On maps almost quasi-conformally close to quasi-isometries

V.M. Miklyukov

Oct 30, 2005, alqus6E.tex

`2000 Mathematics Subject Classification: Primary 26B35, Secondary 26B10`

Key words and phrases: almost quasi-conformal map, quasi-isometry, inner distance, inverse function, implicit function

Abstract

We prove that $W^{1,n}_{\mathrm{loc}}$-maps almost quasi-conformally close to quasi-isometries are quasi-isometric under some assumptions. Estimates of the inner distances and applications to the implicit function theory are given.

1 Main results

We study mappings of the class $W^{1,n}_{\mathrm{loc}}$ that are almost quasi-conformal by Callender [1]. Together with quasi-conformal maps this class contains also quasi-isometric mappings.

We show conditions under which the almost quasi-conformal maps close to quasi-isometries are quasi-isometric. In particular, our results are related to the well-known inverse mapping problem. Estimates of the inner distance and applications to the implicit function theory are given.

Firstly, we remind some notations and definitions. Let $x = (x_1, \dots, x_n)$ be a point of the n-dimensional Euclidean spaces $\mathbf{R}^n$, $n \geq 1$,

$$|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} .$$

We denote by $S(a,r)$ and $B(a,r)$ a sphere and a ball with center at $a \in \mathbf{R}^n$ and radius $0 < r < \infty$, respectively.

Let D be a domain in $\mathbf{R}^n$. A map $f : D \to \mathbf{R}^m$, $m \geq 1$, satisfies the Lipschitz condition on D, if

$$\sup_{x',x'' \in D} \frac{|f(x'') - f(x')|}{|x'' - x'|} = L < \infty .$$

A map f satisfies the Lipschitz condition locally on D, if f satisfies this condition on every subdomain $D' \subset\subset D$ with some Lipschitz constant $L(D')$.

Let C', $C'' > 0$ be some constants. A map $f : D \to \mathbf{R}^n$ is called (C', C'')-*quasi-isometric*, if

$$C' \, |x'' - x'| \le |f(x'') - f(x')| \le C'' \, |x'' - x'| \quad \forall x', x'' \in D. \tag{1.1}$$

A map $f : D \to \mathbf{R}^n$ is called locally quasi-isometric, if it is (C', C'')-quasi-isometric on every subdomain $D' \subset\subset D$ with some constants $0 < C'(D') \le C''(D') < \infty$.

By $W^{1,n}_{\text{loc}}(D)$ we denote the set of the functions f having generalized Sobolew derivatives $\partial f / \partial x_i$ $(i = 1, \ldots, n)$ of the class $L^n_{\text{loc}}(D)$ in a domain $D \subset \mathbf{R}^n$. A vector function $f = (f_1, \ldots, f_m) : D \to \mathbf{R}^n$ belongs to the class $W^{1,n}_{\text{loc}}(D)$, if every function f_i $(i = 1, \ldots, m)$ belong to this class.

By the Rademacher – Stepanoff theorem every locally Lipschitz function $f : D \to \mathbf{R}$ is differentiable almost everywhere on D [3, **3.1.6**] and, as it is easy to see that every locally Lipschitz map $f : D \to \mathbf{R}^m$ belongs to the class $W^{1,n}_{\text{loc}}(D)$.

Let $f : D \subset \mathbf{R}^n \to \mathbf{R}^m$ be a map of the class $W^{1,n}_{\text{loc}}(D)$. We let

$$f'(x) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \cdots & \cdots \\ \dfrac{\partial f_m}{\partial x_1} \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}$$

and,

$$|f'(x)| = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}(x) \right)^2 \right)^{1/2}, \qquad \|f'\|_D = \operatorname{ess\,sup}_{x \in D} |f'(x)| \, .$$

By Callender [1], a map $f : D \subset \mathbf{R}^n \to \mathbf{R}^n$ of the class $W^{1,n}_{\text{loc}}(D)$ is called *almost quasi-conformal* on D with a distortion coefficient $K > 0$ and locally integrable function $\delta(x) : D \to \mathbf{R}$, if a.e. on D the following inequality holds

$$|f'(x)|^n \le K \det\,\big(f'(x)\big) + \delta(x) \, . \tag{1.2}$$

For $\delta \equiv 0$ the condition (1.2) means, that the map f has *bounded distortion* [7, §3 Chapt. I], or is *quasi-regular* [10, Section 14.1].

Observe, that the assumption (1.2) does not imply constancy of the sign of the Jacobian det $(f'(x))$. Thus, almost quasi-conformal maps can change their orientation.

In order to evaluate volume of the considering class of maps, we formulate the following elementary statement. See the proof in the end of this paper.

1.3. Proposition. *Let $f : D \to \mathbf{R}^n$ be a $W^{1,n}_{\rm loc}(D)$-map such that $\|f'\|_D \le q < \infty$. Then f is almost quasi-conformal with a coefficient $K = \varepsilon\, n^{n/2}$ and $\delta = (1+\varepsilon)\, q^n$, where $\varepsilon = \mathrm{const} > 0$ is arbitrary.*

Let $f, g : D \to \mathbf{R}^n$ be maps of the class $W^{1,n}_{\rm loc}(D)$. We say that g is *almost quasi-conformally close* to f on D with a coefficient $K > 0$ and locally integrable function δ, if the map $\varphi = (f - g) : D \to \mathbf{R}^n$ is almost quasi-conformal with the coefficient $K > 0$ and the function δ, i.e. a.e. on D the following inequality holds

$$|f'(x) - g'(x)|^n \le K \det\,(f'(x) - g'(x)) + \delta(x)\,. \tag{1.4}$$

We call the maps f and g *almost quasi-conformally close*, if g is close to f or f is close to g.

If identically constant map $g \equiv \mathrm{const}$ is almost quasi-conformally close on $D \subset \mathbf{R}^n$ with $K > 0$ and a locally integrable function δ to a map f, then f is almost quasi-conformal with the same constant $K > 0$ and function δ.

For $n = 2$ and $K = 2$, $\delta = 0$, the inequality (1.4) means, that $f - g$ is a holomorphic function.

The main result of this paper is the statement:

1.5. Theorem. *Let $a_1, a_2 \in \mathbf{R}^n$ be a pair of points, such that $d = |a_2 - a_1| > 0$. Let $D = B(a_1, d) \cup B(a_2, d)$ be a subdomain of $\mathbf{R}^n$ and $b : D \to \mathbf{R}^n$ be (A', A'')-quasi-isometric.*

Let $f : D \to \mathbf{R}^n$ be a continuous $W^{1,n}_{\rm loc}(D)$-map almost quasi-conformally close to b with a constant $K > 0$ and a function $\delta(x)$ satisfying to the assumption

$$\frac{1}{r} \int\limits_{B(a_i,r)} \delta(x)\, d\mathcal{H}^n \le \lambda \int\limits_{S(a_i,r)} \delta(x)\, d\mathcal{H}^{n-1}\,, \quad 0 < r < d\,, \tag{1.6}$$

for every $i = 1, 2$ and a constant $\lambda \ge n/K$.

Let

$$h(a_1, a_2) \equiv \max_{i=1,2} \left(\frac{n}{|B(0,d)|} \int\limits_{B(a_i,d)} |f'(x) - b'(x)|^n\, d\mathcal{H}^n + \lambda\, d^{-n+\frac{n}{K}} \int\limits_{B(a_i,r)} \delta^+(x)\, d\mathcal{H}^n \right)^{\frac{1}{n}}\,,$$

where

$$|B(0,d)| = \mathcal{H}^n(B(0,d))\,, \quad \delta^+(x) = \max\{0, \delta(x)\}\,,$$

and let

$$\nu(n,K)\, h(a_1,a_2) < A'\,. \tag{1.7}$$

Then $f(a_1) \neq f(a_2)$ and, moreover,

$$C'\,|a_2 - a_1| \ \leq |f(a_2) - f(a_1)| \leq C''\,|a_2 - a_1|\,. \tag{1.8}$$

Here $\mathcal{H}^k(E)$ is the k-dimensional Hausdorff measure of the set $E \subset \mathbf{R}^n$,

$$C' = A' - \nu(n,K)\, h(a_1,a_2)\,, \quad C'' = A'' + \nu(n,K)\, h(a_1,a_2)$$

and

$$\mu_n = \mathcal{H}^n\left(B(\xi_1,1) \cap B(\xi_2,1)\right), \quad |\xi_1 - \xi_2| = 1,$$

$$\nu(n,K) = \frac{2K(nK - K + 1)\omega_{n-1}}{n^{(n-1)/n}(nK+1)\mu_n}\,, \quad \omega_{n-1} = \mathcal{H}^{n-1}(S(0,1))\,.$$

The volume of the class of δ, satisfying (1.6) is not clear. It is easy to see that to (1.6) there satisfy, for example, the functions $\delta \equiv \text{const}$ for $\lambda \geq 1/n$. Using Lemma 5.48, it is not difficult to prove, that to this condition there satisfy the functions $\delta(x) = |\varphi'(x)|$, where $\varphi : D \to \mathbf{R}^n$ is a quasi-regular map. Moreover, if the distortion coefficient of φ equals $K > 0$, then we may choose $\lambda = K/n$. We would like to find other examples.

The reversibility problem of maps is a well-known problem of analysis, see [11], [12], [14], [21, Theorem 4.4.1 Ch. 1] [13], [15], [16], [19], [17], [18, Ch. 2], [20, Section V.2], [9], [4], [5, Ch. 3], [27, Section 6.4] etc. Here we observe only the special case of Theorem 1.5, touching upon the problem of global reversibility of $W^{1,n}_{\text{loc}}(\mathbf{R}^n)$-maps.

Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a map of the class $W^{1,n}_{\text{loc}}(\mathbf{R}^n)$ and D be arbitrary domain of the form $D = B(a_1,r) \cup B(a_2,r)$. Denote by $\mathcal{B}_f = \mathcal{B}_f(D, A', A'', K, \delta)$ the set of the (A', A'')-quasi-isometries $b : D \to \mathbf{R}^n$ almost quasi-conformally close to f with a constant $K > 0$ and an integrable on D function $\delta(x)$ satisfying (1.6).

We put

$$\eta_f(a_1, a_2, \mathcal{B}_f) = \inf_b \max_{i=1,2} \frac{n}{\mathcal{H}^n(B(0,d)} \int\limits_{B(a_i,d)} |f'(x) - b'(x)|^n \, d\mathcal{H}^n\,,$$

where the infimum is taken over the quasi-isometries $b \in \mathcal{B}_f(D, A', A'', K, \delta)$.

1.9. Corollary. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous map of the class $W^{1,n}_{\text{loc}}(\mathbf{R}^n)$. Suppose that for every a_1, $a_2 \in \mathbf{R}^n$ the following statement holds*

$$\text{(1.10)} \quad \eta_f(a_1, a_2, \mathcal{B}_f) + \lambda\, d^{-n+n/K} \max_{i=1,2} \int\limits_{B(a_i,r)} \delta^+(x)\, d\mathcal{H}^n \le \frac{1}{\nu^n(n,K)\,(A')^n}\,.$$

Then f is quasi-isometric on $\mathbf{R}^n$. In particular, f is globally invertible.

2 $W^{1,p}$-closeness

Let $D \subset \mathbf{R}^n$ be a domain and $p \ge 1$ be a constant. Let f, $g : D \to \mathbf{R}^n$ be $W^{1,p}_{\text{loc}}(D)$-maps. We say, that a map g is $W^{1,p}$*-close* to f on D with a nonnegative function $\delta(x) \in L^p_{\text{loc}}(D)$, if

$$\text{(2.11)} \qquad |f'(x) - g'(x)| \le \delta(x) \quad \text{a.e. on } D\,.$$

2.12. Theorem. *Let $a_1, a_2 \in \mathbf{R}^n$ be a pair of points such that $d = |a_2 - a_1| > 0$. Let $D = B(a_1,d) \cup B(a_2,d)$ be a subdomain of $\mathbf{R}^n$ and $f : D \to \mathbf{R}^n$ be a continuous $W^{1,p}_{\text{loc}}(D)$-map.*

Suppose, that there exists an (A', A'')-quasi-isometric map $b : D \to \mathbf{R}^n$, which is $W^{1,p}$-close to f with $\delta(x) > 0$, satisfying

$$\text{(2.13)} \qquad \tau^{-n} \int\limits_{B(a_i,\tau)} \delta^p(x)\, d\mathcal{H}^n \le r^{-n} \int\limits_{B(a_i,r)} \delta^p(x)\, d\mathcal{H}^n\,, \quad 0 < \tau < r < d\,,$$

for every $i = 1, 2$.

Let

$$h_1(a_1, a_2) \equiv \max_{i=1,2} \left(d^{-n} \int\limits_{B(a_i,d)} \delta^p(x)\, d\mathcal{H}^n \right)^{\frac{1}{p}}$$

and let

$$\text{(2.14)} \qquad \nu_1(n,p)\, h_1(a_1, a_2) < A'\,.$$

Then $f(a_1) \neq f(a_2)$. Moreover,

$$\text{(2.15)} \qquad C'\,|a_2 - a_1| \le |f(a_2) - f(a_1)| \le C''\,|a_2 - a_1|\,.$$

Here

$$C' = A' - \nu_1(n,p)\, h_1(a_1,a_2)\,, \quad C'' = A'' + \nu_1(n,p)\, h_1(a_1,a_2)\,,$$

the constants μ_n, ω_{n-1} *are defined as above, and*

$$\nu_1(n,p) = 2p(\omega_{n-1}/\, n)^{(p-1)/p} np \;\, / \, (\mu_n\, p(np+p))\,.$$

The function $\delta \equiv \mathrm{const}$ satisfies (2.13). Setting $b(x)$ to be the identity map, we obtain the known statement [4], [5].

2.16. Corollary. *Let* $f : \mathbf{R}^n \to \mathbf{R}^n$ *be a continuous* $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^n)$*-map. Suppose that*

$$\|f'(x) - E_n\|_{\mathbf{R}^n} \le \delta_0\,, \tag{2.17}$$

where $\delta_0 \equiv \mathrm{const}$ *and* E_n *is the identity matrix.*

If

$$q \equiv \delta_0 \left(\frac{\omega_{n-1}}{n}\right)^{1/p} < 1\,, \tag{2.18}$$

then f *is global invertible. Moreover*

$$\begin{aligned} (1-q)\,|a_2 - a_1| \quad &\le |f(a_2) - f(a_1)| \le \\ &\le (1+q)\,|a_2 - a_1| \quad \forall\;\; a_1,\, a_2 \in \mathbf{R}^n\,. \end{aligned} \tag{2.19}$$

3 Convex and quasi-convex domains

Now let $D \subset \mathbf{R}^n$ be an arbitrary domain. We define the inner distance $r_D(x',\, x'')$ between points x' and x'' in D by setting

$$r_D(x',\, x'') = \inf_{\gamma} \int\limits_{\gamma} |dx|\,,$$

where the infimum is taken over all rectifiable arcs $\gamma \subset D$ joining points x' and x''.

A *distortion* of $D \subset \mathbf{R}^n$ is called the quantity

$$\mathrm{distort}\,(D) = \sup_{\substack{x',\, x'' \in D \\ x' \ne x''}} \frac{r_D(x'',\, x')}{|x'' - x'|}$$

(see [2, Section 1.14]).

Let $D \subset \mathbf{R}^n$ be a domain. Recall that D is convex, if every its two points can be joined by a segment containing in D. The condition $\operatorname{distort}(D) < \infty$ implies that

$$r_D(x'', x') \le Q\,|x'' - x'|\,, \quad Q = \operatorname{distort}(D)\,. \tag{3.20}$$

Such domains $D \subset \mathbf{R}^n$ is called *$Q-$quasi-convex* (see [2, p. 393]).

It is easy to see that every convex domain is 1-quasi-convex.

3.21. Theorem. *Let $D \subset \mathbf{R}^n$ be a domain and let $f : D \to \mathbf{R}^m$ be a $W^{1,n}_{\text{loc}}(D)$-map. Suppose, that there exists an (A', A'')–quasi-isometry $b : D \to \mathbf{R}^m$, almost quasi-conformally close to f with a constant $K > 0$ and a function $\delta(x) : D \to \mathbf{R}$, satisfying* (1.6) *for every ball $B(a,r) \subset D$.*

Suppose also that for every ball $B(a,r) \subset D$ the following property holds

$$\frac{n}{|B(0,r)|} \int\limits_{B(a,r)} |f'(x) - b'(x)|^n \, d\mathcal{H}^n + \lambda\, r^{-n+\frac{n}{K}} \int\limits_{B(a,r)} \delta^+(x)\, d\mathcal{H}^n \le q \tag{3.22}$$

with a constant

$$q < \frac{(A')^n}{\nu^n(n,K)}\,. \tag{3.23}$$

(i) Then $D' = f(D)$ is a domain and for an arbitrary pair of points a', $a'' \in D$ the following statement holds

$$(A' - q^{1/n})\, \rho_D(a', a'') \le \rho_{D'}(f(a'), f(a'')) \le (A'' + q^{1/n})\, \rho_D(a', a'')\,. \tag{3.24}$$

(ii) If $D \subset \mathbf{R}^n$ is convex, then $D' = f(D)$ is quasi-convex with the constant $Q = A'' + q^{1/n}$ and for every pair of points a', $a'' \in D$ we have

$$(A' - q^{1/n})\, |a'' - a'| \le \rho_{D'}(f(a''), f(a')) \le (A'' + q^{1/n})\, |a'' - a'|\,. \tag{3.25}$$

Proof. At first we prove the statement (i). We use Theorem 1.5. The assumption (3.22) implies (1.7) for every pair of points $a_1, a_2 \in D$, satisfying the condition

$$|a_2 - a_1| < \min_{i=1,2} \operatorname{dist}(a_i, \partial D)\,. \tag{3.26}$$

From this we obtain (1.8) with constants

$$C' = A' - q^{1/n}\,, \quad C'' = A'' + q^{1/n}\,.$$

The inequalities (1.8) imply that f is local homeomorphic and, consequently, $D' = f(D)$ is a domain.

Let a', $a'' \in D$ be an arbitrary pair of points. We fix $\varepsilon > 0$ and choose an Jordan arc $\gamma \subset D$ with its endpoints at a' and a'', such that

$$\left|\mathcal{H}^1(\gamma) - \rho_D(a', a'')\right| < \varepsilon/2 .$$

We split the arc γ by points a', x_1, ..., x_n, a'', such that

$$\left|\mathcal{H}^1(\gamma) - (|a' - x_1| + \ldots + |x_n - a''|)\right| < \varepsilon/2 ,$$

and neighboring points satisfy (3.26).

By (1.8), we have

$$|f(a') - f(x_1)| + \ldots + |f(x_n) - f(a'')| \leq C'' \left(|a' - x_1| + \ldots + |x_n - a''|\right)$$

and

$$|f(a') - f(x_1)| + \ldots + |f(x_n) - f(a'')| \leq C'' \left(\rho_D(a', a'') + \varepsilon\right) .$$

Turning fineness of the partition to zero, we obtain

$$\rho_{D'}(f(a'), f(a'')) \leq \mathcal{H}^1(f(\gamma)) \leq C'' \left(\rho_D(a', a'') + \varepsilon\right)$$

and, using arbitrariness of $\varepsilon > 0$, we arrive at the upper estimate in (3.24).

The lower estimate in (3.24) is proved by the same arguments. It is only necessary to suppose that the partition of γ with a', x_1,..., x_n, a'' is satisfies that the length of $f(\gamma)$ is small different from the distance $\rho_{D'}(f(a'), f(a''))$ and the length of the broken line with the tops at points $f(a')$, $f(x_1)$, ..., $f(x_n)$, $f(a'')$ is close enough to the length of $f(\gamma)$. By (1.8), we have

$$|f(a') - f(x_1)| + \ldots + |f(x_n) - f(a'')| \geq C' \left(|a' - x_1| + \ldots + |x_n - a''|\right) ,$$

whence the necessity follows easily.

Now we prove the statement (ii). Fix points a', $a'' \in D$ and denote by $l(a',\, a'')$ the linear segment joining a' and a''. Since D is convex, then $l(a',\, a'')$ lies entirely on D.

We separate $l(a',\, a'')$ with the consecutive one by another points $x_1, x_2, \ldots, x_n$, such that every $l(a',\, x_1)$, $l(x_1,\, x_2)$, ..., $l(x_n,\, a'')$ satisfies (3.26). Thus, for every of these segments we have (1.8) and

$$\begin{aligned} |f(a') - f(x_1)| + \ldots + |f(x_n) - f(a'')| &\leq (A'' + q^{1/n}) \left(|a' - x_1| + \ldots + |x_n - a''|\right) = \\ &= (A'' + q^{1/n})\, |a' - a''| . \end{aligned}$$

Choosing a partition a', x_1, $x_2, \dots$, x_n, a'' of $l(a', a'')$ arbitrarily fine and observing that the left side of this relation will be arbitrarily close to $\mathcal{H}^1 f(l(a', a''))$, we obtain

$$\mathcal{H}^1 f(l(a', a'')) \leq (A'' + q^{1/n})\, |a' - a''|\,.$$

Therefore, we have

$$r_{D'}(a', a'') \leq (A'' + q^{1/n})\, |a' - a''|\,.$$

Thus, the right of the relations (3.25) holds, and the domain D' is quasi-convex with the necessary constant Q.

On the other hand, suppose that $\gamma \subset D'$ is an Jordan arc with the endpoints $f(a')$ and $f(a'')$, for which

$$\Big|\mathcal{H}^1(\gamma) - \rho_{D'}(f(a'), f(a''))\Big| < \varepsilon/2\,,$$

where $\varepsilon > 0$ is an arbitrary constant.

Let $\Gamma = f^{-1}(\gamma)$. We choose the points $x_1, \dots, x_n$ on Γ such that the distances $|a' - x_1|$, $|x_1 - x_2|$, $\dots$, $|x_n - a''|$ were lesser then $\operatorname{dist}(\Gamma, \partial D)$ and

$$\Big|\mathcal{H}^1(\gamma) - (|f(a') - f(x_1)| + |f(x_1) - f(x_2)| + \ldots + |f(x_n) - f(a'')|)\Big| < \varepsilon/2\,.$$

Then, as above,

$$\begin{aligned} |f(a') - f(x_1)| + \ldots + |f(x_n) - f(a'')| \quad &\geq (A' - q^{1/n})\, (|a' - x_1| + \ldots + |x_n - a''|) \geq \\ &\geq (A' - q^{1/n})\, |a' - a''|\,. \end{aligned}$$

Thus,

$$\rho_{D'}(f(a'), f(a'')) \geq (A' - q^{1/n})\, (|a' - a''| - \varepsilon)$$

and, by arbitrariness of $\varepsilon > 0$,

$$(A' - q^{1/n})\, |a' - a''| \leq \rho_{D'}(f(a'), f(a''))\,.$$

The theorem is proved. $\square$

As above, but using Theorem 2.12, we prove the following statement.

3.27. Theorem. *Let $D \subset \mathbf{R}^n$ be a domain and let $f : D \to \mathbf{R}^n$ be a map of the class $W^{1,p}_{\mathrm{loc}}(D)$. Suppose that there exists an (A', A'')–quasi-isometry $b : D \to \mathbf{R}^n$ that is $W^{1,p}$-close to f with a function $\delta(x) : D \to \mathbf{R}$ of the class $L^p_{\mathrm{loc}}(D)$, $p \geq 1$, with the property (2.13) for arbitrary pair of balls $B(a, \tau) \subset B(a, r) \subset D$.*

Suppose that for every ball $B(a,r) \subset D$ the following inequality holds

$$r^{-n} \int\limits_{B(a,r)} \delta^p(x)\, d\mathcal{H}^n \le q \tag{3.28}$$

with a constant

$$q < \frac{(A')^p}{\nu_1^p(n,p)}\,. \tag{3.29}$$

(i) *Then $D' = f(D)$ is a domain and for arbitrary pair of points a', $a'' \in D$ we have*

$$(A' - q^{1/p})\,\rho_D(a',a'') \le \rho_{D'}(f(a'), f(a'')) \le (A'' + q^{1/p})\,\rho_D(a',a'')\,.$$

(ii) *If a domain $D \subset \mathbf{R}^n$ is convex, then the domain $D' = f(D)$ is quasi-convex with a constant $Q = A'' + q^{1/p}$ and for every pair of points a', $a'' \in D$ we have*

$$(A' - q^{1/p})\,|a'' - a'| \le \rho_{D'}(f(a''), f(a')) \le (A'' + q^{1/p})\,|a'' - a'|\,. \tag{3.30}$$

3.31. Corollary. *Let $D \subset \mathbf{R}^n$ be a domain and let $f : D \to \mathbf{R}^n$ be a map of the class $W^{1,p}_{\rm loc}(D)$. Suppose that the identical map is $W^{1,p}$–close to f with a function $\delta(x) : D \to \mathbf{R}$ of a class $L^p_{\rm loc}(D)$, $p \ge 1$, satisfying* (2.13) *for every pair of balls $B(a,\tau) \subset B(a,r) \subset D$.*

Suppose that for every ball $B(a,r) \subset D$ the relation (3.28) *holds with a constant q, satisfying the assumption*

$$q < \frac{1}{\nu_1^p(n,p)}\,. \tag{3.32}$$

Then for every pair of points a', $a'' \in D$ we have

$$\begin{aligned} (1 - q^{1/p}\nu_1(n,p))\,\rho_D(a',a'') &\le \rho_{D'}(f(a'), f(a'')) \le \\ &\le (1 + q^{1/p}\nu_1^p(n,p))\,\rho_D(a',a'')\,. \end{aligned} \tag{3.33}$$

Close by contents questions for quasi-isometric maps between two dimensional surfaces were considered in [5].

4 Implicit functions

Next we show some applications these results to the existence problem of implicit functions. We follow the scheme of the proof of the corresponding statement in [4], where its local version had been proved. On other nonsmooth variants, see Pourciau [23], Warga [25], Cristea [24], Zhuravlev and Igumnov [26].

Let $m, n \geq 1$ be integer and $U \subset \mathbf{R}^n$, $V \subset \mathbf{R}^m$ be domains. Let $F(x,y)$ be a function of the class $W^{1,1}_{\mathrm{loc}}(D)$, where $D = U \times V$. If (x,y) is a point, in which there exist the partial derivatives

$$\partial F/\partial x_i\,, \quad \partial F/\partial y_j \quad (i = 1, \ldots, n\,; j = 1, \ldots, m)\,,$$

then let $F'(x,y)$ be the Jacobi matrix, $F'_x(x,y)$ be the Jacobi matrix with respect to $x = (x_1, \ldots, x_n)$ for fixed $y = (y_1, \ldots, y_m)$ and $F'_y(x,y)$ be the Jacobi matrix with respect to y for a fixed x.

If $P \subset D$ is a set and $\varphi : P \to M_k$, $k \geq 1$, is a matrix function, then we denote by

$$\operatorname{osc}(\varphi, P) = \operatorname*{ess\,sup}_{\xi,\eta \in P} |\varphi(\xi) - \varphi(\eta)|$$

the oscillation of φ on P.

We consider a map $\Phi : D \to \mathbf{R}^{n+m}$ defined by

$$(x,y) \xrightarrow{\Phi} (X,Y) = (x_1, \ldots, x_n, F_1(x,y), \ldots, F_m(x,y))\,.$$

4.34. Theorem. *Let $x_0 \in U \subset \mathbf{R}^n$, $y_0 \in V \subset \mathbf{R}^m$ and $F : D \to V$ be a continuous map. Suppose that one of the following assumptions holds.*

(i) The map $F \in W^{1,n}_{\mathrm{loc}}(D)$ and there exists $\delta(x,y) : D \to \mathbf{R}$ satisfying (1.6) *for every $(m+n)$-dimensional ball $B(a,r) \subset D$ and such that*

$$|F'_x(x,y)|^2 + |F'_y(x,y) - E_m|^2 \leq \delta^{2/(m+n)}(x,y)\,. \tag{4.35}$$

Moreover for every $(m+n)$-dimensional ball $B(a,r) \subset D$,

$$\frac{m+n}{|B(0,r)|} \int\limits_{B(a,r)} \delta(x,y)\, d\mathcal{H}^{m+n} +$$

$$\lambda\, r^{-(m+n)+(m+n)/K} \int\limits_{B(a,r)} \delta(x,y)\, d\mathcal{H}^{m+n} \leq q \tag{4.36}$$

with a constant

$$q < \frac{1}{\nu^{m+n}(m+n,K)} \,. \tag{4.37}$$

(*ii*) *The map $F \in W^{1,p}_{\mathrm{loc}}(D)$ and a.e. on D the following inequality holds*

$$|F'_x(x,y)|^2 + |F'_y(x,y) - E_m|^2 \le \delta^{2/p}(x,y)$$

with a function $\delta(x,y) : D \to \mathbf{R}$ of a class $L^p_{\mathrm{loc}}(D)$, $p \ge 1$, satisfying (2.13) *for every pair of $(m+n)$-dimensional balls $B(a,\tau) \subset B(a,r) \subset D$. Moreover, for every $(m+n)$-dimensional ball $B(a,r) \subset D$,*

$$r^{-n} \int\limits_{B(a,r)} \delta^p(x,y)\, d\mathcal{H}^{m+n} \le q \tag{4.38}$$

with a constant

$$q < \frac{1}{\nu_1^p(m+n,p)} \,. \tag{4.39}$$

(*iii*) *The map $F \in \mathrm{Lip}_{\mathrm{loc}}(D)$ and*

$$\|F'_y - E_m\|_D + \mathrm{osc}\,(F'_x, D) < 1 \,. \tag{4.40}$$

Then there exists a (unique) continuous map

$$G(x) : U \to V \,, \quad G(x_0) = y_0 \,,$$

such that

$$F(x, G(x)) = F(x_0, y_0) \quad \textit{for all} \quad x \in U \,.$$

Moreover, G satisfies the Lipschitz condition globally on U with respect the inner metrics ρ_U and ρ_V.

Proof. Firstly, we consider the case (i). The Jacobi matrix of Φ has the following form

$$\Phi'(x,y) = \begin{pmatrix} E_n & O^n_m \\ F'_x(x,y) & F'_y(x,y) \end{pmatrix} ,$$

where O^n_m is a zero $n \times m$-matrix.

We have

$$\Phi'(x,y) - E_{m+n} = \begin{pmatrix} E_n & O^n_m \\ F'_x(x,y) & F'_y(x,y) \end{pmatrix} - E_{n+m} =$$

$$= \begin{pmatrix} O^n_n & O^n_m \\ F'_x(x,y) & F'_y(x,y) - E_m \end{pmatrix}.$$

Thus,

$$\det\left(\Phi'(x,y) - E_{m+n}\right) = 0\,,$$

$$|\Phi'(x,y) - E_{m+n}|^{m+n} = \left(|F'_x(x,y)|^2 + |F'_y(x,y) - E_m|^2\right)^{(m+n)/2}.$$

The assumption (4.35) implies almost quasi-conformal proximity of Φ to the identity map, and we may use Theorem 3.21.

By (3.24), the set $D' = \Phi(D) \subset \mathbf{R}^{m+n}$ is a domain and for every pair of points a', $a'' \in D$ we have

$$(1 - q^{1/(m+n)})\,\rho_D(a', a'') \le \rho_{D'}(f(a'), f(a'')) \le (1 + q^{1/(m+n)})\,\rho_D(a', a'')\,.$$

The inverse map to $\Phi(x,y)$ has the form

$$x = X, \quad y = \Theta(X,Y).$$

Moreover, the map Φ^{-1} satisfies the Lipschitz condition on D' with the constant

$$\text{Lip}\left(\Phi^{-1},\, D'\right) \le 1/(1 - q^{1/n})\,.$$

For $\Theta(X,Y)$ we obtain

$$\text{Lip}\ (\Theta, D') \le \sqrt{1/(1 - q^{1/(m+n)})^2 - 1}\,.$$

Now we observe that

$$(X,Y) = \Phi(\Phi^{-1}(X,Y)) = (X, F(X, \Theta(X,Y))).$$

This relation implies

$$F(X, \Theta(X,Y)) = Y. \tag{4.41}$$

Denote by Π a connected component of the intersection of the plane

$$Y_1 = F_1(x_0, y_0), \quad \dots, \quad Y_m = F_m(x_0, y_0)$$

with the domain D' containing the point $(X_0, Y_0) = (x_0, F(a))$. Co-dimension of Π equals to m. Let π be an orthogonal projection of $\mathbf{R}^n \times \mathbf{R}^m$ onto $\mathbf{R}^n$. For an arbitrary subset $A \subset \mathbf{R}^n \times \mathbf{R}^m$ we have

$$\pi(A) = \{x \in \mathbf{R}^n : (x, y) \in A\}.$$

By the definition of Φ, we may write

$$\begin{aligned} \pi(\Phi(A')) \quad &= \Phi(\pi(A')) \quad \forall\ A' \subset D, \\ \pi(\Phi^{-1}(A'')) \quad &= \Phi^{-1}(\pi(A'')) \quad \forall\ A'' \subset D'. \end{aligned}$$

The equation of a connected piece of the surface $\Phi^{-1}(\Pi)$ containing $a = (x_0, y_0)$ can be rewritten in the nonparametric form. In fact, let

$$(X, Y) = (x, \Theta(x, Y_0)), \quad x \in \Phi^{-1}(\pi(\Phi(D)))\,.$$

We set $G(x) = \Theta(x, Y_0)$.

By (4.41), we find

$$F(x, G(x)) = Y_0 = F(x_0, y_0)\,,$$

where

$$G(x_0) = \Theta(x_0, Y_0) = \Theta(X_0, Y_0) = y_0\,.$$

Uniqueness of the map G is obvious because $\Phi(x, y)$ is bijective. Indeed, if $(x, y_1), (x, y_2) \in D$ and $F(x, y_1) = F(x, y_2)$, then $\Phi(x, y_1) = \Phi(x, y_2)$. Therefore, $y_1 = y_2$.

In the case (ii) the map Φ is $W^{1,p}$-close to the identity map $E_{m+n}(x, y) : D \to D$ and we may use Theorem 3.27. The next considerations are as above.

Suppose that the assumptions of the case (iii) hold. We need to prove that $\Phi(x, y)$ satisfy to to conditions of Corollary 3.31. Consider an $(n+m) \times (n+m)$-matrix

$$Q(x, y) = \begin{pmatrix} E_n & O^n_m \\ -F'_x(x, y) & E_m \end{pmatrix}.$$

It is easy to see that

$$|Q(x_1, y_1) - Q(x_2, y_2)|_D \leq |F'_x(x_1, y_1) - F'_x(x_2, y_2)|_D \leq \operatorname{osc}(F'_x, D).$$

Thus, we have

$$Q(x,y)\Phi'(x,y) - E_{m+n} = \begin{pmatrix} E_n & O_m^n \\ O_n^m & F_y'(x,y) \end{pmatrix} - E_{n+m} =$$

$$= \begin{pmatrix} O_n^n & O_m^n \\ O_n^m & F_y'(x,y) - E_m \end{pmatrix}.$$

From this relation,

(4.42) $$\|Q(x,y)\Phi'(x,y) - E_{m+n}\|_D = \|F_y'(x,y) - E_m\|_D.$$

For every fixed point $(x^*, y^*) \in D$ we define the map

(4.43) $$\Psi(x,y) = Q(x^*,y^*)\Phi(x,y) : D \to \mathbf{R}^n \times \mathbf{R}^m.$$

Using (4.42) we obtain

$$||\Psi'(x,y) - E_{m+n}||_D = \|Q(x^*,y^*)\Phi'(x,y) - E_{n+m}\| =$$

$$= \|Q(x,y)\Phi'(x,y) - E_{m+n} + (Q(x^*,y^*) - Q(x,y))\Phi'(x,y))\|_D \leq$$

$$\leq \|Q(x,y)\Phi'(x,y) - E_{m+n}\|_D +$$

$$+ \|Q(x^*,y^*) - Q(x,y)\|_D \|\Phi'(x,y)\|_D \leq$$

$$\leq \|F_y' - E_m\|_D + \operatorname{osc}(F_x', D)\, \|\Phi'(x,y)\|_D.$$

Take into consideration that

$$(Q(x^*,y^*) - Q(x,y)\Phi'(x,y)) = \begin{pmatrix} O_n^n & O_m^n \\ & \\ F_x' - F_x'(x^*,y^*) & O_m^m \\ & \end{pmatrix} \begin{pmatrix} E_n & O_m^n \\ & \\ F_x' & F_y' \\ & \end{pmatrix} =$$

$$= \begin{pmatrix} O_n^n & O_m^n \\ & \\ F_x'(x,y) - F_x'(x^*,y^*) & O_m^m \\ & \end{pmatrix}.$$

Thus,

$$\| (Q(x^*, y^*) - Q) \; \Phi'\|_D = \|F'_x - F'_x(x^*, y^*)\|_D \le \operatorname{osc}(F'_x, D)\,. \tag{4.44}$$

Now,

$$\begin{aligned} \Psi'(x,y) - E_{n+m} &= Q^*\, \Phi'(x,y) - E_{n+m} = \\ &= (Q(x^*,y^*) - Q(x,y))\; \Phi'(x,y) + \\ &\quad + Q(x,y)\, \Phi'(x,y) - E_{n+m}\,. \end{aligned} \tag{4.45}$$

Thus, by (4.42), (4.44) and (4.45) we obtain

$$\begin{aligned} \|\Psi'(x,y) - E_{m+n}\|_D &\le \|Q(x,y)\Phi'(x,y) - E_{n+m}\|_D + \|(Q^* - Q)\Phi'\|_D \le \\ &\le \|F'_y - E_m\|_D + \operatorname{osc}(F'_x, D)\,. \end{aligned}$$

From (4.47) we may conclude, that the map $\Psi(x,y) = Q(x^*,y^*)\Phi(x,y)$ is homeomorphic. By Corollary **5.6.16** of [8] from (4.47) we see that the matrix $\Psi'(x,y)$ is nondegenerate. In turn, according to what has been said, $\Phi'(x,y)$ and $Q \equiv Q(x^*,y^*)$ are nondegenerate also. Consequently, the map $\Phi = Q^{-1}\Psi : D \to \mathbf{R}^{n+m}$ is homeomorphic.

We have

$$\begin{aligned} (1-\mu)\, \rho_D\left((x,y),(x_0,y_0)\right) &\le \rho_{\Phi(D)}\left(\Psi(x,y), \Psi(x_0,y_0)\right) \le \\ &\le (1+\mu)\, \rho_D\left((x,y),(x_0,y_0)\right)\,, \end{aligned}$$

where

$$\mu = \|F'_y - E_m\|_D + \operatorname{osc}(F'_x, D)\,.$$

But since $\Psi = Q\Phi$, $Q = Q(x^*,y^*)$, we may write

$$\begin{aligned} \frac{1-\mu}{|Q|}\, \rho_U\left((x,y),(x_0,y_0)\right) &\le \rho_V\left(\Phi(x,y), \Phi(x_0,y_0)\right) \le \\ &\le (1+\mu)\, \rho_U\left((x,y),(x_0,y_0)\right)\,. \end{aligned} \tag{4.46}$$

However,

$$\begin{pmatrix} E_n & O^n_m \\ F'_x(x_0,y_0) & E_m \end{pmatrix} = \begin{pmatrix} E_n & O^n_m \\ O^m_n & E_m \end{pmatrix} + \begin{pmatrix} O^n_n & O^n_m \\ F'_x(x_0,y_0) & O^m_m \end{pmatrix}$$

and, consequently, $|Q| \le 1 + \|F'_x\|_D$.

By (3.33) the map Ψ^{-1} satisfies the Lipschitz condition on the domain $\Psi(D)$ with the constant

$$\operatorname{Lip}\left(\Psi^{-1}, \Psi(D)\right) \le \frac{1}{1-\mu}.$$

Thus, by (4.40) for every fixed point (x^*, y^*) we have

$$||\Psi'(x,y) - E_{m+n}||_D \le \mu < 1. \tag{4.47}$$

The next considerations as in the case (i). In place of Theorem 3.21 it is sufficient to use Corollary 3.31. Theorem 4.34 is proved. □

5 Energy integral

We need the following statement.

5.48. Lemma. *Let $f : D \to \mathbf{R}^n$ be a map of the class $W^{1,n}_{\mathrm{loc}}(D)$. Then for an arbitrary point $a \in D$ and a.e. $r \in (0, R)$, $R = \operatorname{dist}(a, \partial D)$, we have*

$$\left| \int\limits_{B(a,r)} \det f'(x)\, d\mathcal{H}^n \right| \le \frac{r}{n} \int\limits_{S(a,r)} |f'(x)|^n \, d\mathcal{H}^{n-1}. \tag{5.49}$$

See **proof** in [7, Lemma 1.2 Chapt. II].

Let $f : D \to \mathbf{R}^n$ be a map of the class $W^{1,n}_{\mathrm{loc}}(D)$. For every $a \in D$ and $r \in (0, R]$, $R = \operatorname{dist}(a, \partial D)$, we put

$$I(a,r) = \int\limits_{B(a,r)} |f'(x)|^n \, d\mathcal{H}^n .$$

5.50. Lemma. *If $f : D \to \mathbf{R}^n$ is a map of the class $W^{1,n}_{\mathrm{loc}}(D)$, almost quasiconformal with a constant $K > 0$ and a locally integrable function $\delta(x)$ satisfying (1.6), then the quantity*

$$r^{-n/K} I(a,r) + \lambda \int\limits_{B(a,r)} \delta(x)\, d\mathcal{H}^n$$

is nondecreasing on $(0, R]$.

Proof. Because for a.e. $0 < r < R$ the equality

$$I'(a,r) = \int\limits_{S(a,r)} |f'(x)|^n \, d\mathcal{H}^{n-1}$$

holds, then for a.e. $0 < r < R$ we can write

$$\left(r^{-n/K} I(a,r) + \lambda \int\limits_{B(a,r)} \delta(x) \, d\mathcal{H}^n \right)' = -\frac{n}{K} r^{-1-n/K} I(a,r) + r^{-n/K} I'(a,r) +$$

$$+\lambda J'(a,r) = -\frac{n}{K} r^{-1-n/K} I(a,r) +$$

$$+ r^{-n/K} \int\limits_{S(a,r)} |f'(x)|^n d\mathcal{H}^{n-1} + \lambda \int\limits_{S(a,r)} \delta(x) \, \mathcal{H}^{n-1} \, .$$

The relation (5.49) guarantees that

$$\frac{r}{n} \int\limits_{S(a,r)} |f'(x)|^n d\mathcal{H}^{n-1} \geq \left| \int\limits_{B(a,r)} \det \left(f'(x) \right) \, d\mathcal{H}^n \right|$$

and by (1.2), we find

$$\frac{r}{n} \int\limits_{S(a,r)} |f'(x)|^n d\mathcal{H}^{n-1} \geq \frac{1}{K} \int\limits_{B(a,r)} |f'(x)|^n \, d\mathcal{H}^n - \frac{1}{K} \int\limits_{B(a,r)} \delta(x) \, d\mathcal{H}^n \, .$$

Thus, we have

$$\left(r^{-n/K} I(a,r) + \lambda \int\limits_{B(a,r)} \delta(x) \, d\mathcal{H}^n \right)' \geq -\frac{n}{K} r^{-1-n/K} I(a,r) +$$

$$+ \frac{n}{K} r^{-1-n/K} \int\limits_{B(a,r)} |f'(x)|^n \, d\mathcal{H}^n -$$

$$- \frac{n}{rK} \int\limits_{B(a,r)} \delta(x) \, d\mathcal{H}^n + \lambda \int\limits_{S(a,r)} \delta(x) \, \mathcal{H}^{n-1} \, .$$

By using (1.6), we obtain

$$\Big(r^{-n/K}\, I(a,r) + \lambda\, J(a,r)\Big)' \ge \lambda \int\limits_{S(a,r)} \delta(x)\, \mathcal{H}^{n-1} - \frac{n}{r\,K} \int\limits_{B(a,r)} \delta(x)\, \mathcal{H}^{n} = 0\,.$$

The lemma is proved. □

6 Morrey's Lemma

Below we follow [22], where Morrey's Lemma is proved for $W^{1,p}$-functions on Riemannian manifolds.

Let a_1, $a_2 \in \mathbf{R}^n$ and $d = |a_2 - a_1|$. Let $D = B(a_1, d) \cup B(a_2, d)$ be a domain.

Let $\Gamma = \Gamma(a_1, a_2)$ be a family of locally rectifiable arcs $\gamma \subset D$ joining points a_1 and a_2.

6.51. Lemma. *Let $\rho(x) \ge 0$ be a function of a class $L^p_{\mathrm{loc}}(D)$, $p \ge 1$.*
If there exist constants $\alpha, c_1 > 0$ such that

(6.52)
$$\int\limits_{B(a_i,r)} \rho^p\, d\mathcal{H}^n \le c_1\, r^{n-p+\alpha} \quad \text{for every} \quad r \in (0,d), \quad i = 1,2,$$

then

(6.53)
$$\inf_{\gamma \in \Gamma(a_1,\,a_2)} \int\limits_{\gamma} \rho\, d\mathcal{H}^1 \le c_2\, |a_1 - a_2|^{\alpha/p}\,.$$

Moreover we may put

$$c_2 = 2p(\omega_{n-1}/\, n)^{(p-1)/p}(\alpha + np - p)\, c_1^{1/p} \;/\; (\mu_n \alpha(np+\alpha)),$$

$$\omega_{n-1} = \mathcal{H}^{n-1}(S(0,1)).$$

7 Proof of Theorem 1.5

Our purpose is to obtain two-sided estimates of the distortion under mappings almost quasi-conformally close to quasi-isometries.

Let $D = B(a_1, d) \cup B(a_2, d)$ be as above, $b : D \to \mathbf{R}^n$ be an (A', A'')-quasi-isometric map and $f : D \to \mathbf{R}^n$ be a map of the class $W^{1,n}_{\text{loc}}(D)$ almost quasi-conformally close to g with a constant $K > 0$ and a locally integrable function δ satisfying (1.6).

By Lemma 5.50 for every $0 < r \leq d = |a_2 - a_1|$ we have

$$r^{-n/K}\, I(a_i, r) + \lambda \int\limits_{B(a_i,r)} \delta(x)\, d\mathcal{H}^n \leq d^{-n/K}\, I(a_i, d) + \lambda \int\limits_{B(a_i,d)} \delta(x)\, d\mathcal{H}^n \quad (i = 1, 2)\,.$$

Observe that

$$\int\limits_{B(a_i,d)} \delta(x)\, d\mathcal{H}^n - \int\limits_{B(a_i,r)} \delta(x)\, d\mathcal{H}^n \leq \int\limits_{B(a_i,d)} \delta^+(x)\, d\mathcal{H}^n)\,,$$

where

$$\delta^+(x) = \max\{0, \delta(x)\}\,.$$

From this we find

$$\text{(7.54)} \quad I(a_i, r) \leq r^{n/K} \left(d^{-n/K} I(a_i, d) + \lambda \int\limits_{B(a_i,d)} \delta^+(x)\, d\mathcal{H}^n \right) \quad (i = 1, 2)\,.$$

We put

$$J(a, r) = \int\limits_{B(a,r)} \delta^+(x)\, d\mathcal{H}^n \quad (0 < r \leq d)\,.$$

Choose in Lemma 6.51 the function $\rho = |f'(x) - b'(x)|$ and $p = n$. By (7.54) the assumption (6.52) holds with the constants $\alpha = n/K$ and

$$c_1 = d^{-n/K} \max_{i=1,2} \left(I(a_i, d) + \lambda d^{n/K}\, J(a_i, d) \right)\,.$$

The relation (6.53) implies

$$\inf_{\gamma \in \Gamma(a_1,\, a_2)} \int\limits_{\gamma} |f'(x) - b'(x)|\, d\mathcal{H}^1 \leq \omega_{n-1}^{-1/n} \nu(n, K) \max_{i=1,2} \left(I(a_i, d) + \lambda d^{n/K}\, J(a_i, d) \right)^{1/n}\,.$$

However,

$$|(f(a_2) - b(a_2)) - (f(a_1) - b(a_1))| \leq \inf_{\gamma \in \Gamma(a_1,\, a_2)} \int\limits_{\gamma} |f'(x) - b'(x)|\, d\mathcal{H}^1$$

and consequently

$$
\begin{aligned}
&|(f(a_2) - f(a_1)) - (b(a_2) - b(a_1))| \le \\
&\le \omega_{n-1}^{-1/n} \nu(n,K) \max_{i=1,2} \Big(I(a_i,d) + \lambda d^{n/K} J(a_i,d) \Big)^{1/n} .
\end{aligned}
\tag{7.55}
$$

It follows from (7.55), that

$$
\begin{aligned}
|f(a_2) - f(a_1)| &\le |b(a_2) - b(a_1)| + \\
&+ \omega_{n-1}^{-1/n} \nu(n,K) \max_{i=1,2} \Big(I(a_i,d) + \lambda d^{n/K} J(a_i,d) \Big)^{1/n} \le \\
&\le A'' |a_2 - a_1| + \omega_{n-1}^{-1/n} \nu(n,K) \max_{i=1,2} \Big(I(a_i,d) + \lambda\, d^{n/K} J(a_i,d) \Big)^{1/n} .
\end{aligned}
$$

Thus,

$$
|f(a_2) - f(a_1)| \le |a_2 - a_1| \; (A'' + \nu(n,K)\, h(a_1,a_2)) \, . \tag{7.56}
$$

Analogously, if the mapping f is close to a bi-Lipschitz map b, then

$$
|b(a_2) - b(a_1)| - |(f(a_2) - f(a_1)) - (b(a_2) - b(a_1))| \le |f(a_2) - f(a_1)|
$$

and

$$
A' |a_2 - a_1| - \omega_{n-1}^{-1/n} \nu(n,K) \max_{i=1,2} \Big(I(a_i,d) + \lambda\, d^{n/K} J(a_i,r) \Big)^{1/n} \le |f(a_2) - f(a_1)| \, .
$$

Thus,

$$
(A' - \nu(n,K)\, h(a_1,a_2)) \; |a_2 - a_1| \; \le |f(a_2) - f(a_1)| \, . \tag{7.57}
$$

By combining (7.56) and (7.57), we obtain Theorem 1.5. □

8 Proof of Corollary 1.9

We fix a pair of points a_1, $a_2 \in \mathbf{R}^n$ and a constant $\varepsilon > 0$. The assumption (2.14) implies existence of an (A', A'')-quasi-isometry $b : D \to \mathbf{R}^n$ such that

$$
h(a_1,a_2) \le \frac{A' + \varepsilon}{\nu(n,K)} \quad (i = 1,2) \, .
$$

By Theorem 1.5, it guarantees (1.8) and,

$$(A' + \varepsilon - \nu(n,K)h)\,|a_2 - a_1| \ \leq |f(a_2) - f(a_1)| \leq (A'' + \nu(n,K)h)\,|a_2 - a_1|\,.$$

The arbitrariness of $\varepsilon > 0$ implies two-sided estimates (1.8) and global quasi-isometry of $f : \mathbf{R}^n \to \mathbf{R}^n$. $\square$

9 Proof of Proposition 1.3

Let $\varepsilon > 0$. Almost everywhere on D, we have

$$|f'(x)|^n \leq (1+\varepsilon)\,\|f'\|_D^n - \varepsilon\,\|f'\|^n\,.$$

By the Hadamard's inequality for determinants,

$$|\det \mathrm{f}'| \leq \prod_{k=1}^{n} |\nabla f_k|\,.$$

Using the Cauchy inequality

$$\prod_{k=1}^{n} |a_k| \leq n^{-n} \left(\sum_{k=1}^{n} |a_k|\right)^n\,,$$

we obtain

$$|\det \mathrm{f}'|^2 \leq n^{-n} \left(\prod_{k=1}^{n} |\nabla f_k|^2\right)^n$$

or,

$$-n^{n/2} \det \mathrm{f}'(\mathrm{x}) \leq |f'(x)|^n \leq \|f'\|^n\,.$$

Thus, we find

$$|f'(x)|^n \leq (1+\varepsilon)\,\|f'\|_D^n + \varepsilon\, n^{n/2} \det \mathrm{f}'(\mathrm{x})\,.$$

For $K = \varepsilon\, n^{n/2}$ we have $\delta = (1+\varepsilon)\,\|f'\|_D^n \leq (1+\varepsilon)\,q^n$. $\square$

10 Proof of Theorem 2.12

Let

$$I(a,r) = \int\limits_{B(a,r)} \delta^p(x)\,d\mathcal{H}^n \quad (0 < r \leq d)\,.$$

Then by assumptions (2.13), we have

$$I(a_i, r) \le r^n d^{-n} I(a_i, d) \quad (i = 1, 2)\,. \tag{10.58}$$

Choose the function $\rho = |f'(x) - b'(x)|$ in Lemma 6.51 . By (10.58) the assumption (6.52) holds with the constants $\alpha = p$ and

$$c_1 = d^{-n} \max_{i=1,2} I(a_i, d)\,.$$

The relation (6.53) implies that

$$\inf_{\gamma \in \Gamma(a_1, a_2)} \int_\gamma |f'(x) - b'(x)|\, d\mathcal{H}^1 \le \omega_{n-1}^{-1/p} \nu_1(n, p) \max_{i=1,2} I^{1/p}(a_i, d)\,.$$

However,

$$|(f(a_2) - b(a_2)) - (f(a_1) - b(a_1))| \le \inf_{\gamma \in \Gamma(a_1, a_2)} \int_\gamma |f'(x) - b'(x)|\, d\mathcal{H}^1$$

and consequently,

$$\begin{aligned} &|(f(a_2) - f(a_1)) - (b(a_2) - b(a_1))| \le \\ &\le \omega_{n-1}^{-1/p} \nu(n, p) \max_{i=1,2} I^{1/p}(a_i, d)\,. \end{aligned} \tag{10.59}$$

It follows from (10.59) that

$$\begin{aligned} |f(a_2) - f(a_1)| &\le |b(a_2) - b(a_1)| + \\ &+\omega_{n-1}^{-1/p} \nu_1(n, p) \max_{i=1,2} I^{1/p}(a_i, d) \le \\ &\le A''\, |a_2 - a_1| + \omega_{n-1}^{-1/p} \nu_1(n, p) \max_{i=1,2} I^{1/p}(a_i, d)\,. \end{aligned}$$

Thus,

$$|f(a_2) - f(a_1)| \le |a_2 - a_1|\, \left(A'' + \nu_1(n, p)\, h(a_1, a_2)\right). \tag{10.60}$$

Analogously, if f is close to a quasi-isometry b, then

$$|b(a_2) - b(a_1)| - |(f(a_2) - f(a_1)) - (b(a_2) - b(a_1))| \le |f(a_2) - f(a_1)|$$

and

$$A'\,|a_2 - a_1| - \omega_{n-1}^{-1/p}\nu_1(n,p)\,\max_{i=1,2} I^{1/p}(a_i,d) \le |f(a_2) - f(a_1)|\,.$$

Then

(10.61) $$\left(A' - \nu_1(n,p)\,h_1(a_1,a_2)\right)\,|a_2 - a_1|\, \le |f(a_2) - f(a_1)|\,.$$

By combining (10.60) and (10.61), we obtain Theorem 2.12. □

Acknowledgements. The author is indebted to PhD A. Rasila. His remarks were very useful.

Vladimir Mikhaelovich Miklyukov
E-mail: miklyuk@mail.ru
Volgograd State University
Universitetskii prospect 100, Volgograd 400062

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