A note on the connection between Molchan-Golosov- and Mandelbrot-Van Ness representation of fractional Brownian motion

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Abstract

We proof a connection between the generalized Molchan-Golosov integral transform (see [4], Theorem 5.1) and the generalized Mandelbrot-Van Ness integral transform (see [8], Theorem 1.1) of fractional Brownian motion (fBm). The former changes fBm of arbitrary Hurst index K into fBm of index H by integrating over $[0,t]$, whereas the latter requires integration over $(-\infty,t]$ for $t>0$. This completes an argument in [4], where the connection is mentioned without full proof.

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1 Introduction

The *fractional Brownian motion with Hurst index* $H \in (0,1)$, or H*-fBm*, is the continuous, centered Gaussian process $(B_t^H)_{t\in\mathbb{R}}$ with $B_0^H = 0$, a.s, and

$$\mathrm{Cov}_{\mathbb{P}}(B_s^H, B_t^H) \;=\; \frac{1}{2}\left(|s|^{2H} \;+\; |t|^{2H} \;-\; |t-s|^{2H}\right), \quad s,t\in\mathbb{R}.$$

H-fBm is H-self-similar and has stationary increments. For $H=\frac{1}{2}$, fractional Brownian motion is standard Brownian motion and denoted by W. Fix a Hurst index $K \in (0,1)$. Theorem 5.1 in [4] states that there exists a K-fBm $(B_t^K)_{t\in\mathbb{R}}$ such that for all $t\in[0,\infty)$ we have

$$B_t^H \;=\; C(K,H)\int_0^t (t-s)^{H-K}F(1-K-H, H-K, 1+H-K, \frac{s-t}{s})dB_s^K, \;\; a.s. \tag{1.1}$$

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Here $C(K,H) := \frac{C(H)C(K)^{-1}}{\Gamma(H-K+1)}$, $C(H) := \left(\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}\right)^{\frac{1}{2}}$, Γ denotes the gamma function and F is Gauss' hypergeometric function. B^K is unique (up to modification) on $[0,\infty)$. On the other hand, by Theorem 1.1. in [8], there exists a unique (up to modification) K-fBm $(\tilde{B}^K_t)_{t\in\mathbb{R}}$, such that for all $t \in \mathbb{R}$ holds

$$B^H_t \ := \ C(K,H)\int_{\mathbb{R}} \left((t-s)^{H-K}1_{(-\infty,t)}(s) - (-s)^{H-K}1_{(-\infty,0)}(s)\right) d\tilde{B}^K_s, \ a.s. \tag{1.2}$$

The integrals in (1.1) and (1.2) are fractional Wiener integrals. For $K = \frac{1}{2}$, (1.1) corresponds to the Molchan-Golosov representation and (1.2) is the Mandelbrot-Van Ness representation of H-fBm (see [6] and [5], respectively). Consider the generalized Molchan-Golosov integral transform (1.1). B^K has stationary increments, hence for every $s > 0$, the process

$$B^{H,s}_t \ := \ C(K,H)\int_0^t (t-u)^{H-K}\hat{F}(\frac{u-t}{u})dB^K_{u-s}, \ \ t \in [0,\infty), \tag{1.3}$$

where

$$\hat{F}(z) \ := \ F(1-K-H, H-K, 1+H-K, z), \tag{1.4}$$

is an H-fBm. The increments of $B^{H,s}$ are stationary. So by substituting $v := u-s$ in (1.3), we obtain that the time-shifted process

$$\begin{aligned} Z^{H,s}_t \ &:= \ B^{H,s}_{t+s} \ - \ B^{H,s}_s \\ &= \ C(K,H)\left(\int_{-s}^t (t-v)^{H-K}\hat{F}(\frac{v-t}{v+s})dB^K_v - \int_{-s}^0 (-v)^{H-K}\hat{F}(\frac{v}{v+s})dB^K_v\right) \end{aligned}$$

is an H-fBm on $[-s,\infty)$. As $s \to \infty$, we formally obtain

$$Z^H_t := Z^{H,\infty}_t := C(K,H)\int_{\mathbb{R}} \left((t-v)^{H-K}1_{(-\infty,t)}(v) - (-v)^{H-K}1_{(-\infty,0)}(v)\right) dB^K_v$$

for $t \in \mathbb{R}$, which corresponds to the generalized Mandelbrot-Van Ness integral transform.

The aim of this work is modest and consists in proving that for every $K \geq \frac{1}{2}$ and $t \in \mathbb{R}$, there exist constants $C_1 = C_1(K,H,t)$ and $s_1 = s_1(t) > 0$ such that

$$E[Z^{H,s}_t - Z^H_t]^2 \leq C_1 s^{2H-2}, s > s_1. \tag{1.5}$$

Also, we show that for every $K < \frac{1}{2}$ and $t \in \mathbb{R}$, there exist constants $C_2 = C_2(K,H,t)$, $C_3 = C_3(K,H,t)$ and $s_2 = s_2(t) > 0$ with

$$E[Z^{H,s}_t - Z^H_t]^2 \leq C_2 s^{2H-2} + C_3 s^{2K-2}, s > s_2. \tag{1.6}$$

This note can be considered as an appendix of [4] since it completes an argument about the (generalized) Mandelbrot-Van Ness transform emerging as the boundary case of a suitable time-shifted (generalized) Molchan-Golosov transform in section 5.

The work is organized as follows: In Section 2, we first review definitions and facts of the special functions relevant in the proof. Second, we define fractional integrals and derivatives over $\mathbb{R}$ and show the connection to fBm. Third, we recall the definition of the fractional Wiener integral over the real line. Section 3 is dedicated to the proof of (1.5) and (1.6). We specify the constants for $t > 0$.

2 Preliminaries

2.1 The Gamma -, Beta - and Gauss' hypergeometric function

The *gamma function* of $\alpha > 0$ is defined by

$$\Gamma(\alpha) := \int_0^\infty \exp(-u)u^{\alpha-1}du.$$

By partial integration one has $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ which is used to extend Γ recursively to all $\alpha \in \mathcal{A} := \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$. We have $\Gamma(1) = 1$.

The *beta function* of $\alpha, \beta > 0$ is defined by

$$B(\alpha, \beta) \; := \; \int_0^1 (1-v)^{\alpha-1}v^{\beta-1}dv.$$

We have $B(\alpha,\beta) = B(\beta,\alpha)$. For $\beta > 1$, partial integration yields $B(\alpha, \beta-1) = \frac{\alpha+\beta-1}{\beta-1}B(\alpha,\beta)$. Recursion and symmetry are used to extend B to all $\alpha, \beta \in \mathcal{A}$.

The *Gauss hypergeometric function* of parameters a, b, c and variable $z \in \mathbb{R}$ is defined by the formal power series

$$F(a,b,c,z) \;\; := \; {}_2F_1(a,b,c,z) \; := \; \sum_{k=0}^\infty \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},$$

where $(a)_0 := 1$ and $(a)_k := a \cdot (a+1) \cdot \ldots \cdot (a+k-1)$, $k \in \mathbb{N}$. We assume that $c \in \mathcal{A}$ for this to make sense. If $|z| < 1$ or $|z| = 1$ and $c - b - a > 0$, then the series converges absolutely. If furthermore $c > b > 0$ for $z \in [-1, 1)$ and $b > 0$ for $z = 1$, then it can be represented by the Euler integral

$$F(a,b,c,z) \; = \; \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 v^{b-1}(1-v)^{c-b-1}(1-zv)^{-a}dv \qquad (2.1)$$

(see [3], p.59). If $c > b > 0$ then the expression on the right-hand side of (2.1) is well-defined for all $z \in (-\infty, 1)$. For these parameters, we can hence extend the definition of F to all $z \in (-\infty, 1)$ via (2.1). In order to extend F for fixed $z \in (-\infty, 1]$ to more general parameters, we consider Gauss' relations for neighbor functions. Functions of type $F(a \pm m, b, c, z)$, $F(a, b \pm m, c, z)$ and $F(a, b, c \pm m, z)$, $m \in \mathbb{N}$, are called *contiguous* to $F(a, b, c, z)$. If $m = 1$ then they are also called *neighbors*. For any two neighbors $F_1(z)$, $F_2(z)$ of $F(a, b, c, z)$, one has a linear relation of type

$$A(z)F(a, b, c, z) \ + \ A_1(z)F_1(z) \ + \ A_2(z)F_2(z) \ = \ 0, \tag{2.2}$$

where A, A_1 and A_2 are first degree polynomials with coefficients depending on a, b and c. See [1], p.558 for all 15 relations. We use the neighbor relations in order to extend F for $z \in (-\infty, 1)$ to all parameters such that $c \in \mathcal{A}$, and for $z = 1$ to all parameters that satisfy $c, c - b - a \in \mathcal{A}$. Among the most important properties of F are the symmetry

$$F(a, b, c, z) \ = \ F(b, a, c, z)$$

and the reduction formula

$$F(0, b, c, z) \ = \ F(a, b, c, 0) \ = \ 1.$$

Also we have the linear transformation formula (see [1], p.559)

$$F(a, b, c, z) \ = \ (1 - z)^{-a} F(a, c - b, c, \frac{z}{z - 1}), \ z < 1. \tag{2.3}$$

In particular,

$$F(a, b, b, z) \ = \ (1 - z)^{-a}, \ z < 1. \tag{2.4}$$

F is smooth in z and we have (see [1], p.557)

$$\frac{d}{dz} F(a, b, c, z) \ = \ \frac{ab}{c} F(a + 1, b + 1, c + 1, z) \tag{2.5}$$

and

$$\frac{d}{dz} \left(z^a F(a, b, c, \pm z)\right) \ = \ a z^{a-1} F(a + 1, b, c, \pm z). \tag{2.6}$$

F is left-continuous in $z = 1$ and it holds (see [3], p.9)

$$F(a, b, c, 1) \ = \ \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} B(b, c - b - a) \ = \ \frac{\Gamma(c)\Gamma(c - b - a)}{\Gamma(c - b)\Gamma(c - a)}. \tag{2.7}$$

Let $a, b > -1$ and $w < x < y$. Substituting $u := (y - x)v + x$ in (2.1) and using (2.3) implies

$$\begin{aligned} &\textstyle\int_x^y (y - u)^b (u - w)^c (u - x)^a du \\ &= \frac{\Gamma(a + 1)\Gamma(b + 1)}{\Gamma(a + 2 + b)} (y - x)^{1+a+b} (x - w)^c F(-c, a + 1, a + 2 + b, \frac{y - x}{w - x}) \\ &= \frac{\Gamma(a + 1)\Gamma(b + 1)}{\Gamma(a + 2 + b)} (y - x)^{1+a+b} (y - w)^c F(-c, b + 1, a + 2 + b, \frac{y - x}{y - w}). \end{aligned} \tag{2.8}$$

If $x < y < w$ then we have

$$\int_x^y (y-u)^b (w-u)^c (u-x)^a du$$

$$= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+2+b)} (y-x)^{1+a+b} (w-x)^c F(-c, a+1, a+2+b, \frac{y-x}{w-x}) \quad (2.9)$$

$$= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+2+b)} (y-x)^{1+a+b} (w-y)^c F(-c, b+1, a+2+b, \frac{y-x}{y-w}). \quad (2.10)$$

By linearly combining neighbor relations, we obtain relations of type (2.2), where $F_1(z)$ and $F_2(z)$ are contiguous to $F(a,b,c,z)$ and A, A_1 and A_2 are polynomials of higher degree. An example for a contiguity relation is

$$-cF(a,b-1,c,z)+(c-b+zb-za)F(a,b,c+1,z)+b(1-z)F(a,b+1,c+1,z) = 0. \quad (2.11)$$

It can be checked easily by using series.

2.2 Fractional calculus over the real line

For more information on fractional calculus in the context of fractional Brownian motion, see [7] or [4]. See [10] for general information on fractional calculus.

Definition 2.1. Let $\alpha > 0$. The *(right-sided) Riemann-Liouville fractional integral operator of order* α is defined by

$$(\mathcal{I}_-^\alpha f)(s) := \frac{1}{\Gamma(\alpha)} \int_s^\infty f(u)(u-s)^{\alpha-1} du, \ s \in \mathbb{R}.$$

Let $\alpha \in (0,1)$. The *(right-sided) Riemann-Liouville fractional derivative operator of order* α is defined by

$$(\mathcal{D}_-^\alpha f)(s) := \frac{-d}{ds}(\mathcal{I}_-^{1-\alpha} f)(s) = \frac{1}{\Gamma(1-\alpha)} \frac{-d}{ds} \int_s^\infty f(u)(u-s)^{-\alpha} du, \ s \in \mathbb{R}.$$

The *(right-sided) Marchaud fractional derivative operator of order* α is defined by

$$(\mathbf{D}_-^\alpha f)(s) := \lim_{\epsilon \searrow 0} (\mathbf{D}_{-,\epsilon}^\alpha f)(s), \ a.e. \ s \in \mathbb{R},$$

where

$$(\mathbf{D}_{-,\epsilon}^\alpha f)(s) := \frac{\alpha}{\Gamma(1-\alpha)} \int_\epsilon^\infty (f(s) - f(u+s))\, u^{-\alpha-1} du.$$

Moreover,

$$\mathcal{D}_-^0 f := \mathbf{D}_-^0 f := f.$$

We set

$$\mathcal{I}_-^{-\alpha} := \mathbf{D}_-^\alpha, \ \alpha \in [0,1).$$

If $\alpha \in (0,1)$, $p \in [1, \frac{1}{\alpha})$ and $f \in L^p(\mathbb{R})$, then $\mathcal{I}_-^\alpha f$ is well-defined (see [10], p.94). Clearly, $\mathcal{I}_-^1 f$ exists for $f \in L^1(\mathbb{R})$. We have the composition formula

$$\mathcal{I}_-^\alpha \mathcal{I}_-^\beta f \;=\; \mathcal{I}_-^{\alpha+\beta} f,\;\; f \in L^1(\mathbb{R}),\;\; \mathcal{I}_-^\beta f \in L^1(\mathbb{R}),\;\; \alpha, \beta \in (0,1].$$

Furthermore,

$$\mathcal{D}_-^\alpha \mathcal{I}_-^\beta f \;=\; \mathcal{I}_-^{\beta-\alpha} f,\;\; f \in L^1(\mathbb{R}),\;\; \mathcal{I}_-^\beta f \in L^1(\mathbb{R}),\;\; 0 < \alpha \le \beta \le 1. \tag{2.12}$$

If f is piecewise differentiable with $\mathrm{supp}(f') \subseteq (-\infty, y)$ for some $y \in \mathbb{R}$ and so that $\mathcal{D}_-^\alpha f$ exists, then $\mathbf{D}_-^\alpha f = \mathcal{D}_-^\alpha f$.

Let $t \in \mathbb{R} \setminus \{0\}$ and set $1_{[0,t)} := -1_{[t,0)}$ for $t < 0$. For all $H \in (0,1)$ we have

$$(\mathcal{I}_-^{H-\frac{1}{2}} 1_{[0,t)})(s) \;=\; \frac{1}{\Gamma(H+\frac{1}{2})}((t-s)^{H-\frac{1}{2}} 1_{(-\infty,t)}(s) - (-s)^{H-\frac{1}{2}} 1_{(-\infty,0)}(s)).$$

So for $K = \frac{1}{2}$, (1.2) can be written as

$$B_t^H \;=\; C(H) \int_{\mathbb{R}} (\mathcal{I}_-^{H-\frac{1}{2}} 1_{[0,t)})(s) d\tilde{W}_s,\;\; a.s,\;\; t \in \mathbb{R}. \tag{2.13}$$

2.3 Fractional Wiener integrals over the real line

We combine (2.13) with the standard Wiener integral in order to obtain a meaning for the expression $\int_{\mathbb{R}} f(s) dB_s^H$ for suitable deterministic integrands f. For details on this topic, see [9]. For $H > \frac{1}{2}$, the space of integrands is given by

$$\Lambda(H) := \;\{f \in L^1(\mathbb{R}) \mid \int_{\mathbb{R}} (\mathcal{I}_-^{H-\frac{1}{2}} f)(s)^2 ds \;<\; \infty\}.$$

For $H < \frac{1}{2}$, we have

$$\Lambda(H) := \;\{f : \mathbb{R} \to \mathbb{R} \mid \exists\, \phi_f \in L^2(\mathbb{R}) \text{ such that } f \;=\; \mathcal{I}_-^{\frac{1}{2}-H} \phi_f\}.$$

For $f \in \Lambda(H)$, the *(time domain) fractional Wiener integral* with respect to B^H is defined by

$$I^H(f) \;:=\; \int_{\mathbb{R}} f(s) dB_s^H \;:=\; C(H) \int_{\mathbb{R}} (\mathcal{I}_-^{H-\frac{1}{2}} f)(s) d\tilde{W}_s.$$

3 Proof of (1.5) and (1.6)

Clearly, we can assume $H \neq K$ and $t \neq 0$. Moreover, we assume that $t > 0$. The result is derived similarly for $t < 0$. Recall that $\hat{F}$ is defined in (1.4) and denote

$$\Delta f_t^s(v) \;:=\; \left((t-v)^{H-K} \;-\; (-v)^{H-K}\right) 1_{(-\infty,-s)}(v),$$

$$\Delta g^s_t(v) \ := \ (t-v)^{H-K}\hat{F}(\frac{v-t}{v+s})1_{(-s,t)}(v) \ - \ (-v)^{H-K}\hat{F}(\frac{v}{v+s})1_{(-s,0)}(v),$$
$$\Delta h^s_t(v) \ := \ (t-v)^{H-K}1_{(-s,t)}(v) \ - \ (-v)^{H-K}1_{(-s,0)}(v)$$
and $\quad \Delta k^s_t(v) \ := \ k^s_t(v) \ - \ k^s_0(v) \ :=$

$(t-v)^{H-K}(\hat{F}(\frac{v-t}{v+s})-1)1_{(-s,t)}(v) \ - \ (-v)^{H-K}(\hat{F}(\frac{v}{v+s})-1)1_{(-s,0)}(v).$

For continuous G, set ${}^*G = \max_{z\in[-1,0]}|G(z)|$ and $G^* := \max_{z\in[0,1]}|G(z)|$.
For $K=\frac{1}{2}$ we have by independence of increments of $B^K = W$ that

$$\left(\frac{\Gamma(H+\frac{1}{2})}{C(H)}\right)^2 E[Z^{H,s}_t - Z^H_t]^2 \ = \ \int_{\mathbb{R}} \Delta f^s_t(v)^2 dv \ + \ \int_{\mathbb{R}} \Delta k^s_t(v)^2 dv. \tag{3.1}$$

Note that
$$(\sum_{i=1}^n a_i)^2 \ \le \ n\sum_{i=1}^n a_i^2, \ n\in\mathbb{N}. \tag{3.2}$$

So for $K\neq\frac{1}{2}$, we obtain by using (3.2) with $n=2$ that

$$\begin{aligned}\frac{1}{2C(K,H)^2}\cdot E[Z^{H,s}_t - Z^H_t]^2 \ &\le \ E[\int_{\mathbb{R}} \Delta f^s_t(v)dB^K_v]^2 + E[\int_{\mathbb{R}} \Delta k^s_t(v)dB^K_v]^2 \\ &= C(K)^2\left(\int_{\mathbb{R}}(\mathcal{I}^{K-\frac{1}{2}}_-\Delta f^s_t)(v)^2dv + \int_{\mathbb{R}}(\mathcal{I}^{K-\frac{1}{2}}_-\Delta k^s_t)(v)^2dv\right). \end{aligned} \tag{3.3}$$

1. For all $K\in(0,1)$, we show that there exists a constant $c_1(K,H,t)$ such that

$$\int_{\mathbb{R}}(\mathcal{I}^{K-\frac{1}{2}}_-\Delta f^s_t)(v)^2dv \ \le \ c_1(K,H,t)s^{2H-2}, \ s>t. \tag{3.4}$$

First, let $K>\frac{1}{2}$. By using (2.9) we have that $(\mathcal{I}^{K-\frac{1}{2}}_-\Delta f^s_t)(v) \ =$

$$\frac{1}{\Gamma(K+\frac{1}{2})}\frac{(-s-v)^{K-\frac{1}{2}}}{(t-v)^{K-H}}F(K-H,K-\tfrac{1}{2},K+\tfrac{1}{2},\tfrac{-s-v}{t-v})1_{(-\infty,-s)}(v) \ -$$
$$\frac{1}{\Gamma(K+\frac{1}{2})}\frac{(-s-v)^{K-\frac{1}{2}}}{(-v)^{K-H}}F(K-H,K-\tfrac{1}{2},K+\tfrac{1}{2},\tfrac{-s-v}{-v})1_{(-\infty,-s)}(v).$$

Second, let $K<\frac{1}{2}$. Using (2.9) yields $(\mathcal{I}^{K+\frac{1}{2}}_-\Delta f^s_t)(v) \ =$

$$\frac{1}{\Gamma(K+\frac{3}{2})}(t-v)^{H+\frac{1}{2}}\left((\tfrac{-s-v}{t-v})^{K+\frac{1}{2}}F(K-H,K+\tfrac{1}{2},K+\tfrac{3}{2},\tfrac{-s-v}{t-v})\right)1_{(-\infty,-s)}(v) \ -$$
$$\frac{1}{\Gamma(K+\frac{3}{2})}(-v)^{H+\frac{1}{2}}\left((\tfrac{-s-v}{-v})^{K+\frac{1}{2}}F(K-H,K+\tfrac{1}{2},K+\tfrac{3}{2},\tfrac{-s-v}{-v})\right)1_{(-\infty,-s)}(v).$$

By combining (2.6) and (2.11) we obtain

$(\mathcal{I}^{K-\frac{1}{2}}_-\Delta f^s_t)(v) \ = \ (\mathbf{D}^{\frac{1}{2}-K}_-\Delta f^s_t)(v) \ = \ (\mathcal{D}^{\frac{1}{2}-K}_-\Delta f^s_t)(v) \ = \ \frac{-d}{dv}(\mathcal{I}^{K+\frac{1}{2}}_-\Delta f^s_t)(v) \ =$

$$\frac{1}{\Gamma(K+\frac{1}{2})}\frac{(-s-v)^{K-\frac{1}{2}}}{(t-v)^{K-H}}F(K-H,K-\tfrac{1}{2},K+\tfrac{1}{2},\tfrac{-s-v}{t-v})1_{(-\infty,-s)}(v) \ -$$

$$\tfrac{1}{\Gamma(K+\frac{1}{2})}\tfrac{(-s-v)^{K-\frac{1}{2}}}{(-v)^{K-H}}F(K-H,K-\tfrac{1}{2},K+\tfrac{1}{2},\tfrac{-s-v}{-v})1_{(-\infty,-s)}(v).$$

Thus for all $K\in(0,1)$ we have

$$(\mathcal{I}_-^{K-\frac{1}{2}}\Delta f_t^s)(v)=\frac{(-s-v)^{H-\frac{1}{2}}}{\Gamma(K+\frac{1}{2})}\left(G_0(\frac{-s-v}{t-v})\ -\ G_0(\frac{-s-v}{-v})\right)1_{(-\infty,-s)}(v),$$

where

$$G_0(z)\ :=\ z^{K-H}F(K-H,K-\frac{1}{2},K+\frac{1}{2},z).$$

By (2.6) we have

$$\begin{aligned}\frac{dG_0}{dz}(z)\ &=\ (K-H)z^{K-H-1}F(K-H+1,K-\frac{1}{2},K+\frac{1}{2},z)\\ &=:\ (K-H)z^{K-H-1}G_1(z).\end{aligned}\tag{3.5}$$

By the mean value theorem, there exists $\theta_{v,s}\in(\frac{-s-v}{t-v},\frac{-s-v}{-v})\subset(0,1)$ such that

$(\mathcal{I}_-^{K-\frac{1}{2}}\Delta f_t^s)(v)^2\ =$

$(\frac{K-H}{\Gamma(K+\frac{1}{2})})^2(-s-v)^{2H-1}(\frac{t(-v-s)}{-v(t-v)})^2\theta_{v,s}^{2(K-H-1)}G_1^2(\theta_{v,s})1_{(-\infty,-s)}(v)\ \le$

$(\frac{K-H}{\Gamma(K+\frac{1}{2})})^2G_1^{*2}t^2\max(1,(\frac{t+s}{s})^{2(H-K)})(-s-v)^{2K-1}(-v)^{2(H-K-1)}\cdot 1_{(-\infty,-s)}(v).$

From (2.10) and (2.3) we obtain that $\int_{\mathbb{R}}(\mathcal{I}_-^{K-\frac{1}{2}}\Delta f_t^s)(v)^2dv\ \le$

$(\frac{K-H}{\Gamma(K+\frac{1}{2})})^2G_1^{*2}t^2\max(1,(\frac{t+s}{s})^{2(H-K)})\cdot$
$\lim_{x\to-\infty}\int_x^{-s}(-s-v)^{2K-1}(-v)^{2(H-K-1)}dv\ =$
$(\frac{K-H}{\Gamma(K+\frac{1}{2})})^2G_1^{*2}t^2\max(1,(\frac{t+s}{s})^{2(H-K)})\cdot$
$\lim_{x\to-\infty}\frac{1}{2K}(-x-s)^{2K}s^{2(H-K-1)}(\frac{-x}{s})^{-2K}F(2H-1,2K,2K+1,\frac{x+s}{x})\ =$
$(\frac{K-H}{\Gamma(K+\frac{1}{2})})^2G_1^{*2}\frac{t^2}{2K}F(2H-1,2K,2K+1,1)\max(1,(\frac{t+s}{s})^{2(H-K)})s^{2H-2}.$

By using (2.7), we obtain (3.4) with

$$c_1(K,H,t)\ :=\ (\frac{K-H}{\Gamma(K+\frac{1}{2})})^2\frac{G_1^{*2}4\Gamma(2K)\Gamma(2-2H)}{\Gamma(2K-2H+2)}t^2.$$

2. Fix $d>0$. For $K\ge\frac{1}{2}$ we show that there exists a constant $c_2(K,H,t,d)$ such that

$$\int_{\mathbb{R}}(\mathcal{I}_-^{K-\frac{1}{2}}\Delta k_t^s)(v)^2dv\ \le\ c_2(K,H,t,d)s^{2H-2},\ s>2t+4d+1.\tag{3.6}$$

For $K<\frac{1}{2}$ we show that there exist constants $c_3(K,H,t,d)$ and $c_4(K,H,t,d)$ with

$$\int_{\mathbb{R}}(\mathcal{I}_-^{K-\frac{1}{2}}\Delta k_t^s)(v)^2dv\le c_3(K,H,t,d)s^{2H-2}+c_4(K,H,t,d)s^{2K-2},\ s>2t+4d+1.\tag{3.7}$$

Let $s > 2d$. We have

$$\begin{aligned}\Delta k_t^s(v) &= k_t^s(v)1_{[-d,t)}(v) - k_0^s(v)1_{[-d,0)}(v) + \Delta k_t^s(v)1_{[\frac{-s}{2},-d)}(v) \\ &\quad + \Delta g_t^s(v)1_{(-s,\frac{-s}{2})}(v) - \Delta h_t^s(v)1_{(-s,\frac{-s}{2})}(v).\end{aligned}$$

First, let $K = \frac{1}{2}$. Note that $(-s,\frac{-s}{2}) \cap [\frac{-s}{2},-d) \cap [-d,t) = \emptyset$. By using (3.2) with $n = 2$ we have

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}} \Delta k_t^s(v)^2 dv &\leq \int_{-d}^{t} k_t^s(v)^2 dv + \int_{-d}^{0} k_0^s(v)^2 dv + \int_{\frac{-s}{2}}^{-d} \Delta k_t^s(v)^2 dv \\ &\quad + \int_{-s}^{\frac{-s}{2}} \Delta g_t^s(v)^2 dv + \int_{-s}^{\frac{-s}{2}} \Delta h_t^s(v)^2 dv. \qquad (3.8)\end{aligned}$$

Second, let $K > \frac{1}{2}$. Then $\Gamma(K-\frac{1}{2})(\mathcal{I}_-^{K-\frac{1}{2}}\Delta k_t^s)(v) =$

$\int_{-d}^{t} k_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,-d)}(v) + \int_{v}^{t} k_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{[-d,t)}(v)$

$- \int_{-d}^{0} k_0^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,-d)}(v) - \int_{v}^{0} k_0^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{[-d,0)}(v)$

$+ \int_{\frac{-s}{2}}^{-d} \Delta k_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,\frac{-s}{2})}(v) + \int_{v}^{-d} \Delta k_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{[\frac{-s}{2},-d)}(v)$

$+ \int_{-s}^{\frac{-s}{2}} \Delta g_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,-s)}(v) + \int_{v}^{\frac{-s}{2}} \Delta g_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{[-s,\frac{-s}{2})}(v)$

$- \int_{-s}^{\frac{-s}{2}} \Delta h_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,-s)}(v) - \int_{v}^{\frac{-s}{2}} \Delta h_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{[-s,\frac{-s}{2})}(v)$

$=: A_1(v)+B_1(v)+A_2(v)+B_2(v)+A_3(v)+B_3(v)+A_4(v)+B_4(v)+A_5(v)+B_5(v).$

Hence, by using (3.2) with $n = 5$, we obtain

$$\frac{\Gamma(K-\frac{1}{2})^2}{5}\int_{\mathbb{R}} (\mathcal{I}_-^{K-\frac{1}{2}}\Delta k_t^s)(v)^2 dv \leq \sum_{i=1}^{5}\int_{\mathbb{R}} A_i(v)^2 dv + \sum_{i=1}^{5}\int_{\mathbb{R}} B_i(v)^2 dv. \quad (3.9)$$

Third, let $K < \frac{1}{2}$. Let $x \leq v < y$ and f be differentiable with $f' \in L^1[x,y]$. From (2.12) follows that $(\mathcal{I}_-^{K+\frac{1}{2}}\mathcal{D}_-^1 f1_{(x,y)})(v) = (\mathcal{D}_-^{\frac{1}{2}-K}\mathcal{I}_-^1\mathcal{D}_-^1 f1_{(x,y)})(v) = (\mathcal{D}_-^{\frac{1}{2}-K} f1_{(x,y)} - f(y)1_{(x,y)})(v) = (\mathcal{D}_-^{\frac{1}{2}-K} f1_{(x,y)})(v) - \frac{f(y)}{\Gamma(K+\frac{1}{2})}(y-v)^{K-\frac{1}{2}}$.

By using that $\lim_{v\nearrow t} k_t^s(v) = \lim_{v\nearrow 0} k_0^s(v) = 0$, we obtain

$\Gamma(K-\frac{1}{2})(\mathbf{D}_-^{\frac{1}{2}-K}\Delta k_t^s)(v) = \Gamma(K-\frac{1}{2})(\mathcal{D}_-^{\frac{1}{2}-K} k_t^s 1_{(-d,t)})(v) -$

$\Gamma(K-\frac{1}{2})(\mathcal{D}_-^{\frac{1}{2}-K} k_0^s 1_{(-d,0)})(v) + \Gamma(K-\frac{1}{2})(\mathcal{D}_-^{\frac{1}{2}-K}\Delta k_t^s 1_{(\frac{-s}{2},-d)})(v)$

$+ \Gamma(K-\frac{1}{2})(\mathcal{D}_-^{\frac{1}{2}-K}\Delta g_t^s 1_{(-s,\frac{-s}{2})})(v) - \Gamma(K-\frac{1}{2})(\mathcal{D}_-^{\frac{1}{2}-K}\Delta h_t^s 1_{(-s,\frac{-s}{2})})(v) =$

$\int_{-d}^{t} k_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,-d)}(v) +$

$$\Gamma(K-\tfrac{1}{2})(\mathcal{I}_-^{K+\frac{1}{2}}\tfrac{-d}{du}k_t^s 1_{(-d,t)})(v)\cdot 1_{[-d,t)}(v)$$
$$-\ \int_{-d}^{0} k_0^s(u)(u-v)^{K-\frac{3}{2}}du\cdot 1_{(-\infty,-d)}(v)\ -$$
$$\Gamma(K-\tfrac{1}{2})(\mathcal{I}_-^{K+\frac{1}{2}}\tfrac{-d}{du}k_0^s 1_{(-d,0)})(v)\cdot 1_{[-d,0)}(v)$$
$$+\ \int_{\frac{-s}{2}}^{-d}\Delta k_t^s(u)(u-v)^{K-\frac{3}{2}}du\cdot 1_{(-\infty,\frac{-s}{2})}(v)\ +$$
$$\Gamma(K-\tfrac{1}{2})(\mathcal{I}_-^{K+\frac{1}{2}}\tfrac{-d}{du}\Delta k_t^s 1_{(\frac{-s}{2},-d)})(v)\cdot 1_{[\frac{-s}{2},-d)}(v)$$
$$+\ \int_{-s}^{\frac{-s}{2}}\Delta g_t^s(u)(u-v)^{K-\frac{3}{2}}du\cdot 1_{(-\infty,-s)}(v)\ +$$
$$\Gamma(K-\tfrac{1}{2})(\mathcal{I}_-^{K+\frac{1}{2}}\tfrac{-d}{du}\Delta g_t^s 1_{(-s,\frac{-s}{2})})(v)\cdot 1_{[-s,\frac{-s}{2})}(v)$$
$$-\ \int_{-s}^{\frac{-s}{2}}\Delta h_t^s(u)(u-v)^{K-\frac{3}{2}}du\cdot 1_{(-\infty,-s)}(v)\ -$$
$$\Gamma(K-\tfrac{1}{2})(\mathcal{I}_-^{K+\frac{1}{2}}\tfrac{-d}{du}\Delta h_t^s 1_{(-s,\frac{-s}{2})})(v)\cdot 1_{[-s,\frac{-s}{2})}(v)$$
$$+\ \tfrac{\Delta k_t^s(-d)}{K-\frac{1}{2}}(-d-v)^{K-\frac{1}{2}}\cdot 1_{[\frac{-s}{2},-d)}(v)\ +\ \tfrac{\Delta k_t^s(\frac{-s}{2})}{K-\frac{1}{2}}(\tfrac{-s}{2}-v)^{K-\frac{1}{2}}\cdot 1_{[-s,\frac{-s}{2})}(v)$$
$$=:\ A_1(v)+C_1(v)+A_2(v)+C_2(v)+A_3(v)+C_3(v)+A_4(v)+C_4(v)+A_5(v)+C_5(v)+D(v)+E(v).$$

Hence (3.2) with $n=6$ yields

$$\frac{\Gamma(K-\frac{1}{2})^2}{6}\int_{\mathbb{R}}(\mathcal{I}_-^{K-\frac{1}{2}}\Delta k_t^s)(v)^2dv\ \le\ \sum_{i=1}^{5}\int_{\mathbb{R}}A_i(v)^2dv\ +\ \sum_{i=1}^{5}\int_{\mathbb{R}}C_i(v)^2dv + \int_{\mathbb{R}}D(v)^2dv\ +\ \int_{\mathbb{R}}E(v)^2dv. \quad (3.10)$$

Next we estimate the integrals on the right-hand sides of (3.8), (3.9) and (3.10).

1. Estimation of $\int_{-d}^{t}k_t^s(v)^2dv$, $\int_{\mathbb{R}}A_1(v)^2dv$, $\int_{\mathbb{R}}B_1(v)^2dv$ and $\int_{\mathbb{R}}C_1(v)^2dv$.
Let $u\in(-d,t)$. By the mean value theorem, there exists $\theta_{s,u}\in(\frac{u-t}{u+s},0)\subset(\frac{-d-t}{-d+s},0)$ such that

$$|k_t^s(u)|\ =\ (t-u)^{H-K}(\tfrac{t-u}{u+s})|\tfrac{d\hat{F}}{dz}(\theta_{s,u})|\ \le\ (t-u)^{H-K+1}(s-d)^{-1}\max_{z\in(\frac{-d-t}{-d+s},0)}|\tfrac{d\hat{F}(z)}{dz}|.$$

We assume that $s>2d+t$. Set

$$G_2(z)\ :=\ F(2-K-H,H-K+1,2+H-K,z).$$

Then it follows from (2.5) that

$$|k_t^s(u)|\ \le\ (t-u)^{H-K+1}(s-d)^{-1}\frac{|1-K-H||H-K|^*G_2}{|1+H-K|}.$$

So for $K=\frac{1}{2}$ we have

$\int_{-d}^{t} k_t^s(u)^2 du \ \leq \ (\frac{(\frac{1}{2}-H)^2 \cdot {}^*G_2}{|\frac{1}{2}+H|})^2 (s-d)^{-2} \frac{(t+d)^{2H+2}}{2H+2} \ \leq \ \frac{{}^*G_2^2 (t+d)^{2H+2}}{2} s^{-2}.$

Denote

$$G_3(z) \ := \ F(\frac{3}{2}-K, H-K+2, H-K+3, z).$$

For $K\neq\frac{1}{2}$ we obtain by using (2.8) that

$\int_{\mathbb{R}} A_1(v)^2 dv \ = \ \int_{-\infty}^{-d} \left(\int_{-d}^{t} k_t^s(u)(u-v)^{K-\frac{3}{2}} du\right)^2 dv \ \leq$

$(\frac{|1-K-H||H-K|^*G_2}{|1+H-K|})^2 (s-d)^{-2} \int_{-\infty}^{-d} \left(\int_{-d}^{t} (t-u)^{H-K+1}(u-v)^{K-\frac{3}{2}} du\right)^2 dv \ =$

$(\frac{|1-K-H||H-K|^*G_2}{|1+H-K|})^2 (\frac{G_3^*}{H-K+2})^2 (s-d)^{-2} \frac{(t+d)^{2H+2}}{2-2K} \ \leq \frac{2^* G_2^2 G_3^{*2}(t+d)^{2H+2}}{(1+H-K)^2(1-K)} s^{-2}.$

For $K>\frac{1}{2}$ we have

$\int_{\mathbb{R}} B_1(v)^2 dv \ = \ \int_{-d}^{t} \left(\int_{v}^{t} k_t^s(u)(u-v)^{K-\frac{3}{2}} du\right)^2 dv \ \leq$

$(\frac{|1-K-H||H-K|^*G_2}{|1+H-K|})^2 B(H-K+2, K-\frac{1}{2})^2 (s-d)^{-2} \frac{(t+d)^{2H+2}}{2H+2} \ \leq$

$\frac{2^* G_2^2 B(H-K+2, K-\frac{1}{2})^2}{(1+H-K)^2} (t+d)^{2H+2} s^{-2}.$

Furthermore, $|\frac{d}{du} k_t^s(u)| \ \leq$

$|K-H|(t-u)^{H-K-1}(\frac{t-u}{u+s})|\frac{d\hat{F}}{dz}(\theta_{s,u})| \ + \ (t-u)^{H-K}|\frac{d\hat{F}}{dz}(\frac{u-t}{u+s})|\frac{s+t}{(u+s)^2} \ \leq$

$(\frac{|1-K-H||H-K|^*G_2}{|1+H-K|}) \left(\frac{|K-H|}{s-d} \ + \ \frac{s+t}{(s-d)^2}\right) (t-u)^{H-K}.$

So for $K<\frac{1}{2}$ we have by using (3.2) with $n=2$ that

$\int_{\mathbb{R}} C_1(v)^2 dv \ = \ \frac{1}{(K-\frac{1}{2})^2} \int_{-d}^{t} \left(\int_{v}^{t} \frac{-d}{du} k_t^s(u)(u-v)^{K-\frac{1}{2}} du\right)^2 dv \ \leq$

$(\frac{|1-K-H||H-K|^*G_2}{|1+H-K|(K-\frac{1}{2})})^2 \left(\frac{|K-H|}{s-d} + \frac{s+t}{(s-d)^2}\right)^2 \frac{B(H-K+1, K+\frac{1}{2})^2 (t+d)^{2H+2}}{2H+2} \ \leq$

$\frac{68 \cdot {}^* G_2^2 B(H-K+1, K+\frac{1}{2})^2}{(1+H-K)^2 (K-\frac{1}{2})^2} (t+d)^{2H+2} s^{-2}.$

2. Estimation of $\int_{-d}^{0} k_0^s(v)^2 dv$, $\int_{\mathbb{R}} A_2(v)^2 dv$, $\int_{\mathbb{R}} B_2(v)^2 dv$ and $\int_{\mathbb{R}} C_2(v)^2 dv$. We obtain estimates by replacing t by 0 in the calculations of 1.

3. Estimation of $\int_{\frac{-s}{2}}^{-d} \Delta k_t^s(v)^2 dv$, $\int_{\mathbb{R}} A_3(v)^2 dv$, $\int_{\mathbb{R}} B_3(v)^2 dv$, $\int_{\mathbb{R}} C_3(v)^2 dv$, $\int_{\mathbb{R}} D(v)^2 dv$ and $\int_{\mathbb{R}} E(v)^2 dv$.

Denote

$$G_4(z) \;:=\; z^{H-K}\left(\hat{F}(-z) \;-\; 1\right).$$

For $u \in (\frac{-s}{2}, -d)$ we have

$\Delta k_t^s(u) \;=$

$(u+s)^{H-K}\left((\frac{t-u}{u+s})^{H-K}(\hat{F}(\frac{u-t}{u+s})-1) \;-\; (\frac{-u}{u+s})^{H-K}(\hat{F}(\frac{u}{u+s})-1)\right) \;=$

$(u+s)^{H-K}\left(G_4(\frac{t-u}{u+s}) \;-\; G_4(\frac{-u}{u+s})\right).$

By using (2.6) and (2.4) we obtain

$\frac{dG_4}{dz}(z) \;=$

$(H-K)\left(z^{H-K-1}F(H-K+1, 1-K-H, 1+H-K, -z) \;-\; z^{H-K-1}\right) \;=$

$(H-K)z^{H-K-1}((1+z)^{H+K-1}-1).$

By the mean value theorem there exist $\theta_{s,u} \in (\frac{-u}{u+s}, \frac{t-u}{u+s}) \subset (0, \frac{t+\frac{s}{2}}{\frac{s}{2}})$ and $\eta_{s,u} \in (0, \theta_{s,u})$ such that

$$\begin{aligned}\Delta k_t^s(u) \;&=\; (u+s)^{H-K}(\frac{t}{u+s})(H-K)\theta_{s,u}^{H-K-1}((1+\theta_{s,u})^{H+K-1}-1)\\ &=\; (u+s)^{H-K}(\frac{t}{u+s})(H-K)\theta_{s,u}^{H-K-1}(H+K-1)\theta_{s,u}(1+\eta_{s,u})^{H+K-2}.\end{aligned}$$

So $|\Delta k_t^s(u)| \leq |H-K||H+K-1|(u+s)^{H-K-1}t(\frac{-u}{u+s})^{H-K-1}\frac{t-u}{u+s}$.

In particular,

$$|\Delta k_t^s(u)| \quad\leq\quad |H-K||H+K-1|(\frac{s}{2})^{-1}t\frac{t+d}{d}(-u)^{H-K}. \tag{3.11}$$

For $K = \frac{1}{2}$ we obtain

$\int_{\frac{-s}{2}}^{-d} \Delta k_t^s(u)^2 du \;\leq\; (H-\frac{1}{2})^4 t^2(\frac{t+d}{d})^2(\frac{s}{2})^{-2}(\frac{(\frac{s}{2})^{2H}-d^{2H}}{2H}) \;\leq\; \frac{t^2}{8H}(\frac{t+d}{d})^2 s^{2H-2}.$

Also for $u \in (\frac{-s}{2}, -d)$ we have

$$|\Delta k_t^s(u)| \;\leq\; |H-K||H+K-1|t\frac{t+\frac{s}{2}}{\frac{s}{2}}(-u)^{H-K-1}. \tag{3.12}$$

Denote

$$G_5(z) \;:=\; F(\frac{3}{2}-K, H-K, H-K+1, z).$$

Let $K \neq \frac{1}{2}$. For $H > K$ and $s > \max(2t, 4d)$ we obtain by using (3.12) and (2.8) that

$$\int_{\mathbb{R}} A_3(v)^2 dv \;=\; \int_{-\infty}^{\frac{-s}{2}} \left(\int_{\frac{-s}{2}}^{-d} \Delta k_t^s(u)(u-v)^{K-\frac{3}{2}} du\right)^2 dv \;\leq$$
$$(H-K)^2 t^2 (\tfrac{t+\frac{s}{2}}{\frac{s}{2}})^2 \int_{-\infty}^{\frac{-s}{2}} \left(\int_{\frac{-s}{2}}^{-d} (-d-u)^{H-K-1}(u-v)^{K-\frac{3}{2}} du\right)^2 dv \;\leq$$
$$t^2 (\tfrac{t+\frac{s}{2}}{\frac{s}{2}})^2 G_5^{*2} \tfrac{(\frac{s}{2}-d)^{2H-2}}{2-2K} \;\leq\; \tfrac{32 G_5^{*2}}{1-K} t^2 s^{2H-2}.$$

Denote
$$G_6(z) \;:=\; F(\frac{3}{2}-K, H-K+1, H-K+2, z).$$
For $H<K$ we have by using (3.11) and (2.8) that

$$\int_{\mathbb{R}} A_3(v)^2 dv \;=\; \int_{-\infty}^{\frac{-s}{2}} \left(\int_{\frac{-s}{2}}^{-d} \Delta k_t^s(u)(u-v)^{K-\frac{3}{2}} du\right)^2 dv \;\leq$$
$$(\tfrac{s}{2})^{-2} t^2 (\tfrac{t+d}{d})^2 \int_{-\infty}^{\frac{-s}{2}} \left(\int_{\frac{-s}{2}}^{-d} (-d-u)^{H-K}(u-v)^{K-\frac{3}{2}} du\right)^2 dv \;\leq$$
$$\tfrac{t^2}{(H-K+1)^2} (\tfrac{s}{2})^{-2} (\tfrac{t+d}{d})^2 G_6^{*2} \tfrac{(\frac{s}{2}-d)^{2H}}{2-2K} \;\leq\; \tfrac{2 G_6^{*2} t^2}{(H-K+1)^2(1-K)} (\tfrac{t+d}{d})^2 s^{2H-2}.$$

Denote
$$G_7(z) \;:=\; F(2(K-H+1), 1, 2K+1, z).$$
For $K>\frac{1}{2}$ and $s>2t$ we have by using (3.12) and then (2.9) twice that

$$\int_{\mathbb{R}} B_3(v)^2 dv \;=\; \int_{\frac{-s}{2}}^{-d} \left(\int_{v}^{-d} \Delta k_t^s(u)(u-v)^{K-\frac{3}{2}} du\right)^2 dv \;\leq$$
$$t^2 (\tfrac{t+\frac{s}{2}}{\frac{s}{2}})^2 \int_{\frac{-s}{2}}^{-d} \left(\int_{v}^{-d} (-u)^{H-K-1}(u-v)^{K-\frac{3}{2}} du\right)^2 dv \;\leq$$
$$t^2 (\tfrac{t+\frac{s}{2}}{\frac{s}{2}})^2 \tfrac{G_1^{*2}}{(K-\frac{1}{2})^2} \tfrac{1}{2K} (\tfrac{s}{2}-d)^{2K} (\tfrac{s}{2})^{2(H-K-1)} G_7^* \;\leq\; \tfrac{8 G_1^{*2} G_7^*}{(K-\frac{1}{2})^2 K} t^2 s^{2H-2},$$

where G_1 is defined as in (3.5).

Let $K<\frac{1}{2}$. We have

$$\begin{aligned}\frac{d^2 G_4}{d^2 z}(z) \;&=\; (H-K-1)(H-K) z^{H-K-2}\left((1+z)^{H+K-1}-1\right)\\ &\quad + (H-K)(H+K-1) z^{H-K-1}(1+z)^{H+K-2}.\end{aligned}$$

For $u \in (\frac{-s}{2}, -d)$ there exists $\theta_{s,u} \in (\frac{-u}{u+s}, \frac{t-u}{u+s}) \subset (0, \frac{t+\frac{s}{2}}{\frac{s}{2}})$ such that

$$|\tfrac{-d}{du} \Delta k_t^s(u)| \;\leq\; |H-K|(u+s)^{H-K-1} |G_4(\tfrac{t-u}{u+s}) \;-\; G_4(\tfrac{-u}{u+s})| \;+$$
$$(u+s)^{H-K-2} t |\tfrac{dG_4}{dz}(\tfrac{t-u}{u+s})| \;+\; (u+s)^{H-K-2} s \tfrac{t}{u+s} |\tfrac{d^2 G_4}{d^2 z}(\theta_{s,v})| \;\leq$$
$$(H-K)^2 |H+K-1| \tfrac{t+\frac{s}{2}}{\frac{s}{2}} (\tfrac{s}{2})^{-1} t (-u)^{H-K-1} \;+$$

$|H-K||H+K-1|\frac{t+\frac{s}{2}}{\frac{s}{2}}(\frac{s}{2})^{-1}t(t-u)^{H-K-1} +$

$|H-K-1||H-K||H+K-1|\frac{t+\frac{s}{2}}{\frac{s}{2}}st(\frac{s}{2})^{-1}(-u)^{H-K-2} +$

$|H-K||H+K-1|s^{H+K-1}(\frac{s}{2})^{-H-K}t(-u)^{H-K-1}.$

Let

$$\begin{aligned} G_8(z) &:= F(1+K-H, K+\frac{1}{2}, K+\frac{3}{2}, z), \\ G_9(z) &:= F(2(K+1-H), 1, 2K+3, z), \\ G_{10}(z) &:= F(2+K-H, K+\frac{1}{2}, K+\frac{3}{2}, z) \end{aligned}$$

and

$$G_{11}(z) := F(2(K+2-H), 1, 2K+3, z).$$

First, for $\hat{t} \in \{0, t\}$ we have by using (2.9) twice that

$$\int_{\frac{-s}{2}}^{-d}\left(\int_v^{-d}(\hat{t}-u)^{H-K-1}(u-v)^{K-\frac{1}{2}}du\right)^2 dv \;\le\; (\tfrac{G_8^*}{K+\frac{1}{2}})^2\tfrac{G_9^*}{2K+2}(\tfrac{s}{2})^{2H}.$$

Second, in the same way we obtain

$$\int_{\frac{-s}{2}}^{-d}\left(\int_v^{-d}(-u)^{H-K-2}(u-v)^{K-\frac{1}{2}}du\right)^2 dv \;\le\; (\tfrac{G_{10}^*}{K+\frac{1}{2}})^2\tfrac{G_{11}^*}{2K+2}(\tfrac{s}{2})^{2H-2}.$$

By using (3.2) with $n=4$, we obtain for $s>2t$ that

$$\int_{\mathbb{R}} C_3(v)^2dv \;=\; \int_{\frac{-s}{2}}^{-d}\left(\tfrac{1}{K-\frac{1}{2}}\int_v^{-d}\tfrac{-d}{du}\Delta k_t^s(u)(u-v)^{K-\frac{1}{2}}du\right)^2 dv \;\le$$
$$\tfrac{4}{(K-\frac{1}{2})^2}(\tfrac{t+\frac{s}{2}}{\frac{s}{2}})^2t^2(\tfrac{s}{2})^{-2}(\tfrac{G_8^*}{K+\frac{1}{2}})^2\tfrac{G_9^*}{2K+2}(\tfrac{s}{2})^{2H} +$$
$$\tfrac{4}{(K-\frac{1}{2})^2}(\tfrac{t+\frac{s}{2}}{\frac{s}{2}})^2t^2(\tfrac{s}{2})^{-2}\tfrac{G_8^{*2}}{(K+\frac{1}{2})^2}\tfrac{G_9^*}{2K+2}(\tfrac{s}{2})^{2H} +$$
$$\tfrac{4}{(K-\frac{1}{2})^2}(H-K-1)^2(\tfrac{t+\frac{s}{2}}{\frac{s}{2}})^2s^2t^2(\tfrac{s}{2})^{-2}(\tfrac{G_{10}^*}{K+\frac{1}{2}})^2\tfrac{G_{11}^*}{2K+2}(\tfrac{s}{2})^{2H-2} +$$
$$\tfrac{4}{(K-\frac{1}{2})^2}s^{2(H+K-1)}t^2(\tfrac{s}{2})^{-2K-2H}(\tfrac{G_8^*}{K+\frac{1}{2}})^2\tfrac{G_9^*}{2K+2}(\tfrac{s}{2})^{2H} \;\le$$
$$\tfrac{(68G_8^{*2}G_9^*+512G_{10}^{*2}G_{11}^*)}{(K-\frac{1}{2})^2(K+\frac{1}{2})^2}t^2s^{2H-2}.$$

By (3.11) we have $|\Delta k_t^s(-d)| \;\le\; |H-K||H+K-1|(\frac{s}{2})^{-1}t\frac{t+d}{d}d^{H-K}$.

Hence

$$\int_{\mathbb{R}} D(v)^2dv \;\le\; \tfrac{t^2(\frac{t+d}{d})^2d^{2(H-K)}}{(K-\frac{1}{2})^2}(\tfrac{s}{2})^{-2}\tfrac{(\frac{s}{2}-d)^{2K}}{2K} \;\le\; \tfrac{2d^{2(H-K)}}{K(K-\frac{1}{2})^2}(\tfrac{t+d}{d})^2t^2s^{2K-2}.$$

From (3.12) follows $|\Delta k_t^s(\frac{-s}{2})| \;\le\; |H-K||H+K-1|t\frac{t+\frac{s}{2}}{\frac{s}{2}}(\frac{s}{2})^{H-K-1}$.
Hence for $s>2t$ holds

$\int_{\mathbb{R}} E(v)^2 dv \ \le \ (\frac{(H-K)(H+K-1)}{K-\frac{1}{2}})^2 (\frac{t+\frac{s}{2}}{\frac{s}{2}})^2 t^2 \frac{(\frac{s}{2})^{2H-2}}{2K} \ \le \ \frac{8}{K(K-\frac{1}{2})^2} t^2 s^{2H-2}.$

4. Estimation of $\int_{-s}^{\frac{-s}{2}} \Delta g_t^s(v)^2 dv$, $\int_{\mathbb{R}} A_4(v)^2 dv$, $\int_{\mathbb{R}} B_4(v)^2 dv$ and $\int_{\mathbb{R}} C_4(v)^2 dv$. Let

$$G_{12}(z) \ := \ z^{H-K} G_{13}(z) \ := \ z^{H-K} F(2H, H-K, H-K+1, z).$$

By (2.3) we have $\hat{F}(z) = (1-z)^{K-H} G_{13}(\frac{z}{z-1})$. Using (2.6) and (2.4) yields

$$\begin{aligned}\frac{d}{dz} G_{12}(z) &= (H-K) z^{H-K-1} F(2H, H-K+1, H-K+1, z)\\ &= (H-K) z^{H-K-1} (1-z)^{-2H}.\end{aligned}$$

Let $u \in (-s, \frac{-s}{2})$. There exists $\theta_{s,u} \in (\frac{-u}{s}, \frac{t-u}{s+t}) \ \subset \ (\frac{1}{2}, 1)$ such that

$\Delta g_t^s(u) = (t-u)^{H-K} (\frac{s+t}{u+s})^{K-H} G_{13}(\frac{t-u}{s+t}) \ - \ (-u)^{H-K} (\frac{s}{u+s})^{K-H} G_{13}(\frac{-u}{s}) =$

$(u+s)^{H-K} (G_{12}(\frac{t-u}{s+t}) \ - \ G_{12}(\frac{-u}{s})) \ =$

$(u+s)^{H-K} (\frac{(s+u)t}{(s+t)s})(H-K) \theta_{s,u}^{H-K-1} (1-\theta_{s,u})^{-2H}.$

So $|\Delta g_t^s(u)| \ \le \ \frac{|H-K|t}{(s+t)^{1-2H} s} (\frac{1}{2})^{H-K-1} (u+s)^{1-K-H}.$

Thus for $K = \frac{1}{2}$ and $s > t$ we have

$\int_{-s}^{\frac{-s}{2}} \Delta g_t^s(u)^2 du \ \le \ \frac{(H-\frac{1}{2})^2 t^2 (\frac{1}{2})^{2H-3}}{(s+t)^{2-4H} s^2} \frac{(\frac{s}{2})^{2-2H}}{2-2H} \ \le \ \frac{1}{1-H} t^2 s^{2H-2}.$

Denote

$$G_{14}(z) \ := \ F(\frac{3}{2} - K, 1, 3-K-H, z).$$

For $K \ne \frac{1}{2}$ and $s > t$ we obtain by using (2.8) that

$\int_{\mathbb{R}} A_4(v)^2 dv \ = \ \int_{-\infty}^{-s} \left(\int_{-s}^{\frac{-s}{2}} \Delta g_t^s(u)(u-v)^{K-\frac{3}{2}} du \right)^2 dv \ \le$

$(\frac{(H-K)t(\frac{1}{2})^{H-K-1}}{(2-K-H)(s+t)^{1-2H} s} G_{14}^*)^2 (\frac{s}{2})^{2(2-K-H)} \int_{-\infty}^{-s} (-v - \frac{s}{2})^{2K-3} dv \ =$

$(\frac{(H-K)t(\frac{1}{2})^{H-K-1}}{(2-K-H)(s+t)^{1-2H} s} G_{14}^*)^2 (\frac{s}{2})^{2-2H} \frac{1}{2-2K} \ \le \ \frac{2 G_{14}^{*2} t^2}{(1-K)(2-K-H)^2} s^{2H-2}.$

Denote

$$G_{15}(z) \ := \ F(H+K-1, 1, K+\frac{1}{2}, z).$$

Then for $K > \frac{1}{2}$ and $s > t$ we have by using (2.8) that

$\int_{\mathbb{R}} B_4(v)^2 dv \ = \ \int_{-s}^{\frac{-s}{2}} \left(\int_{v}^{\frac{-s}{2}} \Delta g_t^s(u)(u-v)^{K-\frac{3}{2}} du \right)^2 dv \ \le$

$(\frac{(H-K)t(\frac{1}{2})^{H-K-1}G_{15}^*}{(K-\frac{1}{2})(s+t)^{1-2H}s})^2(\frac{s}{2})^{2(1-K-H)}\int_{-s}^{\frac{-s}{2}}(-\frac{s}{2}-v)^{2K-1}dv \ =$

$(\frac{(H-K)t(\frac{1}{2})^{H-K-1}G_{15}^*}{(K-\frac{1}{2})(s+t)^{1-2H}s})^2(\frac{s}{2})^{2-2H}\frac{1}{2K} \ \le \ \frac{2G_{15}^{*2}}{(K-\frac{1}{2})^2K}t^2s^{2H-2}.$

Also, we have $\frac{d^2G_{12}}{d^2z}(z) \ =$

$(H-K)\left((H-K-1)z^{H-K-2}(1-z)^{-2H} \ + \ 2Hz^{H-K-1}(1-z)^{-2H-1}\right).$

So for $u\in(-s,\frac{-s}{2})$ holds

$|\frac{d}{du}\Delta g_t^s(u)| \le \ |H-K|(u+s)^{H-K-1}|G_{12}(\frac{t-u}{s+t}) \ - \ G_{12}(\frac{-u}{s})| \ +$

$(u+s)^{H-K}|\frac{dG_{12}}{dz}(\frac{t-u}{s+t})(\frac{-1}{s+t})-\frac{dG_{12}}{dz}(\frac{-u}{s})\frac{-1}{s}| \ \le$

$(H-K)^2\frac{t(\frac{1}{2})^{H-K-1}}{(s+t)^{1-2H}s}(u+s)^{-H-K} \ + \ (u+s)^{H-K}|\frac{dG_{12}}{dz}(\frac{t-u}{s+t})-\frac{dG_{12}}{dz}(\frac{-u}{s})|\frac{1}{s+t}$

$+ \ (u+s)^{H-K}|\frac{dG_{12}}{dz}(\frac{-u}{s})|\frac{t}{(s+t)s} \ \le$

$\frac{(H-K)^2t(\frac{1}{2})^{H-K-1}}{(s+t)^{1-2H}s}(u+s)^{-H-K} \ + \ \frac{|H-K||H-K-1|t(\frac{1}{2})^{H-K-2}}{s(s+t)^{2-2H}}(u+s)^{1-H-K} \ +$

$\frac{|H-K|2Ht(\frac{1}{2})^{H-K-1}}{s(s+t)^{1-2H}}(u+s)^{-H-K} \ + \ \frac{|H-K|t(\frac{1}{2})^{H-K-1}}{(s+t)s^{1-2H}}(u+s)^{-H-K} \ =$

$\left(\frac{(H-K)^2t(\frac{1}{2})^{H-K-1}}{(s+t)^{1-2H}s} \ + \ \frac{|H-K|2Ht(\frac{1}{2})^{H-K-1}}{s(s+t)^{1-2H}} \ + \ \frac{|H-K|t(\frac{1}{2})^{H-K-1}}{(s+t)s^{1-2H}}\right)(u+s)^{-H-K} \ +$
$\frac{|H-K||H-K-1|t(\frac{1}{2})^{H-K-2}}{s(s+t)^{2-2H}}(u+s)^{1-H-K}.$

Denote

$$G_{16}(z) \ := \ F(K+H,1,K+\frac{3}{2},z)$$

and

$$G_{17}(z) \ := \ F(K+H-1,1,K+\frac{3}{2},z).$$

Then for $K<\frac{1}{2}$ and $s>t$ we obtain by using (2.8) and (3.2) with $n=3$ that

$\int_{\mathbb{R}}C_4(v)^2dv \ = \ \int_{-s}^{\frac{-s}{2}}\left(\frac{1}{K-\frac{1}{2}}\int_v^{\frac{-s}{2}}\frac{-d}{du}\Delta g_t^s(u)(u-v)^{K-\frac{1}{2}}du\right)^2dv \ \le$

$(\frac{1}{K-\frac{1}{2}})^2\left(\frac{t(\frac{1}{2})^{H-K-1}}{(s+t)^{1-2H}s} \ + \ \frac{2t(\frac{1}{2})^{H-K-1}}{s(s+t)^{1-2H}} \ + \ \frac{t(\frac{1}{2})^{H-K-1}}{(s+t)s^{1-2H}}\right)^2\cdot$

$$\int_{-s}^{\frac{-s}{2}}\left(\int_v^{\frac{-s}{2}}(u+s)^{-H-K}(u-v)^{K-\frac{1}{2}}du\right)^2dv \ +$$

$(\frac{1}{K-\frac{1}{2}})^2\left(\frac{|H-K-1|t(\frac{1}{2})^{H-K-2}}{s(s+t)^{2-2H}}\right)^2\int_{-s}^{\frac{-s}{2}}\left(\int_v^{\frac{-s}{2}}(u+s)^{1-H-K}(u-v)^{K-\frac{1}{2}}du\right)^2dv \ \le$

$(\frac{1}{K-\frac{1}{2}})^2\left(\frac{t(\frac{1}{2})^{H-K-1}}{(s+t)^{1-2H}s} \ + \ \frac{2t(\frac{1}{2})^{H-K-1}}{s(s+t)^{1-2H}} \ + \ \frac{t(\frac{1}{2})^{H-K-1}}{(s+t)s^{1-2H}}\right)^2\frac{G_{16}^{*2}}{(K+\frac{1}{2})^2}\frac{(\frac{s}{2})^{2-2H}}{2K+2} \ +$

$(\frac{1}{K-\frac{1}{2}})^2\left(\frac{|H-K-1|t(\frac{1}{2})^{H-K-2}}{s(s+t)^{2-2H}}\right)^2\frac{G_{17}^{*2}}{(K+\frac{1}{2})^2}\frac{(\frac{s}{2})^{4-2H}}{2K+2} \ \le \ \frac{(63G_{16}^{*2}+4G_{17}^{*2})}{(K-\frac{1}{2})^2(K+\frac{1}{2})^2}t^2s^{2H-2}.$

5. Estimation of $\int_{-s}^{\frac{-s}{2}} \Delta h_t^s(v)^2 dv$, $\int_{\mathbb{R}} A_5(v)^2 dv$, $\int_{\mathbb{R}} B_5(v)^2 dv$ and $\int_{\mathbb{R}} C_5(v)^2 dv$.

For $u \in (-s, \frac{-s}{2})$ we have $|\Delta h_t^s(u)| \ \leq \ |H-K| t (\frac{s}{2})^{H-K-1}$.

So for $K = \frac{1}{2}$ follows

$\int_{-s}^{\frac{-s}{2}} \Delta h_t^s(u)^2 du \ \leq \ (H-\frac{1}{2})^2 t^2 (\frac{s}{2})^{2H-2} \ \leq t^2 s^{2H-2}$.

Let

$$G_{18}(z) \ := \ F(\frac{3}{2}-K, 1, 2, z).$$

For $K \neq \frac{1}{2}$, by using (2.8) we obtain

$\int_{\mathbb{R}} A_5(v)^2 dv \ \leq \ \frac{(H-K)^2}{2-2K} t^2 G_{18}^{*2} (\frac{s}{2})^{2H-2} \ \leq \ \frac{2G_{18}^{*2}}{1-K} t^2 s^{2H-2}$.

For $K > \frac{1}{2}$ we have

$\int_{\mathbb{R}} B_5(v)^2 dv \ \leq \ (H-K)^2 t^2 \frac{B(K-\frac{1}{2},1)^2}{2K} (\frac{s}{2})^{2H-2} \ \leq \ \frac{2t^2}{K(K-\frac{1}{2})^2} s^{2H-2}$.

For $u \in (-s, \frac{-s}{2})$ we have $|\frac{d}{du} \Delta h_t^s(u)| \leq |H-K||H-K-1| t (\frac{s}{2})^{H-K-2}$.

So for $K < \frac{1}{2}$ holds

$\int_{\mathbb{R}} C_5(v)^2 dv \ \leq \ (\frac{H-K-1}{K-\frac{1}{2}})^2 t^2 \frac{B(1,K+\frac{1}{2})^2}{2K+2} (\frac{s}{2})^{2H-2} \ \leq \ \frac{8\Gamma(K-\frac{1}{2})^2}{\Gamma(K+\frac{3}{2})^2} t^2 s^{2H-2}$.

By combining (3.8) and (3.9), respectively, with these estimates, we obtain (3.6) where

$$c_2(\frac{1}{2}, H, t, d) \ = \ 2 \cdot^* G_2^2 (t+d)^{2H+2} \ + \ \left(\frac{(t+d)^2}{4Hd^2} + \frac{2}{1-H} + 2 \right) t^2$$

and

$$c_2(K,H,t,d) = \left(\frac{20 \cdot^* G_2^2 G_3^{*2}}{\Gamma(K-\frac{1}{2})^2 (1+H-K)^2 (1-K)} + \frac{20 \cdot^* G_2^2 \Gamma(H-K+1)^2}{\Gamma(H+\frac{3}{2})^2} \right) (t+d)^{2H+2} \ +$$
$$\left(\frac{\max(160 G_5^{*2}, \frac{10 G_6^{*2} (t+d)^2}{(H-K+1)^2 d^2}) + 10 G_{18}^{*2} + \frac{10 G_{14}^{*2}}{(2-K-H)^2}}{(1-K)\Gamma(K-\frac{1}{2})^2} \ + \ \frac{10 G_{15}^{*2} + 40 G_1^{*2} G_7^* + 10}{K \Gamma(K+\frac{1}{2})^2} \right) t^2, \quad K > \frac{1}{2}.$$

In the same way, by combing (3.10) with the estimates, we obtain (3.7) with

$$c_3(K,H,t,d) = \left(\frac{24 \cdot^* G_2^2 G_3^{*2}}{\Gamma(K-\frac{1}{2})^2 (1+H-K)^2 (1-K)} + \frac{816 \cdot^* G_2^2 \Gamma(H-K+1)^2}{(1+H-K)^2 \Gamma(H+\frac{3}{2})^2} \right) (t+d)^{2H+2} \ +$$
$$\left(\frac{\max(192 G_5^{*2}, \frac{12 G_6^{*2} (t+d)^2}{(H-K+1)^2 d^2}) \ + \ \frac{12 G_{14}^{*2}}{(2-H-K)^2} + 12 G_{18}^{*2}}{\Gamma(K-\frac{1}{2})^2 (1-K)} \right) t^2 \ + \ \frac{48}{K \Gamma(K+\frac{1}{2})^2} t^2 \ +$$

$$\left(\frac{408G_8^{*2}G_9^* + 3072G_{10}^{*2}G_{11}^* + 378G_{16}^{*2} + 24G_{17}^{*2} + 48}{\Gamma(K+\frac{3}{2})^2}\right)t^2$$

and

$$c_4(K,H,t,d) \ = \frac{12d^{2(H-K-1)}(t+d)^2}{\Gamma(K+\frac{1}{2})^2K}t^2.$$

Eventually, by combining (3.1) and (3.3) with the results in **1.** and **2.**, we obtain the following:

For $K = \frac{1}{2}$ we have

$$\left(\frac{\Gamma(H+\frac{1}{2})}{C(H)}\right)^2 E[Z_t^{H,s} - Z_t^H]^2 \ \leq \ (c_1(\frac{1}{2},H,t) + c_2(\frac{1}{2},H,t,d))s^{2H-2}, \quad s > 2t+4d+1.$$

For $K > \frac{1}{2}$ we have

$$\frac{\Gamma(H-K+1)^2}{2C(H)^2}E[Z_t^{H,s} - Z_t^H]^2 \ \leq \ (c_1(K,H,t) + c_2(K,H,t,d))s^{2H-2}, \quad s > 2t+4d+1.$$

For $K < \frac{1}{2}$ holds

$$\frac{\Gamma(H-K+1)^2}{2C(H)^2}\cdot E[Z_t^{H,s} - Z_t^H]^2 \ \leq \ (c_1(K,H,t) \ + \ c_3(K,H,t,d))s^{2H-2} \ + \ c_4(K,H,t,d)s^{2K-2}, \ s > 2t+4d+1.$$

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