Homeomorphisms with lower bounds for moduli

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Abstract

We elucidate possibilities of lower estimates of moduli for families of surfaces of dimension $n-1$ under mappings with finite distortion. In particular, it makes possible to investigate the boundary behavior of homeomorphisms of finite area distortion, especially, of finitely bi–Lipschitz homeomorphisms between quasi-extremal distance domains by Gehring–Martio.

1 Introduction

Many classes of the so–called mappings with finite distortion are intensively studied during the last years, see e.g. [AIKM], [FKZ], [GI], [HK$_1$], [HK$_*$], [HK$_1^*$], [HM], [HP], [IKO$_1$]–[IKO$_2$], [IM], [IŠ], [Ka], [KKM$_1$]–[KKM$_2$], [KM], [KKMOZ], [KO], [KOR], [MV$_1$]–[MV$_2$], [On$_1$]–[On$_3$], [Pa], and [Ra$_1$]–[Ra$_4$]. So far the upper estimates of moduli have played the major role in the theory, see e.g. [MRSY$_1$] – [MRSY$_6$], [IR$_1$] – [IR$_2$], [RS] and our previous preprint [KR].

In this paper we consider the lower estimates of moduli. First recall the base concepts. Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \to [1, \infty]$ be a measurable function. A homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is called a **Q−homeomorphism** if

$$M(f\Gamma) \leq \int\limits_D Q(x) \cdot \rho^n(x)\ dm(x) \tag{1.1}$$

for every family Γ of paths in D and every admissible function ρ for Γ, see [MRSY$_3$]–[MRSY$_6$]. Here the notation m refers to the Lebesgue measure in $\mathbb{R}^n$.

Recall that, given a family of paths Γ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called **admissible** for Γ, abbr. $\rho \in adm\,\Gamma$, if

$$\int\limits_\gamma \rho\, ds \ \geq\ 1 \tag{1.2}$$

for each $\gamma \in \Gamma$. The (conformal) **modulus** of Γ is the quantity

$$M(\Gamma) = \inf_{\rho \in adm\,\Gamma} \int\limits_D \rho^n(x)\, dm(x)\ . \tag{1.3}$$

In particular, the homeomorphisms $f : D \to \mathbb{R}^n$, $n \geq 2$, of the class $W^{1,n}_{loc}$ with a locally integrable inner dilatation $K_I(x, f)$ are Q–homeomorphisms with $Q(x) = K_I(x, f)$.

The following localization and extension of the notion of $Q-$homeomorphisms was first introduced in [RSY$_1$] for $n = 2$ and then investigated in [RS] for an arbitrary $n \geq 2$. It was motivated by Gehring's ring definition of quasiconformality in [Ge$_1$].

Given a domain $D \subseteq \mathbb{R}^n$, $n \geq 2$, $x_0 \in D$, $\varepsilon_0 < dist\,(x_0, \partial D)$, a measurable function $Q : B(x_0, \varepsilon_0) \to [0, \infty]$, a homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is called a **ring Q−homeomorphism** at x_0 if

$$M(\Gamma(fS_1, fS_2)) \ \leq \ \int\limits_R Q(x) \cdot \eta^n(|\,x - x_0|)\; dm(x) \tag{1.4}$$

for every ring

$$R = R(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |\,x - x_0| < r_2\}\ , \quad 0 < r_1 < r_2 < \varepsilon_0\ ,$$

and every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \ \geq 1 \tag{1.5}$$

where

$$S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |\,x - x_0| = r_i\}\ , \quad i = 1, 2\,,$$

and $\Gamma(C_1, C_2)$, $C_i = fS_i$, denotes the family of all path $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ which join C_1 and C_2.

We may assume in the above definition of the ring homeomorphism that Q is given in the whole domain D because every measurable function in $B(x_0, \varepsilon_0)$ can be extended to a measurable function in D, as in [RS]. There it was shown that (1.4) is equivalent to the inequality

$$M(\Gamma(fS_1, fS_2)) \ \leq \ \frac{\omega_{n-1}}{I^{n-1}} \tag{1.6}$$

where ω_{n-1} is an area of the unit sphere S^{n-1} in $\mathbb{R}^n$,

$$I = I(x_0, r_1, r_2) = \int\limits_{r_1}^{r_2} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} \tag{1.7}$$

where $q_{x_0}(r)$ is the mean of the function $Q(x)$ over the sphere $|\,x - x_0| = r$. Note that the infimum of the expression from the right in (1.4) is realized for the function

$$\eta_0(r) \ = \ \frac{1}{I} \cdot \frac{1}{r\, q_{x_0}^{\frac{1}{n-1}}(r)}\,.$$

In the present paper, we study a similar notion in terms of modulus for surfaces of the dimension $n - 1$.

Below H^k, $k = 1, ..., n-1$ denotes the **k−dimensional Hausdorff measure** in $\mathbb{R}^n$, $n \geq 2$. More precisely, if A is a set in $\mathbb{R}^n$, then

$$H^k(A) \ = \ \sup_{\varepsilon>0} \ H^k_\varepsilon(A), \tag{1.8}$$

$$H^k_\varepsilon(A) \ = \ V_k \inf \ \sum_{i=1}^{\infty} \left(\frac{\delta_i}{2}\right)^k \tag{1.9}$$

where the infimum is taken over all countable collections of numbers $\delta_i \in (0, \varepsilon)$ such that some sets A_i in $\mathbb{R}^n$ with diameters δ_i cover A. Here V_k denotes the volume of the unit ball in $\mathbb{R}^k$.

Let ω be an open set in $\overline{\mathbb{R}^k}$, $k = 1, ..., n-1$. A (continuous) mapping $S : \omega \to \mathbb{R}^n$ is called a $k-$dimensional surface S in $\mathbb{R}^n$. Sometimes we call the image $S(\omega) \subseteq \mathbb{R}^n$ by the surface S, too. The number of preimages

$$N(S, y) \ = \ N(S, y, \omega) \ = \ card\, S^{-1}(y) \ = \ card\, \{x \in \omega : \ S(x) = y\} \tag{1.10}$$

is said to be a **multiplicity function** of the surface S at a point $y \in \mathbb{R}^n$. In the other words, $N(S, y)$ means the multiplicity of covering of the point y by the surface S. It is known that multiplicity function is lower semi-continuous, i.e.,

$$N(S, y) \ \geq \ \liminf_{m\to\infty} \ N(S, y_m)$$

for every sequence $y_m \in \mathbb{R}^n$, $m = 1, 2, ...$ such that $y_m \to y \in \mathbb{R}^n$ as $m \to \infty$, see e.g. [RR], p. 160. Thus, the function $N(S, y)$ is Borel measurable and hence measurable with respect to every Hausdorff measure H^k, see e.g. [Sa], p. 52.

$k-$dimensional Hausdorff area in $\mathbb{R}^n$ (or simply **area**) associated with a surface $S : \omega \to \mathbb{R}^n$ is given by

$$\mathcal{A}_S(B) \ = \ \mathcal{A}^k_S(B) \ := \ \int\limits_B N(S, y) \ dH^k y \tag{1.11}$$

for every Borel set $B \subseteq \mathbb{R}^n$ and, more generally, for an arbitrary set which is measurable with respect to H^k in $\mathbb{R}^n$. The surface S is **rectifiable** if $S(\mathbb{R}^n) < \infty$.

If $\rho : \mathbb{R}^n \to [0, \infty]$ is a Borel function, then its **integral over** S is defined by the equality

$$\int\limits_S \rho \ d\mathcal{A} \ := \ \int\limits_{\mathbb{R}^n} \rho(y) \ N(S, y) \ dH^k y \ . \tag{1.12}$$

Given a family Γ of $k-$dimensional surfaces S, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called **admissible** for Γ, abbr. $\rho \in adm\, \Gamma$, if

$$\int\limits_S \rho^k \ d\mathcal{A} \ \geq \ 1 \tag{1.13}$$

for every $S \in \Gamma$. Given $p \in (0, \infty)$, the **p−modulus** of Γ is the quantity

$$M_p(\Gamma) \ = \ \inf_{\rho\in adm\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x) \ dm(x) \ . \tag{1.14}$$

We also set

$$M(\Gamma) \;=\; M_n(\Gamma)\;. \tag{1.15}$$

The modulus is itself an outer measure on the collection of all families Γ of $k-$dimensional surfaces.

Sometimes, under proofs, it is more convenient to use the following notion. A Lebesgue measurable function $\rho:\mathbb{R}^n\to[0,\infty]$ is said to be **p−extensively admissible** for a family Γ of $k-$dimensional surfaces S in $\mathbb{R}^n$, abbr. $\rho\in ext_p\, adm\,\Gamma$, if

$$\int\limits_S \rho^k\; d\mathcal{A}\;\ge\;1 \tag{1.16}$$

for $p-$a.e. $S\in\Gamma$. The **p−extensive modulus** $\overline{M}_p(\Gamma)$ of Γ is the quantity

$$\overline{M}_p(\Gamma)\;=\;\inf\int\limits_{\mathbb{R}^n}\rho^p(x)\;dm(x) \tag{1.17}$$

where the infimum is taken over all $\rho\in ext_p\; adm\;\Gamma$. In the case $p=n$, we use notations $\overline{M}(\Gamma)$ and $\rho\in ext\; adm\;\Gamma$, respectively. For every $p\in(0,\infty)$, $k=1,...,n-1$, and every family Γ of $k-$dimensional surfaces in $\mathbb{R}^n$,

$$\overline{M}_p(\Gamma)\;=\;M_p(\Gamma), \tag{1.18}$$

see Corollary 2.16 in [KR]. The same is also true for moduli with weights.

Given a domain $D\subseteq\mathbb{R}^n$, $n\ge2$, $x_0\in\overline{D}\setminus\{\infty\}$, a measurable function $Q:D\to(0,\infty)$, we say that a homeomorphism $f:D\to\overline{\mathbb{R}^n}$ is a **lower Q−homeomorphism at the point** x_0 if

$$M(f\Sigma_\varepsilon)\;\ge\;\inf_{\rho\in adm\Sigma_\varepsilon}\int\limits_{D\cap R_\varepsilon}Q^{-1}(x)\;\varrho^n(x)\;dm(x) \tag{1.19}$$

for every ring

$$R_\varepsilon\;=\;\{x\in\mathbb{R}^n: \varepsilon<|\,x-x_0|<\varepsilon_0\}\;,\quad \varepsilon\in(0,\varepsilon_0)$$

where

$$0<\varepsilon_0<d_0\;=\;\sup_{x\in D}|x-x_0|\;=\;\sup_{x\in\partial D}|x-x_0| \tag{1.20}$$

and Σ_ε denotes the family of all intersections of the spheres

$$S(r)\;=\;S(x_0,r)\;=\;\{x\in\mathbb{R}^n: |\,x-x_0|=r\}\;,\quad r\in(\varepsilon,\varepsilon_0)\,,$$

with $\overline{D}$. Here $adm\;\Sigma_\varepsilon$ consists of Borel functions $\varrho:\mathbb{R}^n\to[0,\infty]$ with

$$\int\limits_{D(r)}\varrho^{n-1}\;d\mathcal{A}\;\ge1\;,\quad\forall\; r\in(\varepsilon,\varepsilon_0) \tag{1.21}$$

where

$$D(r)\;=\;D(x_0,r)\;=\;\{x\in D: |\,x-x_0|=r\}\;=\;D\cap S(x_0,r)\,. \tag{1.22}$$

As usual, the notion can be extended to the case $x_0 = \infty \in \overline{D}$ through applying the inversion T with respect to the unit sphere in $\overline{\mathbb{R}^n}$, $T(x) = x/|\,x|^2$, $T(\infty) = 0$, $T(0) = \infty$.

We also say that a homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is a **lower Q−homeomorphism** in D if f is a lower $Q-$homeomorphism at every point $x_0 \in \overline{D}$.

We show here that the condition (1.19) is equivalent to the inequality:

$$M(f\Sigma_\varepsilon) \;\geq\; \int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{||\,Q||_{\,n-1}(r)} \tag{1.23}$$

where

$$||\,Q||_{\,n-1}(r) = \left(\int\limits_{D(r)} Q^{n-1}\; d\mathcal{A}\right)^{\frac{1}{n-1}} . \tag{1.24}$$

Note that the infimum in (1.19) is attained only for the function

$$\varrho_0(x) \;=\; ||\,Q||^{-1}_{\,n-1}(|\,x|)\cdot Q^{n-1}(x)\,. \tag{1.25}$$

Below we always assume that $Q \equiv 0$ outside of D and take the integrals in (1.24) over the whole spheres $S(r) = S(x_0, r)$.

Let Σ^*_ε be the family of all $(n-1)-$dimensional surfaces in D which separate the spheres $|\,x - x_0| = \varepsilon$ and $|\,x - x_0| = \varepsilon_0$ in D. Note that (1.23) implies the corresponding lower estimate for Σ^*_ε because $\Sigma_\varepsilon \subset \Sigma^*_\varepsilon$ and hence $\operatorname{adm}\Sigma^*_\varepsilon \subset \operatorname{adm}\Sigma_\varepsilon$. However, the inequality (1.23) for Σ^*_ε is not precise. The same is true for Σ^{**}_ε consisting of all closed sets C in $\overline{D}$ which separate the given spheres in D. Indeed, $\Sigma_\varepsilon \subseteq \Sigma^{**}_\varepsilon$ and hence $\operatorname{adm}\Sigma^{**}_\varepsilon \subset \operatorname{adm}\Sigma_\varepsilon$, cf. [Z]. In the case of Σ^{**}_ε, the definitions in the (1.11)–(1.15) are similar with $N(C, y) \equiv 1$. Thus, $M(f\Sigma_\varepsilon)$ is majorized by $M(f\Sigma^*_\varepsilon)$ as well as by $M(f\Sigma^{**}_\varepsilon)$.

This makes possible to find the corresponding estimates of distortion under lower $Q-$homeomorphisms and to investigate the removability of isolated singularities and other problems.

Moreover, here we state that homeomorphisms with finite area distortion studied in [KR] are lower $Q-$homeomorphisms with $Q(x) = K_O(x, f)$ where $K_O(x, f)$ is the outer dilatation of f at x. In particular, this holds for the so–called finitely bi–Lipschitz homeomorphisms which are a natural extension of isometries as well as quasi–isometries, see [K].

Given a mapping $\varphi : E \to \mathbb{R}^n$ and a point $x \in E \subseteq \mathbb{R}^n$, let

$$L(x, \varphi) \;=\; \limsup_{y\to x\; y\in E} \frac{|\varphi(y) - \varphi(x)|}{|y - x|}\;, \tag{1.26}$$

and

$$l(x, \varphi) \;=\; \liminf_{y\to x\; y\in E} \frac{|\varphi(y) - \varphi(x)|}{|y - x|}\;. \tag{1.27}$$

A mapping $f : D \to \mathbb{R}^n$ is said to be of **finitely be–Lipschitz** if

$$0 \ < \ l(x, f) \ \leq \ L(x, f) \ < \ \infty \qquad \forall x \in D\,. \tag{1.28}$$

Recall that **outer dilatation** of f at x is defined by

$$K_O(x, f) = \begin{cases} \frac{|f'(x)|^n}{|J(x,f)|}, & \text{if } J(x, f) \neq 0 \\ 1, & \text{if } f'(x) = 0 \end{cases} \tag{1.29}$$

and otherwise we set $K_O(x, f) = \infty$. Similarly, the **inner dilatation** of f at x is defined as

$$K_I(x, f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^n}, & \text{if } J(x, f) \neq 0 \\ 1, & \text{if } f'(x) = 0 \end{cases} \tag{1.30}$$

and $K_I(x, f) = \infty$ otherwise. Here $f'(x)$ denotes the Jacobian matrix of f, $J(x, f) = det\, f'(x)$ is its Jacobian, $|f'(x)|$ is the operator norm of $f'(x)$, i.e.

$$|\, f'(x)| = \max\{|\, f'(x)h| : h \in \mathbb{R}^n, |\, h| = 1\}\,, \tag{1.31}$$

$$l(f'(x)) = \min\{|\, f'(x)h| : h \in \mathbb{R}^n, |\, h| = 1\}\,. \tag{1.32}$$

2 On mappings with finite area distortion.

Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$. A mapping $f : \Omega \to \mathbb{R}^n$ is said to be of **finite metric distortion**, abbr. $f \in FMD$, if f has $(N)-$property and

$$0 \ < \ l(x, f) \ \leq \ L(x, f) \ < \ \infty \qquad a.e. \tag{2.1}$$

Note that a mapping $f : \Omega \to \mathbb{R}^n$ is of FMD if and only if f is differentiable a.e. and has $(N)-$ and $(N^{-1})-$properties, see Corollary 3.4 in [MRSY$_2$]. Recall that a mapping $f : X \to Y$ between measurable spaces (X, Σ, μ) and (X', Σ', μ') is said to have $(\mathbf{N})-$**property** if $\mu'(f(E)) = 0$ whenever $\mu(E) = 0$. Similarly, f has the $(\mathbf{N^{-1}})-$**property** if $\mu(E) = 0$ whenever $\mu'(f(E)) = 0$.

We say that a mapping $f : \Omega \to \mathbb{R}^n$ has $(\mathbf{A_k})-$**property** if the two conditions hold:

$(A_k^{(1)})$: for a.e. $k-$dimensional surface S in Ω the restriction $f|_S$ has $(N)-$property;

$(A_k^{(2)})$: for a.e. $k-$dimensional surface S_* in $\Omega_* = f(\Omega)$ the restriction $f|_S$ has $(N^{-1})-$property for each lifting S of S_*.

Here a surface S in Ω is a **lifting** of a surface S_* in $\mathbb{R}^n$ under a mapping $f : \Omega \to \mathbb{R}^n$ if $S_* = f \circ S$. We also say that a mapping $f : \Omega \to \mathbb{R}^n$ is **of finite distortion of area in dimension** $k = 1, ..., n-1$, abbr. $f \in FAD_k$, if $f \in FMD$ and has the $(A_k)-$property. Note that analogues of $(A_k)-$properties and the classes FAD_k have been first formulated in the mentioned work [MRSY$_2$] for $k = 1$ where it is additionally requested local rectifiability of S_* and S in $(A_k^{(1)})-$ and $(A_k^{(2)})-$properties, respectively. Thus, the mapping class FLD (finite length distortion) in [MRSY$_2$] is a subclass of FAD_1. Finally, we say that a mapping $f:\Omega\to\mathbb{R}^n$ is of **finite area distortion**, abbr. $f\in FAD$, if $f\in FAD_k$ for every $k=1,...,n-1$, see [KR].

2.2. Lemma. *Let Ω be an open set in $\mathbb{R}^n$, $n\geq 2$, and $f:\Omega\to\mathbb{R}^n$ a FMD homeomorphism with $(A_k^{(1)})-$property for some $k=1,\ldots,n-1$. Then*

$$M(f\Gamma)\ \geq\ \inf_{\varrho\in adm\Gamma}\int\limits_{\Omega}K_O^{-1}(x,f)\ \varrho^n(x)\ dm(x) \tag{2.3}$$

for every family Γ of $k-$dimensional surfaces S in Ω.

Proof. Let B be a (Borel) set of all points x in Ω where f has a differential $f'(x)$ and $J(x,f)=det\,f'(x)\neq 0$. As known, B is the union of a countable collection of Borel sets B_l, $l=1,2,\ldots$ such that $=f_lf|_{B_l}$ is bi–Lipschitz, see e.g. 3.2.2 in [Fe]. Without loss of generality we may assume that B_l are mutually disjoint. Note that $B_0=\Omega\setminus B$ and $f(B_0)$ have the Lebesgue measure zero in $\mathbb{R}^n$ for $f\in FMD$. Thus, by Theorem 2.11 in [KR] $\mathcal{A}_S(B_0)=0$ for a.e. $S\in\Gamma$ and hence by $(A_k^{(1)})-$property $\mathcal{A}_{S_*}(f(B_0))=0$ for a.e. $S\in\Gamma$ where $S_*=f\circ S$.

Let $\varrho_*\in ext\,adm\,f\Gamma$, $\varrho_*\equiv 0$ outside of $f(D)$, and set $\varrho\equiv 0$ outside of D and

$$\varrho(x)\ =\ \varrho_*(f(x))\,||\,f'(x)||\,,\quad x\in D\,.$$

Arguing piecewise on B_l, we have by 3.2.20 and 1.7.6 in [Fe] that

$$\int\limits_S\varrho^k\,d\mathcal{A}\ \geq\ \int\limits_{S_*}\varrho_*^k\,d\mathcal{A}\ \geq\ 1$$

for a.e. $S\in\Gamma$ and, thus, $\varrho\in ext\,adm\,\Gamma$.

By the change of variables for the class FMD, see Proposition 3.7 in [MRSY$_2$],

$$\int\limits_{\Omega}K_O^{-1}(x,f)\ \varrho^n(x)\ dm(x)\ =\ \int\limits_{f(\Omega)}\varrho_*^n(y)\ dm(y)$$

and (2.3) follows.

2.4. Remark. It is easy to see by the well–known Lusin theorem that

$$\inf_{\varrho\in ext\,adm\Gamma}\int\limits_{\Omega}K_O^{-1}(x,f)\ \varrho^n(x)\ dm(x)\ =\ \inf_{\varrho\in adm\Gamma}\int\limits_{\Omega}K_O^{-1}(x,f)\varrho^n(x)\ dm(x)\,, \tag{2.5}$$

see similar arguments to (2.17) in [MRSY$_2$]. The expressions in (2.5) are particular cases of moduli with weights.

Combining Lemma 3.10 in [KR] with Lemma 2.2 we have the following statement.

2.6. Theorem. *Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, and let a homeomorphism $f : \Omega \to \mathbb{R}^n$ belong to FAD_k for some $k = 1, \ldots, n-1$. Then, for every family Γ of $k-$dimensional surfaces S in Ω, f satisfies the double inequality*

$$\text{(2.7)}\quad \inf \int\limits_{\Omega} K_O^{-1}(x,f) \cdot \varrho^n(x) dm(x) \leq M(f\Gamma) \ \leq \ \inf \int K_I(x,f) \cdot \varrho^n(x) dm(x)$$

where the infimums are taken over all $\varrho \in adm\ \Gamma$.

2.8. Corollary. *Every homeomorphism $f : D \to \mathbb{R}^n$ of finite area distortion in the dimension $n-1$ is a lower $Q-$homeomorphism with $Q(x) = K_O(x,f)$.*

3 The main lemma on lower $Q-$homeomorphisms

We start first from the following general statement.

3.1. Lemma. *Let (X, μ) be a measure space, $p \in (1, \infty)$ and let $\varphi : X \to (0, \infty)$ be a measurable function. Set*

$$I(\varphi, p) \ = \ \inf_{\alpha} \int\limits_{X} \varphi\, \alpha^p \, d\mu \tag{3.2}$$

where the infimum is taken over all measurable functions $\alpha : X \to [0, \infty]$ such that

$$\int\limits_{X} \alpha \, d\mu \ = \ 1 \, . \tag{3.3}$$

Then

$$I(\varphi, p) \ = \ \left[\int\limits_{X} \varphi^{-\lambda} \, d\mu \right]^{-\frac{1}{\lambda}} \tag{3.4}$$

where

$$\lambda = \frac{q}{p} \, , \qquad \frac{1}{p} + \frac{1}{q} = 1 \, , \tag{3.5}$$

i.e. $\lambda = 1/(p-1) \in (0, \infty)$. Moreover, the infimum in (3.2) is attained only under the function

$$\alpha_0 \ = \ C \cdot \varphi^{-\lambda} \tag{3.6}$$

where

$$C = \left(\int\limits_{X} \varphi^{-\lambda} \, d\mu \right)^{-1} . \tag{3.7}$$

Proof. Indeed, by the Hölder inequality

$$1 \ = \ \int\limits_{X} \alpha \, d\mu \ = \ \int\limits_{X} (\varphi^{-\frac{q}{p}})^{\frac{1}{q}} \, [\varphi \, \alpha^p]^{\frac{1}{p}} \, d\mu \ \leq \ \left[\int\limits_{X} \varphi^{-\frac{q}{p}} \, d\mu \right]^{\frac{1}{q}} \cdot \left[\int\limits_{X} \varphi \, \alpha^p \, d\mu \right]^{\frac{1}{p}}$$

and the equality holds if and only if

$$c\cdot\varphi^{-\frac{q}{p}} \;=\; \varphi\cdot\alpha^p \quad a.e.,$$

see e.g. [HLP] or [Ru]. $C = c^{\frac{1}{p}}$ in (3.7), i.e.

$$C \;=\; \left(\int\limits_X \varphi^{-\frac{1}{p-1}}\,d\mu\right)^{-1}$$

and

$$\alpha_0(x) \;=\; \left(\int\limits_X \varphi^{-\frac{1}{p-1}}\,d\mu\right)^{-1}\cdot\varphi^{-\frac{1}{p-1}}(x)\,.$$

3.8. Theorem. *Let D be a domain in $\mathbb{R}^n$, $n\geq 2$, $x_0\in\overline{D}$, and let $Q: D\to(0,\infty)$ be a measurable function. A homeomorphism $f: D\to\mathbb{R}^n$ is a lower $Q-$homeomorphism at x_0 if and only if*

$$M(f\Sigma_\varepsilon) \;\geq\; \int\limits_\varepsilon^{\varepsilon_0}\frac{dr}{||\,Q||_{\,n-1}(r)} \qquad \forall\,\varepsilon\in(0,\varepsilon_0) \tag{3.9}$$

where

$$0<\varepsilon_0<d_0 \;=\; \sup_{x\in D}\,|x-x_0| \;=\; \sup_{x\in\partial D}\,|x-x_0|\;, \tag{3.10}$$

Σ_ε denotes the family of all the intersections of D with the spheres $S(r)=\{x\in\mathbb{R}^n: |\,x-x_0|=r\}$, $r\in(\varepsilon,\varepsilon_0)$ and

$$||\,Q||_{\,n-1}(r) = \left(\int\limits_{D(r)} Q^{n-1}\;d\mathcal{A}\right)^{\frac{1}{n-1}} \tag{3.11}$$

is the $L_{n-1}-$norm of Q over $D(r)=\{x\in D: |\,x-x_0|=r\}=D\cap S(r)$. The infimum of the expression from the right in (1.19) is attained only for the function

$$\varrho_0(x) \;=\; ||\,Q||^{-1}_{\,n-1}(|\,x|)\cdot Q^{n-1}(x)\,.$$

Proof. Note that, in view of the Lusin theorem, in (1.19)

$$\inf_{\varrho\in adm\Sigma_\varepsilon}\int\limits_{R_\varepsilon} Q^{-1}(x)\;\varrho^n(x)\;dm(x) \;=\; \inf_{\varrho\in ext\,adm\Sigma_\varepsilon}\int\limits_{R_\varepsilon} Q^{-1}(x)\;\varrho^n(x)\;dm(x)\;,$$

see (1.16) for the definition of $ext\,adm\,\Sigma_\varepsilon$. Moreover, for every $\varrho\in ext\,adm\,\Sigma_\varepsilon$,

$$A(r) \;=\; \int\limits_{D(r)}\varrho^{n-1}\;d\mathcal{A}\;\neq\;0 \qquad a.e.$$

is a measurable function in the parameter r, say by the Fubini theorem. Thus, we may request the equality $A(r)\equiv 1$ a.e. instead of (1.16) and

$$\inf_{\varrho\in ext\,adm\Sigma_\varepsilon}\int\limits_{R_\varepsilon} Q^{-1}(x)\;\varrho^n(x)\;dm(x) \;=\; \int\limits_\varepsilon^{\varepsilon_0}\left(\inf_{\alpha\in I(r)}\int\limits_{D(r)} Q^{-1}(x)\;\alpha^p(x)\;d\mathcal{A}\right)dr$$

where $p = n/(n-1) > 1$ and $I(r)$ denotes the set of all measurable function α on the surface $D(r) = S(r) \cap D$ such that

$$\int\limits_{D(r)} \alpha \; d\mathcal{A} \;=\; 1\,.$$

Hence Theorem 3.8 follows by Lemma 3.1 with $X = D(r)$, the $(n-1)$–dimensional area as a measure μ on X, $\varphi = \frac{1}{Q}|_{D(r)}$ and $p = n/(n-1) > 1$.

3.12. Corollary. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in \overline{D}$, $Q : D \to (0,\infty)$ a measurable function and let $f : D \to \mathbb{R}^n$ be a lower $Q-$homeomorphism at x_0. Then*

$$M(f\Sigma_\varepsilon) \;\geq\; \omega_{n-1}^{\frac{1}{1-n}} \int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{r \cdot q_{n-1}(r)} \qquad \forall\, \varepsilon \in (0, \varepsilon_0) \tag{3.13}$$

where

$$q_{n-1}(r) \;=\; \left(⨍_{S(r)} q^{n-1}\; d\mathcal{A} \right)^{1/(n-1)} \tag{3.14}$$

where

$$q(x) \;=\; \begin{cases} Q(x), & x \in D\,, \\ 0, & x \in \mathbb{R}^n \setminus D\,. \end{cases} \tag{3.15}$$

4 Estimates of distortion under hyper $Q-$homeomorphisms

In what follows, we use the **spherical (chordal) metric** $h(x,y) = |\,\pi(x) - \pi(y)|$ in $\overline{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ where π is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}\,e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$:

$$h(x,\infty) \;=\; \frac{1}{\sqrt{1+|\,x|^2}}\,, \quad h(x,y) = \frac{|\,x-y|}{\sqrt{1+|\,x|^2}\sqrt{1+|\,y|^2}}\,, \quad x \neq \infty \neq y\,. \tag{4.1}$$

Thus, by definition $h(x,y) \leq 1$ for all x and $y \in \overline{\mathbb{R}^n}$. The **spherical (chordal) diameter** of a set $E \subset \overline{\mathbb{R}^n}$ is

$$h(E) \;=\; \sup_{x,\,y\in E} h(x,y). \tag{4.2}$$

Note that

$$h(x,y) \;\leq\; |\,x-y| \tag{4.3}$$

for all x,$y \in \mathbb{R}^n$ and

$$h(x,y) \;\geq\; \frac{1}{2}\,|\,x-y| \tag{4.4}$$

for all x and y in the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ with the equality in (4.4) on $\partial\,\mathbb{B}^n$.

4.5. Lemma. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $f : D \to \mathbb{R}^n$ be a lower $Q-$homeomorphism at $x_0 \in D$ and let $0 < \varepsilon < \varepsilon_0 < dist(x_0, \partial D)$. Then*

$$h(fS_\varepsilon) \ \leq \ \frac{\alpha_n}{h(fS_{\varepsilon_0})} \cdot exp\left(-\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{r\, q_{n-1}(r)}\right) \tag{4.6}$$

where $\alpha_n = 2\lambda_n^2$ with $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$ and $\lambda_n^{\frac{1}{n}} \to e$ as $n \to \infty$,

$$q_{n-1}(r) \ = \ \left(\fint_{|x-x_0|=r} Q^{n-1}(x)\, d\mathcal{A}\right)^{\frac{1}{n-1}} \ , \tag{4.7}$$

S_ε and S_{ε_0} denote the spheres in $\mathbb{R}^n$ centered at x_0 with radii ε and ε_0, correspondingly.

Proof. Set $E = fS_\varepsilon$ and $F = fS_{\varepsilon_0}$. By the known Gehring lemma

$$cap\, R(E, F) \ \geq \ cap\, R_T\left(\frac{1}{h(E)\, h(F)}\right) \tag{4.8}$$

where $h(E)$ and $h(F)$ denote the spherical diameters of E and F, correspondingly, and $R_T(s)$ is the Teichmüller ring

$$R_T(s) \ = \ \mathbb{R}^n \setminus ([-1, 0] \cup [s, \infty])\, , \quad s \ > \ 1\, , \tag{4.9}$$

see e.g. 7.37 in [Vu_1] or [Ge_2]. It is also known that

$$cap\, R_T(s) \ = \ \frac{\omega_{n-1}}{(\log \Psi(s))^{n-1}} \tag{4.10}$$

where the function Ψ admits the good estimates:

$$s + 1 \ \leq \ \Psi(s) \ \leq \ \lambda_n^2 \cdot (s+1) \ < \ 2\lambda_n^2 \cdot s\, , \quad s \ > \ 1\, , \tag{4.11}$$

see e.g. [Ge_2], p. 225–226, and (7.19) and (7.22) in [Vu_1]. Hence the inequality (4.8) implies that

$$cap\, R(E, F) \ \geq \ \frac{\omega_{n-1}}{\left(\log \frac{2\lambda_n^2}{h(E)\, h(F)}\right)^{n-1}}\, . \tag{4.12}$$

By Theorem 3.13 in [Z] and (3.13) we have

$$cap\, R(E, F) \ \leq \ \frac{1}{M^{n-1}(f\Sigma_\varepsilon)} \ \leq \ \frac{\omega_{n-1}}{\left(\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{r \cdot q_{n-1}(r)}\right)^{n-1}} \tag{4.13}$$

because $f\Sigma_\varepsilon \subset \Sigma(fS_\varepsilon, fS_{\varepsilon_0})$ where $\Sigma(fS_\varepsilon, fS_{\varepsilon_0})$ consists of all $(n-1)-$dimensional surfaces which separate fS_ε and fS_{ε_0}.

Finally, combining (4.12) and (4.13) we obtain (4.6).

5 On removability of isolated singularities

By Theorem 3.8 similarly to the proof of Lemma 4.5 we obtain the following statement.

5.1. Theorem. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$, $Q : D \to (0, \infty)$ be a measurable function and let $f : D \setminus \{x_0\} \to \mathbb{R}^n$ be a lower $Q-$homeomorphism. Suppose that*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{r \cdot q_{n-1}(r)} \;=\; \infty \tag{5.2}$$

where $\varepsilon_0 < dist(x_0, \partial D)$ and

$$q_{n-1}(r) \;=\; \left(\fint_{|x-x_0|=r} Q^{n-1}(x)\; d\mathcal{A} \right)^{\frac{1}{n-1}}, \tag{5.3}$$

then f has a homeomorphic extension to D.

5.4. Corollary. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$ and let $f : D \backslash \{x_0\} \to \mathbb{R}^n$ be a lower $Q-$homeomorphism. If*

$$\fint_{|x-x_0|=r} Q^{n-1}(x)\; d\mathcal{A} \;=\; O\left(\log^{n-1} \frac{1}{r} \right) \tag{5.5}$$

as $r \to 0$ then f has a homeomorphic extension to D.

5.6. Corollary. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$ and let $f : D \backslash \{x_0\} \to \mathbb{R}^n$ be a lower $Q-$homeomorphism. If*

$$\fint_{|x-x_0|=r} Q^{n-1}(x)\; d\mathcal{A} = O\left(\left[\log \frac{1}{r} \cdot \log\log \frac{1}{r} \cdot \ldots \cdot \log \ldots \log \frac{1}{r} \right]^{n-1} \right) \tag{5.7}$$

as $r \to 0$ then f has a homeomorphic extension to D.

5.8. Corollary. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$ and $f : D \backslash \{x_0\} \to \mathbb{R}^n$ a homeomorphism of the class FAD_{n-1}. If*

$$\fint_{|x-x_0|=r} K_O^{n-1}(x, f)\; d\mathcal{A} \;=\; O\left(\log^{n-1} \frac{1}{r} \right) \tag{5.9}$$

as $r \to 0$ then f has a homeomorphic extension to D.

5.10. Remark. In particular, (5.9) holds if

$$K_O(x, f) \;=\; O\left(\log \frac{1}{|x - x_0|} \right) \tag{5.11}$$

as $x \to x_0$.

6 On continuous extension to boundary points

Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a domain. ∂D is said to be **strongly accessible** if, for nondegenerate continua E and F in $\overline{D}$,

$$M(\Delta(E,F;D)) > 0 \tag{6.1}$$

and **weakly flat** if, for nondegenerate continua E and F in $\overline{D}$ with $E \cap F \neq \emptyset$,

$$M(\Delta(E,F;D)) = \infty \tag{6.2}$$

where $\Delta(E,F;D)$ is the family of all paths joining E and F in D. It is known that every weakly flat boundary is strongly accessible, see Lemma 5.6 in [MRSY$_6$].

A domain $D \subset \mathbb{R}^n$, $n \geq 2$, is called **locally connected at** $x_0 \in \partial D$ if x_0 has an arbitrarily small neighborhood U such that $U \cap D$ is connected. Every Jordan domain D in $\mathbb{R}^n$ is locally connected at every point of ∂D, see [Wi], p. 66.

6.3. Lemma. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in \partial D$, $Q : D \to (0,\infty)$ be a measurable function and let $f : D \to \mathbb{R}^n$ be a lower Q–homeomorphism at x_0. Suppose that the domain D be locally connected at x_0 and the domain $D' = f(D)$ has a strongly accessible boundary. If*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{||\,Q||_{\,n-1}(r)} \;=\; \infty \tag{6.4}$$

where

$$0 < \varepsilon_0 < d_0 \;=\; \sup_{x\in D} |x - x_0| \;=\; \sup_{x\in\partial D} |x - x_0| \tag{6.5}$$

and

$$||\,Q||_{\,n-1}(r) = \left(\int\limits_{D\cap S(x_0,r)} Q^{n-1}\, d\mathcal{A} \right)^{\frac{1}{n-1}} , \tag{6.6}$$

then f extends by continuity to x_0.

Proof. We must show that the cluster set $E = C(x_0,f) = \{y \in \overline{\mathbb{R}^n} : y = \lim\limits_{k\to\infty} f(x_k), x_k \to x_0, x_k \in D\}$ is a singleton. Note that E is a continuum because D is locally connected at x_0. Let us assume that the continuum E is not degenerate.

Let Γ_ε be a family of all paths joining the spheres $S_\varepsilon = \{x \in \mathbb{R}^n : |x - x_0| = \varepsilon\}$ and $S_0 = \{x \in \mathbb{R}^n : |x - x_0| = \varepsilon_0\}$.

Arguing similarly to the Section 4 and 5 on the base of Theorem 3.8 we have that $M(f\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$ in view of (6.4).

On the other hand, $M(f\Gamma_\varepsilon) \geq M_0 = M(\Delta(fS_0,E;D')$ and by the strong accessibility of $\partial D'$ we have that $M_0 > 0$. The contradiction disproves the above assumption.

7 On quasiextremal distance domains

A domain $D \subset \mathbb{R}^n$, $n \geq 2$, is called a **quasiextremal distance domain**, abbr. a **QED domain**, if

$$M(\Delta(E,F;\overline{\mathbb{R}^n}) \ \leq \ K \cdot M(\Delta(E,F;D)) \tag{7.1}$$

for some $K \geq 1$ and for all pairs of disjoint continua E and F in D, see [GM]. It is known that the inequality (7.1) also holds in a QED domain for every pair of disjoint continua E and F in $\overline{D}$, see Theorem 2.8 in [HK$_2^*$], p. 173, cf. Lemma 6.11 in [MV], p. 35. The latter implies (7.1) for nondegenerate intersecting continua E and F in $\overline{D}$, too. Hence QED domains have weakly flat boundaries, see (6.2), cf. Lemma 3.1 in [HK$_2^*$], p. 196. Every QED domain is **quasiconvex**, i.e., each pair of points x_1 and $x_2 \in D$ can be joined by a rectifiable arc γ in D such that

$$s(\gamma) \ \leq \ a \cdot |x_1 - x_2|, \tag{7.2}$$

see Lemma 2.7 in [GM], p. 184. Hence D is locally connected at ∂D, cf. also Lemma 2.11 in [GM], p. 187, and [HK$_2^*$], p. 190.

A domain $D \subset \mathbb{R}^n$, $n \geq 2$, is said to be **uniform** if the inequalities (7.2) and

$$\min_{i=1,2} \ s(\gamma(x_i,x)) \ \leq \ b \cdot d(x,\partial D) \tag{7.3}$$

hold for some γ and for all $x \in \gamma$ where $\gamma(x_i,x)$ is the part of γ between x_i and x, see [MS]. Every uniform domain is a QED domain but there exist QED domains which are not uniform, see [GM]. Bounded convex domains provide simple examples of uniform domains.

7.4. Theorem. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in \partial D$, $Q : D \to (0,\infty)$ be a measurable function and let $f : D \to \mathbb{R}^n$ be a lower $Q-$homeomorphism at x_0. Suppose that D and $D' = f(D)$ are QED domains. If*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{||\,Q||_{\,n-1}(r)} \ = \ \infty \tag{7.5}$$

where

$$0 < \varepsilon_0 < d_0 \ = \ \sup_{x\in D} \, |x - x_0| \ = \ \sup_{x\in \partial D} \, |x - x_0| \tag{7.6}$$

and

$$||\,Q||_{\,n-1}(r) = \left(\int\limits_{D\cap S(x_0,r)} Q^{n-1} \, d\mathcal{A} \right)^{\frac{1}{n-1}}, \tag{7.7}$$

then f extends by continuity to x_0.

8 On singular null-sets for extremal distances

A closed set $X \subset \mathbb{R}^n$, $n \geq 2$, is called a **null-set for extremal distances,** abbr. a **NED set**, if

$$M(\Delta(E,F;\mathbb{R}^n)) = M(\Delta(E,F;\mathbb{R}^n \setminus X)) \tag{8.1}$$

for every pair of disjoint continua E and $F \subset \mathbb{R}^n \setminus X$.

8.2. Remark. It is known that, if $X \subset \mathbb{R}^n$ is a NED set, then

$$|X| = 0 \tag{8.3}$$

and X does not locally disconnect $\mathbb{R}^n$, i.e.,

$$dim\, X \ \leq \ n-2\,. \tag{8.4}$$

Conversely, if $X \subset \mathbb{R}^n$ is closed and

$$H^{n-1}(X) = 0\,, \tag{8.5}$$

then X is a NED set, see [Va$_2$].

Here $H^{n-1}(X)$ denotes the $(n-1)$-dimensional Hausdorff measure of a subset X in $\mathbb{R}^n$. We also denote by $C(X,f)$ the **cluster set** of a mapping $f : D \to \overline{\mathbb{R}^n}$ in a set $X \subset \overline{D}$,

$$C(X,f) := \{y \in \overline{\mathbb{R}^n} : y = \lim_{k\to\infty} f(x_k),\ x_k \to x_0 \in X,\ x_k \in D\}\,. \tag{8.6}$$

Note that the complements of NED sets in $\mathbb{R}^n$ are a very particular case of QED domains considered in the previous section. Thus, arguing locally, we obtain by Theorem 7.4 the following statement.

8.7. Theorem. *Let D be a domain in $\mathbb{R}^n$ and let $f : D\setminus X \to \overline{\mathbb{R}^n}$, $n \geq 2$, be lower $Q-$homeomorphism at $x_0 \in X$ where $X \subset D$. Suppose that X and $C(X,f)$ are NED sets. If*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{||\,Q||_{\,n-1}(r)} \ = \ \infty \tag{8.8}$$

where $\varepsilon_0 < dist(x_0, \partial D)$ and

$$||\,Q||_{\,n-1}(r) = \left(\ \int\limits_{|\,x-x_0|=r} Q^{n-1}(x)\ d\mathcal{A}\right)^{\frac{1}{n-1}}\,, \tag{8.9}$$

then f extends by continuity to x_0.

9 Lemma on cluster sets under lower $Q-$homeomorphisms

9.1. Lemma. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, z_1 and z_2 distinct points in ∂D and f a lower $Q-$homeomorphism of D onto D' with $Q \in L^{n-1}(D)$. If D is locally connected at z_1 and z_2 and $\partial D'$ is weakly flat, then*

$$C(z_1, f) \cap C(z_2, f) \;=\; \emptyset\,. \tag{9.2}$$

9.3. Remark. In fact, it is sufficient for (9.2) to request in Lemma 9.1 instead of the condition $Q \in L^{n-1}(D)$ that $Q \in L^{n-1}(D \cap U)$ for some neighborhood U of one of the points z_i, $i = 1, 2$.

Furthermore, it follows from our proof below it is sufficient for (9.2) even that Q is integrable on

$$D(r) \;=\; \{x \in D : |\, x - z_1| = r\} \;=\; D \cap S(z_1, r)$$

for some set of $r < |\, z_1 - z_2|$ of a positive linear measure.

Proof. Without loss of generality, we may assume that the domain D is bounded. Let $d = |\, z_1 - z_2|$. By the Fubini theorem the set

$$E \;=\; \{r \in (0, d) : Q|_{D(r)} \in L^{n-1}(D(r))\}$$

has a positive linear measure because $Q \in L^{n-1}(D)$. Choose ε and $\varepsilon_0 \in (0, d)$ such that

$$E_0 \;=\; \{r \in E : r \in (\varepsilon, \varepsilon_0)\}$$

has a positive measure. The choice is possible because of a countable subadditivity of the linear measure and because of the exhaustion

$$E \;=\; \bigcup_{m=1}^{\infty} E_m$$

where

$$E_m \;=\; \{r \in E : r \in (1/m, d - 1/m)\}\,.$$

Note that each of the spheres $S(z_1, r)$, $r \in E_0$, separates the points z_1 and z_2 in $\mathbb{R}^n$ and $D(r)$, $r \in E_0$, in D. Thus, by Theorem 3.8 we have that

$$M(f\Sigma_\varepsilon) \;>\; 0 \tag{9.4}$$

where Σ_ε denotes the family of all intersections of the spheres

$$S(r) \;=\; S(z_1, r) \;=\; \{x \in \mathbb{R}^n : |\, x - z_1| = r\}\;, \quad r \in (\varepsilon, \varepsilon_0)\,,$$

with D.

For $i = 1, 2$, let C_i be the cluster set $C(z_i, f)$ and suppose that $C_1 \cap C_2 \neq \emptyset$. Since D is locally connected at z_1 and z_2, there exist neighborhoods U_i of z_i such that $W_i = D \cap U_i$ is connected and $U_1 \subset B^n(z_1, \varepsilon)$ and $U_2 \subset \mathbb{R}^n \setminus B^n(z_1, \varepsilon_0)$.

Set $\Gamma = \Gamma(\overline{W_1}, \overline{W_2}; D)$. By (9.4)

$$M(f\Gamma) \ \le\ \frac{1}{M^{n-1}(f\Sigma_\varepsilon)} \ <\ \infty\,, \tag{9.5}$$

see Theorem 3.13 in [Z] and Theorem 5.13 in [Ma], cf. also [Ca], [He], [HK$_2$] and [Sh].

However, $\partial D'$ is weakly flat and $\overline{W_i}$, $i = 1, 2$ are non-degenerate continua in $\overline{D'}$ with a non–empty intersection contradicting (9.5). Thus, the assumption $C_1 \cap C_2 \ne \emptyset$ was not true.

As an immediate consequence of Lemma 9.1 we have the following statement.

9.6. Theorem. *Let D and D' be domains in $\mathbb{R}^n$, $n \ge 2$, D be locally connected on ∂D and $\partial D'$ be weakly flat. If f is a lower $Q-$homeomorphism of D onto D' with $Q \in L^{n-1}(D)$, then f^{-1} has a continuous extension to $\overline{D'}$.*

9.7. Remark. In view of Remark 9.3, really it is sufficient to request in Theorem 9.6 that Q is integrable in a neighborhood of ∂D only.

10 On homeomorphic extension to boundaries

Combining results of Sections 6–9 we obtain the following statements.

10.1. Theorem. *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$, $Q : D \to (0, \infty)$ belong to $L^{n-1}(D)$ and let $f : D \to \mathbb{R}^n$ be a lower Q–homeomorphism in D. Suppose that the domain D be locally connected on ∂D and the domain $D' = f(D)$ have a strongly accessible boundary. If at every point $x_0 \in \partial D$*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{||\,Q||\,_{n-1}(r)} \ =\ \infty \tag{10.2}$$

where

$$0 < \varepsilon_0 < d_0 \ =\ \sup_{x\in D}\,|x - x_0| \ =\ \sup_{x\in\partial D}\,|x - x_0| \tag{10.3}$$

and

$$||\,Q||\,_{n-1}(r) = \left(\int\limits_{D\cap S(x_0,r)} Q^{n-1}(x)\; d\mathcal{A} \right)^{\frac{1}{n-1}}\,, \tag{10.4}$$

then f has a homeomorphic extension to $\overline{D}$.

10.5. Theorem. *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$, $Q : D \to (0, \infty)$ belong to $L^{n-1}(D)$ and let $f : D \to \mathbb{R}^n$ be a lower Q–homeomorphism in D. Suppose that D and $D' = f(D)$ are QED domains. If the condition (10.2) holds at every point $x_0 \in \partial D$, then f has a homeomorphic extension to $\overline{D}$.*

10.6. Theorem. *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$, $Q : D \to (0,\infty)$ belong to $L^{n-1}(D)$ and let $f : D\backslash X \to \overline{\mathbb{R}^n}$, $n \ge 2$, $X \subset D$, be lower $Q-$homeomorphism. Suppose that X and $C(X,f)$ are NED sets. If the condition (10.2) holds at every point $x_0 \in X$ for $\varepsilon_0 < dist(x_0, \partial D)$ where*

$$||\,Q||_{\,n-1}(r) = \left(\int\limits_{|x-x_0|=r} Q^{n-1}(x)\ d\mathcal{A} \right)^{\frac{1}{n-1}}, \tag{10.7}$$

then f has homeomorphic extension to D.

10.8. Remark. The results of the section are valid if, instead of the condition $Q \in L^{n-1}(D)$, either $Q \in L^{n-1}(D \cap U)$ where U is a neighborhood of ∂D or $Q \in L^{n-1}(U)$ where U is a neighborhood of X. By Corollary 5.7 in [IR$_1$], the condition $Q \in L^{n-1}(U)$ in Theorem 10.6 can be omitted at all if $dim\, X = 0$, i.e., if the set X is totally disconnected.

10.9. Corollary. *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$ and let $f : D \to \mathbb{R}^n$ be a homeomorphism of the class FAD_{n-1}. Suppose that the domain D be locally connected on ∂D and the domain $D' = f(D)$ have a strongly accessible boundary. If at every point $x_0 \in \partial D$*

$$K_O(x,f) \ = \ O\left(\log \frac{1}{|\,x-x_0|}\right) \tag{10.10}$$

as $x \to x_0$, then f has a homeomorphic extension to $\overline{D}$.

10.11. Corollary. *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$, and let $f : D \to \mathbb{R}^n$ be a homeomorphism of the class FAD_{n-1}. Suppose that D and $D' = f(D)$ are QED domains. If the condition (10.10) holds at every point $x_0 \in \partial D$, then f has a homeomorphic extension to $\overline{D}$.*

10.12. Corollary. *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$, and let $f : D\backslash X \to \overline{\mathbb{R}^n}$ be a homeomorphism of the class FAD_{n-1}. Suppose that X and $C(X,f)$ are NED sets. If the condition (10.10) holds at every point $x_0 \in X$, then f has a homeomorphic extension to D which belongs to the class FAD_{n-1}.*

10.13. Remark. In particular, the conclusion of Theorem 10.6 and Corollary 10.12 is valid if X is closed set with

$$H^{n-1}(X) \ = \ 0 \ = \ H^{n-1}(C(X,f))\,. \tag{10.14}$$

Thus, the results of the paper extend the well–known Gehring–Martio–Vuorinen theorems for quasiconformal mappings to lower Q–homeomorphisms and, in particular, to homeomorphisms with finite area distortion and, especially, to finitely be–Lipschitz homeomorphisms , see [GM], p. 196, and [MV], p. 36, cf. [Na], [Va$_1$], [Vu$_2$] and [Vu$_3$], and also the corresponding results for Q–homeomorphisms in [MRSY$_6$] and [IR$_1$]–[IR$_2$].

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