A transformation formula for fractional Brownian motion

Céline Jost*

Department of Mathematics and Statistics, P.O. Box 68 (Gustaf Hällströmin katu 2b), 00014 University of Helsinki, Finland

Abstract

We derive a Molchan-Golosov-type integral transform which changes fractional Brownian motion of arbitrary Hurst index K into fractional Brownian motion of index H. Integration is carried out over $[0,t]$, $t>0$. The formula is derived in the time domain. Based on this transform, we construct a prelimit which converges in $L^2(\mathbb{P})$-sense to an analogous, already known Mandelbrot-Van Ness-type integral transform, where integration is over $(-\infty,t]$, $t>0$.

Keywords: Fractional Brownian motion; Integral transform; Fractional calculus; Stochastic integration

1 Introduction

The *fractional Brownian motion with Hurst index* $H \in (0,1)$, or H*-fBm*, is the continuous, centered Gaussian process $(B_t^H)_{t\in\mathbb{R}}$ with $B_0^H = 0$, a.s, and

$$Cov_{\mathbb{P}}(B_s^H, B_t^H) \;=\; \frac{1}{2}\left(|s|^{2H} \;+\; |t|^{2H} \;-\; |t-s|^{2H}\right), \quad s,t\in\mathbb{R}.$$

For $H=\frac{1}{2}$, fractional Brownian motion is standard Brownian motion (sBm) and denoted by W. One of the most importent properties of fBm is the stationarity of its increments. For $H>\frac{1}{2}$, increments are positively correlated and form a long-range dependent sequence, whereas for $H<\frac{1}{2}$, increments are negatively correlated and this sequence exhibits short-range dependence. Furthermore, fBm is H-self-similar, meaning that $(B_{at}^H)_{t\in\mathbb{R}} \overset{d}{=} (a^H B_t^H)_{t\in\mathbb{R}}$ for every $a>0$. FBm was first mentioned and studied by A. Kolmogorov in 1940 under the name of *Wiener spiral*. The modern name *fractional Brownian motion* was

*Tel.: +358-9-19151510; Fax: +358-9-19151400; E-mail address: celine.jost@helsinki.fi

proposed by B. Mandelbrot and J. Van Ness in 1968, when they described fBm by a Wiener integral process of a *fractional integral* kernel, namely

$$B_t^H \;=\; \frac{1}{C'(H)}\int_{\mathbb{R}}\left((t-s)_+^{H-\frac{1}{2}}-(-s)_+^{H-\frac{1}{2}}\right)d\tilde{W}_s,\ a.s,\ t\in\mathbb{R}, \tag{1.1}$$

for some sBm $(\tilde{W}_t)_{t\in\mathbb{R}}$. Here $x_+^a := x^a\cdot 1_{(0,\infty)}(x)$,

$$C'(H) := \left(\int_0^\infty\left((1+s)^{H-\frac{1}{2}}-s^{H-\frac{1}{2}}\right)^2 ds+\frac{1}{2H}\right)^{\frac{1}{2}} = \frac{\Gamma(H+\frac{1}{2})}{(\Gamma(2H+1)\sin(\pi H))^{\frac{1}{2}}}$$

and Γ is the gamma function. In 1969, G. Molchan and J. Golosov represented fBm for positive t alternatively by

$$B_t^H \;=\; \frac{C(H)}{\Gamma(H+\frac{1}{2})}\int_0^t (t-s)^{H-\frac{1}{2}}F(\frac{1}{2}-H,H-\frac{1}{2},H+\frac{1}{2},\frac{s-t}{s})dW_s,\ a.s, \tag{1.2}$$

for some (different) sBm $(W_t)_{t\in[0,\infty)}$, where F is Gauss' hypergeometric function and

$$C(H) \;:=\; \left(\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}\right)^{\frac{1}{2}}.$$

Expressing the integrand in (1.2) involving F is due to L. Decreusefond and A. Üstünel in 1999. Proofs for (1.1) and (1.2) can also be found in [14], p.320-325 and [8], respectively. Due to continuity of fBm, the moving averages on the right-hand side of (1.1) and (1.2) are in fact indistinguishable from B^H. Nowadays, they are called *time domain representations* of fBm. FBm has been studied recently in connection to applications in finance and telecommunications. The filtrations generated by $(B_t^H)_{t\in[0,\infty)}$ and $(W_t)_{t\in[0,\infty)}$ do coincide, whereas this is not the case for the natural filtrations of $(B_t^H)_{t\in\mathbb{R}}$ and $(\tilde{W}_t)_{t\in\mathbb{R}}$. Therefore, in financial modelling, (1.2) seems a priori more convenient than (1.1) for representing fBm in terms of sBm. FBm is also interesting from a theoretical point of view, since it is fairly simple but neither Markov nor semimartingale. The latter fact for example makes integration with respect to it challenging.

The aim of this work is to prove a generalization of (1.2) in the sense that for any $K\in(0,1)$, there exists B^K such that for all $t\in[0,\infty)$ we have a.s. that

$$B_t^H = \frac{C(H)C(K)^{-1}}{\Gamma(H-K+1)}\int_0^t (t-s)^{H-K}F(1-K-H,H-K,1+H-K,\frac{s-t}{s})dB_s^K. \tag{1.3}$$

The integral in (1.3) is a *fractional Wiener integral*. This is in the spirit of [10], where the authors prove the real line analogue of (1.3), i.e. a generalization of (1.1). Namely, given $K\in(0,1)$, they show that there exists $\tilde{B}^K$ such that a.s.

$$B_t^H = \frac{C'(K)\Gamma(H+\frac{1}{2})}{\Gamma(K+\frac{1}{2})C'(H)\Gamma(H-K+1)}\int_{\mathbb{R}}\left((t-s)_+^{H-K}-(-s)_+^{H-K}\right)d\tilde{B}_s^K,\ t\in\mathbb{R}. \tag{1.4}$$

The article is organized as follows: In Section 2, we explain the Special functions involved in the transformation. Section 3 is dedicated to the fractional calculus connected to fractional Wiener integration. In Section 4, we review the construction of the fractional Wiener integral in the time domain. The ingredients to derive formula (1.4) will be repeated briefly within sections 3 and 4. Then in Section 5, we prove transformation formula (1.3). Moreover, we show that transformation (1.4) can be obtained from this by a limiting result. The proof of convergence and the specification of its speed is the content of Section 6.

2 Some Special functions

The proof of the transformation formula heavily relies on the understanding of some special Euler integrals and their respective extensions. We review their definitions, clarify how they relate to each other and state well-known properties relevant for us.

The *gamma function* of $\alpha > 0$ is defined by

$$\Gamma(\alpha) := \int_0^\infty \exp(-u)u^{\alpha-1}du.$$

By partial integration we obtain $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. This recursion is used to extend Γ to all $\alpha \in \mathcal{A} := \mathbb{R} \setminus -\mathbb{N}_0$. We have $\Gamma(1) = 1$ and thus

$$\Gamma(\alpha) = (\alpha-1)!,\ \alpha \in \mathbb{N}. \tag{2.1}$$

Furthermore,

$$\frac{1}{\Gamma(\beta)} := \lim_{\substack{\alpha \to \beta \\ \alpha \in \mathcal{A}}} \frac{1}{\Gamma(\alpha)} = 0,\ \beta \in -\mathbb{N}_0. \tag{2.2}$$

We have the functional equation (see [4], p.3)

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)} \tag{2.3}$$

and the duplication formula (see [4], p.5)

$$\Gamma(2\alpha) = 2^{2\alpha-1}\pi^{-\frac{1}{2}}\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2}). \tag{2.4}$$

The *beta function* of $\alpha, \beta > 0$ is defined by

$$B(\alpha,\beta) := \int_0^1 (1-v)^{\alpha-1}v^{\beta-1}dv. \tag{2.5}$$

By substituting $w := 1-v$ we obtain $B(\alpha,\beta) = B(\beta,\alpha)$. For $\beta > 1$, partial integration yields $B(\alpha,\beta) = \frac{\beta-1}{\alpha}B(\alpha+1,\beta-1) = \frac{\beta-1}{\alpha}\left(B(\alpha,\beta-1) - B(\alpha,\beta)\right)$, hence $B(\alpha,\beta-1)=\frac{\alpha+\beta-1}{\beta-1}B(\alpha,\beta)$. This recursion and symmetry are used to extend B to all $\alpha,\beta\in\mathcal{A}$. B can be expressed in terms of Γ via (see [4], p.9)

$$B(\alpha,\beta) \;=\; \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\; \alpha,\beta\in\mathcal{A}. \tag{2.6}$$

Substituting $u:=(t-s)v+s,\ s<t$, in (2.5) implies

$$\int_s^t (t-u)^{\alpha-1}(u-s)^{\beta-1}du \;=\; B(\alpha,\beta)(t-s)^{\alpha+\beta-1},\; \alpha,\beta>0. \tag{2.7}$$

The *Pochhammer symbol* of $a\in\mathbb{R}$ is defined by

$$(a)_0:=1 \quad \text{and} \quad (a)_k:=a\cdot(a+1)\cdot\ldots\cdot(a+k-1),\; k\in\mathbb{N}.$$

We have

$$(a)_k \;=\; \frac{\Gamma(a+k)}{\Gamma(a)},\; a+k\in\mathcal{A}. \tag{2.8}$$

Moreover,

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{(-1)^k(-n)_k}{k!},\; k,n\in\mathbb{N}_0,\; n\geq k. \tag{2.9}$$

The *Gauss hypergeometric function* of parameters a,b,c and variable $z\in\mathbb{R}$ is defined by the formal power series

$$F(a,b,c,z) \;:=\; {}_2F_1(a,b,c,z) \;:=\; \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}. \tag{2.10}$$

It is necessary to assume that $c\in\mathcal{A}$ for this to make sense, unless a (or b) $\in -\mathbb{N}_0$ with $c\leq a$ (or $c\leq b$). In this case, the series reduces to a polynomial if $c<a$, and to the simpler series $\sum_{k=0}^{\infty}(b)_k\frac{z^k}{k!}$ if $c=a$. If $|z|<1$ or $|z|=1$ and $c-b-a>0$, then the series converges absolutely. If furthermore $c>b>0$ for $z\in[-1,1)$ and $b>0$ for $z=1$, then it can be represented by the Euler integral

$$F(a,b,c,z) \;=\; \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 v^{b-1}(1-v)^{c-b-1}(1-vz)^{-a}dv; \tag{2.11}$$

see [4], p.59. If $c>b>0$ then the expression on the right-hand side of (2.11) is well-defined for all $z\in(-\infty,1)$, and is therefore used as extended definition of F. In order to extend F for fixed $z\in(-\infty,1]$ to more general parameters a,b and c, we consider Gauss' relations for neighbor functions. Functions of type $F(a\pm m,b,c,z)$, $F(a,b\pm m,c,z)$ and $F(a,b,c\pm m,z)$, $m\in\mathbb{N}$, are called *contiguous* to $F(a,b,c,z)$. If $m=1$ then they are also called *neighbors*. For any two neighbors $F_1(z)$, $F_2(z)$ of $F(a,b,c,z)$, one has a linear relation of type

$$A(z)F(a,b,c,z) \;+\; A_1(z)F_1(z) \;+\; A_2(z)F_2(z) \;=\; 0, \tag{2.12}$$

where $A(z), A_1(z)$ and $A_2(z)$ are first degree polynomials in z with coefficients depending on a, b and c. An example is

$$c(1-z)F(a,b,c,z) \; - \; cF(a,b-1,c,z) \; + \; (c-a)zF(a,b,c+1,z) \; = \; 0.$$

See [1], p.558 for all $\left(\begin{smallmatrix} 6 \\ 2 \end{smallmatrix} \right) = 15$ relations. They can be checked easily by using (2.10). We use the neighbor relations to extend F for $z \in (-\infty, 1)$ to all parameters $a, b, c \in \mathbb{R}$ such that $c \in \mathcal{A}$, and for $z = 1$ to all parameters that satisfy $c, c-b-a \in \mathcal{A}$. Among the most important properties of F are the symmetry

$$F(a,b,c,z) \; = \; F(b,a,c,z),$$

the reduction formula

$$F(0,b,c,z) \; = F(a,b,c,0) \; = \; 1 \tag{2.13}$$

and the linear transformation formula

$$F(a,b,c,z) \; = \; (1-z)^{-a} F(a, c-b, c, \frac{z}{z-1}), \; z < 1. \tag{2.14}$$

Especially,

$$F(a,b,b,z) \; = \; (1-z)^{-a}, \; z < 1. \tag{2.15}$$

F is smooth in every parameter and in z. It is easy to prove that

$$\frac{d}{dz} F(a,b,c,z) \; = \; \frac{ab}{c} F(a+1, b+1, c+1, z), \tag{2.16}$$

$$\frac{d^n}{d^n z} \left(z^{a+n-1} F(a,b,c,\pm z) \right) \; = \; (a)_n z^{a-1} F(a+n, b, c, \pm z), \; n \in \mathbb{N}, \tag{2.17}$$

and

$$\frac{d^n}{d^n z} \left(z^{c-1} F(a,b,c,z) \right) \; = \; (c-n)_n z^{c-n-1} F(a, b, c-n, z), \; n \in \mathbb{N}. \tag{2.18}$$

F is left-continuous in $z = 1$ and we have

$$F(a,b,c,1) \; = \; \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} B(b, c-b-a) \; = \; \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}. \tag{2.19}$$

Moreover, for $c \in -\mathbb{N}_0$ we have (see [1], p.556)

$$\begin{aligned} \frac{F(a,b,c,z)}{\Gamma(c)} \; &:= \; \lim_{\substack{x \to c \\ x \in \mathcal{A}}} \frac{F(a,b,x,z)}{\Gamma(x)} \\ &= \; \frac{(a)_{1-c}(b)_{1-c}}{\Gamma(2-c)} z^{1-c} F(a+1-c, b+1-c, 2-c, z). \end{aligned} \tag{2.20}$$

Let $a, b > -1$ and $w < x < y$. Substituting $u := (y-x)v + x$ in (2.11) and using (2.14) implies

$$\int_x^y (y-u)^b (u-w)^c (u-x)^a du$$

$$= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+2+b)} (y-x)^{1+a+b} (x-w)^c F(-c, a+1, a+2+b, \frac{y-x}{w-x}) \quad (2.21)$$

$$= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+2+b)} (y-x)^{1+a+b} (y-w)^c F(-c, b+1, a+2+b, \frac{y-x}{y-w}). \quad (2.22)$$

If $x < y < w$ then we have

$$\int_x^y (y-u)^b (w-u)^c (u-x)^a du$$

$$= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+2+b)} (y-x)^{1+a+b} (w-x)^c F(-c, a+1, a+2+b, \frac{y-x}{w-x}) \quad (2.23)$$

By linearly combining the neighbor relations, we obtain relations of type (2.12), where $F_1(z)$ and $F_2(z)$ are contiguous to $F(a, b, c, z)$ and $A(z)$, $A_1(z)$ and $A_2(z)$ are polynomials in z. An example is

$$-cF(a, b-1, c, z) + (c-b+zb-za)F(a, b, c+1, z) + b(1-z)F(a, b+1, c+1, z) = 0. \quad (2.24)$$

The *third hypergeometric function* is one of P. Appell's four generalizations of Gauss' hypergeometric function to two variables. It is defined by the formal series

$$F_3(a, a', b, b', c, z, z') := \sum_{k,l=0}^{\infty} \frac{(a)_k (a')_l (b)_k (b')_l}{(c)_{k+l}} \frac{z^k}{k!} \frac{z'^l}{l!}.$$

We assume $c \in \mathcal{A}$ so that the summands are well-defined. If $|z|, |z'| < 1$ then the series converges absolutely; see [2], p.74. If additionally $b, b', c-b-b' > 0$ then it has the double Euler integral representation (see [2], p.77)

$$F_3(a, a', b, b', c, z, z') = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')} \cdot \int_0^1 \int_0^{1-v} u^{b-1} v^{b'-1} (1-u-v)^{c-b-b'-1} (1-uz)^{-a} (1-vz')^{-a'} du dv. \quad (2.25)$$

For these parameter values, the right-hand side of (2.25) is well-defined for all $z, z' < 1$, and is therefore used as extension of F_3. By substituting $w := \frac{u}{1-v}$ in (2.25) and using (2.11) we obtain

$$F_3(a, a', b, b', c, z, z') = \frac{\Gamma(c)}{\Gamma(b')\Gamma(c-b')} \cdot \int_0^1 v^{b'-1} (1-vz')^{-a'} (1-v)^{c-b'-1} F(a, b, c-b', (1-v)z) dv. \quad (2.26)$$

Also we have the symmetry relations

$$F_3(a, a', b, b', c, z, z') = F_3(a', a, b', b, c, z', z) = F_3(a', b, b', a, c, z', z). \quad (2.27)$$

By using (2.26) and (2.27) one can derive neighbor relations for F_3, where here it is clear what we mean by neighbors and contiguous functions. For example, we have

$$-F_3(a-1,a',b,b',c,z,z') \;+\; (1-z)F_3(a,a',b,b',c,z,z') \;+ \\ \frac{c-b'-b}{c}zF_3(a,a',b,b',c+1,z,z') + \frac{b'}{c}zF_3(a,a',b,b'+1,c+1,z,z') \;=\; 0.$$

Using these, one can extend F_3 for $z, z' \in (-\infty, 1)$ to all parameters such that $c \in \mathcal{A}$. For $a, c > -1$ and $0 < s < t$, the substitution $u := (t-s)v + s$ in (2.26) yields

$$\int_s^t (u-s)^a u^b (t-u)^c F(d,e,c+1,\tfrac{t-u}{t})du \;=\; \\ \frac{\Gamma(a+1)\Gamma(c+1)}{\Gamma(a+c+2)}(t-s)^{a+c+1}s^b F_3(d,-b,e,a+1,a+c+2,\frac{t-s}{t},\frac{s-t}{s}). \quad (2.28)$$

We have the reduction formulae

$$F_3(a,0,b,b',c,z,z') \;=\; F(a,b,c,z) \quad (2.29)$$

and (see [2], p.79-80)

$$F_3(a,a',b,b',a+a',\frac{s-t}{s},\frac{t-s}{t}) \;=\; (\frac{t}{s})^{b'}F(a,b+b',a+a',\frac{s-t}{s}). \quad (2.30)$$

F_3 is smooth in all arguments. In particular, we have the differentiation formula

$$\tfrac{d}{dz}\Big((1-z)^{-a'}z^{c-1}F_3(a,a',b,b',c,z,\tfrac{z}{z-1})\Big) \;= \\ (c-1)z^{c-2}(1-z)^{-a'}F_3(a,a',b,b',c-1,z,\tfrac{z}{z-1}) \;+ \\ a'z^{c-1}(1-z)^{-a'-1}F_3(a,a'+1,b,b',c,z,\frac{z}{z-1}). \quad (2.31)$$

It is easily derived by using series expansion. By combining neighbor relations linearly, we obtain contiguity relations. An example is

$$cF_3(a,a',b,b',c,z,z') \;=\; cF_3(a,a',b,b'+1,c,z,z') \;- \\ z'a'F_3(a,a'+1,b,b'+1,c+1,z,z'). \quad (2.32)$$

Remark 2.1. All four Euler integrals Γ, B, F and F_3 are analytic when considered as functions of complex arguments. The extensions we defined above are the restrictions of the unique analytic extensions to real arguments.

3 Fractional calculus

Next we review basic Riemann-Liouville fractional calculus and its connection to fBm. A general source on fractional calculus is [13]. See also [9] for details on fractional calculus in the context of fBm. We show that the integrand in (1.2) is a power weighted fractional integral of an indicator. Furthermore, we derive a composition formula for power weighted fractional integrals. After that, we will briefly repeat the fractional calculus for the real line case.

Riemann-Liouville fractional calculus is a generalization of usual calculus. It is based on Cauchy's formula for α-fold integration, which says that for $\alpha \in \mathbb{N}$, f measurable and well-behaved, and $s_\alpha \in \mathbb{R}$ holds

$$\int_{s_\alpha}^{\infty} \cdots \int_{s_2}^{\infty} \int_{s_1}^{\infty} f(u)\, du ds_1 \ldots ds_{\alpha-1} \;=\; \frac{1}{\Gamma(\alpha)} \int_{s_\alpha}^{\infty} f(u)(u-s_\alpha)^{\alpha-1} du. \quad (3.1)$$

This follows from Fubini's theorem and (2.1) by induction. The right-hand side of (3.1) is well-defined not only for $\alpha \in \mathbb{N}$, but for all $\alpha > 0$. This motivates the following definition for an integral over a wedge of *fractional* dimension α.

Definition 3.1. Let $\alpha > 0$. The *(right-sided) Riemann-Liouville fractional integral operator of order* α over the real line is defined by

$$(\mathcal{I}_-^\alpha f)(s) \;:=\; \frac{1}{\Gamma(\alpha)} \int_s^\infty f(u)(u-s)^{\alpha-1} du, \;\; s \in \mathbb{R}.$$

Let $T > 0$. The *(right-sided) Riemann-Liouville fractional integral operator of order* α over $[0,T]$ is defined by

$$(\mathcal{I}_{T-}^\alpha f)(s) := (\mathcal{I}_-^\alpha f 1_{[0,T)})(s), \;\; s \geq 0.$$

Note that $\mathcal{I}_{T-}^\alpha f$ depends on f only via $f 1_{[0,T]}$. So $\mathcal{I}_{T-}^\alpha$ is fact operates on functions $f : [0,T] \to \mathbb{R}$.

Remark 3.2. The *right-sided* indicates that we integrate over the right side of the interval $\mathbb{R}$ and $[0,T]$, respectively, contrary to the so-called *left-sided fractional integral operator*, where integration is carried out over the left side. We neglect this supplement here, since right-sided fractional integrals are the only ones under consideration.

Which functions are in the domain of the linear operators $\mathcal{I}_-^\alpha$ and $\mathcal{I}_{T-}^\alpha$?
· If $\alpha \in (0,1)$, $p \in [1, \frac{1}{\alpha})$ and $f \in L^p(\mathbb{R})$, then $\mathcal{I}_-^\alpha f$ is well-defined (see [13], p.94). Furthermore, $\mathcal{I}_-^1 f$ exists for $f \in L^1(\mathbb{R})$.
· If $\alpha > 0$ and $f \in L^1[0,T]$ (so in particular if $f \in L^p[0,T]$, $p > 1$), then triangle inequality and Fubini's theorem imply that $\mathcal{I}_{T-}^\alpha f$ exists.

$\mathcal{I}_{T-}^\alpha$ (and also $\mathcal{I}_-^\alpha$) is injective, i.e.

$$f \in L^1[0,T] \text{ and } \mathcal{I}_{T-}^\alpha f \equiv 0 \;\Rightarrow\; f(s) = 0, \;\; a.e. \; s \in [0,T], \quad (3.2)$$

where *a.e.* stands for almost everywhere in Lebesgue sense (see [13], Lemma 2.5, p.40). Combining (2.7) and (2.6) yields the composition formulae

$$\mathcal{I}_-^\alpha \mathcal{I}_-^\beta f \;=\; \mathcal{I}_-^{\alpha+\beta} f, \;\; f \in L^1(\mathbb{R}), \; \mathcal{I}_-^\beta f \in L^1(\mathbb{R}), \;\; \alpha, \beta \in (0,1], \quad (3.3)$$

and

$$\mathcal{I}^{\alpha}_{T-}\mathcal{I}^{\beta}_{T-}f \;=\; \mathcal{I}^{\alpha+\beta}_{T-}f,\;\; f\in L^1[0,T],\;\; \alpha,\beta>0. \tag{3.4}$$

3.1 Extension and composition of fractional integrals with power weights

For $t \in (0,T]$ we have

$$(\mathcal{I}^{\alpha}_{T-}f1_{[0,t)})(s) = (\mathcal{I}^{\alpha}_{t-}f)(s) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_s^t f(u)(u-s)^{\alpha-1}du & \text{if } s\in[0,t) \\ 0 & \text{if } s \ge t. \end{cases}$$

Assuming $f \in L^1[s,t]$ and $\alpha > 0$ is vital to ensure the existence of the integral. However, by using extended Euler integrals, we are able to define $\mathcal{I}^{\alpha}_{t-}f$ for certain parametric, non-integrable functions f on the one hand and for negative α on the other hand. Therefore, fix $s \in (0,t)$ and consider first functions of type

$$f_\gamma(u) \;:=\; u^\gamma,\;\; \gamma\in\mathbb{R}.$$

f_γ is bounded on $[s,t]$ and hence $(\mathcal{I}^{\alpha}_{t-}f_\gamma)(s)$ is well-defined for $\alpha > 0$. Using (2.21) and (2.22) we obtain

$$\begin{aligned} (\mathcal{I}^{\alpha}_{t-}f_\gamma)(s) \;&=\; \frac{1}{\Gamma(\alpha)}\int_s^t u^\gamma(u-s)^{\alpha-1}du \\ &=\; \frac{1}{\Gamma(1+\alpha)}(t-s)^\alpha s^\gamma F(-\gamma,\alpha,1+\alpha,\frac{s-t}{s}) \qquad (3.5) \\ &=\; \frac{1}{\Gamma(1+\alpha)}(t-s)^\alpha t^\gamma F(-\gamma,1,1+\alpha,\frac{t-s}{t}). \qquad (3.6) \end{aligned}$$

For $\alpha \le 0$, we define the (improper) fractional integral $(\mathcal{I}^{\alpha}_{t-}f_\gamma)(s)$ by the right-hand side of (3.5) or (3.6), respectively. By (2.20) and (2.15) we have especially

$$(\mathcal{I}^{\alpha}_{t-}\cdot^\gamma)(s) \;=\; (-\gamma)_{-\alpha}s^{\gamma+\alpha},\;\; \alpha\in-\mathbb{N}.$$

Furthermore, (2.13) implies

$$(\mathcal{I}^{\alpha}_{t-}1)(s) \;=\; \frac{1}{\Gamma(1+\alpha)}(t-s)^\alpha,\;\; \alpha\in\mathbb{R}. \tag{3.7}$$

Next, we consider more general functions of type

$$f^{\beta}_{t,\gamma}(u) \;:=\; u^\gamma(\mathcal{I}^{\beta}_{t-}1)(u),\;\; \gamma,\beta\in\mathbb{R}.$$

The integral $(\mathcal{I}^{\alpha}_{t-}f^{\beta}_{t,\gamma})(s)$ converges for $\alpha, \beta+1 > 0$. By (2.21) and (2.22) we have

$$\begin{aligned} (\mathcal{I}^{\alpha}_{t-}f^{\beta}_{t,\gamma})(s) \;&=\; \frac{1}{\Gamma(\alpha)\Gamma(1+\beta)}\int_s^t (t-u)^\beta u^\gamma(u-s)^{\alpha-1}du \\ &=\; \frac{s^\gamma(t-s)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}F(-\gamma,\alpha,\alpha+\beta+1,\frac{s-t}{s}) \qquad (3.8) \\ &=\; \frac{t^\gamma(t-s)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}F(-\gamma,\beta+1,\alpha+\beta+1,\frac{t-s}{t}). \qquad (3.9) \end{aligned}$$

For ($\alpha \leq 0$ or $\beta + 1 \leq 0$) and $\beta \notin -\mathbb{N}$, we define the (improper) fractional integral $(\mathcal{I}^{\alpha}_{t-} f^{\beta}_{t,\gamma})(s)$ by the right-hand side of (3.8) or (3.9), respectively. Let $\beta \in -\mathbb{N}$. By using (3.7) and (2.2) we have $(\mathcal{I}^{\alpha}_{t-} f^{\beta}_{t,\gamma})(s) = 0$ for $\alpha > 0$. For $\alpha \leq 0$ we thus set $(\mathcal{I}^{\alpha}_{t-} f^{\beta}_{t,\gamma})(s) := 0$. Finally, we consider functions of type

$$f^{\beta}_{t,\gamma,\lambda}(u) \ := \ u^{\gamma}(\mathcal{I}^{\beta}_{t-} \cdot^{\lambda})(u), \ \gamma, \beta, \lambda \in \mathbb{R}.$$

If $\beta+1 > 0$ then $f^{\beta}_{t,\gamma,\lambda} \in L^1[s,t]$. Thus $(\mathcal{I}^{\alpha}_{t-} f^{\beta}_{t,\gamma,\lambda})(s)$ converges for $\alpha, \beta+1 > 0$. By using (3.6) and (2.28) we obtain

$$\begin{aligned}(\mathcal{I}^{\alpha}_{t-} f^{\beta}_{t,\gamma,\lambda})(s) \ &= \ \frac{t^{\lambda}}{\Gamma(\alpha)\Gamma(1+\beta)} \int_s^t (u-s)^{\alpha-1} u^{\gamma} (t-u)^{\beta} F(-\lambda, 1, \beta+1, \frac{t-u}{t}) du \\ &= \ \frac{t^{\lambda} s^{\gamma} (t-s)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} F_3(-\lambda, -\gamma, 1, \alpha, \alpha+\beta+1, \frac{t-s}{t}, \frac{s-t}{s}). \qquad (3.10)\end{aligned}$$

For ($\alpha \leq 0$ or $\beta+1 \leq 0$) and $\beta, \alpha+\beta \notin -\mathbb{N}$, we define the (improper) fractional integral $(\mathcal{I}^{\alpha}_{t-} f^{\beta}_{t,\gamma,\lambda})(s)$ by the right-hand side of (3.10). By combining (2.29) and (3.6) we obtain on the one hand

$$(\mathcal{I}^{0}_{t-} f^{\beta}_{t,\gamma,\lambda})(s) \ = \ f^{\beta}_{t,\gamma,\lambda}(s), \ \beta \notin -\mathbb{N}, \qquad (3.11)$$

and on the other hand

$$(\mathcal{I}^{\alpha}_{t-} \mathcal{I}^{\beta}_{t-} f_{\lambda})(s) \ = \ (\mathcal{I}^{\alpha+\beta}_{t-} f_{\lambda})(s) \ = \ (\mathcal{I}^{\beta}_{t-} \mathcal{I}^{\alpha}_{t-} f_{\lambda})(s), \ \alpha, \beta, \alpha+\beta \notin -\mathbb{N}. \qquad (3.12)$$

We can now easily state and prove a reduction formula for the composition of suitably power weighted fractional integrals:

Lemma 3.3 (Composition formula). *Let $\nu, \alpha, \beta \in \mathbb{R}$ with $\beta, \alpha+\beta \notin -\mathbb{N}$. Then*

$$s^{\beta}(\mathcal{I}^{\alpha}_{t-} \cdot^{-\beta-\alpha} \mathcal{I}^{\beta}_{t-} \cdot^{\alpha+\nu})(s) \ = \ (\mathcal{I}^{\alpha+\beta}_{t-} \cdot^{\nu})(s), \ s \in (0,t).$$

Proof. Combine (3.10), (2.27), (2.30) and (3.5). □

3.2 Fractional derivatives

In 3.1, we obtained a definition for fractional integrals of negative order of certain parametric functions in a very natural way. Observing (3.11) and (3.12) suggests that $\mathcal{I}^{-\alpha}_{t-}$ inverts the operator $\mathcal{I}^{\alpha}_{t-}$, i.e. $\mathcal{I}^{-\alpha}_{t-} f$ is indeed a kind of derivative of f. A fractional derivative for arbitrary functions is defined by the formal solution in g of *Abel's integral equation* $\mathcal{I}^{\alpha}_{T-} g = f$ (see [13], p.29-30):

Definition 3.4. Let $\alpha \in (0,1)$. The *Riemann-Liouville fractional derivative operator of order* α over the real line is defined by

$$(\mathcal{D}^{\alpha}_{-} f)(s) := \frac{-d}{ds}(\mathcal{I}^{1-\alpha}_{-} f)(s) = \frac{1}{\Gamma(1-\alpha)} \frac{-d}{ds} \int_s^{\infty} f(u)(u-s)^{-\alpha} du, \ a.e. \ s \in \mathbb{R}.$$

Moreover,

$$\mathcal{D}^0_- f \ := \ f. \tag{3.13}$$

For $\alpha = [\alpha] + \{\alpha\}$, where $[\alpha]$ and $\{\alpha\}$ correspond to integer and fractional part of α, define

$$(\mathcal{D}^\alpha_- f)(s) \ := \ (-1)^{[\alpha]} \frac{d^{[\alpha]}}{ds^{[\alpha]}} (\mathcal{D}^{\{\alpha\}}_- f)(s), \ \textit{a.e. } s \in \mathbb{R}.$$

The *Riemann-Liouville fractional derivative operator of order* $\alpha \geq 0$ over $[0,T]$ is defined by

$$(\mathcal{D}^\alpha_{T-} f)(s) \ := \ (\mathcal{D}^\alpha_- f 1_{[0,T)})(s), \ \textit{a.e. } s > 0.$$

Functions in the domain of $\mathcal{D}^\alpha_-$ must, besides satisfying an integrability condition, be suitably smooth. If $0 < \alpha \leq \beta$ and f is such that $\mathcal{I}^\beta_- f$ exists as a proper fractional integral, then $\mathcal{D}^\alpha_- \mathcal{I}^\beta_- f$ exists. In particular, from (3.3) we obtain

$$\mathcal{D}^\alpha_- \mathcal{I}^\beta_- f \ = \ \begin{cases} \mathcal{I}^{\beta-\alpha}_- f & \text{if } f \in L^1(\mathbb{R}) \text{ and } \alpha < \beta \leq 1 \\ f & \text{if } f \in L^1(\mathbb{R}) \text{ and } \alpha = \beta \leq 1. \end{cases} \tag{3.14}$$

Furthermore,

$$\mathcal{D}^\alpha_{T-} \mathcal{I}^\beta_{T-} f \ = \ \begin{cases} \mathcal{I}^{\beta-\alpha}_{T-} f & \text{if } f \in L^1[0,T] \text{ and } \alpha < \beta \\ f & \text{if } f \in L^1[0,T] \text{ and } \alpha = \beta. \end{cases} \tag{3.15}$$

This is true for $\alpha, \beta \in \mathbb{N}$ simply by (3.1). Using (3.4) we obtain that the claim is true for general fractional parameters as well. As for ordinary calculus, fractional integral and fractional derivative operator do not commute. For details on the composition of $\mathcal{D}^\alpha_-$ and $\mathcal{I}^\beta_-$, see [13], subparagraphs 2.6 and 2.7.

For $t \in (0,T]$ holds $\mathcal{D}^\alpha_{T-} f 1_{[0,t)} = \mathcal{D}^\alpha_{t-} f$. The fractional derivative $(\mathcal{D}^\alpha_{t-} f^\beta_{t,\gamma,\lambda})(s)$, $s \in (0,t)$, is (properly or improperly) defined for $\beta, 1 - \{\alpha\} + \beta \notin -\mathbb{N}$ via (3.10). The following shows that fractional derivatives and fractional integrals of negative order coincide.

Lemma 3.5. *Let $\alpha \geq 0$. For $\gamma, \lambda \in \mathbb{R}$, $\beta, 1 - \{\alpha\} + \beta, \beta - \alpha \notin -\mathbb{N}$, we have*

$$(\mathcal{D}^\alpha_{t-} f^\beta_{t,\gamma,\lambda})(s) \ = \ (\mathcal{I}^{-\alpha}_{t-} f^\beta_{t,\gamma,\lambda})(s), \ \ s \in (0,t).$$

Proof. For $\alpha = 0$, the claim is true by (3.11) and (3.13). For $\alpha \in \mathbb{N}$, we have by using (2.18), (2.9), (2.8) and (3.10) that

$$(\mathcal{D}^\alpha_{t-} f^\beta_{t,\gamma,\lambda})(s) \ = \ \tfrac{t^{\lambda+\beta}(-1)^\alpha}{\Gamma(\beta+1)} \tfrac{d^\alpha}{ds^\alpha} \left(s^\gamma (\tfrac{t-s}{t})^\beta F(-\lambda, 1, 1+\beta, \tfrac{t-s}{t}) \right) \ =$$

$$\tfrac{t^{\lambda+\beta}(-1)^\alpha}{\Gamma(\beta+1)} \textstyle\sum_{k=0}^\alpha \binom{\alpha}{k} (-\gamma)_k (-1)^k s^{\gamma-k} (1+\beta-\alpha+k)_{\alpha-k} \cdot$$

$$(\tfrac{t-s}{t})^{\beta-\alpha+k} F(-\lambda, 1, 1+\beta-\alpha+k, \tfrac{t-s}{t}) \tfrac{(-1)^{\alpha-k}}{t^{\alpha-k}} \ =$$

$\frac{t^\lambda (t-s)^{\beta-\alpha} s^\gamma}{\Gamma(\beta+1-\alpha)} F_3(-\alpha,-\lambda,-\gamma,1,1+\beta-\alpha,\frac{s-t}{s},\frac{t-s}{t}) \;=\; (\mathcal{I}^{-\alpha}_{t-} f^\beta_{t,\gamma,\lambda})(s).$

Using (2.31) and (2.32) yields for $\alpha \in (0,1)$

$(\mathcal{D}^\alpha_{t-} f^\beta_{t,\gamma,\lambda})(s) \;=\; \frac{-d}{ds}(\mathcal{I}^{1-\alpha}_{t-} f^\beta_{t,\gamma,\lambda})(s) \;=$

$\frac{t^{\lambda+\gamma+1-\alpha+\beta}}{\Gamma(2-\alpha+\beta)} \frac{-d}{ds}\left((\frac{s}{t})^\gamma (\frac{t-s}{t})^{1-\alpha+\beta} F_3(-\lambda,-\gamma,1,1-\alpha,2-\alpha+\beta,\frac{t-s}{t},\frac{s-t}{s})\right) \;=$

$\frac{t^{\lambda-\alpha+\beta+\gamma}}{\Gamma(2-\alpha+\beta)}((1-\alpha+\beta)(\frac{t-s}{t})^{-\alpha+\beta}(\frac{s}{t})^\gamma F_3(-\lambda,-\gamma,1,1-\alpha,1-\alpha+\beta,\frac{t-s}{t},\frac{s-t}{s}) +$

$(\frac{t-s}{t})^{1-\alpha+\beta}(\frac{s}{t})^{\gamma-1}(-\gamma) F_3(-\lambda,-\gamma+1,1,1-\alpha,2-\alpha+\beta,\frac{t-s}{t},\frac{s-t}{s})) \;=$

$\frac{t^\lambda s^\gamma (t-s)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)} F_3(-\lambda,-\gamma,1,-\alpha,\beta-\alpha+1,\frac{t-s}{t},\frac{s-t}{s}) \;=\; (\mathcal{I}^{-\alpha}_{t-} f^\beta_{t,\gamma,\lambda})(s).$

By iterating, we obtain the claim for all α with $\{\alpha\} > 0$. $\square$

In view of Lemma 3.5 it is consistent to set

$$\mathcal{I}^{-\alpha}_{T-} \;:=\; \mathcal{D}^\alpha_{T-},\; \alpha \geq 0.$$

3.3 The connection to fBm

For $H \in (0,1)$, define weighted fractional integral operators by

$$(\mathbf{K}^H f)(s) \;:=\; (\mathbf{K}^H_T f)(s) \;:=\; C(H) s^{\frac{1}{2}-H} (\mathcal{I}^{H-\frac{1}{2}}_{T-} \cdot^{H-\frac{1}{2}} f)(s),\; s \in (0,T)$$

and

$$(\mathbf{K}^{H,*} f)(s) \;:=\; (\mathbf{K}^{H,*}_T f)(s) \;:=\; C(H)^{-1} s^{\frac{1}{2}-H} (\mathcal{I}^{\frac{1}{2}-H}_{T-} \cdot^{H-\frac{1}{2}} f)(s),\; s \in (0,T).$$

Note that $\mathbf{K}^H$ is a true weighted integral operator, i.e. of positive order, if and only if $H > \frac{1}{2}$. In the same way, $\mathbf{K}^{H,*}$ is a true weighted integral operator if and only if $H < \frac{1}{2}$. $\mathbf{K}^{\frac{1}{2}}$ and $\mathbf{K}^{\frac{1}{2},*}$ are identity operators. By (3.5) we have

$$(\mathbf{K}^H 1_{[0,t)})(s) \;=\; \frac{C(H)}{\Gamma(H+\frac{1}{2})}(t-s)^{H-\frac{1}{2}} F(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, \frac{s-t}{s}) 1_{(0,t)}(s)$$

and

$$(\mathbf{K}^{H,*} 1_{[0,t)})(s) \;=\; \frac{C(H)^{-1}}{\Gamma(\frac{3}{2}-H)}(t-s)^{\frac{1}{2}-H} F(\frac{1}{2}-H, \frac{1}{2}-H, \frac{3}{2}-H, \frac{s-t}{s}) 1_{(0,t)}(s). \tag{3.16}$$

So for $t \in [0,T]$, (1.2) can be written as

$$B^H_t \;=\; \int_0^T (\mathbf{K}^H 1_{[0,t)})(s) dW_s,\; a.s. \tag{3.17}$$

Lemma 3.6. *Let $H, K \in (0,1)$. Then*

$$(\mathbf{K}^{K,*}\mathbf{K}^{H}1_{[0,t)})(s) = \frac{C(H)}{C(K)}\ s^{1-H-K}(\mathcal{I}^{H-K}_{t-}\cdot^{H+K-1})(s),\ s \in (0,T). \tag{3.18}$$

Moreover,

$$(\mathbf{K}^{K}\mathbf{K}^{K,*}\mathbf{K}^{H}1_{[0,t)})(s) \ = \ (\mathbf{K}^{H}1_{[0,t)})(s),\ s \in (0,T). \tag{3.19}$$

Proof. Let $s \in (0,t)$. By setting $\alpha := \frac{1}{2} - K, \beta := H - \frac{1}{2}$ and $\nu := H + K - 1$ in Lemma 3.3 follows

$$\begin{aligned}(\mathbf{K}^{K,*}\mathbf{K}^{H}1_{[0,t)})(s) &= \frac{C(H)}{C(K)} s^{\frac{1}{2}-K}(\mathcal{I}^{\frac{1}{2}-K}_{t-}\cdot^{K-\frac{1}{2}}\cdot^{\frac{1}{2}-H}\mathcal{I}^{H-\frac{1}{2}}_{t-}\cdot^{H-\frac{1}{2}})(s) \\ &= \frac{C(H)}{C(K)}\ s^{1-H-K}(\mathcal{I}^{H-K}_{t-}\cdot^{H+K-1})(s).\end{aligned}$$

Using this and furthermore setting $\alpha := K - \frac{1}{2}, \beta := H - K$ and $\nu := H - \frac{1}{2}$ in Lemma 3.3 implies

$$\begin{aligned}(\mathbf{K}^{K}\mathbf{K}^{K,*}\mathbf{K}^{H}1_{[0,t)})(s) &= C(H)s^{\frac{1}{2}-K}(\mathcal{I}^{K-\frac{1}{2}}_{t-}\cdot^{K-\frac{1}{2}}\cdot^{1-H-K}\mathcal{I}^{H-K}_{t-}\cdot^{H+K-1})(s) \\ &= C(H)s^{\frac{1}{2}-H}(\mathcal{I}^{H-\frac{1}{2}}_{t-}\cdot^{H-\frac{1}{2}})(s) \\ &= (\mathbf{K}^{H}1_{[0,t)})(s). \quad \square\end{aligned}$$

In particular, Lemma 3.6 says that $\mathbf{K}^{K,*}$ is the inverse operator of $\mathbf{K}^{K}$ on the linear space generated by the indicators. Indeed, for $f := 1_{[0,t)}$, $t \in (0,T]$, we obtain by setting $H = K$ in (3.18) that

$$(\mathbf{K}^{K,*}\mathbf{K}^{K}f)(s) \ = \ f(s),\ s \in (0,T). \tag{3.20}$$

Furthermore, setting $H = \frac{1}{2}$ in (3.19) implies

$$(\mathbf{K}^{K}\mathbf{K}^{K,*}f)(s) \ = \ f(s),\ s \in (0,T). \tag{3.21}$$

Remark 3.7. Let $K > \frac{1}{2}$. If $f \in L^2[0,T]$ then Hölder's inequality implies that $\cdot^{K-\frac{1}{2}}f(\cdot) \in L^1[0,T]$, i.e. $\mathbf{K}^{K}f$ is well-defined. From (3.15) follows that (3.20) holds for all $f \in L^2[0,T]$. If $K < \frac{1}{2}$, we obtain in the same way that (3.21) is true for all $f \in L^2[0,T]$.

3.4 Fractional calculus for the real line analogue

Let $t \in \mathbb{R} \setminus \{0\}$ and set

$$1_{[0,t)} \ := \ -1_{[t,0)},\ t < 0.$$

For $\alpha > 0$ we have

$$(\mathcal{I}^{\alpha}_{-}1_{[0,t)})(s) \ = \ \frac{1}{\Gamma(\alpha+1)}\left((t-s)^{\alpha}1_{(-\infty,t)}(s) \ - \ (-s)^{\alpha}1_{(-\infty,0)}(s)\right). \tag{3.22}$$

For $\alpha \leq 0$, define the (improper) fractional integral $(\mathcal{I}_-^\alpha 1_{[0,t)})(s)$ by the right-hand side of (3.22). If $\alpha, \beta+1 > 0$ then (2.7) and (2.6) imply

$$(\mathcal{I}_-^\alpha \mathcal{I}_-^\beta 1_{[0,t)})(s) \;=\; (\mathcal{I}_-^{\alpha+\beta} 1_{[0,t)})(s), \;\; s \in \mathbb{R}. \tag{3.23}$$

For $\alpha \leq 0$ and $\beta + 1 > 0$, define the (improper) integral $(\mathcal{I}_-^\alpha \mathcal{I}_-^\beta 1_{[0,t)})(s)$ by the right-hand side of (3.23).

Let $\alpha \in (0,1)$. When dealing with fractional Wiener integrals, we need to invert $g = \mathcal{I}_-^\alpha f$ for square-integrable f. However, (3.14) is in general not true for such f. Therefore, the operator $\mathcal{D}_-^\alpha$ must be modified to a more convenient form. Note that $(\mathcal{D}_-^\alpha f)(s) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_0^\infty f(v+s) v^{-\alpha} dv$. We assume that f is differentiable with f' being so well-behaved that

$$\begin{aligned}
(\mathcal{D}_-^\alpha f)(s) &= \frac{-1}{\Gamma(1-\alpha)} \int_0^\infty f'(v+s) v^{-\alpha} dv \\
&= \frac{-\alpha}{\Gamma(1-\alpha)} \int_0^\infty f'(v+s) \int_v^\infty u^{-\alpha-1} du dv \\
&= \frac{-\alpha}{\Gamma(1-\alpha)} \int_0^\infty (f(u+s) - f(s)) u^{-\alpha-1} du.
\end{aligned}$$

This motivates the definition of the following alternative operator:

Definition 3.8. Let $\alpha \in (0,1)$. The *Marchaud fractional derivative operator of order* α over the real line is defined by

$$(\mathbf{D}_-^\alpha f)(s) \;:=\; \lim_{\epsilon \searrow 0} (\mathbf{D}_{-,\epsilon}^\alpha f)(s), \;\; a.e.\ s \in \mathbb{R},$$

where

$$(\mathbf{D}_{-,\epsilon}^\alpha f)(s) \;:=\; \frac{\alpha}{\Gamma(1-\alpha)} \int_\epsilon^\infty (f(s) - f(u+s))\, u^{-\alpha-1} du.$$

Furthermore,

$$\mathbf{D}_-^0 f \;:=\; f.$$

$\mathbf{D}_-^\alpha$ has indeed a different domain than $\mathcal{D}_-^\alpha$: for example $\mathbf{D}_-^\alpha 1 \equiv 0$, whereas $\mathcal{D}_-^\alpha 1$ is not defined. If f is piecewise continuously differentiable with $\mathrm{supp}(f') \subseteq (-\infty, y)$ for some $y \in \mathbb{R}$ and so that $\mathcal{D}_-^\alpha f$ exists, then $\mathbf{D}_-^\alpha f = \mathcal{D}_-^\alpha f$. In particular,

$$\mathbf{D}_-^\alpha \mathcal{I}_-^\beta 1_{[0,t)} = \mathcal{D}_-^\alpha \mathcal{I}_-^\beta 1_{[0,t)}, \;\; \alpha \in [0,1), \;\; \beta > -1. \tag{3.24}$$

The following is taken from [13], Theorem 6.1, p.125.

Theorem 3.9. *Let* $\alpha \in (0,1)$, $p \in [1, \frac{1}{\alpha})$ *and* $f \in L^p(\mathbb{R})$. *If* $g = \mathcal{I}_-^\alpha f$, *then* $\mathbf{D}_-^\alpha g = f$, *i.e.* $\mathbf{D}_-^\alpha \mathcal{I}_-^\alpha f = f$.

This yields the following version of (3.14) for square-integrable functions:

$$\mathbf{D}_-^\alpha \mathcal{I}_-^\alpha f \;=\; f, \;\; \alpha \in (0, \frac{1}{2}), \;\; f \in L^2(\mathbb{R}).$$

Lemma 3.10. *For $\alpha \in [0,1)$ and $\beta > -1$ we have*

$$\mathbf{D}_{-}^{\alpha}\mathcal{I}_{-}^{\beta}1_{[0,t)} \;=\; \mathcal{I}_{-}^{-\alpha}\mathcal{I}_{-}^{\beta}1_{[0,t)}. \tag{3.25}$$

Proof. For $\alpha = 0$ this is trivial. Let $\alpha \in (0,1)$. Clearly, $\mathcal{D}_{-}^{\alpha}\mathcal{I}_{-}^{\beta}1_{[0,t)} = \mathcal{I}_{-}^{\beta-\alpha}1_{[0,t)}$. The claim follows from (3.24). □

Remark 3.11. For $\alpha \in (0,1)$ and $\beta - \alpha \in (-1,1)$, (3.25) can be obtained directly from Theorem 3.9 with $p = 1$, $f = \mathcal{I}_{-}^{\beta-\alpha}1_{[0,t)}$ and $g = \mathcal{I}_{-}^{\beta}1_{[0,t)}$.

In view of Lemma 3.10, we may set

$$\mathcal{I}_{-}^{-\alpha} \;:=\; \mathbf{D}_{-}^{\alpha},\;\; \alpha \in [0,1).$$

Hence (1.1) is equivalent to

$$B_t^H \;=\; \frac{\Gamma(H+\frac{1}{2})}{C'(H)}\int_{\mathbb{R}}(\mathcal{I}_{-}^{H-\frac{1}{2}}1_{[0,t)})(s)d\tilde{W}_s,\;\; a.s,\;\; t \in \mathbb{R}. \tag{3.26}$$

4 Fractional Wiener integrals

First, we review - by combining time domain representation (3.17) with standard Wiener integral - how one can identify a space of deterministic functions f for which the expression $\int_0^T f(s)dB_s^H$ is well-defined. Second, we explain the analogous procedure to obtain integrals over the real line. For details on fractional Wiener integration over $[0,T]$ and $\mathbb{R}$, see [11] and [12], respectively.

4.1 Fractional Wiener integrals over $[0,T]$

Let $\mathcal{E}_T$ denote the space of elementary functions on $[0,T]$, i.e. functions of type

$$f(s) \;:=\; \sum_{i=1}^{m} a_i 1_{[0,t_i)}(s),$$

where $t_1,\ldots,t_m \in [0,T]$ and $a_1,\ldots,a_m \in \mathbb{R}$. Furthermore, let

$$\mathcal{H}_T(B^H) \;:= \overline{\operatorname{span}_{\mathbb{R}}\{B_s^H \mid s \in [0,T]\}} \;\subset\; L^2(\mathbb{P}).$$

This Hilbert space is called *fractional Wiener chaos of first order* over $[0,T]$. Its elements are centered and Gaussian. We define the *fractional Wiener integral* of an elementary function f with respect to B^H in the natural way by the linear injection

$$\begin{aligned} I_T^H \;:\; \mathcal{E}_T \;&\to\; \mathcal{H}_T(B^H) \\ f \;&\mapsto\; I_T^H(f) \;:=\; \int_0^T f(s)dB_s^H \;:=\; \sum_{i=1}^{m} a_i B_{t_i}^H. \end{aligned}$$

Obviously, $I_T^H(\mathcal{E}_T) = \mathrm{span}_{\mathbb{R}}\{B_s^H \mid s \in [0,T]\}$. We want to extend I_T^H to more general f. Formula (3.17) and the linearity of $\mathbf{K}^H$ yield

$$I_T^H(f) \;=\; \int_0^T f(s)dB_s^H \;=\; \int_0^T (\mathbf{K}^H f)(s)dW_s,\ a.s,\ f \in \mathcal{E}_T. \tag{4.1}$$

By the classical Wiener-Itô isometry, the standard Wiener integral $\int_0^T g(s)dW_s$ is well-defined for every $g \in L^2[0,T]$. In view of (4.1), for $H > \frac{1}{2}$ it is natural to define a linear space by

$$\Lambda_T(H) := \;\{f : [0,T] \to \mathbb{R} \mid \cdot^{H-\frac{1}{2}} f(\cdot) \in L^1[0,T] \text{ and } \int_0^T (\mathbf{K}^H f)(s)^2 ds \;<\; \infty\}.$$

For $H < \frac{1}{2}$, $\mathbf{K}^H$ is a weighted derivative operator, thus its arguments must be sufficiently smooth. In this case, we set

$$\Lambda_T(H) := \;\{f : [0,T] \to \mathbb{R} \mid \exists\ \phi_f \in L^2[0,T] \text{ such that } f = \mathbf{K}^{H,*}\phi_f\}.$$

By Remark 3.7 we have $\mathbf{K}^H f = \phi_f$. Clearly, for all H we have $\mathcal{E}_T \subseteq \Lambda_T(H)$. We define a scalar product on $\Lambda_T(H)$ by

$$\langle f, g\rangle_{\Lambda_T(H)} \;:=\; (\mathbf{K}^H f, \mathbf{K}^H g)_{L^2[0,T]}. \tag{4.2}$$

The positive definiteness follows from (3.2) for $H > \frac{1}{2}$ and from the linearity of $\mathbf{K}^{H,*}$ for $H < \frac{1}{2}$. $\mathcal{E}_T$ is dense in $\Lambda_T(H)$ (see [11], Theorem 4.1, p.879 and Theorem 4.2, p.881). Hence for $f \in \Lambda_T(H)$ there exists $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}_T$ such that $\lim_{n\to\infty} |f_n - f|_{\Lambda_T(H)} = 0$, where $|\cdot|^2_{\Lambda_T(H)} := \langle\cdot,\cdot\rangle_{\Lambda_T(H)}$. We define the *(time domain) fractional Wiener integral* of f with respect to B^H by

$$I_T^H(f) \;:=\; \int_0^T f(s)dB_s^H \;:=\; L^2(\mathbb{P})-\lim_{n\to\infty}\int_0^T f_n(s)dB_s^H \;=\; \int_0^T (\mathbf{K}^H f)(s)dW_s.$$

The equality follows from Wiener-Itô isometry and (4.2). By construction, $\int_0^T f(s)dB_s^H$ is centered and Gaussian. Furthermore, the linear map $I_T^H : \Lambda_T(H) \to \mathcal{H}_T(B^H)$ preserves distance, or equivalently,

$$E[\int_0^T f(s)dB_s^H]^2 \;=\; |f|^2_{\Lambda_T(H)},\ f \in \Lambda_T(H).$$

For $H > \frac{1}{2}$, we have $\Lambda_T(H) \subsetneq \overline{\mathcal{E}_T}$, i.e. $\Lambda_T(H)$ is not complete. This is due to the fact that $\mathbf{K}^H f$ is a true weighted fractional integral, i.e. more smooth than a general square integrable function. With other words, $\mathbf{K}^H(\Lambda_T(H)) \subsetneq L^2[0,T]$. For an exact argument, see [11], Section 5. Therefore, $\Lambda_T(H)$ is isometric not to $\mathcal{H}_T(B^H)$ but to a strict subspace. Hence, there exist $X \in \mathcal{H}_T(B^H)$ that can not be represented by a fractional Wiener integral with respect to B^H.

For $H < \frac{1}{2}$, $\Lambda_T(H)$ is isometric to $L^2[0,T]$ and hence complete. Indeed, we have $\mathbf{K}^H(\Lambda_T(H)) = L^2[0,T]$ (see [11], p.881) and $\mathbf{K}^{H,*}(L^2[0,T]) = \Lambda_T(H)$. In particular, $\Lambda_T(H)$ is isometric to $\mathcal{H}_T(B^H)$. Hence, every $X \in \mathcal{H}_T(B^H)$ can be represented by a fractional Wiener integral with respect to B^H.

Example 4.1. An important fractional Wiener integral process is the autoregressive representation of fBm, which shows how to transform the complicated fBm into the simpler sBm. By (3.16) we have a.s. for all $t \in [0,T]$

$$\begin{aligned} W_t &= \int_0^T (\mathbf{K}^H \mathbf{K}^{H,*} 1_{[0,t)})(s) dW_s = \int_0^T (\mathbf{K}^{H,*} 1_{[0,t)})(s) dB_s^H \\ &= \int_0^t \frac{(t-s)^{\frac{1}{2}-H}}{C(H)\Gamma(\frac{3}{2}-H)} F(\frac{1}{2}-H, \frac{1}{2}-H, \frac{3}{2}-H, \frac{s-t}{s}) dB_s^H. \quad (4.3) \end{aligned}$$

Representation (1.2) and its reciprocal (4.3) together imply that $\mathbb{F}^{B^H} = \mathbb{F}^W$, where $\mathbb{F}^{B^H} := (\sigma^c\{B_s^H, s \in [0,t]\})_{t\in[0,T]}$ is the completed natural filtration of $(B_t^H)_{t\in[0,T]}$. Furthermore, $\mathcal{H}_t(B^H) = \mathcal{H}_t(W)$, $t \in [0,T]$.

4.2 Fractional Wiener integrals over $\mathbb{R}$

As in 4.1, let $\mathcal{E}$ denote the space of elementary functions on $\mathbb{R}$, so functions

$$f(s) := \sum_{i=1}^m a_i 1_{[0,t_i)}(s),$$

where $t_1, \ldots, t_m, a_1, \ldots, a_m \in \mathbb{R}$. Moreover, define the *fractional Wiener chaos of first order* over $\mathbb{R}$ by

$$\mathcal{H}(B^H) := \overline{\mathrm{span}_{\mathbb{R}}\{B_s^H \mid s \in \mathbb{R}\}} \subset L^2(\mathbb{P}).$$

We define the *fractional Wiener integral* of elementary f with respect to B^H by the linear injection

$$\begin{aligned} I^H : \mathcal{E} &\to \mathcal{H}(B^H) \\ f &\mapsto I^H(f) := \int_{\mathbb{R}} f(s) dB_s^H := \sum_{i=1}^m a_i B_{t_i}^H. \end{aligned}$$

From (3.26) follows

$$I^H(f) = \int_{\mathbb{R}} f(s) dB_s^H = \frac{\Gamma(H+\frac{1}{2})}{C'(H)} \int_{\mathbb{R}} (\mathcal{I}_-^{H-\frac{1}{2}} f)(s) d\tilde{W}_s, \ a.s.$$

For $H > \frac{1}{2}$, we define the linear space of integrands by

$$\Lambda(H) := \{f \in L^1(\mathbb{R}) \mid \int_{\mathbb{R}} (\mathcal{I}_-^{H-\frac{1}{2}} f)(s)^2 ds < \infty\},$$

whereas for $H < \frac{1}{2}$, we set

$$\Lambda(H) := \{f : \mathbb{R} \to \mathbb{R} \mid \exists\, \phi_f \in L^2(\mathbb{R}) \text{ such that } f = \mathcal{I}_-^{\frac{1}{2}-H} \phi_f\}.$$

We define a scalar product on $\Lambda(H)$ by

$$\langle f, g\rangle_{\Lambda(H)} \ := \ \left(\frac{\Gamma(H+\frac{1}{2})}{C'(H)}\right)^2 (\mathcal{I}_-^{H-\frac{1}{2}} f, \mathcal{I}_-^{H-\frac{1}{2}} g)_{L^2(\mathbb{R})}.$$

As for the finite interval analogue, the elementary functions are dense in $\Lambda(H)$. Moreover, $\Lambda(H)$ is complete and hence isometric to $L^2(\mathbb{R})$ if and only if $H < \frac{1}{2}$ (see [12], Theorem 3.2, p.260 and Theorem 3.3, p.263). Define the *(time domain) fractional Wiener integral* of $f \in \Lambda(H)$ with respect to B^H by

$$I^H(f) \ := \ \int_{\mathbb{R}} f(s) dB_s^H \ := \ \frac{\Gamma(H+\frac{1}{2})}{C'(H)} \int_{\mathbb{R}} (\mathcal{I}_-^{H-\frac{1}{2}} f)(s) d\tilde{W}_s.$$

Then $\int_{\mathbb{R}} f(s) dB_s^H$ is centered and Gaussian with $E[\int_{\mathbb{R}} f(s) dB_s^H]^2 = |f|^2_{\Lambda(H)}$.

Example 4.2. By using (3.23) and (3.22) we obtain the autoregressive representation of fBm on the real line. For all $t \in \mathbb{R}$ we have a.s.

$$\begin{aligned} \tilde{W}_t &= \frac{C'(H)}{\Gamma(H+\frac{1}{2})} \int_{\mathbb{R}} (\mathcal{I}_-^{\frac{1}{2}-H} 1_{[0,t)})(s) dB_s^H \\ &= \frac{C'(H)}{\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)} \int_{\mathbb{R}} \left((t-s)_+^{\frac{1}{2}-H} - (-s)_+^{\frac{1}{2}-H}\right) dB_s^H. \end{aligned} \tag{4.4}$$

Let $\mathbb{G}^{B^H} := (\mathcal{G}_t^{B^H})_{t\in\mathbb{R}} := (\sigma^c\{B_s^H, s \in (-\infty, t]\})_{t\in\mathbb{R}}$ be the completed filtration generated by $(B_t^H)_{t\in\mathbb{R}}$. For $t > 0$, the integrals in (1.1) and (4.4) are over $(-\infty, t]$ and so $\mathcal{G}_t^{B^H} = \mathcal{G}_t^{\tilde{W}}$. For $t < 0$, one integrates over $(-\infty, 0]$ and thus $\mathcal{G}_t^{B^H} \neq \mathcal{G}_t^{\tilde{W}}$. In particular, $\mathbb{G}^{B^H} \neq \mathbb{G}^{\tilde{W}}$.

5 The transformation formula

We are able now to state and prove the main result.

Theorem 5.1. *Let $(B_t^H)_{t\in[0,T]}$ be an H-fBm and let $K \in (0,1)$. Then there exists a unique K-fBm $(B_t^K)_{t\in[0,T]}$ such that for all $t \in [0,T]$ we have*

$$B_t^H = C(K,H) \int_0^t (t-s)^{H-K} F(1-K-H, H-K, 1+H-K, \frac{s-t}{s}) dB_s^K, \ a.s,$$

where

$$C(K,H) \ := \ \frac{C(H)C(K)^{-1}}{\Gamma(H-K+1)}.$$

Moreover, $\mathbb{F}^{B^H} = \mathbb{F}^{B^K}$ and $\mathcal{H}_t(B^H) = \mathcal{H}_t(B^K)$, $t \in [0,T]$.

Proof. Let $(W_t)_{t\in[0,T]}$ be the sBm that satisfies

$$B_t^H \ = \ \int_0^T (\mathbf{K}^H 1_{[0,t)})(s) dW_s, \ a.s, \ t \in [0,T].$$

There exists a K-fBm $(B_t^K)_{t\in[0,T]}$ such that

$$W_t \;=\; \int_0^T (\mathbf{K}^{K,*}1_{[0,t)})(s)dB_s^K,\ a.s,\ t\in[0,T].$$

In fact, this is the K-fBm given by

$$B_t^K := \int_0^T (\mathbf{K}^K 1_{[0,t)})(s)dW_s = \int_0^T (\mathbf{K}^{H,*}\mathbf{K}^K 1_{[0,t)})(s)dB_s^H,\ t\in[0,T], \quad (5.1)$$

where the equality follows from (3.19). By Example 4.1 we have $\mathbb{F}^{B^H} = \mathbb{F}^W = \mathbb{F}^{B^K}$ and $\mathcal{H}_t(B^H) = \mathcal{H}_t(W) = \mathcal{H}_t(B^K)$, $t\in[0,T]$. Combining (3.19), (3.18) and (3.5) implies for all $t\in[0,T]$ that a.s.

$B_t^H \;=\; \int_0^T(\mathbf{K}^K\mathbf{K}^{K,*}\mathbf{K}^H 1_{[0,t)})(s)dW_s \;=\; \int_0^T(\mathbf{K}^{K,*}\mathbf{K}^H 1_{[0,t)})(s)dB_s^K \;=$

$C(K,H)\int_0^t(t-s)^{H-K}F(1-K-H,H-K,1+H-K,\frac{s-t}{s})dB_s^K.$

The uniqueness of the K-fBm is trivial. □

In particular, we obtain an easy formula for the transformation of an fBm with positively correlated increments into an fBm with negatively correlated increments and vice-versa:

Corollary 5.2. *Let $(B_t^H)_{t\in[0,T]}$ be an H-fBm. Then there exists a unique $(1-H)$-fBm $(B_t^{1-H})_{t\in[0,T]}$ such that*

$$B_t^H \;=\; \left(\frac{2H}{\Gamma(2H)\Gamma(3-2H)}\right)^{\frac{1}{2}}\int_0^t (t-s)^{2H-1}dB_s^{1-H},\ a.s,\ t\in[0,T].$$

Proof. Set $K:=1-H$ in Theorem 5.1 and use (2.13). □

Remark 5.3. In Theorem 5.1 and Corollary 5.2, the finite interval $[0,T]$ can be replaced by $[0,\infty)$. This is due to the fact that the kernel $(\mathbf{K}^{K,*}\mathbf{K}^H 1_{[0,t)})(s)$ and its reciprocal $(\mathbf{K}^{H,*}\mathbf{K}^K 1_{[0,t)})(s)$ - so especially B^K defined in (5.1) - do not depend on T.

Remark 5.4. The Malliavin derivative of B_t^H with respect to B^K is given by $(D^{B^K}B_t^H)(\omega) \;=\; C(K,H)(t-\cdot)^{H-K}F(1-K-H,H-K,1+H-K,\frac{\cdot-t}{\cdot})1_{[0,t)}(\cdot)$.

Remark 5.5. The *Riemann-Liouville process* with index $H>0$ is defined by

$$U_t^H \;:=\; \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}}dW_s,\ t\in[0,\infty).$$

We have $Cov_{\mathbb{P}}(U_s^H,U_t^H) \;=\; \frac{2H}{H+\frac{1}{2}}s^{H+\frac{1}{2}}t^{H-\frac{1}{2}}F(\frac{1}{2}-H,1,H+\frac{3}{2},\frac{s}{t})$ for $s\le t$. U^H was already mentioned in [6] as an alternative form of fBm. However, contrary to fBm, its increments are not stationary. Furthermore, we have $U^H_{at} \stackrel{d}{=} a^H U^H_t$ for every $a, t > 0$, but U^H is not H-self-similar. By (3.7) we have $U^H_t = \sqrt{2H}\Gamma(H+\frac{1}{2})\int_0^T (I^{H-\frac{1}{2}}_{T-} 1_{[0,t)})(s)dW_s,\ t \in [0,T]$, so it is clear how one can define a fractional Wiener integral with respect to U^H. It is easy to show that for the Riemann-Liouville process $U^K_t := \sqrt{2K}\int_0^t (t-s)^{K-\frac{1}{2}} dW_s$, $t \geq 0$, holds

$$U^H_t \ = \ \frac{\sqrt{H}\Gamma(H+\frac{1}{2})}{\sqrt{K}\Gamma(K+\frac{1}{2})\Gamma(H-K+1)} \int_0^t (t-s)^{H-K} dU^K_s,\ t \in [0,\infty).$$

By Corollary 5.2, the process B^H is hence obtained from B^{1-H} by the same transformation (modulo a constant) as the process U^H from U^{1-H}.

The real line transformation formula (1.4), where $\tilde{B}^K$ is the K-fBm defined by

$$\tilde{B}^K_t \ := \ \frac{\Gamma(K+\frac{1}{2})}{C'(K)} \int_{\mathbb{R}} (\mathcal{I}^{K-\frac{1}{2}}_{-} 1_{[0,t)})(s) d\tilde{W}_s,\ t \in \mathbb{R},$$

is obtained by combining (3.26) with the composition formula (3.23), and then using (3.22). This is shown in detail in [10]. However, (1.4) can also be obtained directly from Theorem 5.1. In order to see how, let B^K be a K-fBm. By Theorem 5.1 and Remark 5.3, the process defined by

$$B^H_t \ := \ C(K,H)\int_0^t (t-u)^{H-K}\hat{F}(\frac{u-t}{u}) dB^K_u,\ t \in [0,\infty), \tag{5.2}$$

where for convenience we set

$$\hat{F}(z) := F(1-K-H, H-K, 1+H-K, z),$$

is an H-fBm. B^H has stationary increments and so for every $s > 0$, the shifted process

$$(B^{H,s}_t)_{t\in[-s,\infty)} \ := \ (B^H_{t+s} - B^H_s)_{t\in[-s,\infty)}$$

is an H-fBm. We substitute $v := u - s$ in (5.2) and obtain $B^{H,s}_t =$ $C(K,H)\left(\int_0^{t+s}(t+s-u)^{H-K}\hat{F}(\frac{u-(t+s)}{u})dB^K_u - \int_0^s (s-u)^{H-K}\hat{F}(\frac{u-s}{u})dB^K_u\right) =$ $C(K,H)\left(\int_{-s}^t (t-v)^{H-K}\hat{F}(\frac{v-t}{v+s})dB^K_{v+s} - \int_{-s}^0 (-v)^{H-K}\hat{F}(\frac{v}{v+s})dB^K_{v+s}\right)$.

The increments of B^K are stationary, hence

$$Z^{H,s}_t := C(K,H)\left(\int_{-s}^t (t-v)^{H-K}\hat{F}(\frac{v-t}{v+s})dB^K_v - \int_{-s}^0 (-v)^{H-K}\hat{F}(\frac{v}{v+s})dB^K_v\right)$$

is an H-fBm on $[-s,\infty)$. The pointwise limit of the integrands as $s \to \infty$ suggests the following:

Corollary 5.6. *Let $(B_t^K)_{t\in\mathbb{R}}$ be a K-fBm. Then the process*

$$Z_t^H \ := \ C(K,H)\int_{\mathbb{R}} \left((t-v)^{H-K}1_{(-\infty,t)}(v) - (-v)^{H-K}1_{(-\infty,0)}(v)\right) dB_v^K$$

is an H-fBm on the real line.

In Section 6, we show that for every $t\in\mathbb{R}$ holds

$$\lim_{s\to\infty} E[Z_t^{H,s} - Z_t^H]^2 \ = \ 0. \tag{5.3}$$

Hence $E[Z_t^H \cdot Z_{t'}^H] = \lim_{s\to\infty} E[Z_t^{H,s} Z_{t'}^{H,s}] = \frac{1}{2}\left(|t|^{2H} + |t'|^{2H} - |t'-t|^{2H}\right)$ for all $t,t'\in\mathbb{R}$. This proves Corollary 5.6. Note that by combining (2.3) with (2.4) we have $\frac{C(H)}{\Gamma(H+\frac{1}{2})} = C'(H)^{-1}$. So the constant $C(K,H)$ in Corollary 5.6 indeed coincides with the constant in (1.4).

6 Proof of convergence

We prove (5.3) for $H\neq K$ and specify the speed of convergence. We assume that $t>0$. The result is derived similarly for $t<0$. Denote

$$\Delta f_t^s(v) \ := \ \left((t-v)^{H-K} \ - \ (-v)^{H-K}\right) 1_{(-\infty,-s)}(v),$$

$$\Delta g_t^s(v) \ := \ (t-v)^{H-K}\hat{F}(\frac{v-t}{v+s})1_{(-s,t)}(v) \ - \ (-v)^{H-K}\hat{F}(\frac{v}{v+s})1_{(-s,0)}(v),$$

$$\Delta h_t^s(v) \ := \ (t-v)^{H-K}1_{(-s,t)}(v) \ - \ (-v)^{H-K}1_{(-s,0)}(v)$$

and $\qquad \Delta k_t^s(v) \ := \ k_t^s(v) \ - \ k_0^s(v) \ :=$

$(t-v)^{H-K}(\hat{F}(\frac{v-t}{v+s})-1)1_{(-s,t)}(v) \ - \ (-v)^{H-K}(\hat{F}(\frac{v}{v+s})-1)1_{(-s,0)}(v).$

For continuous G we set ${}^*G := \max_{z\in[-1,0]}|G(z)|$ and $G^* := \max_{z\in[0,1]}|G(z)|$.

For $K=\frac{1}{2}$ we have by independence of increments of B^K that

$$\left(\frac{\Gamma(H+\frac{1}{2})}{C(H)}\right)^2 E[Z_t^{H,s} - Z_t^H]^2 \ = \ \int_{\mathbb{R}}\Delta f_t^s(v)^2 dv \ + \ \int_{\mathbb{R}}\Delta k_t^s(v)^2 dv. \tag{6.1}$$

For $K\neq\frac{1}{2}$ we have by using $(a-b)^2\leq 2(a^2+b^2)$ that

$$\begin{aligned}\frac{1}{2C(K,H)^2}\cdot E[Z_t^{H,s} - Z_t^H]^2 \ &\leq \ E[\int_{\mathbb{R}}\Delta f_t^s(v)dB_v^K]^2 + E[\int_{\mathbb{R}}\Delta k_t^s(v)dB_v^K]^2 \\ &= C(K)^2\left(\int_{\mathbb{R}}(\mathcal{I}_-^{K-\frac{1}{2}}\Delta f_t^s)(v)^2dv + \int_{\mathbb{R}}(\mathcal{I}_-^{K-\frac{1}{2}}\Delta k_t^s)(v)^2dv\right). \end{aligned} \tag{6.2}$$

1. For $K \in (0,1)$ we show that there exists a constant $c_1(K,H,t)$ such that

$$\int_{\mathbb{R}} (\mathcal{I}_-^{K-\frac{1}{2}} \Delta f_t^s)(v)^2 dv \ \le \ c_1(K,H,t) s^{2H-2}, \ s > t. \tag{6.3}$$

For all $K \in (0,1)$ we have

$$(\mathcal{I}_-^{K-\frac{1}{2}} \Delta f_t^s)(v) = \frac{(-s-v)^{H-\frac{1}{2}}}{\Gamma(K+\frac{1}{2})} \left(G_0(\frac{-s-v}{t-v}) \ - \ G_0(\frac{-s-v}{-v}) \right) 1_{(-\infty,-s)}(v),$$

where $G_0(z) := z^{K-H} F(K-H, K-\frac{1}{2}, K+\frac{1}{2}, z)$. This is trivial for $K = \frac{1}{2}$. For $K > \frac{1}{2}$ it follows from (2.23). For $K < \frac{1}{2}$ it follows from combining identity $(\mathcal{I}_-^{K-\frac{1}{2}} \Delta f_t^s)(v) = \frac{-d}{dv} (\mathcal{I}_-^{K+\frac{1}{2}} \Delta f_t^s)(v)$ with (2.23), (2.17) and (2.24). By (2.17) we have that $\frac{d}{dz} G_0(z) = (K-H) z^{K-H-1} G_1(z) := (K-H) z^{K-H-1} F(K-H+1, K-\frac{1}{2}, K+\frac{1}{2}, z)$. Combining mean value theorem, (2.21), (2.14) and (2.19) yields (6.3) with

$$c_1(K,H,t) \ = \ \frac{4 G_1^{*2} \Gamma(2K) \Gamma(2-2H) t^2}{\Gamma(K+\frac{1}{2})^2 \Gamma(2K-2H+2)}.$$

2. Fix $d > 0$. For $K \ge \frac{1}{2}$ we show that there exists a constant $c_2(K,H,t,d)$ such that

$$\int_{\mathbb{R}} (\mathcal{I}_-^{K-\frac{1}{2}} \Delta k_t^s)(v)^2 dv \ \le \ c_2(K,H,t,d) s^{2H-2}, \ s > 2t + 4d. \tag{6.4}$$

For $K < \frac{1}{2}$ we show that there exist $c_3(K,H,t,d)$ and $c_4(K,H,t,d)$ such that

$$\int_{\mathbb{R}} (\mathcal{I}_-^{K-\frac{1}{2}} \Delta k_t^s)(v)^2 dv \le c_3(K,H,t,d) s^{2H-2} \ + \ c_4(K,H,t,d) s^{2K-2}, \ s > 2t + 4d. \tag{6.5}$$

Let $s > 2d$. We have

$$\begin{aligned} \Delta k_t^s(v) \ &= \ k_t^s(v) 1_{[-d,t)}(v) \ - \ k_0^s(v) 1_{[-d,0)}(v) \ + \ \Delta k_t^s(v) 1_{[\frac{-s}{2},-d)}(v) \\ &\quad + \ \Delta g_t^s(v) 1_{(-s,\frac{-s}{2})}(v) \ - \ \Delta h_t^s(v) 1_{(-s,\frac{-s}{2})}(v). \end{aligned}$$

First, let $K = \frac{1}{2}$. Then

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}} \Delta k_t^s(v)^2 dv \ &\le \ \int_{-d}^{t} k_t^s(v)^2 dv \ + \ \int_{-d}^{0} k_0^s(v)^2 dv \ + \ \int_{\frac{-s}{2}}^{-d} \Delta k_t^s(v)^2 dv \\ &\quad + \ \int_{-s}^{\frac{-s}{2}} \Delta g_t^s(v)^2 dv \ + \ \int_{-s}^{\frac{-s}{2}} \Delta h_t^s(v)^2 dv. \end{aligned} \tag{6.6}$$

Second, let $K > \frac{1}{2}$. Then $\Gamma(K-\frac{1}{2}) (\mathcal{I}_-^{K-\frac{1}{2}} \Delta k_t^s)(v) \ =$

$\int_{-d}^{t} k_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,-d)}(v) \ + \ \int_{v}^{t} k_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{[-d,t)}(v)$

$- \int_{-d}^{0} k_0^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,-d)}(v) \; - \; \int_{v}^{0} k_0^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{[-d,0)}(v)$

$+ \int_{\frac{-s}{2}}^{-d} \Delta k_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,\frac{-s}{2})}(v) + \int_{v}^{-d} \Delta k_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{[\frac{-s}{2},-d)}(v)$

$+ \int_{-s}^{\frac{-s}{2}} \Delta g_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,-s)}(v) + \int_{v}^{\frac{-s}{2}} \Delta g_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{[-s,\frac{-s}{2})}(v)$

$- \int_{-s}^{\frac{-s}{2}} \Delta h_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,-s)}(v) - \int_{v}^{\frac{-s}{2}} \Delta h_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{[-s,\frac{-s}{2})}(v)$

$=: \; A_1(v)+B_1(v)+A_2(v)+B_2(v)+A_3(v)+B_3(v)+A_4(v)+B_4(v)+A_5(v)+B_5(v).$

Hence

$$\frac{\Gamma(K-\frac{1}{2})^2}{8} \int_{\mathbb{R}} (\mathcal{I}_-^{K-\frac{1}{2}} \Delta k_t^s)(v)^2 dv \; \le \; \sum_{i=1}^{5} \int_{\mathbb{R}} A_i(v)^2 dv \; + \; \sum_{i=1}^{5} \int_{\mathbb{R}} B_i(v)^2 dv. \quad (6.7)$$

Third, let $K < \frac{1}{2}$. Let $x \le v < y$ and $f \in \mathcal{C}^1(x,y)$ with $f' \in L^1[x,y]$. By (3.14) we have $(\mathcal{I}_-^{K+\frac{1}{2}} \mathcal{D}_-^1 f 1_{(x,y)})(v) = (\mathcal{D}_-^{\frac{1}{2}-K} f 1_{(x,y)})(v) \; - \; \frac{f(y)}{\Gamma(K+\frac{1}{2})}(y-v)^{K-\frac{1}{2}}$.

So $\Gamma(K-\frac{1}{2})(\mathbf{D}_-^{\frac{1}{2}-K} \Delta k_t^s)(v) \; = \; \Gamma(K-\frac{1}{2})(\mathcal{D}_-^{\frac{1}{2}-K} \Delta k_t^s)(v) \; =$

$$\begin{aligned}
&\int_{-d}^{t} k_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,-d)}(v) + \\
&\qquad\qquad \Gamma(K-\tfrac{1}{2})(\mathcal{I}_-^{K+\frac{1}{2}} \tfrac{-d}{du} k_t^s 1_{(-d,t)})(v) \cdot 1_{[-d,t)}(v) \\
&- \int_{-d}^{0} k_0^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,-d)}(v) - \\
&\qquad\qquad \Gamma(K-\tfrac{1}{2})(\mathcal{I}_-^{K+\frac{1}{2}} \tfrac{-d}{du} k_0^s 1_{(-d,0)})(v) \cdot 1_{[-d,0)}(v) \\
&+ \int_{\frac{-s}{2}}^{-d} \Delta k_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,\frac{-s}{2})}(v) + \\
&\qquad\qquad \Gamma(K-\tfrac{1}{2})(\mathcal{I}_-^{K+\frac{1}{2}} \tfrac{-d}{du} \Delta k_t^s 1_{(\frac{-s}{2},-d)})(v) \cdot 1_{[\frac{-s}{2},-d)}(v) \\
&+ \int_{-s}^{\frac{-s}{2}} \Delta g_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,-s)}(v) + \\
&\qquad\qquad \Gamma(K-\tfrac{1}{2})(\mathcal{I}_-^{K+\frac{1}{2}} \tfrac{-d}{du} \Delta g_t^s 1_{(-s,\frac{-s}{2})})(v) \cdot 1_{[-s,\frac{-s}{2})}(v) \\
&- \int_{-s}^{\frac{-s}{2}} \Delta h_t^s(u)(u-v)^{K-\frac{3}{2}} du \cdot 1_{(-\infty,-s)}(v) - \\
&\qquad\qquad \Gamma(K-\tfrac{1}{2})(\mathcal{I}_-^{K+\frac{1}{2}} \tfrac{-d}{du} \Delta h_t^s 1_{(-s,\frac{-s}{2})})(v) \cdot 1_{[-s,\frac{-s}{2})}(v) \\
&+ \frac{\Delta k_t^s(-d)}{K-\frac{1}{2}}(-d-v)^{K-\frac{1}{2}} \cdot 1_{[\frac{-s}{2},-d)}(v) \; + \; \frac{\Delta k_t^s(\frac{-s}{2})}{K-\frac{1}{2}}(\tfrac{-s}{2}-v)^{K-\frac{1}{2}} \cdot 1_{[-s,\frac{-s}{2})}(v)
\end{aligned}$$

$=: \; A_1(v) + C_1(v) + A_2(v) + C_2(v) + A_3(v) + C_3(v) + A_4(v) + C_4(v) + A_5(v) + C_5(v) + D(v) + E(v).$

Hence

$$\begin{aligned}
\frac{\Gamma(K-\frac{1}{2})^2}{8} \int_{\mathbb{R}} (\mathcal{I}_-^{K-\frac{1}{2}} \Delta k_t^s)(v)^2 dv \quad &\le \quad \sum_{i=1}^{5} \int_{\mathbb{R}} A_i(v)^2 dv \; + \; \sum_{i=1}^{5} \int_{\mathbb{R}} C_i(v)^2 dv \\
&\quad + \; \int_{\mathbb{R}} D(v)^2 dv \; + \; \int_{\mathbb{R}} E(v)^2 dv. \quad (6.8)
\end{aligned}$$

We estimate the integrals on the right-hand sides of (6.6), (6.7) and (6.8):

1. Estimation of $\int_{-d}^{t} k_t^s(v)^2 dv$, $\int_{\mathbb{R}} A_1(v)^2 dv$, $\int_{\mathbb{R}} B_1(v)^2 dv$ and $\int_{\mathbb{R}} C_1(v)^2 dv$. For $u \in (-d,t)$, mean value theorem and (2.16) yield $|k_t^s(u)| \leq \frac{4 \cdot^* G_2 s^{-1}}{1+H-K}(t-u)^{H-K+1}$, $s > 2d+t$, with $G_2(z) := F(2-K-H, H-K+1, H-K+2, z)$. Hence for $K = \frac{1}{2}$ we obtain $\int_{-d}^{t} k_t^s(v)^2 dv \leq \frac{8 \cdot^* G_2^2 (t+d)^{2H+2}}{(\frac{1}{2}+H)^2} s^{-2}$, $s > 2d+t$. By using (2.22) we get $\int_{\mathbb{R}} A_1(v)^2 dv \leq \frac{8 \cdot^* G_2^2 G_3^{*2} (t+d)^{2H+2}}{(1+H-K)^2 (1-K)} s^{-2}$, $s > 2d+t$, where $G_3(z) := F(\frac{3}{2}-K, H-K+2, H-K+3, z)$. For $K > \frac{1}{2}$ we have $\int_{\mathbb{R}} B_1(v)^2 dv \leq \frac{8 \cdot^* G_2^2 \Gamma(H-K+1)^2 \Gamma(K-\frac{1}{2})^2 (t+d)^{2H+2}}{\Gamma(H+\frac{3}{2})^2} s^{-2}$, $s > 2d+t$. Furthermore, for $K < \frac{1}{2}$ the mean value theorem yields $\int_{\mathbb{R}} C_1(v)^2 dv \leq \frac{128 \cdot^* G_2^2 \Gamma(H-K+1)^2 \Gamma(K-\frac{1}{2})^2 (t+d)^{2H+2}}{(1+H-K)^2 \Gamma(H+\frac{3}{2})^2} s^{-2}$, $s > \max(2d+t, 2t)$.

2. Estimation of $\int_{-d}^{0} k_0^s(v)^2 dv$, $\int_{\mathbb{R}} A_2(v)^2 dv$, $\int_{\mathbb{R}} B_2(v)^2 dv$ and $\int_{\mathbb{R}} C_2(v)^2 dv$. We obtain estimates by replacing t by 0 in the results of 1.

3. Estimation of $\int_{\frac{-s}{2}}^{-d} \Delta k_t^s(v)^2 dv$, $\int_{\mathbb{R}} A_3(v)^2 dv$, $\int_{\mathbb{R}} B_3(v)^2 dv$, $\int_{\mathbb{R}} C_3(v)^2 dv$, $\int_{\mathbb{R}} D(v)^2 dv$ and $\int_{\mathbb{R}} E(v)^2 dv$. Let $u \in (\frac{-s}{2}, -d)$. Then $\Delta k_t^s(u) = (u+s)^{H-K} \cdot \left(G_4(\frac{t-u}{u+s}) - G_4(\frac{-u}{u+s})\right)$, where $G_4(z) := z^{H-K}\left(\hat{F}(-z) - 1\right)$. Combining (2.17) and (2.15) yields $\frac{dG_4}{dz}(z) = (H-K) z^{H-K-1}((1+z)^{H+K-1} - 1)$. So

$$|\Delta k_t^s(u)| \;\leq\; 2t \frac{t+d}{d} s^{-1} (-u)^{H-K}. \tag{6.9}$$

On the other hand

$$|\Delta k_t^s(u)| \;\leq\; 2t(-u)^{H-K-1}, \; s > 2t. \tag{6.10}$$

For $K = \frac{1}{2}$, (6.9) yields $\int_{\frac{-s}{2}}^{-d} \Delta k_t^s(u)^2 du \leq \frac{2}{H} t^2 (\frac{t+d}{d})^2 s^{2H-2}$. If $H > K$ and $s > \max(2t, 4d)$ then (6.10) and (2.22) imply $\int_{\mathbb{R}} A_3(v)^2 dv \leq \frac{32 G_5^{*2} t^2}{(H-K)^2 (1-K)} s^{2H-2}$, with $G_5(z) := F(\frac{3}{2}-K, H-K, H-K+1, z)$. For $H < K$, (6.9) and (2.22) yield $\int_{\mathbb{R}} A_3(v)^2 dv \leq \frac{2 G_6^{*2} t^2 (\frac{t+d}{d})^2}{(H-K+1)^2 (1-K)} s^{2H-2}$, where $G_6(z) := F(\frac{3}{2}-K, H-K+1, H-K+2, z)$. For $K > \frac{1}{2}$ we obtain from (6.10) and (2.23) that $\int_{\mathbb{R}} B_3(v)^2 dv \leq \frac{8 G_1^{*2} G_7^{*} t^2}{K(K-\frac{1}{2})^2} s^{2H-2}$, $s > 2t$, where $G_7(z) := F(2K-2H+2, 1, 2K+1, z)$. Let $K < \frac{1}{2}$. Triangle inequality and mean value theorem yield $|\frac{-d}{du} \Delta k_t^s(u)| \leq 16 t s^{-1} (-u)^{H-K-1} + 8t(-u)^{H-K-2}$, $s > 2t$. Hence (2.23) yields $\int_{\mathbb{R}} C_3(v)^2 dv \leq \frac{256 (G_8^{*2} G_9^{*} + G_{10}^{*2} G_{11}^{*}) t^2}{K(K+\frac{1}{2})^2 (K-\frac{1}{2})^2} s^{2H-2}$, $s > 2t$, where $G_8(z) := F(1+K-H, K+\frac{1}{2}, K+\frac{3}{2}, z)$, $G_9(z) := F(2K+2-2H, 1, 2K+3, z)$, $G_{10}(z) := F(2+K-H, K+\frac{1}{2}, K+\frac{3}{2}, z)$ and $G_{11}(z) := F(2K+4-2H, 1, 2K+3, z)$. By (6.9) we have $\int_{\mathbb{R}} D(v)^2 dv \leq \frac{2 t^2 d^{2(H-K-2)} (t+d)^2}{(K-\frac{1}{2})^2 K} s^{2K-2}$. From (6.10) follows $\int_{\mathbb{R}} E(v)^2 dv \leq \frac{8t^2}{(K-\frac{1}{2})^2 K} s^{2H-2}$, $s > 2t$.

4. Estimation of $\int_{-s}^{\frac{-s}{2}} \Delta g_t^s(v)^2 dv$, $\int_{\mathbb{R}} A_4(v)^2 dv$, $\int_{\mathbb{R}} B_4(v)^2 dv$ and $\int_{\mathbb{R}} C_4(v)^2 dv$. Let $u \in (-s, \frac{-s}{2})$. (2.14) implies $\Delta g_t^s(u) = (u+s)^{H-K}\left(G_{12}(\frac{t-u}{s+t}) - G_{12}(\frac{-u}{s})\right)$,

where $G_{12}(z) := z^{H-K}F(2H, H-K, H-K+1, z)$. Using (2.17) and (2.15) yields $\frac{d}{dz}G_{12}(z) = (H-K)z^{H-K-1}(1-z)^{-2H}$. Thus $|\Delta g_t^s(u)| \leq \frac{16t}{s^{2-2H}}(u+s)^{1-K-H}$, $s > t$. So for $K = \frac{1}{2}$ we have $\int_{-s}^{\frac{-s}{2}} \Delta g_t^s(u)^2 du \leq \frac{128t^2}{1-H}s^{2-2H}$, $s > t$. By using (2.22) we obtain $\int_{\mathbb{R}} A_4(v)^2 dv \leq \frac{128G_{13}^{*2}t^2}{(1-K)(2-K-H)^2}s^{2H-2}$, $s > t$, where $G_{13}(z) := F(\frac{3}{2} - K, 1, 3-K-H, z)$. For $K > \frac{1}{2}$ we have $\int_{\mathbb{R}} B_4(v)^2 dv \leq \frac{128G_{14}^{*2}t^2}{K(K-\frac{1}{2})^2}s^{2-2H}$, $s > t$, where $G_{14}(z) := F(H+K-1, 1, K+\frac{1}{2}, z)$. Let $K < \frac{1}{2}$. Combining triangle inequality and mean value theorem yields $|\frac{d}{du}\Delta g_t^s(u)| \leq \frac{96t}{s^{2-2H}}(u+s)^{-H-K} + \frac{64t}{s^{3-2H}}(u+s)^{1-K-H}$, $s > t$. So by using (2.22) we get $\int_{\mathbb{R}} C_4(v)^2 dv \leq \frac{(9216G_{15}^{*2} + 2048G_{16}^{*2})t^2}{K(K-\frac{1}{2})^2(K+\frac{1}{2})^2}s^{2H-2}$, $s > t$, where $G_{15}(z) := F(K+H, 1, K+\frac{3}{2}, z)$ and $G_{16}(z) := F(K+H-1, 1, K+\frac{3}{2}, z)$.

5. Estimation of $\int_{-s}^{\frac{-s}{2}} \Delta h_t^s(v)^2 dv$, $\int_{\mathbb{R}} A_5(v)^2 dv$, $\int_{\mathbb{R}} B_5(v)^2 dv$ and $\int_{\mathbb{R}} C_5(v)^2 dv$. Elementary calculations involving the mean value theorem yield $\int_{-s}^{\frac{-s}{2}} \Delta h_t^s(v)^2 dv \leq 4t^2 s^{2H-2}$ for $K = \frac{1}{2}$. Also $\int_{\mathbb{R}} A_5(v)^2 dv \leq \frac{2G_{17}^{*2}t^2}{1-K}s^{2H-2}$, where $G_{17}(z) := F(\frac{3}{2} - K, 1, 2, z)$. For $K > \frac{1}{2}$ we have $\int_{\mathbb{R}} B_5(v)^2 dv \leq \frac{2t^2}{K(K-\frac{1}{2})^2}s^{2H-2}$. Furthermore, for $K < \frac{1}{2}$ holds $\int_{\mathbb{R}} C_5(v)^2 dv \leq \frac{8t^2}{K(K-\frac{1}{2})^2(K+\frac{1}{2})^2}s^{2H-2}$.

By combining (6.6) and (6.7), respectively, with these estimates, we obtain (6.4) where

$$c_2(\frac{1}{2}, H, t, d) = \frac{32 \cdot^* G_2^2}{(\frac{1}{2}+H)^2}(t+d)^{2H+2} + \left(\frac{4(t+d)^2}{Hd^2} + \frac{256}{1-H} + 8\right)t^2$$

and

$$c_2(K, H, t, d) = \left(\frac{128 \cdot^* G_2^2 G_3^{*2}}{\Gamma(K-\frac{1}{2})^2(1+H-K)^2(1-K)} + \frac{128 \cdot^* G_2^2 \Gamma(H-K+1)^2}{\Gamma(H+\frac{3}{2})^2}\right)(t+d)^{2H+2} +$$
$$\left(\frac{\max(\frac{256G_5^{*2}}{(H-K)^2}, \frac{16G_6^{*2}(t+d)^2}{(H-K+1)^2d^2}) + \frac{1024G_{13}^{*2}}{(2-K-H)^2} + 16G_{17}^{*2}}{(1-K)\Gamma(K-\frac{1}{2})^2} + \frac{1024G_{14}^{*2} + 64G_1^{*2}G_7^* + 16}{K\Gamma(K+\frac{1}{2})^2}\right)t^2, \quad K > \frac{1}{2}.$$

In the same way, by using (6.8) we obtain (6.5) with

$$c_3(K, H, t, d) = \left(\frac{128 \cdot^* G_2^2 G_3^{*2}}{\Gamma(K-\frac{1}{2})^2(1+H-K)^2(1-K)} + \frac{2048 \cdot^* G_2^2 \Gamma(H-K+1)^2}{(1+H-K)^2\Gamma(H+\frac{3}{2})^2}\right)(t+d)^{2H+2} +$$
$$\left(\frac{\max(\frac{256G_5^{*2}}{(H-K)^2}, \frac{16G_6^{*2}(t+d)^2}{(H-K+1)^2d^2}) + \frac{1024G_{13}^{*2}}{(2-H-K)^2} + 16G_{17}^{*2}}{\Gamma(K-\frac{1}{2})^2(1-K)}\right)t^2 +$$
$$\left(\frac{2048G_8^{*2}G_9^* + 2048G_{10}^{*2}G_{11}^* + 73728G_{15}^{*2} + 16384G_{16}^{*2} + 64}{K\Gamma(K+\frac{3}{2})^2}\right)t^2$$

and

$$c_4(K, H, t, d) = \frac{16d^{2(H-K-2)}(t+d)^2t^2}{\Gamma(K+\frac{1}{2})^2K}.$$

Eventually, by combining (6.1) and (6.2), respectively, with the results in **1.** and **2.**, we obtain (5.3) with the following speed of convergence:

Lemma 6.1. *For $K = \frac{1}{2}$ we have*

$$\left(\frac{\Gamma(H+\frac{1}{2})}{C(H)}\right)^2 E[Z_t^{H,s} - Z_t^H]^2 \le (c_1(\frac{1}{2}, H, t) + c_2(\frac{1}{2}, H, t, d))s^{2H-2}, \ s > 2t+4d.$$

For $K > \frac{1}{2}$ we have

$$\frac{\Gamma(H-K+1)^2}{2C(H)^2} E[Z_t^{H,s} - Z_t^H]^2 \le (c_1(K,H,t) + c_2(K,H,t,d))s^{2H-2}, \ s > 2t+4d.$$

For $K < \frac{1}{2}$ holds

$$\frac{\Gamma(H-K+1)^2}{2C(H)^2} \cdot E[Z_t^{H,s} - Z_t^H]^2 \le (c_1(K,H,t) + c_3(K,H,t,d))s^{2H-2} + c_4(K,H,t,d)s^{2K-2}, \ s > 2t+4d.$$

Remark 6.2. The constants in Lemma 6.1 are not optimal and obviously very bulky. Note however, that a hypergeometric function that exists as an Euler integral is either increasing or decreasing in z, so its absolute maximum over $[0,1]$ can be estimated via (2.13) or (2.19). Approximating values for $\Gamma(\alpha)$ can be found in integral tables; see for example [1], p.267. For $K < \frac{1}{2}$, I conjecture that more sophisticated methods than the mean value theorem lead to a simplified speed of convergence of s^{2H-2} (modulo constant). Moreover, for all $K \in (0,1)$, I conjecture that the speed of convergence of s^{2H-2} (modulo constant) is optimal.

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