Dirichlet problem at infinity for $\mathcal{A}$-harmonic functions*

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Abstract

We study the Dirichlet problem at infinity for $\mathcal{A}$-harmonic functions on a Cartan-Hadamard manifold M and give a sufficient condition for a point at infinity $x_0 \in M(\infty)$ to be $\mathcal{A}$-regular. This condition is local in the sense that it only involves sectional curvatures of M in a set $U \cap M$, where U is an arbitrary neighborhood of x_0 in the cone topology. The results apply to the Laplacian and p-Laplacian, $1 < p < \infty$, as special cases.

1 Introduction

Throughout this paper M is a Cartan-Hadamard n-manifold, $n \geq 2$, and $o \in M$ is fixed. Recall that a Cartan-Hadamard manifold is a complete, simply connected Riemannian manifold with non-positive sectional curvatures. We denote $\rho = d(\cdot, o)$. In this paper c will denote an arbitrary positive constant that may vary even within a line.

We are interested in the following question: Are there bounded non-constant harmonic functions on M? For example, if $M = \mathbb{R}^n$, the answer is no by Liouville's theorem but if M is the Poincaré disk, then the answer is yes. Greene and Wu [5] conjectured that M has a non-constant bounded harmonic function if the sectional curvatures of M satisfy

$$K_M \leq -\frac{A}{\rho^2}$$

for some constant $A > 0$ outside a compact set. This is still open in general but has been verified in the case $n = 2$ (see [12]).

Let us recall the definition of cone topology. For details and proofs, see [4]. We say that two unit speed geodesics $\gamma, \sigma : \mathbb{R} \to M$ are *asymptotic* if $\sup_{t \geq 0} d\big(\gamma(t), \sigma(t)\big) < \infty$. This defines an equivalence relation. Denote the equivalence class of γ by $\gamma(\infty)$ and the set of all equivalence classes by $M(\infty)$. We call elements of $M(\infty)$ *points at infinity* and denote $\bar{M} = M \cup M(\infty)$. For every $x \in M$ and $y \in \bar{M} \setminus \{x\}$ there exists a unique unit speed geodesic $\gamma^{x,y}$ such that $\gamma^{x,y}(0) = x$ and $y \in \gamma^{x,y}(0, \infty]$. Given $x \in M$, $v \in T_xM \setminus \{0\}$, $\delta > 0$, and $r > 0$, we define a *cone*

$$C(v, \delta) = \{y \in \bar{M} \setminus \{x\} : \sphericalangle(v, \dot{\gamma}_0^{x,y}) < \delta\}$$

and a *truncated cone*

$$T(v, \delta, r) = C(v, \delta) \setminus \bar{B}(x, r).$$

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It can be shown that the collection {open balls} $\cup$ {cones} is a basis for a topology on $\bar{M}$. This is called the *cone topology*. Equipping $\bar{M}$ with the cone topology, $\bar{M}$ is homeomorphic to the closed unit ball $\bar{B}(0,1)$ in $\mathbb{R}^n$ and $M(\infty)$ is mapped onto $\mathbb{S}^{n-1}$ in this homeomorphism. From now on we always equip $\bar{M}$ with the cone topology.

A function $u \in W^{1,p}_{\text{loc}}(M)$ is $\mathcal{A}$-harmonic if it is continuous and a weak solution of the equation

$$-\operatorname{div} \mathcal{A}(\nabla u) = 0.$$

Here $\langle \mathcal{A}(\nabla u), \nabla u \rangle \approx |\nabla u|^p$ and $1 < p < \infty$. See Section 2 for precise assumptions on $\mathcal{A}$. We say that the *Dirichlet problem at infinity* (or *asymptotic Dirichlet problem*) for $\mathcal{A}$-harmonic functions (or the operator $\mathcal{A}$) is solvable if for every $f \in C(M(\infty))$ there exists a function $u \in C(\bar{M})$ such that $u|M$ is $\mathcal{A}$-harmonic and $u|M(\infty) = f$. It is easy to see that such u is always unique by compactness of $\bar{M}$. Solvability of this Dirichlet problem implies that M has a lot of bounded non-constant $\mathcal{A}$-harmonic functions. Therefore one way to approach the Greene-Wu conjecture is to consider the Dirichlet problem at infinity for the Laplacian.

The Dirichlet problem at infinity for the Laplacian has been solved under various assumptions on the manifold, see Section 1 in [10] for the history of the problem. In the case of the p-Laplacian, Pansu [13] showed the existence of non-constant bounded p-harmonic functions with finite p-energy on Cartan-Hadamard manifolds of pinched curvature $-b^2 \le K_M \le -a^2$ for $p > (n-1)b/a$. In [8] Holopainen solved the Dirichlet problem at infinity for the p-Laplacian on Cartan-Hadamard manifolds of pinched curvature. This result was generalized in [10] to the setting of Gromov hyperbolic metric measure spaces.

In this paper we consider the Dirichlet problem at infinity for a general operator $\mathcal{A}$. Associated with the Dirichlet problem at infinity are Perron's method and $\mathcal{A}$-regular points at infinity. In order to show that a given $x_0 \in M(\infty)$ is $\mathcal{A}$-regular, we construct a barrier-like function w. In fact, in the case of the Laplacian, we construct for any given $\delta > 0$ a function $-w$ that is a barrier at $\dot{\gamma}_0^{o,x_0}$ with angle δ using Choi's terminology in [3]. The problem is to verify that $w(x) \to 0$ as $x \to x_0$. For this we use Cheng's ideas from [2] adapted into our setting.

2 Preliminaries

In this section we define $\mathcal{A}$-harmonic functions and $\mathcal{A}$-regular points at infinity.

Let N be a Riemannian manifold and $1 < p < \infty$. Suppose that $\mathcal{A}\colon TN \to TN$ is an operator that satisfies the following assumptions for some $0 < \alpha \le \beta < \infty$: the mapping $\mathcal{A}_x = \mathcal{A}|T_xN \colon T_xN \to T_xN$ is continuous for almost every $x \in N$ and the mapping $x \mapsto \mathcal{A}_x(V_x)$ is measurable for all measurable vectorfields V on M; for almost every $x \in N$ and every $v \in T_xN$:

$$\langle \mathcal{A}_x(v), v \rangle \ge \alpha |v|^p,$$

$$|\mathcal{A}_x(v)| \le \beta |v|^{p-1},$$

$$\big\langle \mathcal{A}_x(v) - \mathcal{A}_x(w), v - w \big\rangle > 0,$$

whenever $w \in T_xN \setminus \{v\}$, and

$$\mathcal{A}_x(\lambda v) = \lambda |\lambda|^{p-2} \mathcal{A}_x(v)$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$. We denote the set of all such operators by $\mathcal{A}^p(N)$. The numbers α and β are called *structure constants* of $\mathcal{A}$.

Suppose that $U \subset N$ is an open set and $\mathcal{A} \in \mathcal{A}^p(N)$. A function $u \in C(U) \cap W^{1,p}_{\text{loc}}(U)$ is *$\mathcal{A}$-harmonic* in U if it is a weak solution of the equation

$$-\operatorname{div} \mathcal{A}(\nabla u) = 0, \tag{2.1}$$

in other words, if

$$\int_U \langle \mathcal{A}(\nabla u), \nabla \varphi \rangle = 0 \tag{2.2}$$

for every test function $\varphi \in C_0^\infty(U)$. If $|\nabla u| \in L^p(U)$, then it is equivalent to require (2.2) for all $\varphi \in W_0^{1,p}(U)$ by approximation.

A lower semicontinuous function $u : U \to (-\infty, \infty]$ is $\mathcal{A}$-*superharmonic* if $u \not\equiv \infty$ in each component of U, and for each open $D \subset\subset U$ and each $h \in C(\bar{D})$, $\mathcal{A}$-harmonic in D, $h \le u$ on ∂D implies $h \le u$ in D.

In the case of the *p-Laplacian*

$$\mathcal{A}(v) = |v|^{p-2} v,$$

the continuous weak solutions of (2.1) are called *p-harmonic functions*. In this case $\alpha = \beta = 1$. A function $u \in C(U) \cap W^{1,2}_{\mathrm{loc}}(U)$ is 2-harmonic if and only if it belongs to $C^\infty(U)$ and $\Delta u \equiv 0$ in U, i.e. u is harmonic in the usual sense.

The $\mathcal{A}$-harmonic functions have many features in common with harmonic functions. See [6] for properties and theory of $\mathcal{A}$-harmonic and $\mathcal{A}$-superharmonic functions in $\mathbb{R}^n$.

2.3 Perron's method and regular points at infinity

We approach the Dirichlet problem at infinity using Perron's method. Our definitions of the upper and lower Perron solutions follow [6]. Fix $p \in (1, \infty)$ and $\mathcal{A} \in \mathcal{A}^p(M)$.

2.4 Definition. A function $u : M \to (-\infty, \infty]$ belongs to the *upper class* $\mathcal{U}_f$ of $f : M(\infty) \to [-\infty, \infty]$ if

(i) u is $\mathcal{A}$-superharmonic in M,

(ii) u is bounded below, and

(iii) $\liminf_{x \to x_0} u(x) \ge f(x_0)$ for all $x_0 \in M(\infty)$.

The function

$$\overline{H}_f = \inf\{u : u \in \mathcal{U}_f\}$$

is called the *upper Perron solution.*

2.5 Theorem. *One of the following is true:*

(i) $\overline{H}_f$ *is* $\mathcal{A}$*-harmonic in* M*,*

(ii) $\overline{H}_f \equiv \infty$ *in* M*,*

(iii) $\overline{H}_f \equiv -\infty$ *in* M*.*

Proof. As in [6, Theorem 9.2]. □

Note that if f is bounded, then also $\overline{H}_f$ is bounded and by Theorem 2.5 it is then $\mathcal{A}$-harmonic in M. The upper Perron solution is a good candidate to the solution of the Dirichlet problem at infinity.

2.6 Definition. A point $x_0 \in M(\infty)$ is $\mathcal{A}$-*regular*, if

$$\lim_{x \to x_0} \overline{H}_f(x) = f(x_0)$$

for each continuous $f : M(\infty) \to \mathbb{R}$.

Define the *lower class* $\mathcal{L}_f = -\mathcal{U}_{-f}$ and the *lower Perron solution* $\underline{H}_f = -\overline{H}_{-f}$. It then holds that $\overline{H}_f \geq \underline{H}_f$.

2.7 Remark. If two points $x_1, x_2 \in M(\infty)$, $x_1 \neq x_2$, are $\mathcal{A}$-regular, then there exists a non-constant bounded $\mathcal{A}$-harmonic function on M. On the other hand, the Dirichlet problem at infinity for $\mathcal{A}$-harmonic functions is solvable if and only if all points at infinity are $\mathcal{A}$-regular.

3 Estimates involving Jacobi fields

In this section we establish some elementary geometric estimates that we will need later on.

If $K : [0,\infty) \to (-\infty, 0]$ is a smooth function, let $F_K \in C^\infty\big([0,\infty)\big)$ be the solution to the initial value problem

$$\begin{cases} F_K(0) = 0, \\ F_K'(0) = 1, \\ F_K'' + K F_K = 0. \end{cases}$$

3.1 Example. Suppose that $\phi > 1$ and $t_0 > 0$ are constants and $K : [0,\infty) \to (-\infty, 0]$ is a smooth function such that $K(t) = -\phi(\phi-1)/t^2$ when $t \geq t_0$. It is easy to verify that then

$$F_K(t) = c_1 t^\phi + c_2 t^{1-\phi}$$

for all $t \geq t_0$, where

$$c_1 = t_0^{-\phi} \frac{F_K(t_0)(\phi-1) + t_0 F_K'(t_0)}{2\phi - 1}$$

and

$$c_2 = t_0^{\phi-1} \frac{F_K(t_0)\phi - t_0 F_K'(t_0)}{2\phi - 1}.$$

3.2 Lemma. *Let $k, K : [0,\infty) \to (-\infty, 0]$ be smooth functions that are constant in some neighborhood of 0. Suppose that $v \in T_oM$ is a unit vector and $\gamma = \gamma^v : \mathbb{R} \to M$ is the unit speed geodesic with $\dot\gamma_0 = v$. Suppose that for every $t > 0$ we have*

$$k(t) \leq K_M(P) \leq K(t)$$

for every 2-dimensional subspace $P \subset T_{\gamma(t)}M$ that contains the radial vector $\dot\gamma_t$.

(a) If W is a Jacobi field along γ with $W_0 = 0$, $|W_0'| = 1$, and $W_0' \perp v$, then

$$F_K(t) \leq |W(t)| \leq F_k(t)$$

for every $t \geq 0$.

(b) For every $t > 0$ we have

$$(n-1)\frac{F_K'(t)}{F_K(t)} \leq \Delta\rho\big(\gamma(t)\big) \leq (n-1)\frac{F_k'(t)}{F_k(t)}.$$

Proof. Let M_K be $\mathbb{R}^n$ equipped with the Riemannian metric $dr^2 + F_K(r)^2 d\theta^2$, where r is the distance function from 0 and $d\theta^2$ is the standard metric on $\mathbb{S}^{n-1}$. Note that since K is constant in a neighborhood of 0, the metric $dr^2 + F_K(r)^2 d\theta^2$ extends smoothly over 0. Now M_K is a rotationally symmetric manifold with radial curvature function K. Similarly, let M_k be $\mathbb{R}^n$ equipped with the the metric $dr^2 + F_k(r)^2 d\theta^2$.

(a) Let $\tilde v \in T_0 M_K$, $|\tilde v| = 1$, and let $\tilde W$ be a Jacobi field along the unit speed geodesic $\gamma^{\tilde v}$ with $\tilde W_0 = 0$, $\tilde W_0' \perp \tilde v$, and $|\tilde W_0'| = 1$. Then we then have that

$$|\tilde W(t)| = F_K(t)$$

for every $t \geq 0$. Applying the Rauch comparison theorem shows that $|W(t)| \geq |\tilde{W}(t)| = F_K(t)$ for every $t \geq 0$. The other inequality follows similarly but using k instead of K.

(b) On the manifold M_K we have that

$$\Delta r = (n-1)\frac{F_K' \circ r}{F_K \circ r}$$

in $M_K \setminus \{0\}$ by [5, Proposition 2.20]. It follows from the Hessian comparison theorem [5, Theorem A] that $\Delta\rho(\gamma(t)) \geq \Delta r(\tilde{\gamma}(t)) = (n-1)F_K'(t)/F_K(t)$ for every $t > 0$. The other inequality follows similarly but using k instead of K. □

Write $S_oM = \{v \in T_oM : |v| = 1\}$ and define $\varphi : (0,\infty) \times S_oM \to M \setminus \{o\}$,

$$\varphi(r,\xi) = \exp_o(r\xi).$$

We denote $\lambda_M = |J_\varphi|$, the absolute value of the Jacobian. Let $\xi \in S_oM$ and $W_1, W_2, \ldots, W_{n-1}$ be Jacobi fields along the geodesic $\gamma : t \mapsto \exp_o(\xi t)$ such that $W_i(0) = 0$ for every i and $\big(W_1'(0), \ldots, W_{n-1}'(0), \dot{\gamma}_0\big)$ is an orthonormal basis of T_oM. Then

$$\lambda_M(r,\xi) = \big|\det\big(W_1(r), \ldots, W_{n-1}(r), \dot{\gamma}_r\big)\big|, \tag{3.3}$$

where the determinant is taken with respect to any orthonormal basis of $T_{\gamma(r)}M$. So, if the the radial curvatures along the geodesic $\varphi(\cdot,\xi)$ are bounded from below by $k \circ \rho$, we have

$$\lambda_M(r,\xi) \leq F_k(r)^{n-1} \tag{3.4}$$

by (3.3) and Lemma 3.2(a).

3.5 Lemma. *Let $x_0 \in M \setminus \{o\}$, $U = M \setminus \gamma^{o,x_0}(\mathbb{R})$, and define $\theta : U \to [0,\pi]$, $\theta(x) = \sphericalangle_o(x_0, x) := \arccos\langle \dot{\gamma}_0^{o,x_0}, \dot{\gamma}_0^{o,x}\rangle$. Let $x \in U$ and $\gamma = \gamma^{o,x}$. Then*

$$|\nabla\theta(x)| \leq \frac{1}{j\big(\rho(x)\big)},$$

where $j(t) = \inf\big\{|W(t)| : W$ is a Jacobi field along γ with $W(0) = 0$, $W_0' \perp \dot{\gamma}_0$, and $|W_0'| = 1\big\}$.

Proof. Let $\varphi : M \to \mathbb{R}^n$ be a normal chart at o. Let $X \in T_xM$ be a unit vector. We want to prove that $|X\theta| \leq 1/j(\rho(x))$. If $X = \dot{\gamma}_{\rho(x)}$, then $X\theta = 0$ so without loss of generality we can assume that $X \perp \dot{\gamma}_{\rho(x)}$. Now

$$\theta = \tilde{\theta} \circ \varphi | U,$$

where $\tilde{\theta} : \varphi U \to [0,\pi]$,

$$\tilde{\theta}(z) = \sphericalangle_0\big(\varphi(x_0), z\big) = \arccos\Big(\frac{v \cdot z}{|z|}\Big),$$

and $v = \frac{\varphi(x_0)}{|\varphi(x_0)|}$. We see that $|\nabla\tilde{\theta}(z)| = |z|^{-1}$.

Let $w = \frac{\varphi(x)}{|\varphi(x)|}$ and define

$$W_t = (\varphi^{-1})_{*tw}\big(t\tfrac{\varphi_* X}{|\varphi_* X|}\big).$$

Then W is a Jacobi field along the geodesic γ with $W_0 = 0$, $|W_0'| = 1$, and $W_{\rho(x)} = \rho(x)\frac{X}{|\varphi_* X|}$. Since $W_0 = 0$ and $W_{\rho(x)} \perp \dot{\gamma}_{\rho(x)}$, also $W \perp \dot{\gamma}$. Now

$$|X\theta| = |X(\tilde{\theta} \circ \varphi)| = \big|\nabla\tilde{\theta}\big(\varphi(x)\big) \cdot \varphi_* X\big| \leq \frac{|\varphi_* X|}{\rho(x)} = \frac{1}{|W_{\rho(x)}|} \leq \frac{1}{j\big(\rho(x)\big)}$$

as we wanted. □

The next lemma is a modification of [2, Lemma 3.2].

3.6 Lemma. *Let $v \in T_oM$ be a unit vector and $\gamma = \gamma^v$. Suppose that $r_0 > 0$ and $k < 0$ are constants such that $K_M(P) \ge k$ for every 2-dimensional subspace $P \subset T_xM$, $x \in B(o, r_0)$. Suppose also that there exists a constant $C \ge 1$ such that*

$$|K_M(P)| \le C|K_M(P')|$$

whenever $t \ge r_0$ and $P, P' \subset T_{\gamma(t)}M$ are 2-dimensional subspaces containing the radial vector $\dot\gamma_t$. Let V and $\bar V$ be two Jacobi fields along γ such that $V_0 = 0 = \bar V_0$, $V_0' \perp \dot\gamma_0 \perp \bar V_0'$, and $|V_0'| = 1 = |\bar V_0'|$. Then there exists a constant $c_0 = c_0(C, r_0, k) > 0$ such that

$$|V_r|^C \ge c_0 |\bar V_r|$$

for every $r \ge r_0$.

Proof. Let $t_1 \ge r_0$. Denote $W = V/|V_{t_1}|$. As in the proof of [2, Lemma 3.2], there exist plane sections σ and $\hat\sigma$ along γ containing the radial vectors such that

$$\frac{1}{2}\Big(\log \frac{|V|^{2C}}{|\bar V|^2}\Big)'(t_1) \ge \int_0^{t_1} \big(K_M(\hat\sigma) - CK_M(\sigma)\big)|W|^2.$$

We continue from this and estimate

$$\int_0^{t_1} \big(K_M(\hat\sigma) - CK_M(\sigma)\big)|W|^2 \ge k \int_0^{r_0} |W|^2.$$

We note that if $0 \le t \le r_0$, then the Rauch comparison theorem implies that

$$|W_t| = \frac{|V_t|}{|V_{t_1}|} \le \frac{\sinh\big(r_0\sqrt{-k}\big)/\sqrt{-k}}{t_1}.$$

Combining the above yields

$$\Big(\log \frac{|V|^{2C}}{|\bar V|^2}\Big)'(t_1) \ge -c_1/t_1^2.$$

where $c_1 = c_1(r_0, k) > 0$. We integrate with respect to t_1 and get

$$\Big(\log \frac{|V|^{2C}}{|\bar V|^2}\Big)(r) \ge \Big(\log \frac{|V|^{2C}}{|\bar V|^2}\Big)(r_0) - \int_{r_0}^{r} \frac{c_1}{t^2}\, dt$$

for every $r \ge r_0$. The Rauch comparison theorem implies that the right hand side is bounded from below by a constant $c_2 = c_2(C, r_0, k) \in \mathbb{R}$ and the claim follows. □

4 The main theorem

Fix $1 < p < \infty$ and $\mathcal{A} \in \mathcal{A}^p(M)$. We are ready to formulate our main result.

4.1 Theorem. *Let $x_0 \in M(\infty)$ and $\phi > 1$. Suppose that x_0 has a neighborhood U (in the cone topology) such that*

$$K_M(P) \le -\frac{\phi(\phi-1)}{\rho(x)^2} \tag{4.2}$$

for every $x \in U \cap M$ and every 2-dimensional subspace $P \subset T_xM$ that contains the radial vector $\nabla\rho(x)$. Suppose also that there exists a constant $C \ge 1$ such that

$$|K_M(P)| \le C|K_M(P')| \tag{4.3}$$

whenever $x \in U \cap M$ and $P, P' \subset T_xM$ are 2-dimensional subspaces containing $\nabla\rho(x)$. Suppose that

$$1 < p < \frac{\alpha}{\beta}\big(1 + (n-1)\phi\big), \tag{4.4}$$

where α and β are the structure constants of $\mathcal{A}$. Then x_0 is $\mathcal{A}$-regular.

When $n = 2$ the condition (4.3) is trivially valid and we immediately obtain the following special case.

4.5 Corollary. *Suppose that $n = 2$ and that there exists a constant $\phi > 1$ such that*

$$K_M \le -\frac{\phi(\phi - 1)}{\rho^2}$$

outside a compact set. Suppose also that

$$1 < p < \frac{\alpha}{\beta}(1 + \phi).$$

Then the Dirichlet problem at infinity is solvable for $\mathcal{A}$-harmonic functions.

The proof of Theorem 4.1 is a variant of the proof of [2, Theorem 3.1]. We start with the following lemma, which is our main tool in using $\mathcal{A}$-harmonicity.

4.6 Lemma. *Let $\eta : M \to \mathbb{R}$ be a non-negative Lipschitz function and $\theta \in C(M) \cap W^{1,p}_{\text{loc}}(M)$. Suppose that $\Omega \subset\subset M$ is an open set and u is a bounded $\mathcal{A}$-harmonic function in Ω with $|\nabla u| \in L^p(\Omega)$. Denote $h = |u - \theta|$. Suppose that $q \ge p$ and that $\eta^p h^{q-p}(u - \theta) \in W^{1,p}_0(\Omega)$. Then*

$$\Big(\int_\Omega \eta^p h^{q-p} |\nabla h|^p\Big)^{1/p} \le \Big(1 + \frac{\beta}{\alpha}\Big)\Big(\int_\Omega \eta^p h^{q-p}|\nabla\theta|^p\Big)^{1/p} + \frac{p\beta}{(q-p+1)\alpha}\Big(\int_\Omega h^q |\nabla\eta|^p\Big)^{1/p}.$$

Proof. Denote

$$\varphi = \eta^p h^{q-p}(u - \theta).$$

Then $\varphi \in W^{1,p}_0(\Omega)$ by assumption and

$$\nabla\varphi = (q - p + 1)\eta^p h^{q-p}\nabla(u - \theta) + p h^{q-p}(u - \theta)\eta^{p-1}\nabla\eta.$$

Testing the $\mathcal{A}$-harmonicity of u with the test function φ gives

$$\begin{aligned}
\int_\Omega \eta^p h^{q-p}|\nabla u|^p &\le \frac{1}{\alpha}\int_\Omega \big\langle \mathcal{A}(\nabla u), \eta^p h^{q-p}\nabla u\big\rangle \\
&= \frac{1}{\alpha}\int_\Omega \big\langle \mathcal{A}(\nabla u), \eta^p h^{q-p}\nabla\theta\big\rangle - \frac{p}{(q-p+1)\alpha}\int_\Omega \big\langle \mathcal{A}(\nabla u), h^{q-p}(u-\theta)\eta^{p-1}\nabla\eta\big\rangle \\
&\le \frac{\beta}{\alpha}\int_\Omega \eta^p h^{q-p}|\nabla u|^{p-1}|\nabla\theta| + \frac{p\beta}{(q-p+1)\alpha}\int_\Omega h^{q-p+1}|\nabla u|^{p-1}\eta^{p-1}|\nabla\eta| \\
&\le \frac{\beta}{\alpha}\Big(\int_\Omega \eta^p h^{q-p}|\nabla u|^p\Big)^{(p-1)/p}\Big(\int_\Omega \eta^p h^{q-p}|\nabla\theta|^p\Big)^{1/p} \\
&\quad + \frac{p\beta}{(q-p+1)\alpha}\Big(\int_\Omega \eta^p h^{q-p}|\nabla u|^p\Big)^{(p-1)/p}\Big(\int_\Omega h^q|\nabla\eta|^p\Big)^{1/p}.
\end{aligned}$$

The claim follows now from Minkowski's inequality. □

The next lemma corresponds to [2, Proposition 1.1] but instead of assuming $\lambda_1(M) > 0$, we assume that the radial curvatures have a weak upper bound.

4.7 Lemma. *Assume that* $\phi > 1$ *and*

$$1 < p < \frac{\alpha}{\beta}\big(1 + (n-1)\phi\big).$$

Let $v \in T_oM \setminus \{0\}$, $\delta > 0$, *and* $r_0 > 0$ *be such that*

$$K_M(P) \le -\frac{\phi(\phi-1)}{\rho(x)^2}$$

whenever $x \in T(v,\delta,r_0) \cap M$ *and* $P \subset T_xM$ *is a 2-dimensional subspace containing the radial vector* $\nabla\rho(x)$. *Then there exist constants* $r_1 = r_1(\phi,p,n,\beta/\alpha,r_0) > r_0$, $c_0 = c_0(\phi,p,n,\beta/\alpha) > 0$, *and* $q_0 = q_0(\phi,p,n,\beta/\alpha) > p$ *such that if* $\theta \in C(M) \cap W^{1,p}_{\text{loc}}(M)$, $R > r \ge r_1$, $\Omega = T(v,\delta,r) \cap B(o,R)$, *and* u *is the unique* $\mathcal{A}$*-harmonic function in* Ω *that satisfies* $u - \theta \in W^{1,p}_0(\Omega)$, *then*

$$\int_\Omega |u-\theta|^q \le (c_0 q)^q \int_\Omega \rho^q |\nabla\theta|^q$$

whenever $q \ge q_0$.

Proof. Choose $\varepsilon > 0$ such that

$$2(1+\varepsilon)^2 = 1 + \frac{1+(n-1)\phi}{p\beta/\alpha}$$

and choose

$$q_0 = \frac{1+\varepsilon}{\varepsilon} p.$$

Let $K\colon [0,\infty) \to (-\infty, 0]$ be a smooth function such that $K(t) = 0$ for $t \in [0, r_0]$, $K(t) \ge -\phi(\phi-1)/t^2$ for $t \in [r_0, r_0+1]$, and $K(t) = -\phi(\phi-1)/t^2$ for $t \ge r_0+1$. Then the radial curvatures in $C(v,\delta)\cap M$ are bounded from above by $K \circ \rho$. By Lemma 3.2(b) we get

$$\Delta\rho \ge (n-1)\frac{F_K' \circ \rho}{F_K \circ \rho}$$

in $C(v,\delta) \cap M$. By Example 3.1 there exist constants $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that $F_K(t) = c_1 t^\phi + c_2 t^{1-\phi}$ for all $t \ge r_0 + 1$. We conclude that there exists a constant $r_1 \ge r_0 + 1$ such that

$$\Delta\rho \ge \frac{(n-1)\phi}{(1+\varepsilon)\rho} \tag{4.8}$$

in $T(v,\delta,r_1) \cap M$.

Fix an arbitrary $q \ge q_0$. Fix $\theta \in C(M) \cap W^{1,p}_{\text{loc}}(M)$ and $R > r \ge r_1$. Denote $\Omega = T(v,\delta,r) \cap B(o,R)$. Let u be the $\mathcal{A}$-harmonic function in Ω that satisfies $u - \theta \in W^{1,p}_0(\Omega)$. Denote $h = |u-\theta|$. Then $u \in W^{1,p}(\Omega)$ and $\rho^p h^{q-p}(u-\theta) \in W^{1,p}_0(\Omega)$ so that we can apply Lemma 4.6 with $\eta = \rho$ to obtain

$$\Big(\int_\Omega \rho^p h^{q-p} |\nabla h|^p\Big)^{1/p} \le \Big(1 + \frac{\beta}{\alpha}\Big)\Big(\int_\Omega \rho^p h^{q-p}|\nabla\theta|^p\Big)^{1/p} + \frac{p\beta}{(q-p+1)\alpha}\Big(\int_\Omega h^q |\nabla\rho|^p\Big)^{1/p}. \tag{4.9}$$

We use (4.8) and Green's formula to estimate

$$\begin{aligned}\int_\Omega h^q &\le \frac{1+\varepsilon}{(n-1)\phi}\int_\Omega h^q \rho\Delta\rho = -\frac{1+\varepsilon}{(n-1)\phi}\int_\Omega \langle \nabla(h^q\rho), \nabla\rho\rangle \\ &\le -\frac{1+\varepsilon}{(n-1)\phi}\int_\Omega h^q + \frac{(1+\varepsilon)q}{(n-1)\phi}\int_\Omega \rho h^{q-1}|\nabla h|,\end{aligned}$$

which implies

$$\big(1+(n-1)\phi\big)\int_\Omega h^q \le (1+\varepsilon)q\Big(\int_\Omega h^q\Big)^{(p-1)/p}\Big(\int_\Omega \rho^p h^{q-p}|\nabla h|^p\Big)^{1/p}.$$

Next we use (4.9) to obtain

$$\begin{aligned}\big(1+(n-1)\phi\big)\Big(\int_\Omega h^q\Big)^{1/p} &\le (1+\varepsilon)q\Big(\int_\Omega \rho^p h^{q-p}|\nabla h|^p\Big)^{1/p}\\ &\le cq\Big(\int_\Omega \rho^p h^{q-p}|\nabla\theta|^p\Big)^{1/p} + (1+\varepsilon)q\Big(\frac{p\beta}{(q-p+1)\alpha}\Big)\Big(\int_\Omega h^q\Big)^{1/p}.\end{aligned}$$

Our choices of q_0 and ε allow us to estimate the constant that appears here by writing

$$\begin{aligned}(1+\varepsilon)q\Big(\frac{p\beta}{(q-p+1)\alpha}\Big) &\le (1+\varepsilon)\Big(\frac{q_0}{q_0-p+1}\Big)\Big(\frac{p\beta}{\alpha}\Big) \le (1+\varepsilon)^2\Big(\frac{p\beta}{\alpha}\Big)\\ &= \frac{(p\beta/\alpha)+\big(1+(n-1)\phi\big)}{2} < \big(1+(n-1)\phi\big).\end{aligned}$$

Using Hölder's inequality then gives

$$\begin{aligned}\int_\Omega h^q &\le cq^p\int_\Omega \rho^p h^{q-p}|\nabla\theta|^p\\ &\le cq^p\Big(\int_\Omega h^q\Big)^{(q-p)/q}\Big(\int_\Omega \rho^q|\nabla\theta|^q\Big)^{p/q}\end{aligned}$$

so that

$$\int_\Omega h^q \le (c_0 q)^q\int_\Omega \rho^q|\nabla\theta|^q.$$

This is the result we came looking for. □

In order to obtain pointwise estimates from L^p-estimates we need the following lemma, which is a modification of [2, Theorem 2.2].

4.10 Lemma. *Let $\theta\in C(M)\cap W^{1,p}_{\text{loc}}(M)$. Let $x\in M$, $R>0$, and u be an $\mathcal{A}$-harmonic function in $B(x,2R)$. Suppose that $A>0$ is a constant such that*

$$|u-\theta|, |\nabla\theta| \le A$$

in $B(x,2R)$. Let $Q\ge p$. Then

$$\sup_{B(x,R)}|u-\theta|^{Q+np} \le C\int_{B(x,2R)}|u-\theta|^Q,$$

where $C=C(n,p,\beta/\alpha,R,A,Q)$.

Proof. Let $q\ge Q$. Denote $h=|u-\theta|$ and $\lambda=n/(n-1)$. Let $\Omega\subset\subset B(x,2R)$ be an open set and suppose that $\eta: M\to\mathbb{R}$ is a non-negative Lipschitz function such that $\eta|\Omega^c\equiv 0$. Then $\eta^p h^{q-p}(u-\theta)\in W^{1,p}_0(\Omega)$ and hence Lemma 4.6 and the inequality $(a+b)^p\le 2^{p-1}a^p+2^{p-1}b^p$, with $a,b\ge 0$, imply

$$\int_\Omega \eta^p h^{q-p}|\nabla h|^p \le c_1\int_\Omega \eta^p h^{q-p}|\nabla\theta|^p + c_1 q^{-p}\int_\Omega h^q|\nabla\eta|^p,$$

where $c_1=c_1(p,\beta/\alpha)$. It follows that

$$\int_\Omega \eta^p|\nabla(h^{q/p})|^p \le c_1 q^p\int_\Omega \eta^p h^{q-p}|\nabla\theta|^p + c_1\int_\Omega h^q|\nabla\eta|^p. \tag{4.11}$$

For $j = 0, 1, \dots$, write $r_j = R + \lambda^{-j} R/2$, $B_j = B(x, r_j)$, and

$$q_j = \lambda^j (Q + np) - np.$$

Then $q_0 = Q$ and $q_{j+1} = \lambda(q_j + p)$ for all j. Define $\eta_j : M \to \mathbb{R}$ by

$$\eta_j(y) = \begin{cases} 1, & d(x,y) \le r_{j+1}, \\ \frac{r_j - d(x,y)}{r_j - r_{j+1}}, & r_{j+1} \le d(x,y) \le r_j, \\ 0, & d(x,y) \ge r_j. \end{cases}$$

We write $\varepsilon = \lambda^{-j} R$. Then

$$|\nabla \eta_j| \le 2n\lambda^j / R = 2n\varepsilon^{-1}.$$

Use Young's inequality

$$ab \le \frac{1}{p}\varepsilon^{p-1} a^p + \frac{p-1}{p}\varepsilon^{-1} b^{p/(p-1)}, \qquad a, b \ge 0,$$

to estimate

$$\begin{aligned}
|\nabla(\eta^p h^q)| &\le p\eta^{p-1} h^q |\nabla \eta| + q\eta^p h^{q-1} |\nabla h| \\
&= p\eta^{p-1} h^q |\nabla \eta| + p\eta^p h^{q-q/p} |\nabla (h^{q/p})| \\
&\le \Big(\varepsilon^{p-1} h^q |\nabla \eta|^p + (p-1)\varepsilon^{-1} \eta^p h^q\Big) + \Big(\varepsilon^{p-1} \eta^p |\nabla (h^{q/p})|^p + (p-1)\varepsilon^{-1}\eta^p h^q\Big) \\
&= \varepsilon^{p-1} h^q |\nabla \eta|^p + 2(p-1)\varepsilon^{-1} \eta^p h^q + \varepsilon^{p-1} \eta^p |\nabla (h^{q/p})|^p.
\end{aligned} \tag{4.12}$$

Recall that since M is a Cartan-Hadamard manifold, it admits a Sobolev inequality, see [7]. This means that there exists a positive constant $C_S = C_S(n)$ such that

$$\Big(\int_M |\varphi|^\lambda\Big)^{1/\lambda} \le C_S \int_M |\nabla \varphi|$$

for all $\varphi \in W_0^{1,1}(\Omega)$. Using the Sobolev inequality, (4.12) and (4.11) we get

$$\begin{aligned}
\Big(\int_{B_{j+1}} h^{\lambda q}\Big)^{1/\lambda} &\le \Big(\int_{B_j} |\eta_j^p h^q|^\lambda\Big)^{1/\lambda} \le c \int_{B_j} \big|\nabla(\eta_j^p h^q)\big| \\
&\le c\varepsilon^{p-1} \int_{B_j} h^q |\nabla \eta_j|^p + c\varepsilon^{-1} \int_{B_j} \eta_j^p h^q + c\varepsilon^{p-1} \int_{B_j} \eta_j^p |\nabla (h^{q/p})|^p \\
&\le c\varepsilon^{p-1} \int_{B_j} h^q |\nabla \eta_j|^p + c\varepsilon^{-1} \int_{B_j} \eta_j^p h^q + c\varepsilon^{p-1} q^p \int_{B_j} \eta_j^p h^{q-p} |\nabla \theta|^p \\
&\le c\varepsilon^{-1} \int_{B_j} h^q + c\varepsilon^{p-1} q^p \int_{B_j} h^{q-p} |\nabla \theta|^p \\
&\le c_2 A^p \varepsilon^{-1} \int_{B_j} h^{q-p} + c_2 A^p \varepsilon^{-1} (\varepsilon q)^p \int_{B_j} h^{q-p},
\end{aligned}$$

where $c_2 = c_2(p, n, \beta/\alpha, C_S)$. We apply this with $q = q_j + p$ to obtain

$$\begin{aligned}
\Big(\int_{B_{j+1}} h^{q_{j+1}}\Big)^{1/\lambda} &\le c_2 \varepsilon^{-1} A^p \big(1 + \big(\varepsilon(q_j + p)\big)^p\big) \int_{B_j} h^{q_j} \\
&\le c_2 2^j R^{-1} A^p \big(1 + R^p (Q + np)^p\big) \int_{B_j} h^{q_j}.
\end{aligned}$$

Define

$$I_j = \Big(\int_{B_j} h^{q_j} \Big)^{1/\lambda^j}.$$

Then

$$I_{j+1} \le 2^{j/\lambda^j} c_3^{1/\lambda^j} I_j,$$

where $c_3 = c_2 R^{-1} A^p \big(1 + R^p (Q+np)^p\big)$. As $j \to \infty$,

$$I_j = \|h\|_{L^{q_j}(B_j)}^{q_j/\lambda^j} \to \|h\|_{L^\infty(B(x,R))}^{Q+np}.$$

Hence

$$\sup_{B(x,R)} h^{Q+np} = \lim_{j\to\infty} I_j \le 2^{\sum_{k=0}^\infty j/\lambda^j} c_3^n I_0 \le c c_3^n \int_{B(x,2R)} h^Q$$

as wanted. □

We are now ready to prove Theorem 4.1.

Proof. (Proof of Theorem 4.1.) Let $f : M(\infty) \to \mathbb{R}$ be any continuous function. We have to prove that

$$\lim_{x\to x_0} \overline{H}_f(x) = f(x_0).$$

Fix an arbitrary $\varepsilon > 0$. Denote $v = \dot\gamma_0^{o,x_0}$. Let $\delta \in (0,\pi)$ and $r_0 > 0$ be such that $T(v,\delta,r_0) \subset U$ and that $|f(x_1) - f(x_0)| < \varepsilon$ for all $x_1 \in C(v,\delta) \cap M(\infty)$. Then the assumptions of Lemma 4.7 are satisfied. Let $r_1 > r_0$, $c_0 > 0$, and $q_0 > p$ be the constants described in Lemma 4.7.

Define $\theta : \bar M \to \mathbb{R}$,

$$\theta(x) = \min\Big(1, \max\big(0, r_1 + 1 - \rho(x), \delta^{-1} \sphericalangle_o(x_0,x)\big)\Big).$$

For $j \in \mathbb{N} \cap (r_1,\infty)$, let u_j be the $\mathcal{A}$-harmonic function in $\Omega_j := T(v,\delta,r_1) \cap B(o,j)$ that satisfies $u_j - \theta \in W_0^{1,p}(\Omega_j)$. Since $\exp_o^{-1}(\Omega_j) = T\big((\exp_o^{-1})_* v, \delta, r_1\big) \cap B(0,j) \subset T_oM$ satisfies the external cone condition and $\exp_o^{-1} | B(o,j+1)$ is bilipschitz, we see that Ω_j is a regular domain for the Dirichlet problem. Write $\Omega = T(v,\delta,r_1) \cap M$. Since $(u_j)_j$ is a uniformly bounded sequence of $\mathcal{A}$-harmonic functions, there exists a subsequence $(u_{i_j})_j$ and an $\mathcal{A}$-harmonic limit function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that $u_{i_j} \to u$ locally uniformly in Ω as $j \to \infty$.

Let $y_0 \in M \cap \partial\Omega$. We claim that

$$\lim_{y\to y_0,\ y\in\Omega} u(y) = 1. \tag{4.13}$$

Choose $j_0 \in \mathbb{N}$ such that $j_0 > \rho(y_0)$. Let $\eta \in C_0^\infty(M)$ be such that $0 \le \eta \le 1$, $\eta | B\big(o,\rho(y_0)\big) \equiv 1$, and $\operatorname{supp}\eta \subset B(0,j_0)$. Let w be the $\mathcal{A}$-harmonic function in Ω_{j_0} such that $w - \eta\theta \in W_0^{1,p}(\Omega_{j_0})$. Since Ω_j is regular for every j, we have

$$\lim_{y\to y_1,\ y\in\Omega_{j_0}} w(y) = \eta(y_1)\theta(y_1) \le \lim_{y\to y_1,\ y\in\Omega_{j_0}} u_j(y)$$

for every $j \ge j_0$ and $y_1 \in \partial\Omega_{j_0}$. It then follows from the comparison principle that $w \le u_j$ in Ω_{j_0} for every $j \ge j_0$. Hence

$$\liminf_{y\to y_0,\ y\in\Omega} u(y) \ge \liminf_{y\to y_0,\ y\in\Omega_{j_0}} w(y) = \eta(y_0)\theta(y_0) = 1.$$

Equation (4.13) follows from this since $u \le 1$ everywhere.

Next we claim that

$$\lim_{y\to x_0,\ y\in\Omega} u(y) = 0. \tag{4.14}$$

For $x \in \Omega$, let $J(x)$ be the supremum and $j(x)$ the infimum of $|V(\rho(x))|$ over Jacobi fields V along $\gamma^{o,x}$ that satisfy $V_0 = 0$, $|V_0'| = 1$, and $V_0' \perp \dot\gamma_0^{o,x}$. By Lemma 3.6 and the assumption (4.3) there exists a constant $c_1 > 0$ such that

$$J(x) \le c_1 j(x)^C \tag{4.15}$$

for every $x \in \Omega$. By Lemma 3.5 there exists a constant $c_2 > 0$ such that

$$|\nabla\theta(x)| \le \frac{c_2}{j(x)} \tag{4.16}$$

for all $x \in \Omega$. Let $K : [0,\infty) \to (-\infty, 0]$ be a smooth function such that $K(t) = 0$ for $t \in [0, r_0]$, $K(t) \ge -\phi(\phi-1)/t^2$ for $t \in [r_0, r_0+1]$, and $K(t) = -\phi(\phi-1)/t^2$ for $t \ge r_0+1$. Then the radial curvatures in $C(v,\delta)\cap M$ are bounded from above by $K \circ \rho$ by the assumption (4.2). By Lemma 3.2(a) and Example 3.1 we get

$$j(x) \ge (F_K \circ \rho)(x) \ge c\rho(x)^\phi \tag{4.17}$$

for all $x \in \Omega$. Using the inequality (4.16), the equation (3.3) and inequalities (4.15) and (4.17) we get

$$\begin{aligned}
\int_\Omega \rho^q |\nabla\theta|^q &\le c \int_{T(v,\delta,r_1)\cap M} \rho(x)^q j(x)^{-q}\, dm_M(x) \\
&= c \int_{r_1}^\infty \int_{S_oM\cap C((\exp_o^{-1})_* v,\delta)} r^q j(r,\xi)^{-q} \lambda_M(r,\xi)\, d\xi\, dr \\
&\le c \int_{r_1}^\infty \int_{S_oM\cap C((\exp_o^{-1})_* v,\delta)} r^q j(r,\xi)^{-q} J(r,\xi)^{n-1}\, d\xi\, dr \\
&\le c \int_{r_1}^\infty \int_{S_oM\cap C((\exp_o^{-1})_* v,\delta)} r^q j(r,\xi)^{-q+C(n-1)}\, d\xi\, dr \\
&\le c \int_{r_1}^\infty r^{(1-\phi)q+\phi C(n-1)}\, dr
\end{aligned}$$

for all $q > C(n-1)$. We conclude that there exists a constant $q_1 > 0$ such that $\int_\Omega \rho^q|\nabla\theta|^q < \infty$ for every $q \ge q_1$. Fix $q \ge \max\{q_0, q_1\}$. By Fatou's lemma and Lemma 4.7 we get

$$\int_\Omega |u-\theta|^q \le \liminf_{j\to\infty} \int_{\Omega_{i_j}} |u_{i_j} - \theta|^q \le (c_0 q)^q \liminf_{j\to\infty} \int_{\Omega_{i_j}} \rho^q|\nabla\theta|^q = (c_0q)^q \int_\Omega \rho^q |\nabla\theta|^q < \infty.$$

Lemma 4.10 implies

$$|u(x)-\theta(x)| \le \sup_{B(x,1)} |u-\theta| \le c\Big(\int_{B(x,2)} |u-\theta|^q\Big)^{1/(q+np)}$$

whenever $x \in \Omega$ is such that $B(x,2) \subset \Omega$. We showed above that $\int_\Omega |u-\theta|^q < \infty$ and hence

$$|u(x)-\theta(x)| \to 0$$

as $x \to x_0$, $x \in M$. Thus $u(x) \to \theta(x_0) = 0$ as $x \to x_0$, $x \in M$. This proves the claim (4.14).

Now we define $w : M \to \mathbb{R}$,

$$w(x) = \begin{cases} \min(1, 2u)(x), & \text{if } x \in \Omega, \\ 1, & \text{if } x \in M \setminus \Omega. \end{cases}$$

Since the minimum of two $\mathcal{A}$-superharmonic functions is $\mathcal{A}$-superharmonic and (4.13) holds for every $y_0 \in M \cap \partial\Omega$, we see that w is continuous and $\mathcal{A}$-superharmonic in some neighborhood of each point in M. Since $\mathcal{A}$-superharmonicity is a local property, w is $\mathcal{A}$-superharmonic. Now

$$\overline{H}_f \leq f(x_0) + \varepsilon + 2(\sup |f|)w$$

by the definition of $\overline{H}_f$. By (4.14) we get $\limsup_{x \to x_0} \overline{H}_f(x) \leq f(x_0) + \varepsilon$. Similarly one proves that $\liminf_{x \to x_0} \underline{H}_f(x) \geq f(x_0) - \varepsilon$. Taking into account $\overline{H}_f \geq \underline{H}_f$ and that $\varepsilon > 0$ is arbitrary, we get $\lim_{x \to x_0} \overline{H}_f(x) = f(x_0)$. □

5 Discussion of the condition $p < 1 + (n-1)\phi$

If $\mathcal{A}$ is the p-Laplacian or more generally if $\alpha = \beta$, the condition (4.4) in Theorem 4.1 simplifies to $p < 1 + (n-1)\phi$. In this section we give an example of a manifold for which this bound is sharp.

A Riemannian manifold N is called *p-parabolic*, with $1 < p < \infty$, if

$$\operatorname{cap}_p(K, N) = 0$$

for every compact $K \subset M$. Equivalently, N is p-parabolic if every non-negative supersolution of

$$-\operatorname{div} \mathcal{A}(\nabla u) = 0$$

on N is constant for all $\mathcal{A} \in \mathcal{A}^p(N)$. It is known that a sufficient condition for p-parabolicity of a complete connected non-compact Riemannian manifold is the Ahlfors type condition

$$\int_1^\infty \Big(\frac{t}{V(t)}\Big)^{1/(p-1)} dt = \infty,$$

where $V(\cdot) = m_N\big(B(x_o, \cdot)\big)$ and $x_0 \in N$ is fixed, see [9, Theorem 5.16]. As an application we get the following result.

5.1 Proposition. *Suppose that there exists a constant $\phi > 1$ and a compact set $K \subset M$ such that*

$$K_M(P) \geq -\frac{\phi(\phi - 1)}{\rho(x)^2}$$

for every $x \in M \setminus K$ and every 2-dimensional subspace $P \subset T_x M$ that contains the radial vector $\nabla\rho(x)$. Suppose that

$$p \geq 1 + (n-1)\phi.$$

Then M is p-parabolic.

Proof. Let $R > 1$ be so large that $K \subset B(o, R-1)$ and let

$$B = \inf\big\{K_M(P) : P \subset T_x M \text{ is a 2-dimensional subspace and } x \in \bar{B}(o, R-1)\big\}.$$

Then $B > -\infty$ by compactness. Let $k : [0, \infty) \to (-\infty, 0]$ be a smooth function such that k is constant in some neighborhood of 0, $k(t) \leq B$ for every $t \in [0, R-1]$, $k(t) \leq -\phi(\phi-1)/t^2$ for every

$t \in [R-1, R]$, and $k(t) = -\phi(\phi-1)/t^2$ for every $t \geq R$. Then all the radial curvatures on $M \setminus \{o\}$ are bounded from below by $k \circ \rho$. By (3.4) we have

$$\lambda_M(r, \xi) \leq F_k(r)^{n-1}.$$

for all $r > 0$ and $\xi \in S_oM$. By Example 3.1 there exist constants $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$F_k(t) = c_1 t^{\phi} + c_2 t^{1-\phi}$$

for all $t \geq R$. Thus the function $V(\cdot) = m_M\big(B(o, \cdot)\big)$ satisfies

$$V(r) = \int_0^r \int_{S_oM} \lambda_M(t, \xi)\, d\xi\, dt \leq c \int_0^r (1+t)^{(n-1)\phi}\, dt \leq c\, r^{1+(n-1)\phi}$$

for all $r \geq 1$. It follows that

$$\int_1^\infty \Big(\frac{t}{V(t)}\Big)^{1/(p-1)} dt \geq c \int_1^\infty t^{-(n-1)\phi/(p-1)}\, dt = \infty$$

since $(n-1)\phi/(p-1) \leq 1$. Hence M is p-parabolic. □

The following example shows that if $\alpha = \beta$, then the condition $p < 1 + (n-1)\phi$ in Theorem 4.1 is sharp in some cases.

5.2 Example. Fix $\phi > 1$ and let $K : [0, \infty) \to (-\infty, 0]$ be a smooth function such that K is constant in some neighborhood of 0 and there exists $t_0 > 0$ such that $K(t) = -\phi(\phi-1)/t^2$ for every $t \geq t_0$. Let M_K be $\mathbb{R}^n$ equipped with the Riemannian metric $dr^2 + F_K(r)^2 d\theta^2$. Then M_K is a rotationally symmetric manifold with radial curvature function $K \leq 0$. A computation shows that the sectional curvature of a 2-dimensional subspace $P \subset T_xM_K$, $x \in M_K \setminus \{0\}$, is

$$K_{M_K}(P) = K\big(r(x)\big)\cos^2\theta + \frac{1 - F_K'\big(r(x)\big)^2}{F_K\big(r(x)\big)^2}\sin^2\theta,$$

where θ is the angle between P and $\nabla r(x)$. It follows that M_K is a Cartan-Hadamard manifold.

Suppose now that $M = M_K$ and $\alpha = \beta$. Theorem 4.1 and Proposition 5.1 then imply that the Dirichlet problem at infinity is solvable for the operator $\mathcal{A}$ if and only if $p < 1 + (n-1)\phi$. Moreover, M is p-parabolic if $p \geq 1 + (n-1)\phi$.

The previous example also shows that at least an additional condition

$$p < 1 + (n-1)\phi$$

is needed if we want to generalize Greene-Wu's conjecture for $\mathcal{A}$-harmonic functions of type $p > n$. We are led to the following question, which in the case $p = 2$ comes back to Greene-Wu's conjecture.

5.3 Question. *Suppose that $\phi > 1$ and*

$$1 < p < 1 + (n-1)\phi.$$

Assume that the sectional curvatures on M satisfy

$$K_M \leq -\frac{\phi(\phi-1)}{\rho^2}$$

outside a compact set. Does it follow that there exists a non-constant bounded p-harmonic function on M?

It follows from results in this paper that the answer to this question is yes for $n = 2$. Indeed, if $n = 2$, then Corollary 4.5 states that the Dirichlet problem at infinity is solvable for p-harmonic functions under the assumptions of Question 5.3. In the case $n \geq 3$ and $p = 2$ the assumptions in Question 5.3 are not enough to guarantee that the Dirichlet problem at infinity is solvable, see [1].

References

[1] Ancona, A. *Convexity at infinity and Brownian motion on manifolds with unbounded negative curvature.* - Rev. Mat. Iberoamericana 10, 1 (1994), 189-220.

[2] Cheng, S. Y. *The Dirichlet problem at infinity for nonpositively curved manifolds.* - Comm. Anal. Geom. 1, 1 (1993), 101-112.

[3] Choi, H. I. *Asymptotic Dirichlet problems for harmonic functions on Riemannian manifolds.* - Trans. Amer. Math. Soc. 281, 2 (1984), 691-716.

[4] Eberlein, P. and O'Neill, B. *Visibility manifolds.* - Pacific J. Math. 46 (1973), 45-109.

[5] Greene, R. E. and Wu, H. *Function theory on manifolds which possess a pole, vol. 699 of Lecture notes in Mathematics* - Springer, Berlin, 1979.

[6] Heinonen, J., Kilpeläinen, T. and Martio, O. *Nonlinear potential theory of degenerate elliptic equations.* - Oxford University Press, Oxford, 1993.

[7] Hoffman, D. and Spruck, J. *Sobolev and isoperimetric inequalities for Riemannian submanifolds.* - Comm. Pure Appl. Math. 27 (1974) 715-727. See also "A correction to: Sobolev and isoperimetric inequalities for Riemannian submanifolds", Comm. Pure Appl. Math., 28 (1975) 765-766.

[8] Holopainen, I. *Asymptotic Dirichlet problem for the p-Laplacian on Cartan-Hadamard manifolds.* - Proc. Amer. Math. Soc. 130, 11 (2002), 3393-3400.

[9] Holopainen, I. *Quasiregular mappings and the p-Laplace operator.* - Contemporary Mathematics Vol. 338 (2003), 219-239.

[10] Holopainen, I., Lang, U. and Vähäkangas, A. *Dirichlet problem at infinity on Gromov hyperbolic metric measure spaces.* - (To appear).

[12] Hsu, E. P., Kendall, W. S. *Limiting angle of Brownian motion in certain two-dimensional Cartan-Hadamard manifolds.* - Ann. Fraculté Sci. Toulouse Math. (sér. 6) 1, 169-186.

[13] Pansu, P. *Cohomologie L^p des variétés à courbure négative, cas du degré 1.* - Rend. Sem. Mat. Univ. Politec. Torino, Special Issue (1989), 95-120 (1990) Conference on Partial Differential Equations and Geometry (Torino, 1988).