On conformal moduli of polygonal quadrilaterals

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Abstract. The change of conformal moduli of polygonal quadrilaterals under some geometric transformations is studied. We consider the motion of one vertex when the other vertices remain fixed, the rotation of sides, polarization, symmetrization, and averaging transformation of the quadrilaterals. Some open problems are formulated.

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1 Introduction

The notion of the conformal modulus of a quadrilateral has found very important applications to several questions in Geometric Function Theory and its applications (cf. [Ah], [AB], [LV], [K], [RO] and the references therein). A simply connected domain of hyperbolic type in the extended complex plane with four distinct marked accessible boundary points a, b, c, d (vertices) is called a quadrilateral. The points define, in this order, a positive orientation with respect to the domain. Following [H], we denote the quadrilateral by $(Q; a, b, c, d)$ and call its sides the boundary arcs between neighbouring vertices, the vertices themselves excluded, and denote these arcs by $(a, b), (b, c), (c, d)$ and (d, a), respectively. Let the function $w = f(z)$ be a one-to-one conformal map of the domain Q onto the rectangle $0 < u < 1, 0 < v < M (w = u + iv)$ with the vertices a, b, c, d corresponding to $0, 1, 1 + iM, iM$. The number M is called the (conformal) modulus of the quadrilateral $(Q; a, b, c, d)$ and we will denote it $M(Q; a, b, c, d)$. Note that $M(Q; a, b, c, d) = 1/M(Q; b, c, d, a)$. The modulus of a quadrilateral is a conformal invariant and agrees with the extremal length of the family of curves joining the sides (a, b) and (c, d) in the domain Q [Ah]. In Physics, the modulus means, for example, the reciprocal electrical resistance (up to a constant multiple) of Q as a metallic plate or an electrical conductor with electrodes $\overline{(b, c)}$ and $\overline{(d, a)}$ (with a constant potential there). We will also need the definition of the modulus of a quadrilateral as the capacity of a condenser [PS]. In what follows we write

$$I(v, B) := \iint_B |\nabla v|^2 dxdy .$$

It is well-known that

$$M(Q; a, b, c, d) = \min I(v, Q), \tag{1.1}$$

where the minimum is taken over all admissible functions v, i.e. real-valued functions, continuous in $\overline{Q}$, satisfying a Lipschitz condition in a neighborhood of every finite point of Q, equal to zero on the side (d, a) and equal to one on the side (b, c). The function $u(z) = Ref(z)$ is called the potential function of the quadrilateral

$(Q; a, b, c, d)$. In the case of smooth sides (a, b) and (c, d) the potential function may be characterized as the function continuous in $\overline{Q}$, harmonic in Q, equal to zero on (d, a), equal to one on (b, c) and such that the normal derivative $\partial u/\partial n = 0$ on the other sides of the quadrilateral. From the Dirichlet principle it follows that

$$M(Q; a, b, c, d) = I(u, Q), \tag{1.2}$$

and $M(Q; a, b, c, d) < I(v, Q)$, if $v \not\equiv u$ (cf. [H, p. 434]). The minimum in (1.1) is called the capacity of the condenser with the plates $\overline{(d, a)}$ and $\overline{(b, c)}$ and the field Q. More general condensers are studied in [D1]. Note that the next two monotonicity properties of the modulus of a quadrilateral follow easily from (1.1) and (1.2) (cf. [D1, p.40], [H, p.436], [B, p.128]).

Property 1. *If for two quadrilaterals $(Q; a, b, c, d)$ and $(Q; a', b', c', d')$ the inclusions $(b, c) \subset (b', c')$ and $(d, a) \subset (d', a')$ hold, then*

$$M(Q; a, b, c, d) \le M(Q; a', b', c', d')$$

with equality if and only if the quadrilaterals coincide.

Property 2. *If for two quadrilaterals $(Q; a, b, c, d)$ and $(Q'; a, b, c, d)$ the domain Q' is an extension of Q through the sides (a, b) and (c, d), then*

$$M(Q; a, b, c, d) \le M(Q'; a, b, c, d)$$

with equality only in the case $Q = Q'$.

A consequence of the second property is that extension of the domain Q across the sides (b, c) and (d, a) decreases the modulus of the quadrilateral $(Q; a, b, c, d)$. In what follows, unless otherwise stated, we will understand that the quadrilaterals are polygonal, i.e. quadrilaterals with (linear) intervals as sides. In this case the vertices a, b, c, d uniquely define a domain Q however, it will be convenient to adhere to the above notation $(Q; a, b, c, d)$. In spite of the considerable interest of the modulus of quadrilateral, and the apparent simplicity of a polygonal quadrilateral, the behavior of the conformal modulus under geometric transformations has been investestigated very little in the literature (cf. [H, p.428], [K, p.115], [PS, p.188], [B, p.128].) The purpose of this work is to partially fill this gap. The motivation of this paper is also connected with the well-known Schwarz-Christoffel formula [AQVV]. Numerical implementation of this formula in [HVV] enabled one to explore the values of the modulus of a quadrilateral and led to a number of conjectures which will be settled here. In particular, we study the variation of the modulus of a quadrilateral when one of its vertices is moving, the decrease of the modulus under polarization, symmetrization, and averaging transformation of the quadrilaterals.

2 The motion of one vertex

It is clear that every transformation of a quadrilateral may be considered as a motion of its vertices. In this section we consider some of the simplest motions of one of the vertices, when the other vertices remain fixed.

2.1. Theorem. *Let $(Q; a, b, c, d)$ be a quadrilateral for which the interior angle at the vertex a is at most equal to π and let the point a' be a point of the side (a, b) such that the segment $(d, a') \subset Q$. Then*

$$M(Q; a, b, c, d) < M(Q'; a', b, c, d)\,. \tag{2.2}$$

If the quadrilateral $(Q; a, b, c, d)$ is convex and the point a', $a' \neq a$, lies in the closed set, whose boundary consists of the side (a, b) and the linear extensions of the sides (d, a) and (b, c) over the points a, b, respectively, up to the point of their intersection (which may be ∞), then the inequality (2.2) is again valid.

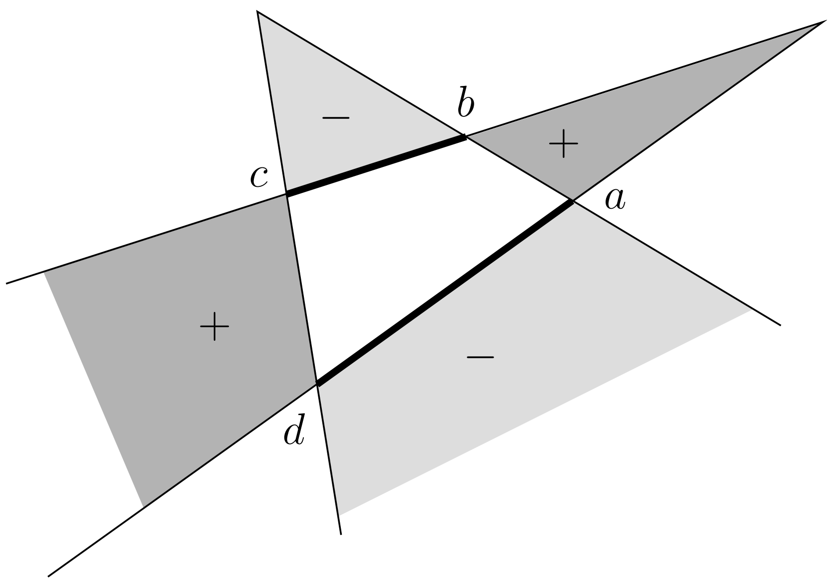

Figure 1: The modulus increases (decreases), when a vertex moves into a region marked with + (-).

Proof. In the first case we consider the nonpolygonal quadrilateral $(Q; a', b, c, d)$ and the polygonal quadrilateral $(Q'; a', b, c, d)$. Applying Properties 1 and 2 we have

$$M(Q; a, b, c, d) < M(Q; a', b, c, d) < M(Q'; a', b, c, d)\,.$$

In the second case we denote by Q'' the domain obtained by attaching to the domain Q the triangle $aa'b$ with its side (a, b). Then $(Q''; a, b, c, d)$ is not polygonal. Again by Properties 1 and 2

$$M(Q; a, b, c, d) < M(Q''; a, b, c, d) < M(Q''; a', b, c, d)$$
$$< M(Q'; a', b, c, d)\,.$$

□

Figure 1 displays the four convex sets, each bounded by a side of the quadrilateral and by the linear extensions of its two neighboring sides, and marked by + or −. In accordance with Theorem 2.1 the modulus decreases (increases) if the vertex moves from its current location into a region marked with − (+).

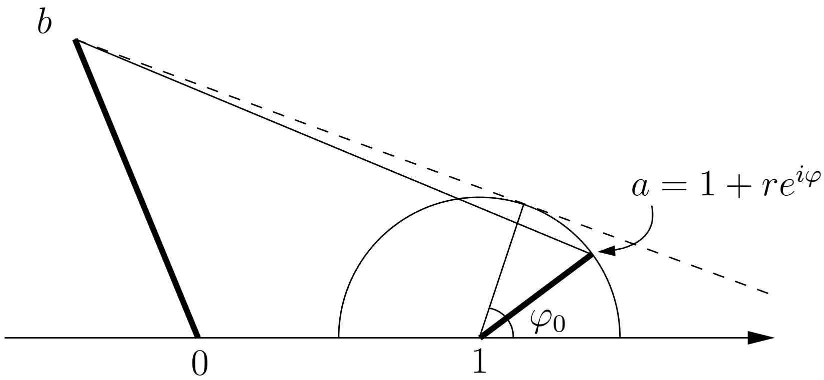

Figure 2: The modulus strictly increases on $[0, \varphi_0]$.

2.3. Corollary. *Let $0 < r < 1, Re b < 1 + r, Im b > 0, |b - 1| > r$, and let $1 + re^{i\varphi_0}$ be the point of tangency of the circle $|z - 1| = r$, closest to $1 + r$, with the line going through the point b. Then the function $M(Q; 1 + re^{i\varphi}, b, 0, 1)$ strictly increases on $[0, \varphi_0]$.*

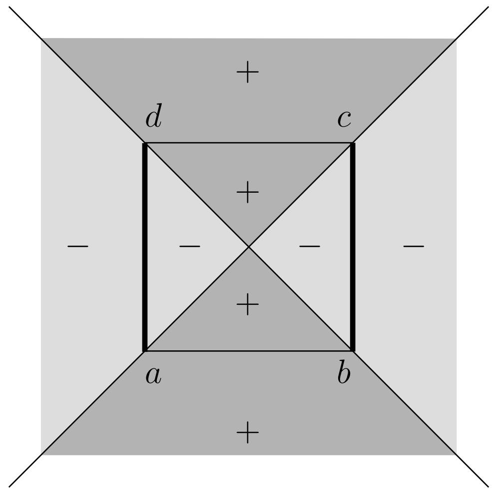

Figure 3: The regions of increase (+) and decrease (-) of modulus in the case of a square.

In the particular case, when Q is a square, Theorem 2.1 together with the well-known observation that the modulus of a quadrilateral, symmetric with respect to a diagonal, is equal to one [H, p. 433], give a complete description of those sets, into which the displacement of one of the vertices increases or decreases the numerical value of the conformal modulus of the square (Figure 3).

In the general case, the description of such sets seems to be difficult. We only give the following result.

2.4. Theorem. *For fixed $M > 0, \varphi \in (0, \pi/2)$ we have the formula*

$$M(Q(\varphi, h); M, M + i, i, he^{i\varphi}) = M + \frac{1}{2}(M \sin\varphi - \cos\varphi)h + o(h) \tag{2.5}$$

when $h \to 0$.

Proof. Let $h \in (0, \min\{1, M\})$, $l_1 = (he^{i\varphi}, M)$, $l_2 = (i, he^{i\varphi})$; let Δ_1 be a triangle with vertices at the points $0, M, he^{i\varphi}$, let Δ_2 be a triangle with vertices at the points $0, he^{i\varphi}, i$; let $u(x,y)$ be the potential function of the quadrilateral $(Q(\varphi,h); M, M+i, i, he^{i\varphi})$ and let M_1 be the modulus of this quadrilateral; and finally let $\partial/\partial n$ stand for the differentiation in the direction orthogonal to the boundary and pointing into the domain. Green's formula gives

$$0 = \int_{\partial Q(\varphi,h)} \left(y\frac{\partial u}{\partial n} - u\frac{\partial y}{\partial n} \right) ds$$

$$= \int_{l_1} y\frac{\partial u}{\partial n} ds - M_1 - \int_{l_2} u\frac{\partial y}{\partial n} ds + M$$

$$= \int_{l_1} y\frac{\partial (u-y)}{\partial n} ds + I(y, \Delta_1) - M_1 - \int_{l_2} (u-y)\frac{\partial y}{\partial n} ds - I(y, \Delta_2) + M + O(h)$$

$$= \frac{1}{2} M\, h\, \sin\varphi - M_1 - \frac{1}{2} h \cos\varphi + M + O(h)$$

when $h \to 0$. □

From Theorem 2.4 it follows that in the case of a rectangle in a neighborhood of the vertex $z = 0$ we have the situation depicted in Figure 4.

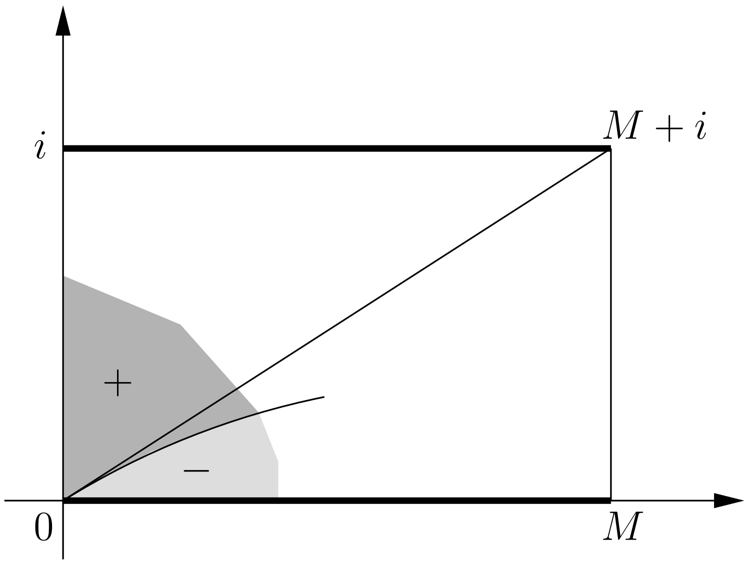

Figure 4: The change of modulus in a neighborhood of the vertex $z = 0$ in the case of a rectangle.

3 Polarization of quadrilaterals

The title of this section refers to a transformation of condensers. In the particular case of quadrilaterals it is expedient to define this transformation directly, not via transformation of the plates as it was carried out in [D2], for example. Let the vertices a, b of the quadrilateral $(Q; a, b, c, d)$ be located on a ray emanating from the origin and the vertices c, d on the ray obtained from the first one by reflection in the real axis. We also allow for the limiting case when the angle between the rays is zero; then both lines are parallel to the x -axis, one of them in the upper, the other one in the lower half plane (Figure 5).

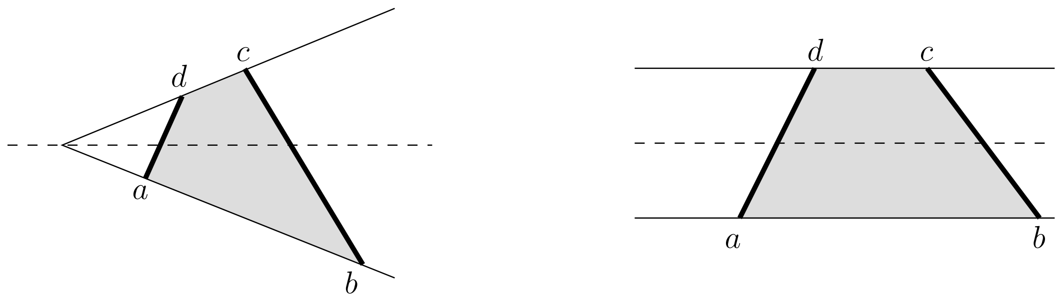

Figure 5: Before polarization.

We say that the result of the polarization of the quadrilateral $(Q; a, b, c, d)$ (with respect to real axis) is the quadrilateral $(PQ; a, \overline{c}, \overline{b}, d)$. Figure 6 displays the resulting polarized quadrilaterals in the both cases.

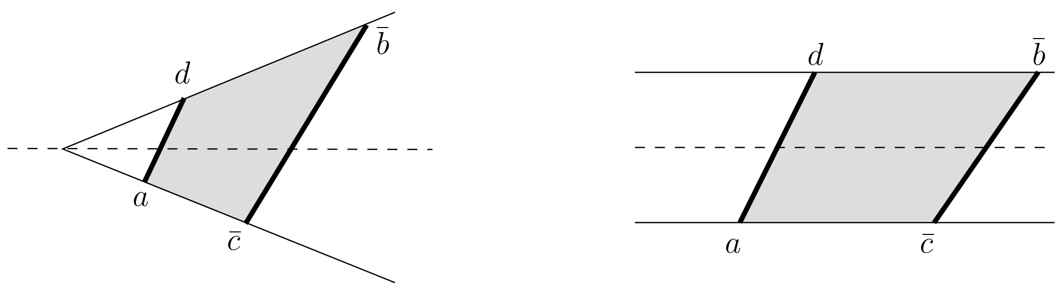

Figure 6: After polarization.

3.1. Theorem. *If the points a, b, c, d are as above, and if $Re a \le Re d$, $Re c < Re b$, (or $Re a < Re d$, $Re c \le Re b$,) then the following inequalities hold*

$$M(Q; a, b, c, d) > M(PQ; a, \overline{c}, \overline{b}, d) ,$$

$$M(Q'; d, c, b, a) > M(PQ'; d, \overline{b}, \overline{c}, a) .$$

Proof. It is enough to consider the case when $Im a < 0$, $Im b < 0$, and $Re a < Re b$ (Fig. 5) . Let $u(z)$ be the potential function of the quadrilateral $(Q; a, b, c, d)$. We extend it by continuity to be equal to zero or one in the intersection with the domain $Re a \le Re z \le Re b$, respectively, with the sector or the horizontal strip, on the boundary of which the vertices a, b, c, d are located. Following the proof of Theorem 1.1 of [D2] we introduce an auxiliary function

$$Pu(z) = \begin{cases} \min\{u(z), u(\overline{z})\}, & Im z \ge 0 , \\ \max\{u(z), u(\overline{z})\}, & Im z \le 0 , \end{cases}$$

for $z \in \overline{PQ}$. This function is admissible for the quadrilateral $(PQ; a, \overline{c}, \overline{b}, d)$. Therefore we obtain from (1.1) and (1.2) as in [D2]

$$M(Q; a, b, c, d) = I(u, Q) = I(Pu, PQ) \geq M(PQ; a, \overline{c}, \overline{b}, d)\,.$$

The equality is possible only in the case when the function $Pu(z)$ coincides with the potential function of the quadrilateral $(PQ; a, \overline{c}, \overline{b}, d)$. In this case, by virtue of uniqueness $Pu(z) \equiv u(z), z \in Q$, which contradicts the hypothesis of the theorem. Thus, the first inequality of Theorem 3.1 is proved. The proof of the second inequality is analogous, the only difference being that the potential function must be extended in the intersection of the set $\{z : Rez \leq Red\} \cup \{z : Rez \geq Rec\}$ correspondingly with the sector or the horizontal strip, and in the definition of the function $Pu(z)$ the places of the max and min must be exchanged. The theorem is proved. □

4 Continuous symmetrization

The symmetrization of quadrilaterals was introduced by Polya and Szegö [PS, p. 188]. They also consider continuous symmetrization of ring condensers with the additional property that the domains bounded by the level curves of the potential functions are convex [PS, p. 200]. Combining these notions we are led to continuous symmetrization of quadrilaterals. Let $\alpha, \beta, \gamma, \delta, \lambda$ be real numbers with $\alpha < \beta, \gamma > \delta, 0 \leq \lambda \leq 1$. We define

$$\alpha(\lambda) = \alpha - \lambda(\alpha + \beta)/2, \quad \beta(\lambda) = \beta - \lambda(\alpha + \beta)/2,$$

$$\gamma(\lambda) = \gamma - \lambda(\gamma + \delta)/2, \quad \delta(\lambda) = \delta - \lambda(\gamma + \delta)/2,$$

$$\mathbb{Q}(\lambda) = (Q(\lambda); 1 + i\alpha(\lambda), 1 + i\beta(\lambda), i\gamma(\lambda), i\delta(\lambda)).$$

When the parameter λ varies from 0 to 1 the quadrilateral $\mathbb{Q}(0)$ continuously transforms onto the trapezoid $\mathbb{Q}(1)$ which is the result of the Steiner symmetrization of $\mathbb{Q}(0)$ with respect to the real axis.

The following assertion is due to Polya and Szegö.

4.1. Theorem. *The function $M\mathbb{Q}(\lambda)$ monotoneously decreases on the segment* $[0, 1]$.

Proof. It is enough to prove that $M\mathbb{Q}(0) \geq M\mathbb{Q}(\lambda)$ for every $\lambda \in [0, 1]$. Following [PS, p. 188] we translate the domain $Q(0)$ up along the imaginary axis such that the image of the domain under the translation, $\widetilde{Q}(0)$, does not intersect with $Q(0)$. Let $u(z)$ be the potential function of the quadrilateral $\mathbb{Q}(0)$, and $\widetilde{u}(z)$ be the potential function corresponding to the quadrilateral $\widetilde{\mathbb{Q}}(0)$. Let the function $v(z) = u(z)$ for $z \in Q(0), v(z) = 1 - \widetilde{u}(z)$ for $z \in \widetilde{Q}(0)$ and let $v(z)$ be defined by continuity, equal to zero or one in the strip $S = \{z : 0 \leq Rez \leq 1\}$, respectively. Finally let $v_\lambda(z)$ be the result of the continuous symmetrization of $v(z)$ [PS, p. 203]. We have

$$2M\mathbb{Q}(0) = I(u, Q(0)) + I(\widetilde{u}, \widetilde{Q}(0)) = I(v, S)\,.$$

According to Polya and Szegö

$$I(v, S) \geq I(v_\lambda, S). \tag{4.2}$$

It remains to observe that the function v_λ is admissible for two nonintersecting quadrilaterals, which are obtained from $\mathbb{Q}(\lambda)$ by translation and therefore have moduli equal to its modulus. The proof is complete. □

4.3. Remark. For the proof of (4.2) it is essential that every line $Re z = x, 0 < x < 1$, intersects every level curve of the function $u(z)$ only in one point. This property of the level curve for polygonal quadrilaterals is easily established by use of standard methods of the theory of functions [N, p. 221].

4.4. Remark. Polarization of a trapezoid with two opposite sides of equal length yields a parallelogram. Therefore the superposition of continuous symmetrization and polarization gives a continuous transformation of a quadrilateral of the form $\mathbb{Q}(0)$ onto a parallelogram having an area equal to the area of $Q(0)$. If this parallelogram is again symmetrized we obtain a rectangle. At each step the modulus of the quadrilateral does not increase. The fact that the modulus $M\mathbb{Q}(0)$ is greater than or equal to the modulus of the corresponding rectangle with the same area coincides with the classical isoperimetric inequality.

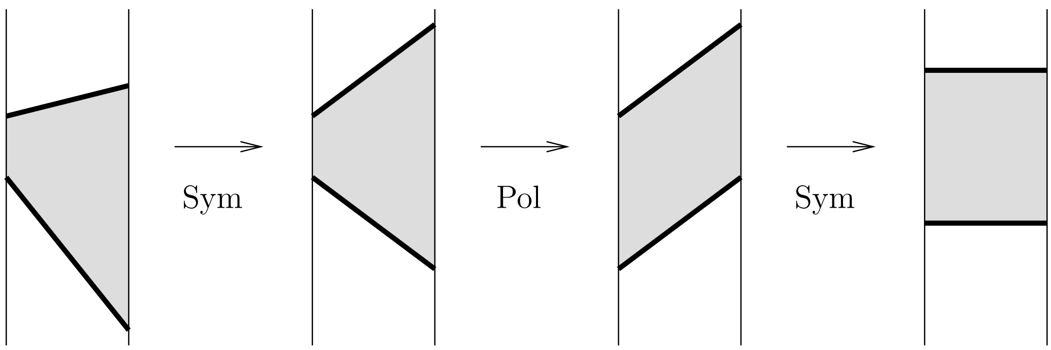

Figure 7: At each step the modulus of the quadrilateral does not increase.

In 1998 E. Reich made an interesting observation that the Polya -Szegö method yields a convex function $M\mathbb{Q}(\lambda)$ in the case when $Q(0)$ is a parallelogram [R]. In fact, even a more general assertion holds, which follows from the linear averaging transformation of Marcus [M].

5 Averaging transformation and rotation of sides

Let $\{\mathbb{Q}_k\}_{k=1}^n$ be a collection of quadrilaterals of the form

$$\mathbb{Q}_k = (Q_k; 1 + i\alpha_k, 1 + i\beta_k, i\gamma_k, i\delta_k)$$

with $\alpha_k < \beta_k, \gamma_k > \delta_k, k = 1, ..., n$, and let $A = \{a_k\}, k = 1, ..., n$, be a set of positive numbers with $\sum_{k=1}^n a_k = 1$. The result of linear averaging transformations of the family of quadrilaterals $\{\mathbb{Q}_k\}_{k=1}^n$, is, by definition, the quadrilateral

$$\mathcal{L}_A(\{\mathbb{Q}_k\}_{k=1}^n) = (Q; 1 + i\sum_{k=1}^n a_k\alpha_k, 1 + i\sum_{k=1}^n a_k\beta_k, i\sum_{k=1}^n a_k\gamma_k, i\sum_{k=1}^n a_k\delta_k) .$$

5.1. Theorem. *The following inequality holds*

$$M\mathcal{L}_A(\{\mathbb{Q}_k\}_{k=1}^n) \leq \sum_{k=1}^n a_k M\mathbb{Q}_k .$$

Proof. We may assume that $\alpha_k \geq 0$, $\delta_k \geq 0$ for $k = 1, ..., n$. Denote by $u_k, k = 1, ..., n$, the potential function corresponding to the quadrilateral $\mathbb{Q}_k$ extended by continuity to the half strip $\{z : 0 \leq Re z \leq 1, Im z \geq 0\}$ with values zero and one. Let

$$u^* = \mathcal{L}_A(\{u_k\}_{k=1}^n) ,$$

be the result of the linear averaging transformation of Marcus of the family of functions $\{u_k\}_{k=1}^n$. By [M, Theorem 1.1]

$$I(u^*, Q) \leq \sum_{k=1}^n a_k I(u_k, Q_k) = \sum_{k=1}^n a_k M\mathbb{Q}_k .$$

It remains to observe that the function u^* is admissible for the quadrilateral $\mathcal{L}_A(\{\mathbb{Q}_k\}_{k=1}^n)$. The proof is complete. □

5.2. Corollary. *The function* $M(y) = M(Q(y); 1 + i\alpha, 1 + iy, i\gamma, i\delta), \gamma > \delta$, *is a decreasing and convex function on the interval* (α, ∞).

5.3. Theorem. *For real numbers* γ *and* $\beta, 0 < \gamma \leq \beta$, *the function*

$$q_1(y) = M(Q_1(y); 1 + iy, 1 + i\beta, i\gamma, -iy)$$

has the following properties:

1) $q_1(y)$ *decreases on* $(-\gamma, 0)$,

2) for every $y \in (-\gamma, 0)$ *we have* $q_1(y) > q_1(-y)$,

3) $q_1(y)$ *is convex on* $(-\gamma, \beta)$.

Proof. The property 3) follows from Theorem 5.1. The property 2) is a consequence of Theorem 3.1. Finally, the property 1) is a corollary to 2) and 3). □

Note that when y increases from $-\gamma$ to β the side $(-iy, 1+iy)$ of the quadrilateral $(Q_1(y); 1 + iy, 1 + i\beta, i\gamma, -iy)$ turns around the point $z = 1/2$ counterclockwise.

5.4. Theorem. *For real numbers* φ, r_1, r_2 *with* $0 < \varphi < \pi/2, 1/\cos\varphi < r_1 \leq r_2$, *the function*

$$q_2(y) = M\,\mathbb{Q}(r) , \quad \mathbb{Q}(r) = (Q_2(r); re^{-i\varphi}, r_2e^{-i\varphi}, r_1e^{i\varphi}, (2\cos\varphi - 1/r)^{-1}e^{i\varphi})$$

has the following properties:

1) $q_2(y)$ *decreases on* $((2\cos\varphi - 1/r_1)^{-1}, 1/\cos\varphi)$,

2) for every $r \in ((2\cos\varphi - 1/r)^{-1}, 1/\cos\varphi)$ *, we have* $q_2(r) > q_2((2\cos\varphi - 1/r)^{-1})$ *,*
3) the function $q_2(1/p)$ *is convex on* $(1/r_2, 2\cos\varphi - 1/r_1)$ *.*

Proof. Let $w = F(z)$ be a regular branch of the function $w = \frac{i}{2\varphi}\ln z + \frac{1}{2}$, which maps the angle $|\arg z| < \varphi$ onto the strip $0 < Rew < 1$. It is clear what is understood by the image of the quadrilateral $\mathbb{Q}(r)$ under the mapping $w = F(z)$. Applying to the potential functions of the (nonpolygonal) quadrilaterals $F\mathbb{Q}(r')$, $F\mathbb{Q}(r'')$ the linear averaging transformation of Marcus [M], in the same way as in the proof of Theorem 5.1 we obtain the inequality

$$MF\mathbb{Q}(((1/r' + 1/r'')/2)^{-1}) \leq \frac{1}{2}(MF\mathbb{Q}(r') + MF\mathbb{Q}(r'')) .$$

This inequality reduces to property 3). Property 2) follows from Theorem 3.1 and property 1) is a consequence of 2) and 3). The proof is complete. □

This result characterizes the change of the modulus of the quadrilateral $\mathbb{Q}(r)$ under the rotation of one of its sides around the point $z = 1$.

6 Examples and open problems

The properties of the moduli of quadrilaterals studied in Sections 1-5 enable us to answer many questions about the behavior of these moduli under geometric transformations. Complementing our earlier examples we now a few additional remarks.

6.1. Question . Let $a, b \in C$ with $\Im a > 0, \Im b > 0$ and assume that $(a, b, 0, 1)$ determines the vertices of a quadrilateral and $\arg b \in (\pi/2, \pi)$, $\arg(a-1) \in (0, \pi/2)$. Is it true that

$$M(Q; a, b, 0, 1) \leq M(Q'; 1 + i|a-1|, i|b|, 0, 1)?$$

Properties 1 and 2 in Section 1 give an affirmative answer to this question.

6.2. Question . Let $h, k > 0, t \in R$ and

$$g(t) = M(Q; 1 + i(t + 2k), ih, -ih, 1 + it).$$

Is it true that g attains its largest value when $t = -k$?

An affimative answer to this question follows from the Steiner symmetrization (see Section 4).

On the other hand, some problems for the moduli of quadrilaterals, involving length and area remain open. We formulate two of them.

6.3. Open problem . Let $a, b \in C$ with $\Im a > 0, \Im b > 0$ and assume that $(a, b, 0, 1)$ determines the vertices of a quadrilateral and $\beta \equiv \arg b > \alpha \equiv \arg a$ Then the area of the quadrilateral $(Q; a, b, 0, 1)$ is $\text{area}(a, b) \equiv |a|(\sin\alpha + |b|\sin(\beta - \alpha))/2$. Let $2h = |b|, 2k = |a-1|$.

(a) Define t by the condition that

$$(Q; a, b, 0, 1), \text{ and } (Q'; t + ik, ih, -ih, t - ik)$$

have equal areas, i.e. that

$$\text{area}(a,b) = (|b| + |a-1|)t/2$$

Is it true that modulus of $(Q; a, b, 0, 1)$ is smaller than that of $(Q'; t+ik, ih, -ih, t-ik)$?

(b) We could also consider various other choices for t. For instance, we could define t as the distance of the segments $[0,b]$ and $[1,a]$ and ask the same question as in (a).

6.4. Open problem . Let $A = 1 + r\exp(i\alpha)$, $B = s\exp(i\beta)$, with $\arg B > \arg A$ and $\alpha, \beta \in (0,\pi)$. Let

$$c_1 = M(Q; A, B, 0, 1), c_2 = M(Q_1; A, A-1, 0, 1), c_3 = M(Q_3; A, B, 0, A-B)\,.$$

Is it true that

$$\min\{c_2, c_3\} \le c_1 \le \max\{c_2, c_3\}\,?$$

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