathbb{T}$.

We will provide yet another characterization of the compact composition operators on BMOA by replacing (2.2) by a condition which involves the Nevanlinna counting function. This result will be useful in our study of C_φ in the vector-valued setting. The following result, which is the main result of this section, gives the necessity of this condition for the compactness of C_φ on BMOA.

Theorem 2.1. *Let $\varphi\colon \mathbb{D}\to\mathbb{D}$ be analytic. If C_φ is compact on* BMOA*, then*

$$\lim_{|w|\to 1} \sup_{\{a\in\mathbb{D}\colon |\varphi(a)|\le R\}} \frac{N(\varphi\circ\sigma_a, w)}{\log(1/|w|)} = 0, \tag{2.3}$$

for every $R\in(0,1)$, where $\sigma_a(z)=(a-z)/(1-\overline{a}z)$ for $a,z\in\mathbb{D}$.

We will observe below that conditions (2.1) and (2.3) together are also sufficient for the compactness of C_φ on BMOA (see Corollary 4.5).

The main idea for the proof of Theorem 2.3 comes from the work of Bourdon, Cima and Matheson [BCM, Theorem 4.1], where it was shown that the compactness of C_φ on BMOA implies its compactness on H^2. The proof in [BCM] is based on an integral criterion [BCM, Theorem 3.1] which in our argument will be replaced by an equivalent criterion due to Wirths and Xiao [WX]. The counting function will be controlled using certain methods from the proof due to Shapiro [S, Theorem 2.3] of the fact that C_φ is compact on the Hardy space H^2 if and only if

$$\lim_{|w|\to 1} \frac{N(\varphi, w)}{\log(1/|w|)} = 0. \tag{2.4}$$

Note that condition (2.3) clearly implies (2.4).

We recall next some auxiliary results. We will use frequently the following easy identities concerning the automorphisms $\sigma_a\colon z\mapsto (a-z)/(1-\overline{a}z)$: It holds that $(\sigma_a\circ\sigma_a)(z)=z$ and $1-|\sigma_a(z)|^2=(1-|z|^2)|\sigma_a'(z)|$ for all $a,z\in\mathbb{D}$ (see [G, I.1] for example). The relevance of the Nevanlinna counting function is seen from the change of variables formula

$$\int_{\mathbb{D}} (\lambda\circ\varphi)(z)|\varphi'(z)|^2 \log\frac{1}{|z|} dA(z) = \int_{\mathbb{D}} \lambda(z) N(\varphi, z) dA(z), \tag{2.5}$$

for positive measurable functions $\lambda\colon \mathbb{D}\to\mathbb{R}$, where A denotes the Lebesgue measure on $\mathbb{D}$ (see [S, 4.3]). Combined with the Littlewood-Paley identity (see [G, Lemma VI.3.1] or [CoM, Theorem 2.30])

$$\|f-f(0)\|_{H^2}^2 = 2\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|}\frac{dA(z)}{\pi}, \tag{2.6}$$

formula (2.5) yields the identity

$$\|f\circ\varphi - f(\varphi(0))\|_{H^2}^2 = 2\int_{\mathbb{D}} |f'(z)|^2 N(\varphi,z)\frac{dA(z)}{\pi},$$

for analytic functions $f\colon \mathbb{D}\to\mathbb{C}$ and $\varphi\colon\mathbb{D}\to\mathbb{D}$. We will also need the following estimate for the integral in (2.6): There is a constant c such that

$$\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|} dA(z) \le c\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)dA(z) \tag{2.7}$$

for all analytic functions $f\colon\mathbb{D}\to\mathbb{C}$ (see e.g. [G, Lemma VI.3.2]). On the other hand, it is easy to check that $(1-|z|^2)\le 2\log(1/|z|)$ for all $z\in\mathbb{D}$. Finally, we need the "only if"-part of the following result from [WX].

Theorem 2.2 ([WX, Theorem 5.1])**.** *Let $\varphi\colon\mathbb{D}\to\mathbb{D}$ be analytic. The composition operator C_φ is compact on* BMOA *if and only if*

$$\lim_{r\to 1}\sup_{\|f\|_{\mathrm{BMOA}}\le 1}\sup_{a\in\mathbb{D}}\int_{\{z\in\mathbb{D}\colon |\varphi(z)|>r\}} |(f\circ\varphi)'(z)|^2(1-|\sigma_a(z)|^2)dA(z) = 0.$$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Assume that C_φ is compact on BMOA. Let $0<R<1$ and $\varepsilon>0$. Recall that $\sup_{w\in\mathbb{D}}\|f_w\|_{\mathrm{BMOA}}<\infty$, where the functions $f_w\in$ BMOA are given by $f_w(z)=\log(1-\overline{w}z)$ for $w,z\in\mathbb{D}$. By Theorem 2.2, there is a number $t_0\in(0,1)$ such that

$$\sup_{a,b,w\in\mathbb{D}}\int_{\{z\in\mathbb{D}\colon |\varphi(z)|>t_0\}} |(f_w\circ\varphi)'(u)|^2(1-|(\sigma_a\circ\sigma_b)(u)|^2)dA(u)<\varepsilon,$$

since $|(\sigma_a\circ\sigma_b)(u)|=|\sigma_c(u)|$ for some $c\in\mathbb{D}$. Let us abbreviate $\Omega(b)=\{z\in\mathbb{D}\colon |(\varphi\circ\sigma_b)(z)|>t_0\}$ for $b\in\mathbb{D}$. By using the change of variable $u=\sigma_b(z)$ and the identities $(\sigma_b\circ\sigma_b)(z)=z$ and $1-|\sigma_a(z)|^2=(1-|z|^2)|\sigma_a'(z)|$, we get that

$$\begin{aligned}\varepsilon &> \sup_{a,b,w\in\mathbb{D}}\int_{\Omega(b)} |(f_w\circ\varphi)'(\sigma_b(z))|^2|(1-|\sigma_a(z)|^2)|\sigma_b'(z)|^2dA(z)\\ &= \sup_{b,w\in\mathbb{D}}\sup_{a\in\mathbb{D}}\int_{\Omega(b)} |(f_w\circ\varphi\circ\sigma_b)'(z)|^2(1-|z|^2)|\sigma_a'(z)|dA(z).\end{aligned}$$

Hence the measures $\mu_{b,w}$ given by

$$d\mu_{b,w}(z) = 1_{\Omega(b)}\frac{|w|^2|(\varphi\circ\sigma_b)'(z)|^2}{|1-\overline{w}(\varphi\circ\sigma_b)(z)|^2}(1-|z|^2)dA(z),$$

are Carleson measures for $b,w\in\mathbb{D}$. In particular, by Carleson's theorem (see [G, Lemma VI.3.3] or [CoM, Theorem 2.33]), there is a constant C so that

$$\sup_{b,w\in\mathbb{D}}\int_{\mathbb{D}} |g|^2 d\mu_{b,w}\le C\varepsilon\|g\|_{H^2}^2, \tag{2.8}$$

for all $g\in H^2$.

Consider next $b\in\mathbb{D}$ such that $|\varphi(b)|\le R$. Let k_w denote the analytic function given by $k_w(z)=\frac{\sqrt{1-|w|^2}}{1-\overline{w}z}$ for $w,z\in\mathbb{D}$, so that $\|k_w\|_{H^2}=1$. Recall that $\|C_\psi\colon H^2\to H^2\|^2\le 2/(1-|\psi(0)|^2)$ for all analytic maps $\psi\colon\mathbb{D}\to\mathbb{D}$ (see [CoM, Corollary 3.7], for instance). Consequently, $\|k_w\circ\varphi\circ\sigma_b\|_{H^2}^2\le$

$2/(1-R^2)$ for all $w \in \mathbb{D}$. By choosing $g = k_w \circ \varphi \circ \sigma_b$ in (2.8) and abbreviating $d\nu(z) = (1-|z|^2)dA(z)$ for $z \in \mathbb{D}$, we get that

$$\begin{aligned}\int_{\Omega(b)} |(k_w \circ \varphi \circ \sigma_b)'(z)|^2 d\nu(z) &= \int_{\Omega(b)} \frac{|w|^2(1-|w|^2)|(\varphi \circ \sigma_b)'(z)|^2}{|1-\overline{w}(\varphi \circ \sigma_b)(z)|^4} d\nu(z) \\ &= \int_{\mathbb{D}} |(k_w \circ \varphi \circ \sigma_b)(z)|^2 d\mu_{b,w}(z) \\ &\leq C\varepsilon \|k_w \circ \varphi \circ \sigma_b\|_{H^2}^2 \leq 2C\varepsilon/(1-R^2),\end{aligned}$$

for $b, w \in \mathbb{D}$ such that $|\varphi(b)| \leq R$. Choose next a number $r_0 \in (0,1)$ so that $\frac{|w|^2(1-|w|^2)}{(1-|w|t_0)^4} < \varepsilon$ for all $w \in \mathbb{D}$ with $|w| > r_0$. Then $|(k_w \circ \varphi \circ \sigma_b)'(z)|^2 \leq \varepsilon|(\varphi \circ \sigma_b)'(z)|^2$ for such w and $z \in \mathbb{D} \setminus \Omega(b) = \{z \in \mathbb{D} \colon |(\varphi \circ \sigma_b)(z)| \leq t_0\}$. Since $\|\varphi \circ \sigma_b - \varphi(b)\|_{H^2}^2 \leq 4$, we get from (2.6) that

$$\int_{\mathbb{D}\setminus\Omega(b)} |(k_w \circ \varphi \circ \sigma_b)'(z)|^2 d\nu(z) \leq 2\varepsilon \int_{\mathbb{D}} |(\varphi \circ \sigma_b)'(z)|^2 \log \frac{1}{|z|} dA(z) \leq 4\pi\varepsilon,$$

for all $w \in \mathbb{D}$ such that $|w| > r_0$. By applying (2.5) to the function $\lambda(z) = |k_w'(z)|^2$, using (2.7), and combining the above estimates we get that

$$\begin{aligned}\int_{\mathbb{D}} |k_w'(z)|^2 N(\varphi \circ \sigma_b, z) dA(z) &= \int_{\mathbb{D}} |(k_w \circ \varphi \circ \sigma_b)'(z)|^2 \log \frac{1}{|z|} dA(z) \\ \leq c \int_{\mathbb{D}} &|(k_w \circ \varphi \circ \sigma_b)'(z)|^2 d\nu(z) \leq c(\frac{2C}{1-R^2} + 4\pi)\varepsilon,\end{aligned}$$

for all $b, w \in \mathbb{D}$ such that $|\varphi(b)| \leq R$ and $|w| > r_0$. Hence we conclude that

$$\lim_{|w|\to 1} \sup_{\{b \colon |\varphi(b)| \leq R\}} \int_{\mathbb{D}} |k_w'(z)|^2 N(\varphi \circ \sigma_b, z) dA(z) \to 0, \tag{2.9}$$

as $|w| \to 1$.

We recall finally how condition (2.3) can be obtained from (2.9) by applying some methods from [S, 5.4] (see also [CoM, p. 138]). Put $s = \max\{\frac{1}{2}, \frac{R+1}{2}\} \in (0,1)$ and $h = \frac{1-R}{4} \in (0,1)$. Since $\sigma_w^{-1} = \sigma_w$, we get that

$$|\sigma_w^{-1}((\varphi \circ \sigma_b)(0))| = \left| \frac{w - \varphi(b)}{1 - \overline{w}\varphi(b)} \right| \geq \frac{1}{2}(|w| - |\varphi(b)|) > h, \tag{2.10}$$

for all $w, b \in \mathbb{D}$ such that $|w| > s$ and $|\varphi(b)| \leq R$. Fix next $w \in \mathbb{D}$ such that $|w| > s$. By using the identity $(1-|w|^2)|k_w'(z)|^2 = |w|^2|\sigma_w'(z)|^2$ and the change of variable $u = \sigma_w(z)$, we get that

$$\begin{aligned}\int_{\mathbb{D}} |k_w'(z)|^2 N(\varphi \circ \sigma_b, z) \frac{dA(z)}{\pi} &= \frac{|w|^2}{1-|w|^2} \int_{\mathbb{D}} N(\varphi \circ \sigma_b, z) |\sigma_w'(z)|^2 \frac{dA(z)}{\pi} \\ &= \frac{|w|^2}{1-|w|^2} \int_{\mathbb{D}} N(\varphi \circ \sigma_b, \sigma_w(u)) \frac{dA(u)}{\pi}.\end{aligned}$$

Moreover, (2.10) and the sub-mean value property of $N(\varphi, \cdot)$ (see [S, 4.6] or [CoM, p. 137]) give that

$$\int_{h\mathbb{D}} N(\varphi \circ \sigma_b, \sigma_w(u)) \frac{dA(u)}{\pi} \geq h^2 N(\varphi \circ \sigma_b, w).$$

Thus

$$\int_{\mathbb{D}} |k_w'(z)|^2 N(\varphi\circ\sigma_b, z)\frac{dA(z)}{\pi} \geq \frac{|w|^2 h^2 N(\varphi\circ\sigma_b, w)}{(1-|w|^2)} \geq \frac{h^2}{8}\frac{N(\varphi\circ\sigma_b, w)}{\log(1/|w|)},$$

for all $w \in \mathbb{D}$ such that $|w| > s$ and $|\varphi(b)| \leq R$. Condition (2.3) follows now from (2.9). $\square$

3. Vector-valued BMOA and composition operators

In the sequel $X = (X, \|\cdot\|_X)$ will always be a complex Banach space. We will consider the following versions of X-valued BMOA (see [Bl], [Bl2], [L]).

Definition 3.1. (1) The space $\mathrm{BMOA}(X)$ consists of the analytic functions $f\colon \mathbb{D} \to X$ such that $\|f\|_{*,X} = \sup_{a\in\mathbb{D}} \|f\circ\sigma_a - f(a)\|_{H^1(X)} < \infty$, where $\|\cdot\|_{H^1(X)}$ denotes the norm on the X-valued Hardy space $H^1(X)$ given by $\|f\|_{H^1(X)} = \sup_{0<r<1}\int_0^{2\pi}\|f(re^{i\theta})\|_X\frac{d\theta}{2\pi}$. We equip $\mathrm{BMOA}(X)$ with the complete norm

$$\|f\|_{\mathrm{BMOA}(X)} = \|f(0)\| + \|f\|_{*,X}.$$

(2) The space $w\mathrm{BMOA}(X)$, a weak vector-valued version of BMOA, consists of the analytic functions $f\colon \mathbb{D}\to X$ such that $x^*\circ f \in \mathrm{BMOA}$ for every functional $x^* \in X^*$. The complete norm on $w\mathrm{BMOA}(X)$ is given by

$$\|f\|_{w\mathrm{BMOA}(X)} = \sup_{\|x^*\|\leq 1}\|x^*\circ f\|_{\mathrm{BMOA}}.$$

(3) The space $\mathrm{BMOA}_{\mathcal{C}}(X)$ consists of the analytic functions $f\colon \mathbb{D}\to X$ such that

$$\|f\|^2_{\mathcal{C},X} = \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\|f'(z)\|_X^2(1-|\sigma_a(z)|^2)\frac{dA(z)}{\pi} < \infty.$$

We equip $\mathrm{BMOA}_{\mathcal{C}}(X)$ with the complete norm $\|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)} = \|f(0)\| + \|f\|_{\mathcal{C},X}$.

Note that the space $\mathrm{BMOA}_{\mathcal{C}}(X)$ can be characterized in terms of certain Carleson measures. In fact, by using the identity $1-|\sigma_a(z)|^2 = (1-|z|^2)|\sigma_a'(z)|$ and a theorem of Carleson (see [G, Lemma VI.3.3] or [CoM, Theorem 2.33]) we get that $f \in \mathrm{BMOA}_{\mathcal{C}}(X)$ if and only if the measure $d\mu_f(z) = \|f'(z)\|_X^2(1-|z|^2)dA(z)$ is a Carleson measure.

It is known that the seminorms $\|\cdot\|_{*,\mathbb{C}}$ and $\|\cdot\|_{\mathcal{C},\mathbb{C}}$ are comparable in the special case where $X = \mathbb{C}$ (one checks this fact from (2.6) and (2.7) using a change of variables). In fact, $\mathrm{BMOA} = \mathrm{BMOA}(\mathbb{C}) = w\mathrm{BMOA}(\mathbb{C}) = \mathrm{BMOA}_{\mathcal{C}}(\mathbb{C})$ with equivalent norms. In the general case, however, these spaces are usually different. By [Bl2, Corollary 1.1] the spaces $\mathrm{BMOA}(X)$ and $\mathrm{BMOA}_{\mathcal{C}}(X)$ coincide, and the respective norms are equivalent, if and only if X is isomorphic to a Hilbert space. It is also known that $\mathrm{BMOA}(X) = w\mathrm{BMOA}(X)$, and the respective norms are equivalent, if and only if X is finite-dimensional (see e.g. [L, Example 15]). The following result complements these facts.

Proposition 3.2. *The spaces $\mathrm{BMOA}_{\mathcal{C}}(X)$ and $w\mathrm{BMOA}(X)$ coincide, and the respective norms are equivalent, if and only if X is finite-dimensional.*

Proof. Let X be any complex Banach space. We get from (2.6), (2.7) and the change of variables $w = \sigma_a(z)$ that

$$\begin{aligned}\|x^* \circ f \circ \sigma_a - x^*(f(a))\|_{H^2}^2 &\leq 2c \int_{\mathbb{D}} |(x^* \circ f \circ \sigma_a)'(z)|^2 (1-|z|^2) dA(z) \\ &= 2c \int_{\mathbb{D}} |(x^* \circ f)'(w)|^2 (1-|\sigma_a(w)|^2) dA(w) \leq 2c \|x^*\|_{X^*}^2 \|f\|_{\mathcal{C},X}^2,\end{aligned}$$

for $f \in \mathrm{BMOA}_{\mathcal{C}}(X)$ and $x^* \in X^*$, where we also used the identity $(\sigma_a \circ \sigma_a)(w) = w$. Thus $\|f\|_{w\mathrm{BMOA}(X)} \leq \sqrt{2c}\|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)}$ for $f \in \mathrm{BMOA}_{\mathcal{C}}(X)$. Moreover, if $\dim(X) = n < \infty$, then it is not difficult to find a constant C (depending on n) such that $\|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)} \leq C\|f\|_{w\mathrm{BMOA}(X)}$ for all $f \in w\mathrm{BMOA}(X)$.

Assume next that X is infinite-dimensional. Let $n \in \mathbb{N}$. By Dvoretzky's theorem (see e.g. [DJT, Theorem 19.1]) there exists an n-dimensional subspace $E_n \subset X$ and a linear isomorphism $T_n \colon \ell_2^n \to E_n$ so that $\|T_n\| \leq 2$ and $\|T_n^{-1}\| = 1$. Define the analytic function $f_n \colon \mathbb{D} \to X$ by

$$f_n(z) = \sum_{k=1}^{n} \frac{(T_n e_k) z^k}{\sqrt{k}}$$

for $z \in \mathbb{D}$, where $(e_1, \ldots, e_n)$ is an orthonormal basis of ℓ_2^n. Then the argument in [L, p. 744] shows that $\sup_{n\in\mathbb{N}} \|f_n\|_{w\mathrm{BMOA}(X)} < \infty$. On the other hand, since

$$\|f_n'(z)\|_X^2 = \|\sum_{k=1}^{n} \sqrt{k}(T_n e_k) z^{k-1}\|_X^2 \geq \|\sum_{k=1}^{n} \sqrt{k} e_k z^{k-1}\|_{\ell_2^n}^2 = \sum_{k=1}^{n} k|z|^{2(k-1)},$$

we get that

$$\|f_n\|_{\mathcal{C},X}^2 \geq 2\sum_{k=1}^{n} k \int_0^1 r^{2(k-1)}(1-r^2) r\,dr = \sum_{k=1}^{n} \frac{1}{k+1} \geq \frac{\log n}{2}.$$

Thus $\|f_n\|_{\mathrm{BMOA}_{\mathcal{C}}(X)} \to \infty$ as $n \to \infty$, which shows that the norms are not equivalent. Moreover, by using the open mapping theorem we get that $\mathrm{BMOA}_{\mathcal{C}}(X) \subsetneq w\mathrm{BMOA}(X)$. □

We consider next the composition operators $C_\varphi \colon f \mapsto f \circ \varphi$ on the space $\mathrm{BMOA}_{\mathcal{C}}(X)$. It is known that for every analytic map $\varphi \colon \mathbb{D} \to \mathbb{D}$ the operator C_φ is bounded on $\mathrm{BMOA}(X)$ and $w\mathrm{BMOA}(X)$ (see [L, Proposition 3] and e.g. [LT, Theorem 5.2]). We sketch here for completeness a proof that C_φ is bounded on $\mathrm{BMOA}_{\mathcal{C}}(X)$ for any complex Banach space X. We need first a vector-valued version of (2.7): It holds that

$$\int_{\mathbb{D}} \|f'(z)\|_X^2 \log\frac{1}{|z|} dA(z) \leq c \int_{\mathbb{D}} \|f'(z)\|_X^2 (1-|z|^2) dA(z), \tag{3.1}$$

for any complex Banach space X and analytic function $f \colon \mathbb{D} \to X$. In fact, the proof of (3.1) in [G, Lemma VI.3.2] remains valid also in the vector-valued setting, since the map $z \mapsto \|f'(z)\|_X^2$ is subharmonic. Moreover, by the change of variable $w = \sigma_a(z)$ and the identity $(\sigma_a \circ \sigma_a)(z) = z$ we get that

$$\int_{\mathbb{D}} \|f'(w)\|_X^2 (1-|\sigma_a(w)|^2) dA(w) = \int_{\mathbb{D}} \|(f \circ \sigma_a)'(z)\|_X^2 (1-|z|^2) dA(z), \tag{3.2}$$

for all analytic functions $f\colon \mathbb{D} \to X$. By using the estimate $(1-|z|^2) \le 2\log(1/|z|)$, we get from (3.1) and (3.2) that

$$\|f\|_{\mathcal{C},X}^2 \le 2\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \|(f\circ\sigma_a)'(z)\|_X^2 \log\frac{1}{|z|}\frac{dA(z)}{\pi} \le 2c\|f\|_{\mathcal{C},X}^2. \tag{3.3}$$

Recall also that by an inequality due to Littlewood it holds that $N(\varphi\circ\sigma_a, z) \le N(\sigma_{\varphi(a)}, z)$ for all $z \in \mathbb{D}\setminus\{\varphi(a)\}$ and $a \in \mathbb{D}$ (see [S, p. 380] or [CoM, p. 33]). The fact that C_φ is bounded on $\mathrm{BMOA}_{\mathcal{C}}(X)$ can then be seen from (3.3) and the formula (2.5) applied to the function $\lambda(z) = \|f'(z)\|_X^2$. Indeed, we have that

$$\begin{aligned}
\|f\circ\varphi\|_{\mathcal{C},X}^2 &\le 2\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \|(f\circ\varphi\circ\sigma_a)'(z)\|_X^2 \log\frac{1}{|z|}\frac{dA(z)}{\pi}\\
&= 2\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \|f'(z)\|_X^2 N(\varphi\circ\sigma_a, z)\frac{dA(z)}{\pi}\\
&\le 2\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \|f'(z)\|_X^2 N(\sigma_{\varphi(a)}, z)\frac{dA(z)}{\pi}\\
&= 2\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \|(f\circ\sigma_{\varphi(a)})'(z)\|_X^2 \log\frac{1}{|z|}\frac{dA(z)}{\pi} \le 2c\|f\|_{\mathcal{C},X}^2,
\end{aligned}$$

for all $f \in \mathrm{BMOA}_{\mathcal{C}}(X)$. The upper bound

$$\|C_\varphi\colon \mathrm{BMOA}_{\mathcal{C}}(X) \to \mathrm{BMOA}_{\mathcal{C}}(X)\| \le \sqrt{2c} + \frac{1}{\sqrt{2}}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} \tag{3.4}$$

can be calculated from the above estimate and the following lemma, which will be useful in the sequel.

Lemma 3.3. *Let $f \in \mathrm{BMOA}_{\mathcal{C}}(X)$ and $R \in (0,1)$ be arbitrary. Then*

$$\sup_{a\in\mathbb{D}}\int_0^{2\pi} \|(f\circ\sigma_a)'(Re^{i\theta})\|_X^2 \frac{d\theta}{2\pi} \le \frac{2\|f\|_{\mathcal{C},X}^2}{(1-R^2)^2} \tag{3.5}$$

and

$$\|f(z)\|_X \le \|f(0)\|_X + \frac{1}{\sqrt{2}}\|f\|_{\mathcal{C},X}\log\frac{1+|z|}{1-|z|}, \tag{3.6}$$

for every $z \in \mathbb{D}$.

Proof. Let $R \in (0,1)$, $a \in \mathbb{D}$ and $f \in \mathrm{BMOA}_{\mathcal{C}}(X)$. Recall that since the function $z \mapsto \|(f\circ\sigma_a)'(z)\|_X^2$ is subharmonic on $\mathbb{D}$, the integral $\int_0^{2\pi} \|(f\circ\sigma_a)'(\rho e^{i\theta})\|_X^2 d\theta$ increases with $\rho \in (0,1)$. By using (3.2) we get that

$$\begin{aligned}
\|f\|_{\mathcal{C},X}^2 &\ge \int_{\mathbb{D}} \|(f\circ\sigma_a)'(z)\|_X^2(1-|z|^2)\frac{dA(z)}{\pi}\\
&\ge \frac{1}{\pi}\int_R^1\int_0^{2\pi} \|(f\circ\sigma_a)'(re^{i\theta})\|_X^2 d\theta(1-r^2)rdr\\
&\ge \frac{1}{\pi}\int_0^{2\pi} \|(f\circ\sigma_a)'(Re^{i\theta})\|_X^2 d\theta\int_R^1(1-r^2)rdr\\
&= \frac{(1-R^2)^2}{4\pi}\int_0^{2\pi} \|(f\circ\sigma_a)'(Re^{i\theta})\|_X^2 d\theta.
\end{aligned}$$

This proves (3.5). From the Hölder inequality we get that

$$(1-|z|^2)\|f'(z)\|_X = \|(f\circ\sigma_z)'(0)\|_X \le \left(\int_0^{2\pi} \|(f\circ\sigma_z)'(Re^{i\theta})\|_X^2 \frac{d\theta}{2\pi}\right)^{1/2},$$

for every $z \in \mathbb{D}$ and $R \in (0,1)$. Thus (3.5) gives that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\|f'(z)\|_X \le \sqrt{2}\|f\|_{\mathcal{C},X}, \tag{3.7}$$

for every $f \in \mathrm{BMOA}_{\mathcal{C}}(X)$. Since $f(z) - f(0) = e^{i\theta}\int_0^{|z|} f'(te^{i\theta})dt$ for every $z = |z|e^{i\theta} \in \mathbb{D}$, this yields that

$$\|f(z)-f(0)\|_X \le \sqrt{2}\|f\|_{\mathcal{C},X}\int_0^{|z|}\frac{1}{1-t^2}dt = \frac{1}{\sqrt{2}}\|f\|_{\mathcal{C},X}\log\frac{1+|z|}{1-|z|},$$

which proves (3.6). □

4. Weakly compact composition operators on $\mathrm{BMOA}_{\mathcal{C}}(X)$

Recall that a bounded linear map T on a Banach space E is weakly compact if $\overline{TB_E}$ is a weakly compact set, where B_E is the closed unit ball of E. We note that if the composition operator $C_\varphi\colon f \mapsto f\circ\varphi$ is weakly compact on $\mathrm{BMOA}_{\mathcal{C}}(X)$, then X is reflexive and C_φ is weakly compact also on BMOA. In fact, since $C_\varphi(f_x) = f_x$ for the constant functions $f_x \equiv x$ (where $x \in X$), the weak compactness of C_φ on $\mathrm{BMOA}_{\mathcal{C}}(X)$ yields that $\overline{B_X}$ is weakly compact so that X is reflexive. Moreover, given some nonzero $x_0 \in X$, we get that C_φ is weakly compact on the closed subspace $x_0\mathrm{BMOA}_{\mathcal{C}}(\mathbb{C}) = \{x_0 f\colon f \in \mathrm{BMOA}_{\mathcal{C}}(\mathbb{C})\}$ of $\mathrm{BMOA}_{\mathcal{C}}(X)$. Since BMOA is obviously isomorphic to $x_0\mathrm{BMOA}_{\mathcal{C}}(\mathbb{C})$, we deduce that C_φ is weakly compact on BMOA. Note also that if X is infinite-dimensional, then composition operators C_φ are never compact on $\mathrm{BMOA}_{\mathcal{C}}(X)$.

Our main result provides a sufficient condition for the weak compactness of composition operators on $\mathrm{BMOA}_{\mathcal{C}}(X)$.

Theorem 4.1. *Let X be a reflexive Banach space and suppose that $\varphi\colon \mathbb{D} \to \mathbb{D}$ is an analytic map such that $C_\varphi\colon \mathrm{BMOA} \to \mathrm{BMOA}$ is compact. Then $C_\varphi\colon \mathrm{BMOA}_{\mathcal{C}}(X) \to \mathrm{BMOA}_{\mathcal{C}}(X)$ is weakly compact.*

Theorem 4.1 complements [L, Theorem 7] and [BDL, Proposition 11] where it is shown that if X is reflexive and C_φ is compact on BMOA, then C_φ is weakly compact on both $\mathrm{BMOA}(X)$ and $w\mathrm{BMOA}(X)$. In the case of $w\mathrm{BMOA}(X)$ this result follows from a general theorem for composition operators on a large class of vector-valued spaces of weak type. In the case of $\mathrm{BMOA}(X)$ the proof is essentially a vector-valued modification of Smith's characterization of the compact composition operators on BMOA (see conditions (2.1) and (2.2)). We start the proof of Theorem 4.1 by combining (2.1) and Theorem 2.1: If C_φ is compact on BMOA, then

$$\lim_{r\to 1}\sup_{\{a\in\mathbb{D}\colon |\varphi(a)|>r\}}\sup_{0<|w|<1}|w|^2 N(\sigma_{\varphi(a)}\circ\varphi\circ\sigma_a, w) = 0 \tag{4.1}$$

and

$$\lim_{|w|\to 1}\sup_{\{a\in\mathbb{D}\colon |\varphi(a)|\le R\}}\frac{N(\varphi\circ\sigma_a, w)}{\log(1/|w|)} = 0, \tag{4.2}$$

for every $R \in (0,1)$. The remaining parts of the argument are essentially contained in the following two lemmas which will be proved below. Here C_r denotes the linear operator given by $(C_r f)(z) = f(rz)$ for analytic functions $f\colon \mathbb{D} \to X$ and $r \in (0,1)$.

Lemma 4.2. *The operators $C_r\colon \mathrm{BMOA}_{\mathcal{C}}(X) \to \mathrm{BMOA}_{\mathcal{C}}(X)$ satisfy the following properties for $r \in (0,1)$.*

(1) $\sup_{0<r<1} \|C_r\| < \infty$.

(2) *For every $0 < R < 1$, one has*

$$\sup_{\|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)}\le 1} \sup_{|z|\le R} \max\{\|(f - C_r f)'(z)\|_X, \|(f - C_r f)(z)\|_X\} \to 0,$$

as $r \to 1$.

(3) *If X is reflexive, then C_r is weakly compact on* $\mathrm{BMOA}_{\mathcal{C}}(X)$.

Lemma 4.3. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic map such that conditions (4.1) and (4.2) hold. Then*

$$\|C_\varphi - C_\varphi C_r\colon \mathrm{BMOA}_{\mathcal{C}}(X) \to \mathrm{BMOA}_{\mathcal{C}}(X)\| \to 0,$$

as $r \to 1$.

We note that the proof of Theorem 4.1 is easy to complete by using Lemmas 4.2 and 4.3. Indeed, assume that X is reflexive and C_φ is compact on BMOA so that (4.1) and (4.2) hold. Let $r_n = \frac{n}{n+1}$ and consider the linear operators $T_n = C_\varphi C_{r_n}$ for $n \in \mathbb{N}$. By parts (1) and (3) of Lemma 4.2 the operators T_n are bounded and weakly compact on $\mathrm{BMOA}_{\mathcal{C}}(X)$. Since $\|C_\varphi - T_n\| \to 0$ as $n \to \infty$ by Lemma 4.3, the operator C_φ is weakly compact on $\mathrm{BMOA}_{\mathcal{C}}(X)$. This proves Theorem 4.1.

We prove next Lemmas 4.2 and 4.3.

Proof of Lemma 4.2. The assertion (1) follows from the fact that C_r is the composition operator induced by the mapping $z \mapsto rz$. In fact, from (3.4) we get that $\|C_r\| \le \sqrt{2c}$ for every $r \in (0,1)$ (where c is the constant from (2.7)).

We prove next (2). Let $0 < r, R < 1$. Consider an analytic function $f\colon \mathbb{D} \to X$ and a point $z \in \mathbb{D}$. Put $\rho = (|z|+1)/2$ so that $|rz| < |z| < \rho < 1$. Using the Cauchy integral formula we obtain that

$$\begin{aligned}\|f'(z) - rf'(rz)\|_X &= \left\| \int_0^{2\pi} \left(\frac{\rho f'(\rho e^{i\theta})}{\rho - ze^{-i\theta}} - \frac{\rho r f'(\rho e^{i\theta})}{\rho - rze^{-i\theta}} \right) \frac{d\theta}{2\pi} \right\|_X \\ &\le \int_0^{2\pi} \frac{(1-r)\|f'(\rho e^{i\theta})\|_X}{|\rho - ze^{-i\theta}||\rho - rze^{-i\theta}|} \frac{d\theta}{2\pi} \le \frac{4(1-r)}{(1-|z|)^2} \int_0^{2\pi} \|f'(\rho e^{i\theta})\|_X \frac{d\theta}{2\pi}.\end{aligned}$$

From the Hölder inequality and Lemma 3.3 we get that

$$\|(f - C_r f)'(z)\|_X \le \frac{4\sqrt{2}(1-r)}{(1-|z|)^2(1-\rho^2)} \|f\|_{\mathcal{C},X} \le \frac{16(1-r)}{(1-|z|)^3} \|f\|_{\mathcal{C},X}. \tag{4.3}$$

Moreover, since $(f - C_r f)(z) = e^{i\theta} \int_0^{|z|} (f - C_r f)'(te^{i\theta}) dt$ where $z = |z|e^{i\theta}$, we have that

$$\|(f - C_r f)(z)\|_X \le 16(1-r)\|f\|_{\mathcal{C},X} \int_0^{|z|} \frac{dt}{(1-t)^3} \le \frac{8(1-r)}{(1-|z|)^2} \|f\|_{\mathcal{C},X}. \tag{4.4}$$

We obtain (2) by taking the supremum over all $z \in \mathbb{D}$ and f satisfying $|z| \le R$ and $\|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)} \le 1$ in (4.3) and (4.4), and letting $r \to 1$.

Finally we prove (3). We will approximate C_r using the truncation operators P_n, where $(P_n f)(z) = \sum_{k=0}^{n} x_k z^k$ for $f(z) = \sum_{k=0}^{\infty} x_k z^k$ in $\mathrm{BMOA}_{\mathcal{C}}(X)$ and $n \ge 0$. We note first that the operators P_n are bounded on $\mathrm{BMOA}_{\mathcal{C}}(X)$. Indeed, for any analytic function $f\colon \mathbb{D} \to X$ with $f(z) = \sum_{k=0}^{\infty} x_k z^k$ we have that $\|x_0\|_X = \|f(0)\|_X \le \|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)}$. Moreover, there is a constant K such that $\sup_{k\ge 1} \|x_k\|_X \le K \sup_{z\in\mathbb{D}} (1-|z|^2)\|f'(z)\|_X$ for all $f \in \mathrm{BMOA}_{\mathcal{C}}(X)$. Here one may apply the familiar scalar-valued argument (see [Bl3, p. 101], for example). By applying (3.7) we get that $\sup_{k\ge 1} \|x_k\|_X \le \sqrt{2}K\|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)}$. Since $\|z^n\|_{\mathrm{BMOA}_{\mathcal{C}}(\mathbb{C})} \le 1$ for $n \ge 1$, we obtain that $\|P_n\| \le \sqrt{2}K(n+1)$.

Let next $\varepsilon > 0$ and fix n_0 so that $\sum_{k=n_0+1}^{\infty} kr^k < \varepsilon$. For any $z \in \mathbb{D}$ and $f \in \mathrm{BMOA}_{\mathcal{C}}(X)$ with $f(z) = \sum_{k=0}^{\infty} x_k z^k$ we get that

$$\|((C_r - P_{n_0}C_r)f)'(z)\|_X \le \sum_{k=n_0+1}^{\infty} \|x_k\|_X r^k k|z|^{k-1} \le \sqrt{2}K\varepsilon\|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)}.$$

Since $\|(C_r - P_{n_0}C_r)f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)} \le \sup_{z\in\mathbb{D}} \|((C_r - P_{n_0}C_r)f)'(z)\|_X$ by the definition of the $\mathrm{BMOA}_{\mathcal{C}}(X)$ norm, we get that $\|C_r - P_n C_r\| \to 0$ as $n \to \infty$. The proof of (3) is completed by noting that for every $n \in \mathbb{N}$ the operator P_n is weakly compact on $\mathrm{BMOA}_{\mathcal{C}}(X)$ since it factors through the reflexive direct sum $\ell_2^{n+1}(X)$ (see the proof of [LST, Proposition 2]). □

For the proof of Lemma 4.3 we need a refinement of condition (2.1) due to Smith [Sm, Lemma 2.1]. For convenience, we use the following technical modification of Smith's result from [L].

Lemma 4.4 ([L, Lemma 10]). *Let $\psi\colon \mathbb{D} \to \mathbb{D}$ be an analytic function with $\psi(0) = 0$. Suppose that there is $\varepsilon \in (0, \frac{1}{e})$ such that*

$$\sup_{0<|w|<1} |w|^2 N(\psi, w) \le \varepsilon^2.$$

Then $N(\psi, z) \le 2\varepsilon \log(1/|z|)$ for all $z \in \mathbb{D}$ with $\sqrt{\varepsilon} \le |z| < 1$.

We are now ready to prove Lemma 4.3.

Proof of Lemma 4.3. For $r \in (0,1)$ let S_r denote the linear operator $f \mapsto f - C_r f$ so that $\|S_r\| \le K := 1 + \sqrt{2c}$, by Lemma 4.2(1). Since

$$\lim_{r\to 1} \sup_{\|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)}\le 1} \|(f - C_r f)(\varphi(0))\|_X = 0,$$

by Lemma 4.2(2), it suffices to show that

$$\lim_{r\to 1} \sup_{\|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)}\le 1} \sup_{a\in\mathbb{D}} M_a(C_\varphi S_r f) = 0, \tag{4.5}$$

where we denote

$$M_a(g) = \int_{\mathbb{D}} \|g'(z)\|_X^2 (1 - |\sigma_a(z)|^2) \frac{dA(z)}{\pi},$$

for $g \in \mathrm{BMOA}_{\mathcal{C}}(X)$ and $a \in \mathbb{D}$. Let $\varepsilon \in (0, \frac{1}{e})$ and let $f \in \mathrm{BMOA}_{\mathcal{C}}(X)$ be arbitrary. We will abbreviate $\varphi_a = \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a$ and $g_{r,a} = (S_r f) \circ$

$\sigma_{\varphi(a)}$ for all $a \in \mathbb{D}$ and $r \in (0,1)$. By (4.1) there is $R \in (0,1)$ such that $\sup_{0<|w|<1} |w|^2 N(\varphi_a, w) < \varepsilon^2$ for all $a \in \mathbb{D}$ with $|\varphi(a)| > R$. Since $\varphi_a(0) = 0$, we get from Lemma 4.4 that

$$N(\varphi_a, z) \le 2\varepsilon \log(1/|z|) \tag{4.6}$$

for all $a, z \in \mathbb{D}$ such that $|\varphi(a)| > R$ and $\sqrt{\varepsilon} \le |z| < 1$. Using (3.2) and the identity $(C_\varphi S_r f) \circ \sigma_a = g_{r,a} \circ \varphi_a$ we get that

$$M_a(C_\varphi S_r f) = \int_{\mathbb{D}} \|(g_{r,a} \circ \varphi_a)'(z)\|_X^2 (1 - |z|^2) \frac{dA(z)}{\pi}.$$

Thus the estimate $(1 - |z|^2) \le 2 \log(1/|z|)$ and the formula (2.5) applied to the function $\lambda(z) = \|g'_{r,a}(z)\|_X^2$ give that

$$M_a(C_\varphi S_r f) \le 2 \int_{\mathbb{D}} \|g'_{r,a}(z)\|_X^2 N(\varphi_a, z) \frac{dA(z)}{\pi}, \tag{4.7}$$

for all $r \in (0,1)$. By applying (4.6), (3.1) and (3.2), we get that

$$\begin{aligned}
\int_{\sqrt{\varepsilon} \le |z| < 1} \|g'_{r,a}(z)\|_X^2 N(\varphi_a, z) \frac{dA(z)}{\pi} &\le 2\varepsilon \int_{\mathbb{D}} \|g'_{r,a}(z)\|_X^2 \log \frac{1}{|z|} \frac{dA(z)}{\pi} \\
&\le 2c\varepsilon \int_{\mathbb{D}} \|((S_r f) \circ \sigma_{\varphi(a)})'(z)\|_X^2 (1 - |z|^2) \frac{dA(z)}{\pi} \le 2c\varepsilon \|S_r f\|_{\mathcal{C},X}^2,
\end{aligned}$$

for $a \in \mathbb{D}$ such that $|\varphi(a)| > R$. On the other hand, recall that $N(\varphi_a, z) \le \log(1/|z|)$ for $z \in \mathbb{D} \setminus \{0\}$ by Littlewood's inequality (see [S, p. 380] or [CoM, p. 33]). Thus we get from Lemma 3.3 that

$$\begin{aligned}
\int_{|z| < \sqrt{\varepsilon}} \|g'_{r,a}(z)\|_X^2 N(\varphi_a, z) \frac{dA(z)}{\pi} &\le \int_{|z| < \sqrt{\varepsilon}} \|g'_{r,a}(z)\|_X^2 \log \frac{1}{|z|} \frac{dA(z)}{\pi} \\
&= 2 \int_0^{\sqrt{\varepsilon}} \int_0^{2\pi} \|((S_r f) \circ \sigma_{\varphi(a)})'(\rho e^{i\theta})\|_X^2 \frac{d\theta}{2\pi} \left(\log \frac{1}{\rho} \right) \rho d\rho \\
&\le \frac{4\|S_r f\|_{\mathcal{C},X}^2}{(1-\varepsilon)^2} \int_0^{\sqrt{\varepsilon}} \left(\log \frac{1}{\rho} \right) \rho d\rho \le \frac{4\sqrt{\varepsilon}}{(1 - \frac{1}{e})^2} \|S_r f\|_{\mathcal{C},X}^2.
\end{aligned}$$

By combining these estimates with (4.7) we get that

$$\sup_{\{a \in \mathbb{D}\colon |\varphi(a)| > R\}} M_a(C_\varphi S_r f) \le C(\varepsilon + \sqrt{\varepsilon}) \|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)}^2. \tag{4.8}$$

for all $r \in (0,1)$, where C is a constant.

We consider next $a \in \mathbb{D}$ such that $|\varphi(a)| \le R$. By (4.2) there is $t_0 \in (0,1)$ such that

$$N(\varphi \circ \sigma_a, z) \le \varepsilon \log(1/|z|), \tag{4.9}$$

for every $a, z \in \mathbb{D}$ satisfying $|\varphi(a)| \le R$ and $|z| > t_0$. Using Lemma 4.2(2) we choose $r_0 \in (0,1)$ so that

$$\sup_{|z| \le t_0} \|(S_r f)'(z)\|_X^2 \le \varepsilon \|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)}^2 \tag{4.10}$$

for all $r \geq r_0$. Using (3.2), the estimate $(1-|z|^2) \leq 2\log(1/|z|)$ and the formula (2.5) applied to the function $\lambda(z) = \|(S_r f)'(z)\|_X^2$ we get that

$$\begin{aligned} M_a(C_\varphi S_r f) &= \int_{\mathbb{D}} \|((S_r f) \circ \varphi \circ \sigma_a)'(z)\|_X^2 (1-|z|^2) \frac{dA(z)}{\pi} \\ &\leq 2 \int_{\mathbb{D}} \|(S_r f)'(z)\|_X^2 N(\varphi \circ \sigma_a, z) \frac{dA(z)}{\pi}. \end{aligned} \tag{4.11}$$

From (4.9) and (3.1) we get that

$$\begin{aligned} \int_{t_0<|z|<1} \|(S_r f)'(z)\|_X^2 N(\varphi \circ \sigma_a, z) \frac{dA(z)}{\pi} &\leq \varepsilon \int_{\mathbb{D}} \|(S_r f)'(z)\|_X^2 \log \frac{1}{|z|} \frac{dA(z)}{\pi} \\ &\leq c\varepsilon \int_{\mathbb{D}} \|(S_r f)'(z)\|_X^2 (1-|z|^2) dA(z) \leq K^2 c\varepsilon \|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)}^2. \end{aligned}$$

Moreover, by using (4.10) we get that

$$\int_{|z|\leq t_0} \|(S_r f)'(z)\|_X^2 N(\varphi \circ \sigma_a, z) \frac{dA(z)}{\pi} \leq 2\varepsilon \|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)}^2,$$

for $r \geq r_0$, since $2\int_{\mathbb{D}} N(\varphi \circ \sigma_a, z) \frac{dA(z)}{\pi} = \|\varphi \circ \sigma_a - \varphi(a)\|_{H^2}^2 \leq 4$ by (2.5) and (2.6). By combining the preceding estimates with (4.11) we get that

$$\sup_{\{a\in\mathbb{D}\colon |\varphi(a)|\leq R\}} M_a(C_\varphi S_r f) \leq 2(K^2 c + 2)\varepsilon \|f\|_{\mathrm{BMOA}_{\mathcal{C}}(X)}^2,$$

for all $r \geq r_0$. Finally, by taking (4.8) together with the above estimate, we get (4.5). This proves the lemma and finishes the proof of Theorem 4.1. □

We record separately the special case $X = \mathbb{C}$ of Theorem 4.1, where C_φ is compact on BMOA, since the operators C_r are compact on BMOA for $r \in (0,1)$.

Corollary 4.5. *The composition operator C_φ is compact on* BMOA *if and only if (4.1) and (4.2) hold.*

A complete characterization of the weakly compact composition operators on $\mathrm{BMOA}_{\mathcal{C}}(X)$ depends on the question whether all weakly compact composition operators on BMOA are compact or not. Unfortunately this question is open for arbitrary composition operators on BMOA (see e.g. [CM]). However, there are some partial positive results in the literature, which in combination with Theorem 4.1 lead to characterizations of weakly compact composition operators on $\mathrm{BMOA}_{\mathcal{C}}(X)$ in some cases. By applying [Sm, Theorem 4.1], [CM, Theorem 1] and [MT, Corollary 5.4] we obtain the following partial characterization. Assume that $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic and satisfies one of the following conditions:

(1) φ is univalent, or
(2) $\varphi \in$ VMOA and $\varphi(\mathbb{D})$ lies inside a polygon inscribed in the unit circle.

Then C_φ is weakly compact on $\mathrm{BMOA}_{\mathcal{C}}(X)$ if and only if X is reflexive and C_φ is compact on BMOA. See [L, p. 741] for the details.

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References

[AFP] J. Arazy, S.D. Fisher and J. Peetre, *Möbius invariant function spaces*, J. Reine Angew. Math. 363 (1985) 110–145.

[B] A. Baernstein II, *Analytic functions of bounded mean oscillation*, in: Aspects of contemporary complex analysis, Proc. NATO Adv. Study Inst., Durham, 1979, Academic Press, London, 1980, pp. 3–36

[Bl] O. Blasco, *Vector-valued analytic functions of bounded mean oscillation and geometry of Banach spaces*, Illinois J. Math. 41 (1997) 532–558.

[Bl2] O. Blasco, *Remarks on vector-valued BMOA and vector-valued multipliers*, Positivity 4 (2000) 339–356.

[Bl3] O. Blasco, *On coefficients of vector-valued Bloch functions*, Studia Math. 165 (2004) 101–110.

[BP] O. Blasco and M. Pavlović, *Complex convexity and vector-valued Littlewood-Paley inequalities*, Bull. London Math. Soc. 35 (2003) 749–758.

[BDL] J. Bonet, P. Domański and M. Lindström, *Weakly compact composition operators on analytic vector-valued function spaces*, Ann. Acad. Sci. Fenn. Math. 26 (2001) 233–248.

[BCM] P.S. Bourdon, J.A. Cima and A.L. Matheson, *Compact composition operators on BMOA*, Trans. Amer. Math. Soc. 351 (1999) 2183–2196.

[CM] J.A. Cima and A.L. Matheson, *Weakly compact composition operators on VMO*, Rocky Mountain J. Math. 32 (2002) 937–951.

[CoM] C.C. Cowen and B.D. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Raton, 1995.

[DJT] J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators*, Cambridge Univ. Press, Cambridge, 1995.

[G] J.B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.

[L] J. Laitila, *Weakly compact composition operators on vector-valued BMOA*, J. Math. Anal. Appl. 308 (2005) 730–745.

[LT] J. Laitila and H.-O. Tylli, *Composition operators on vector-valued harmonic functions and Cauchy transforms*, Indiana Univ. Math. J. (to appear)

[LST] P. Liu, E. Saksman and H.-O. Tylli, *Small composition operators on analytic vector-valued function spaces*, Pacific J. Math. 184 (1998) 295–309.

[MT] S. Makhmutov and M. Tjani, *Composition operators on some Möbius invariant Banach spaces*, Bull. Austral. Math. Soc. 62 (2000) 1–19.

[S] J.H. Shapiro, *The essential norm of a composition operator*, Ann. Math. 125 (1987) 375–404.

[Sm] W. Smith, *Compactness of composition operators on BMOA*, Proc. Amer. Math. Soc. 127 (1999) 2715–2725.

[St] K. Stephenson, *Weak subordination and stable classes of meromorphic functions*, Trans. Amer. Math. Soc. 262 (1980) 565–577.

[WX] K.J. Wirths and J. Xiao, *Global integral criteria for composition operators*, J. Math. Anal. Appl. 269 (2002) 702–715.

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