GENERALIZED ELLIPTIC INTEGRALS

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Abstract

Jacobi's elliptic integrals and elliptic functions arise naturally from the Schwarz-Christoffel conformal transformation of the upper half plane onto a rectangle. In this paper we study generalized elliptic integrals which arise from the analogous mapping of the upper half plane onto a quadrilateral and obtain sharp monotonicity and convexity properties for certain combinations of these integrals, thus generalizing analogous well-known results for classical conformal capacity and quasiconformal distortion functions.

1 Introduction

Given complex numbers a, b, and c with $c \neq 0, -1, -2, \ldots$, the *Gaussian hypergeometric function* is the analytic continuation to the slit plane $\mathbb{C} \setminus [1, \infty)$ of the series

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!}, \quad |z| < 1 . \tag{1.1}$$

Here $(a, 0) = 1$ for $a \neq 0$, and (a, n) is the *shifted factorial function* or the *Appell symbol*

$$(a, n) = a(a + 1)(a + 2) \cdots (a + n - 1)$$

for $n \in \mathbb{N} \setminus \{0\}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

A *generalized modular equation of order (or degree)* $p > 0$ is

$$\frac{F(a, b; c; 1 - s^2)}{F(a, b; c; s^2)} = p \frac{F(a, b; c; 1 - r^2)}{F(a, b; c; r^2)}, \quad 0 < r < 1 . \tag{1.2}$$

Sometimes we just call this an (a, b, c)-modular equation of order p and we usually assume that $a, b, c > 0$ with $a + b \geq c$, in which case this equation uniquely defines s, see Lemma 4.5.

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Many particular cases of (1.2) have been studied in the literature on both analytic number theory and geometric function theory, [Be], [BB], [BBG], [AVV], [AQVV], [LV]. The classical case $(a, b, c) = (\frac{1}{2}, \frac{1}{2}, 1)$ was studied already by Jacobi and many others in the nineteenth century, see [Be]. In 1995 B. Berndt, S. Bhargava, and F. Garvan published an important paper [BBG] in which they studied the case $(a, b, c) = (a, 1-a, 1)$ and p an integer. For several rational values of a such as $a = \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and integers p (e.g. $p = 2, 3, 5, 7, 11, \ldots$) they were able to give proofs for numerous algebraic identities stated by Ramanujan in his unpublished notebooks. These identities involve r and s from (1.2). After the publication of [BBG] many papers have appeared on modular equations, see e.g. [AQVV], [Be], [BCKZ], [CLT], [Q], and [S].

To rewrite (1.2) in a slightly shorter form, we use the decreasing homeomorphism $\mu_{a,b,c} : (0, 1) \to (0, \infty)$, defined by

$$\mu_{a,b,c}(r) = \frac{B(a, b)}{2} \frac{F(a, b; c; r'^2)}{F(a, b; c; r^2)}, \quad r \in (0, 1) \tag{1.3}$$

for $a, b, c > 0$, $a + b \geq c$, where B is the beta function, see (3.5) below. We call $\mu_{a,b,c}$ the *generalized modulus*, cf. [LV, (2.2)]. We can now write (1.2) as

$$\mu_{a,b,c}(s) = p\, \mu_{a,b,c}(r), \quad 0 < r < 1. \tag{1.4}$$

With $p = 1/K$, $K > 0$, the solution of (1.2) is then given by

$$s = \varphi_K^{a,b,c}(r) = \mu_{a,b,c}^{-1}(\mu_{a,b,c}(r)/K). \tag{1.5}$$

We call $\varphi_K^{a,b,c}$ *the* (a, b, c)*-modular function with degree* $p = 1/K$ [BBG], [AQVV, (1.5)].

In the case $a < c$ we also use the notation

$$\mu_{a,c} = \mu_{a,c-a,c}, \quad \varphi_K^{a,c} = \varphi_K^{a,c-a,c}.$$

For $0 < a < \min\{c, 1\}$ and $0 < b < c \leq a + b$, define the *generalized complete elliptic integrals of the first and second kinds* (cf. [AQVV, (1.9), (1.10), (1.3), and (1.5)]) on $[0, 1]$ by

$$\mathcal{K} = \mathcal{K}_{a,b,c} = \mathcal{K}_{a,b,c}(r) = \frac{B(a, b)}{2} F(a, b; c; r^2), \tag{1.6}$$

$$\mathcal{E} = \mathcal{E}_{a,b,c} = \mathcal{E}_{a,b,c}(r) = \frac{B(a, b)}{2} F(a-1, b; c; r^2), \tag{1.7}$$

$$\mathcal{K}' = \mathcal{K}'_{a,b,c} = \mathcal{K}_{a,b,c}(r'), \text{ and } \mathcal{E}' = \mathcal{E}'_{a,b,c} = \mathcal{E}_{a,b,c}(r') \tag{1.8}$$

for $r \in (0, 1)$, $r' = \sqrt{1 - r^2}$. The end values are defined by limits as r tends to $0+$ and $1-$, respectively. In particular, we denote $\mathcal{K}_{a,c} = \mathcal{K}_{a,c-a,c}$ and $\mathcal{E}_{a,c} = \mathcal{E}_{a,c-a,c}$. Thus, by (3.9) below,

$$\mathcal{K}_{a,b,c}(0) = \mathcal{E}_{a,b,c}(0) = \frac{B(a, b)}{2}$$

and

$$\mathcal{E}_{a,b,c}(1) = \frac{1}{2} \frac{B(a, b) B(c, c + 1 - a - b)}{B(c + 1 - a, c - b)}, \quad \mathcal{K}_{a,b,c}(1) = \infty.$$

Note that the restrictions on a, b and c ensure that the function $\mathcal{K}_{a,b,c}$ is increasing and unbounded whereas $\mathcal{E}_{a,b,c}$ is decreasing and bounded, as in the classical case $a = b = \frac{1}{2}, c = 1$. Note also that our terminology differs from that of [BB, Section 5.5], where generalized elliptic integrals refer to the particular case $c = 1$.

In this paper we study the modular function $\varphi_K^{a,b,c}$ and the generalized modulus $\mu_{a,b,c}$ as well as the generalized elliptic integrals $\mathcal{K}_{a,b,c}$ and $\mathcal{E}_{a,b,c}$. In the case $b = 1-a$, $c = 1$, these functions coincide with the special cases φ_K^a, μ_a, $\mathcal{K}_a$, and $\mathcal{E}_a$ which were studied in [AQVV].

In Section 2 we construct a conformal mapping from a quadrilateral with internal angles $b\pi$, $(c-b)\pi$, $(1-a)\pi$, and $(1-c+a)\pi$ onto the upper half plane. We denote this mapping by $\mathrm{sn}_{a,b,c}$. If $b = 1-a$ and $c = 1$, this mapping reduces to the generalized elliptic sine function sn_a in [AQVV, 2.1].

In Section 3 we recall some basic properties of the hypergeometric, gamma, and beta functions, that are used in the sequel.

Section 4 contains our main results: differentiation formulas and monotone properties of the generalized elliptic integrals and of several combinations of these functions.

Throughout this paper we denote $r' = \sqrt{1-r^2}$ whenever $r \in (0,1)$. The standard symbols $\mathbb{C}, \mathbb{R}, \mathbb{Z}$, and $\mathbb{N}$ denote the sets of complex numbers, real numbers, integers, and natural numbers (with zero included), respectively.

2 The Schwarz-Christoffel map onto a quadrilateral

For $0 < a, b < 1, \max\{a+b, 1\} \le c \le \min\{a,b\}+1$, $r \in (0,1)$, let $g_r(t) = t^{b-1}(1-t)^{c-b-1}(1-r^2t)^{-a}$, $t \in \mathbb{C}$, $\operatorname{Im} t \ge 0$, denote the analytic branch for which the argument of each of the factors t, $1-t$, and $1-r^2t$ is π whenever it is real and negative. Denote $C = C(b,c) = 1/B(b, c-b)$. We define the generalized Jacobi sine function $\mathrm{sn}_{a,b,c}(w) = \mathrm{sn}_{a,b,c}(w,r)$ as the inverse of the function f given on the closed upper half plane by

$$\begin{aligned} w = f(z) = f_{a,b,c}(z) &= C\int_0^z g_r(t)dt \\ &= C\int_0^z t^{b-1}(1-t)^{c-b-1}(1-r^2t)^{-a}dt \qquad (2.1)\\ &= e^{i(a+b+1-c)\pi}Cr^{-2a}\int_0^z t^{b-1}(t-1)^{c-b-1}(t-1/r^2)^{(1-a)-1}dt\,. \end{aligned}$$

Recall the Euler integral representation [AAR, Theorem 2.2.1] [AS, 15.3.1]

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \qquad (2.2)$$

for $\operatorname{Re} c > \operatorname{Re} b > 0$ and $z \in \mathbb{C} \setminus \{u \in \mathbb{R} \mid u \ge 1\}$.

2.3. **Theorem.** *Let H denote the closed upper half-plane $\{z \in \mathbb{C} \,|\, \operatorname{Im} z \ge 0\}$ and let $0 < a, b < 1, \max\{a+b, 1\} \le c \le 1 + \min\{a,b\}$, $r \in (0,1)$. Then the function f in (2.1) is a homeomorphism of H onto the quadrilateral Q with vertices*

$$f(0) = 0\,,\;\; f(1) = F(a,b;c;r^2)\,,$$

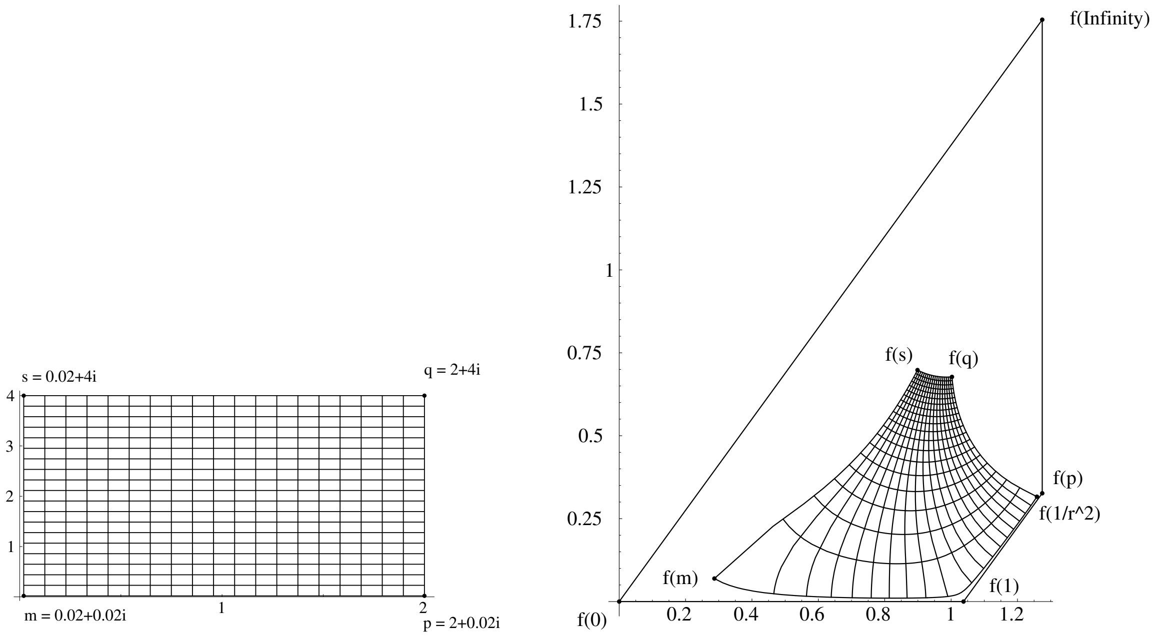

Figure 1: The image quadrilateral and the image of a grid under the mapping $f_{a,b,c}$ with $a = 0.2, b = 0.3, c = 1.0, r = 0.7$

$$f(1/r^2) = f(1) + \frac{B(c-b, 1-a)}{B(b, c-b)} e^{(b+1-c)i\pi} r'^{2(c-a-b)} F(c-a, c-b; c+1-a-b; r'^2)\,,$$

and

$$f(\infty) = f(1/r^2) + \frac{B(1-a, a+1-c)}{B(b, c-b)} e^{i(a+b+1-c)\pi} r^{2(1-c)} r'^{2(c-a-b)} F(1-b, 1-a; 2-c; r^2)\,,$$

and interior angles $b\pi, (c-b)\pi, (1-a)\pi$ and $(a+1-c)\pi$, respectively, at these vertices. It is conformal in the interior of H.

Proof. It is clear that $f(0) = 0$ and by (2.2) $f(1) = F(a, b; c; r^2)$. Next we evaluate

$$\begin{aligned} f(1/r^2) &= C\int_0^{1/r^2} g_r(t)dt \\ &= C\int_0^1 g_r(t)dt + C\int_1^{1/r^2} g_r(t)dt \\ &= F(a, b; c; r^2) + C\int_1^{1/r^2} g_r(t)dt\,. \end{aligned}$$

To evaluate the second integral above, we make the change of variable $t = 1/(1 - r'^2u)$ for which $dt = (1 - r'^2u)^{-2}r'^2du$. We observe that this change of variable is simply the restriction to reals of the plane Möbius transformation taking the ordered triple $(1, 1/r^2, 0)$ onto the ordered triple $(0, 1, \infty)$. Then

$$g_r(t)dt \;=\; (1-r'^2u)^{1-b}\left(1 - \frac{1}{1-r'^2u}\right)^{c-b-1}\left(1 - \frac{r^2}{1-r'^2u}\right)^{-a}(1-r'^2u)^{-2}r'^2du$$

$$
\begin{aligned}
&= (-1)^{b+1-c}\,(r')^{2(c-a-b)}\,u^{c-b-1}(1-u)^{-a}(1-r'^2u)^{a-c}du \\
&= e^{(b+1-c)i\pi}\,(r')^{2(c-a-b)}\,u^{c-b-1}(1-u)^{-a}(1-r'^2u)^{a-c}du
\end{aligned}
$$

and by (2.2) we get

$$
\begin{aligned}
C\int_1^{1/r^2} g_r(t)dt &= e^{(b+1-c)i\pi}\,(r')^{2(c-a-b)}\,\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 u^{c-b-1}(1-u)^{-a}(1-r'^2u)^{a-c}du \\
&= e^{(b+1-c)i\pi}\,(r')^{2(c-a-b)}\,\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{\Gamma(c-b)\Gamma(1-a)}{\Gamma(c+1-a-b)}\,\cdot \\
&\quad \cdot F(c-a,c-b;c+1-a-b;r'^2) \\
&= \frac{B(c-b,1-a)}{B(b,c-b)}e^{(b+1-c)i\pi}\,(r')^{2(c-a-b)}\,F(c-a,c-b;c+1-a-b;r'^2) \\
&= \frac{B(c-b,1-a)}{B(b,c-b)}e^{(b+1-c)i\pi}r^{2(1-c)}\,(r')^{2(c-a-b)}\,\cdot \\
&\quad \cdot F(1-a,1-b;c+1-a-b;r'^2)\,,
\end{aligned}
$$

where the last expression follows from (3.9) below. Hence $f(1/r^2)$ has the value claimed. We proceed to evaluate the remaining value, namely

$$
f(\infty) = C\int_0^\infty g_r(t)dt = C\int_0^{1/r^2} g_r(t)dt + C\int_{1/r^2}^\infty g_r(t)dt\,.
$$

The first integral above equals $f(1/r^2)$. To compute the second one, we apply the change of variable $t = (1-r^2v)/(r^2(1-v))$. We observe that this change of variable is simply the restriction to reals of the plane Möbius transformation taking the ordered triple $(1/r^2,\infty,1)$ onto the ordered triple $(0,1,\infty)$. Then $dt = (1-r^2)/(r^2(1-v)^2)dv$, $t=1/r^2$ gives $v=0$, and $t=\infty$ gives $v=1$. We get

$$
\begin{aligned}
g_r(t)dt &= \left(\frac{1-r^2v}{r^2(1-v)}\right)^{b-1}\left(1-\frac{1-r^2v}{r^2(1-v)}\right)^{c-b-1}\cdot \\
&\quad \cdot\left(1-\frac{r^2(1-r^2v)}{r^2(1-v)}\right)^{-a}\left(\frac{1-r^2}{r^2(1-v)^2}\right)dv \\
&= r^{2(1+a-b+b+1)}(1-v)^{a-c}(1-r^2v)^{b-1}(r^2-1)^{c-b-1}(r^2(r^2-1)v)^{-a}(1-r^2)dv \\
&= (-1)^{a+b+1-c}(r')^{2(c-a-b)}r^{2(1-c)}v^{-a}(1-v)^{a-c}(1-r^2v)^{b-1}dv
\end{aligned}
$$

so that

$$
\begin{aligned}
C\int_{1/r^2}^\infty g_r(t)dt &= (-1)^{a+b+1-c}(r')^{2(c-a-b)}r^{2(1-c)}C\int_0^1 v^{-a}(1-v)^{a-c}(1-r^2v)^{b-1}dv \\
&= (-1)^{a+b+1-c}(r')^{2(c-a-b)}r^{2(1-c)}\frac{B(1-a,1+a-c)}{B(b,c-b)}\,\cdot \\
&\quad \cdot F(1-b,1-a;2-c;r^2)\,.
\end{aligned}
$$

The claimed value for $f(\infty)$ follows.

It follows form the formula (2.1) that (see e.g. [M, pp. 128–134]) f is a Schwarz-Christoffel transformation which maps H onto a quadrilateral Q with vertices $f(0)$, $f(1)$, $f(1/r^2)$, and $f(\infty)$ and interior angles $b\pi$, $(c-b)\pi$, $(1-a)\pi$, and $(1-c+a)\pi$ in counterclockwise order. Since the boundary of Q is a Jordan curve (none of its sides intersect except at the end points), f is univalent and hence a homeomorphism. $\square$

2.4. **Remark.** The quadrilateral Q reduces to a trapezoid if and only if $c = 1$ or $c = a+b$, to a parallelogram if and only if $c = 1 = a+b$, [AQVV] and to a rectangle (the classical case) if and only if $a = b = \frac{1}{2}$ and $c = 1$, [Bo].

2.5. **Corollary.** *Let* $0 < a, b < 1, \max\{a+b, 1\} \le c \le 1 + \min\{a, b\}$, *and let* Q *be a quadrilateral in the upper half plane with vertices* $0, 1, A$ *and* B, *the interior angles at which are, respectively,* $b\pi, (c-b)\pi, (1-a)\pi$ *and* $(1+a-c)\pi$. *Then the conformal modulus of* Q *(cf. [LV]) is given by*

$$\operatorname{mod}(Q) = \mathcal{K}(r')/\mathcal{K}(r),$$

where $r \in (0,1)$ *satisfies the equation*

$$A = 1 + \frac{L r'^{2(c-a-b)} F(c-a, c-b; c+1-a-b; r'^2)}{F(a,b;c;r^2)} = G(r)\,, \tag{2.6}$$

say, and

$$L = \frac{B(c-b, 1-a)}{B(b, c-b)} e^{(b+1-c)i\pi}.$$

Proof. Clearly, $\arg(A-1) = (b+1-c)\pi = \arg(L)$. Since $G(0) = \infty$ and $G(1) = 0$, it follows that a unique solution $r \in (0,1)$ of equation (2.6) exists. Let f be as in Theorem 2.3 and let $g = f/f(1)$. Then g maps the upper half plane H onto Q, with $g(0) = 0$, $g(1) = 1$, $g(1/r^2) = A$, and $g(\infty) = B$. The function $h = \mathrm{sn}^{-1}$ maps the first quadrant conformally [Bo] onto the standard rectangle R, with $h(0) = 0$, $h(1) = \mathcal{K}(r)$, $h(1/r) = \mathcal{K}(r) + i\mathcal{K}(r')$, and $h(\infty) = i\mathcal{K}(r')$. Hence the function $k = h(\sqrt{\ })$ maps H conformally onto R. Thus, by conformal invariance, $mod(Q) = \mathcal{K}(r')/\mathcal{K}(r)$. $\square$

2.7. **Remark.** Bowman [Bo, pp. 103-104] gives a formula for the conformal modulus of the quadrilateral with vertices 0, 1, $1+hi$, and $(h-1)i$ when $h > 1$ as $q = \mathcal{K}(r)/\mathcal{K}(r')$ where

$$r = \frac{t_1 - t_2}{t_1 + t_2}\,, \quad t_1 = \mu^{-1}\left(\frac{\pi}{2c}\right)\,, \quad t_2 = \mu^{-1}\left(\frac{\pi c}{2}\right)\,, \quad c = 2h-1\,.$$

Therefore, the quadrilateral can be conformally mapped onto the rectangle 0, 1, $1+qi$, qi with vertices corresponding.

2.8. **Computational discovery.** We have written a Mathematica® function that computes the modulus of the quadrilateral with vertices at $0, 1, A, B$ where $\mathrm{Im}\, A > 0, \mathrm{Im}\, B > 0$. This led to the following discovery for symmetric quadrilaterals:

$$\text{If } |B| = 1 \text{ and } 2\arg A = \arg B, \text{ then the modulus is equal to } 1\,.$$

It is not difficult to prove this analytically. The following figure illuminates the variation of the modulus of the quadrilateral with vertices $0, 1, x+iy, i$ in the first quadrant.

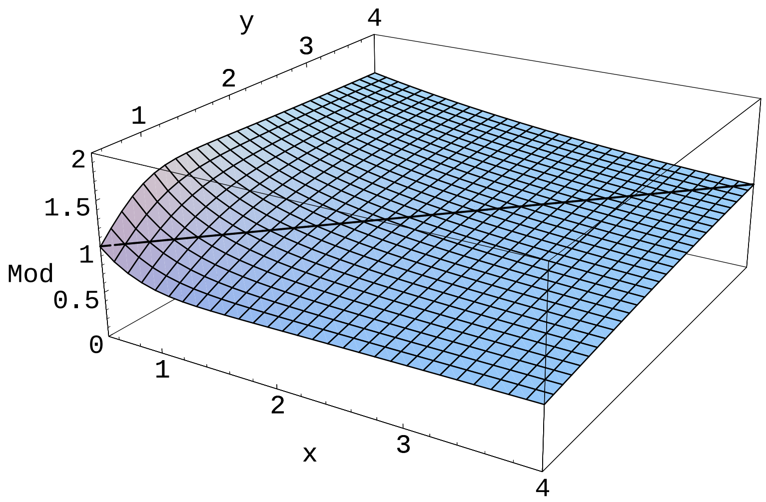

Figure 2: Modulus of the quadrilateral with vertices $0, 1, x+iy, i$ and the line $(x, x, 1)$.

3 Hypergeometric functions

Let Γ denote Euler's *gamma function* and let Ψ be its logarithmic derivative (also called the *digamma function*), $\Psi(z) = \Gamma'(z)/\Gamma(z)$. By [Ah, p. 198] the function Ψ and its derivative have the series expansions

$$\Psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}\,, \quad \Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2}\,, \tag{3.1}$$

where $\gamma = -\Psi(1) = \lim_{n\to\infty}(\sum_{k=1}^{n} 1/k - \log n) = 0.57721\ldots$ is the *Euler-Mascheroni constant*. From (3.1) it is seen that Ψ is strictly increasing on $(0, \infty)$ and that Ψ' is strictly decreasing there, so that Ψ is concave. Moreover, $\Psi(z+1) = \Psi(z) + 1/z$ and $\Psi(\frac{1}{2}) = -\gamma - 2\log 2$, see [AS, Ch. 6].

For all $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and for all $n \in \mathbb{N}$ we have

$$\Gamma(z+n) = (z, n)\Gamma(z)\,, \tag{3.2}$$

a fact which follows by induction [WW, 12.12]. This enables us to extend the Appell symbol for all complex values of a and $a+t$, except for non-positive integer values, by

$$(a,t) = \frac{\Gamma(a+t)}{\Gamma(a)}\,. \tag{3.3}$$

Furthermore, the gamma function satisfies the reflection formula [WW, 12.14]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{3.4}$$

for all $z \notin \mathbb{Z}$. In particular, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

The *beta function* is defined for $\mathrm{Re}\, x > 0$, $\mathrm{Re}\, y > 0$ by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,. \tag{3.5}$$

We use the standard notation for contiguous hypergeometric functions (cf. [R])

$$F = F(a,b;c;z),\;\; F(a+) = F(a+1,b;c;z),\;\; F(a-) = F(a-1,b;c;z)\,,$$

etc. We also let

$$v = v(z) = F\,,\;\; u = u(z) = F(a-)\,,\;\; v_1 = v_1(z) = v(1-z)\,,\;\; \text{and}\;\; u_1 = u_1(z) = u(1-z)\,.$$

The derivative of F can be written in the following several different forms which will be useful in deriving the fifteen important Gauss contiguous relations [R]

$$\begin{aligned}
\frac{dv}{dz} = \frac{dF}{dz} &= \frac{a}{z}(F(a+)-F) = \frac{b}{z}(F(b+)-F) = \frac{c-1}{z}(F(c-)-F) \\
&= \frac{ab}{c}F(a+,b+;c+) = \frac{1}{1-z}\left((a+b-c)F + \frac{(c-a)(c-b)}{c}F(c+)\right) \\
&= \frac{(c-a)u + (a-c+bz)v}{z(1-z)}
\end{aligned} \tag{3.6}$$

and

$$\frac{du}{dz} = \frac{dF(a-)}{dz} = (a-1)\left(F + \frac{b-c}{c}F(c+)\right) = \frac{a-1}{z}(v-u)\,. \tag{3.7}$$

In particular, from (3.6) it follows that (cf. [AQVV, Theorem 3.12 (3)])

$$\frac{ab}{c}z(1-z)F(a+1,b+1;c+1;z) = (c-a)u(z) + (a-c+bz)v(z)\,. \tag{3.8}$$

The behavior of the hypergeometric function near $z=1$ in the three cases $\mathrm{Re}\,(a+b-c) < 0$, $a+b=c$, and $\mathrm{Re}\,(a+b-c) > 0$, respectively, is given by

$$\begin{cases}
F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,, \\
B(a,b)F(a,b;a+b;z) + \log(1-z) = R(a,b) + \mathrm{O}((1-z)\log(1-z))\,, \\
F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z)\,,
\end{cases} \tag{3.9}$$

where $R(a,b) = -\Psi(a) - \Psi(b) - 2\gamma$. The above asymptotic formula for the *zero-balanced* case $a+b=c$ is due to Ramanujan (see [Ask], [Be]). This formula is implied by [AS, 15.3.10]. Note that $R(\frac{1}{2},\frac{1}{2}) = \log 16$.

For complex a, b, c, and z, with $|z| < 1$, we let

$$\mathcal{M}(z) = \mathcal{M}(a,b,c,z) = z(1-z)\left(v_1(z)\frac{dv}{dz} - v(z)\frac{dv_1}{dz}\right). \tag{3.10}$$

From (3.6) it is easy to see that

$$\begin{aligned} \mathcal{M} &= (c-a)(uv_1 + u_1 v) + (2(a-c)+b)vv_1 \\ &= (c-a)(uv_1 + u_1 v - vv_1) + (a+b-c)vv_1. \end{aligned} \tag{3.11}$$

It follows from [AQVV, Corollary 3.13(5)] and (3.4) that

$$\mathcal{M}(a, 1-a, 1, r) = \frac{1-a}{\Gamma(a)\Gamma(2-a)} = \frac{\sin(\pi a)}{\pi} \tag{3.12}$$

for $0 < a < 1$ and $0 \le r < 1$. In particular, we get the classical Legendre relation ([AAR], [BF])

$$\mathcal{M}(1/2, 1/2, 1, r) = \frac{1}{\pi}. \tag{3.13}$$

The next result generalizes [AQVV, Theorem 3.9].

3.14. **Theorem.** *For $0 < a, b < c$, let the function f be defined on $[0,\infty)$ by $f(x) = F(a,b;c;1-e^{-x})$ and let $g(x) = f(x)\exp(-(a+b-c)x)$. Then f and g are increasing, with $f(0) = g(0) = 1$. If $a+b > c$, then $f(\infty) = \infty$. If $a+b = c$, then $f(\infty) = g(\infty) = \infty$. If $a+b < c$, then $f(\infty) = B(c, c-a-b)/B(c-a, c-b)$ and $g(\infty) = \infty$. Moreover, $h(x) = f'(x)e^{-x(a+b-c)}$ is also increasing, with $h(0) = ab/c$ and $h(\infty) = \Gamma(c)\Gamma(a+b+1-c)/(\Gamma(a)\Gamma(b))$ or $h(\infty) = \infty$ according as $a+b+1 > c$ or $a+b+1 \le c$.*

Proof. The assertions $f(0) = g(0) = 1$ and that f is increasing, so that if $a+b < c$, then g is increasing, are all clear. In the three cases, $a+b < c, a+b = c$ and $a+b > c$, the limiting values at ∞ are clear by (3.9). Next, by (3.9), $g(x) = F(c-a, c-b; c; 1-e^{-x})$, which is clearly increasing. Next, by differentiation,

$$c/(ab)f'(x) = e^{-x}F(a+1, b+1; c+1; 1-e^{-x}).$$

Hence, by (3.9)

$$\begin{aligned} c/(ab)f'(x) &= (e^{-x})(e^{-x(c-a-b-1)})F(c-a, c-b; c+1; 1-e^{-x}) \\ &= e^{x(a+b-c)}F(c-a, c-b; c+1; 1-e^{-x}), \end{aligned}$$

so that $h(x) = (ab/c)F(c-a, c-b; c+1; 1-e^{-x})$, which is increasing, with boundary values $h(0) = ab/c$, and by (3.9) and (3.2),

$$h(\infty) = (ab/c)\frac{\Gamma(c+1)\Gamma(a+b+1-c)}{\Gamma(a+1)\Gamma(b+1)} = \frac{\Gamma(c)\Gamma(a+b+1-c)}{\Gamma(a)\Gamma(b)},$$

if $a+b+1 > c$, and $= \infty$ if $a+b+1 \le c$. $\square$

3.15. **Theorem.** *For $a, b, c, d > 0$, with $a + b > c > max\{a, b\}$, let the function f be defined on $[0, \infty)$ by $f(x) = F(a, b; c; 1 - (1 + x)^{-1/d})$, and let*

$$g(x) = (1 + x)^{((c+d)-(a+b))/d} f'(x) .$$

Then g is increasing, with

$$g(0) = \frac{ab}{cd} \text{ and } g(\infty) = \frac{(a + b - c)\Gamma(c)\Gamma(a + b - c)}{d\,\Gamma(a)\Gamma(b)} .$$

Proof. With $u = 1 - (1 + x)^{-1/d}$, by (3.6) and (3.9),

$$\begin{aligned} f'(x) &= \frac{ab}{cd}(1 + x)^{-1-(1/d)} F(a + 1, b + 1; c + 1; u) \\ &= \frac{ab}{cd}(1 + x)^{(1/d)(a+b-c-d)} F(c - a, c - b; c + 1; u) , \end{aligned}$$

so that $g(x) = (ab/(cd))F(c - a, c - b; c + 1; u)$, which is clearly positive and increasing. The boundary value $g(0) = ab/(cd)$ is clear, while the value of $g(\infty)$ follows from (3.9). □

3.16. **Remark.** (1) With $c = a+b$, Theorem 3.14 reduces to [AQVV, Theorem 3.9].
(2) For $a + b = c + d$, Theorem 3.15 reduces to [AQVV, Theorem 3.10.].

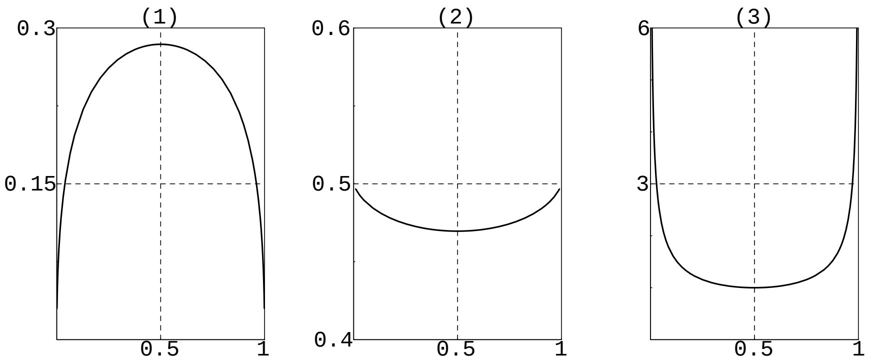

Figure 3: (1) $\mathcal{M}(0.5, 1.0, 2.0, \cdot)$, (2) $\mathcal{M}(0.5, 1.0, 1.5, \cdot)$, (3) $\mathcal{M}(0.5, 1.0, 1.0, \cdot)$

3.17. **Theorem.** *For positive constants a, b, c, the restriction to $(0, 1)$ of the continuous function $\mathcal{M}$ in (3.10) has the following properties. Denote $\mathcal{M}(x) = \mathcal{M}(a, b, c, x)$.*

(1) $\mathcal{M}(x) = \mathcal{M}(1 - x) > 0$ for all $x \in (0, 1)$.

(2) If $a+b \leq c$, then $\mathcal{M}(x)$ is bounded and extends continuously to $[0, 1]$. In particular, if $a + b = c = 1$, then $\mathcal{M}(x)$ equals the constant $\sin(\pi a)/\pi$.

(3) If $a+b>c$, then $\mathcal{M}$ is unbounded on $(0,1)$, with $\mathcal{M}(0+)=\mathcal{M}(1-)=\infty$.
(4) If $a+b<c<a+b+1$, then

$$\lim_{x\to 0+} x^{a+b-c}\mathcal{M}(x) = \lim_{x\to 1-}(1-x)^{a+b-c}\mathcal{M}(x) = \frac{\Gamma(c)\Gamma(a+b+1-c)}{\Gamma(a)\Gamma(b)}\,.$$

(5) If $a+b+1=c$, then

$$\lim_{x\to 0+}\frac{\mathcal{M}(x)}{x\log(1/x)} = \lim_{x\to 1-}\frac{\mathcal{M}(x)}{(1-x)\log(1/(1-x))} = \frac{a+b}{B(a,b)}\,.$$

(6) If $a+b+1<c$, then

$$\lim_{x\to 0+}\frac{\mathcal{M}(x)}{x} = \lim_{x\to 1-}\frac{\mathcal{M}(x)}{1-x} = \frac{ab(2c-a-b-1)B(c,c-a-b)}{c(c-a-b-1)B(c-a,c-b)}\,.$$

(7) If $a+b>c$, then

$$\lim_{x\to 0+} x^{a+b-c}\mathcal{M}(x) = \lim_{x\to 1-}(1-x)^{a+b-c}\mathcal{M}(x) = \frac{(a+b-c)B(c,a+b-c)}{B(a,b)}\,.$$

Proof. (1) From (3.8) and (3.10), we get

$$\begin{aligned}\mathcal{M}(x) &= \frac{abx(1-x)}{c}(F(a+1,b+1;c+1;x)v(1-x)\\ &\quad +F(a+1,b+1;c+1;1-x)v(x))\\ &= G(x)+H(x)>0\,.\end{aligned}$$

Here

$$G(x)=\frac{abx(1-x)}{c}F(a+1,b+1;c+1;x)v(1-x)$$

and

$$H(x)=\frac{abx(1-x)}{c}F(a+1,b+1;c+1;1-x)v(x)=G(1-x)\,.$$

(2) First, if $a+b<c$, then from (3.9) and (3.10),

$$\mathcal{M}(0+)=\mathcal{M}(1-)=(c-a)u(1)+(a+b-c)v(1)=0\,.$$

Next, if $a+b=c$, then from (3.9), $G(0+)=0$ and

$$H(x)=\frac{ab(1-x)}{c}F(a,b;a+b+1;1-x)v(x)\,,$$

so that $H(0+)=(ab/c)F(a,b;a+b+1;1)$. Next, $H(1-)=0$ and

$$G(x)=\frac{abx}{c}F(a,b;a+b+1;x)v(1-x)\,,$$

so that $G(1-)=(ab/c)F(a,b;a+b+1;1)$. Thus,

$$\mathcal{M}(0+)=\mathcal{M}(1-)=(ab/c)F(a,b;a+b+1;1)=1/B(a,b)\,.$$

(3) Let $a+b>c$. Then

$$H(x)=\frac{ab}{c}(1-x)x^{c-a-b}F(c-a,c-b;c+1;1-x)v(x)\,,$$

so that $H(0+)=\infty$. Similarly, $G(1-)=\infty$.

(4) By (3.9),

$$\begin{aligned}\frac{c}{ab}\mathcal{M}(x) &= x(1-x)F(a+1,b+1;c+1;x)F(a,b;c;1-x;)\\ &\quad +x^{c-a-b}(1-x)F(c-a,c-b;c+1;1-x)F(a,b;c;x)\,,\end{aligned}$$

so that the result follows by (1).

(5) By (3.9),

$$\begin{aligned}\frac{c\mathcal{M}(x)}{abx\log(1/x)} &= x(1-x)F(a+1,b+1;c+1;x)F(a,b;c;1-x)\\ &\quad +\frac{1-x}{\log(1/x)}F(a+1,b+1;c+1;1-x)F(a,b;c;x)\,,\end{aligned}$$

so that the result follows from (1) and (3.9).

(6) By (3.9),

$$\begin{aligned}\frac{c\mathcal{M}(x)}{abx} &= (1-x)F(a+1,b+1;c+1;x)F(a,b;c;1-x)\\ &\quad +(1-x)F(a+1,b+1;c+1;x)\,,\end{aligned}$$

so that by (1) and (3.9),

$$\begin{aligned}\lim_{x\to 0+}\frac{\mathcal{M}(x)}{x} &= \lim_{x\to 1-}\mathcal{M}(x)/(1-x)\\ &= \frac{ab}{c}\left(\frac{B(c,c-a-b)}{B(c-a,c-b)}+\frac{B(c+1,c-a-b-1)}{B(c-a,c-b)}\right)\\ &= \frac{ab(2c-a-b-1)}{(c(c-a-b-1)}\frac{B(c,c-a-b)}{B(c-a,c-b)}.\end{aligned}$$

(7) By (3.9),

$$\begin{aligned}\frac{c}{ab}x^{a+b-c}\mathcal{M}(x) &= x(1-x)F(a+1,b+1;c+1;x)F(a,b;c;1-x)\\ &\quad +(1-x)F(c-a,c-b;c+1;1-x)F(a,b;c;x)\,,\end{aligned}$$

so that by (1) and (3.9),

$$\begin{aligned}\lim_{x\to 0+}x^{a+b-c}\mathcal{M}(x) &= \lim_{x\to 1-}(1-x)^{a+b-c}\mathcal{M}(x)\\ &= \frac{ab}{c}F(c-a,c-b;c+1;1)\\ &= \frac{ab}{c}\frac{B(c+1,a+b+1-c)}{B(a+1,b+1)}\\ &= \frac{(a+b-c)B(c,a+b-c)}{B(a,b)}\,. \qquad \square\end{aligned}$$

3.18. **Lemma.** *The function $\mathcal{M}$ in (3.10) satisfies the differentiation formula*

$$\begin{aligned}\frac{d\mathcal{M}(a,b,c,z)}{dz} &= \frac{1}{z(1-z)}\Big((c-a)[(1-c+(a+b-1)z)u(z)v_1(z)\\ &\quad +(-a-b+c+(a+b-1)z)u_1(z)v(z)]\\ &\quad +(1-2z)[(c-a)(a+2b-1)-b^2]v(z)v_1(z)\Big)\,.\end{aligned}$$

Proof. Denote $D=d/dz$. Then, by (3.6), (3.7), and the chain rule, we get

$$zDu=(a-1)(v-u),\;\; z(1-z)Dv=(c-a)u+(a-c+bz)v$$

$$(1-z)Du_1=(1-a)(v_1-u_1),\;\; z(1-z)Dv_1=-((c-a)u_1+(a-c+b(1-z))v_1).$$

Hence, by the product rule, after simplification, we get

$$z(1-z)D(uv_1)=-(c-a)uu_1+(a-1)(1-z)vv_1+((1-a)(1-z)+(c-a-b(1-z)))uv_1,$$

$$z(1-z)D(u_1v)=(c-a)uu_1+(1-a)zvv_1+(a-c+(a+b-1)z)u_1v,$$

and

$$z(1-z)D(vv_1)=(c-a)(uv_1-u_1v)+b(2z-1)vv_1.$$

Substituting these we get

$$\begin{aligned}z(1-z)D\mathcal{M}(a,b,c,z) &= (c-a)[(1-c+(a+b-1)z)uv_1\\ &\quad +(-a-b+c+(a+b-1)z)u_1v]\\ &\quad +(1-2z)[(c-a)(a+2b-1)-b^2]vv_1. \qquad \square\end{aligned}$$

3.19. **Remark.** If we put $c=a+b=1$ in Lemma 3.18 above, then we get the familiar fact that

$$\frac{d}{dz}\mathcal{M}(a,1-a,1,z)=0.$$

We also observe that $D\mathcal{M}(a,b,c,1/2)=0$.

4 Generalized elliptic integrals

The following two important Theorems, 4.1 and 4.3, are indispensable in simplified proofs for monotonicity of the quotient of two functions. The first one, called *L'Hôpital's Monotone Rule*, appears in [AVV, Theorem 1.25], while a more general version of the second one appears in [BK] and [PV, Lemma 2.1].

4.1. **Theorem.** *Let $-\infty < a < b < \infty$, and let f, g : $[a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b). Let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotoneity in the conclusion is also strict.

4.2. **Lemma.** *Let $\{a_n\}$ and $\{b_n\}$ be real sequences with $b_n > 0$ for all n. If the sequence $\{a_n/b_n\}$ is increasing (decreasing), then*

$$T_n = \sum_{k=0}^{n} (n-k)(a_{n-k}b_k - a_k b_{n-k})$$

is positive (negative) for $n = 1, 2, \ldots$.

Proof. It is enough to prove the case $\{a_n/b_n\}$ is increasing, since the other one is similar. Clearly

$$T_1 = a_1 b_0 - a_0 b_1 = b_0 b_1 \left(\frac{a_1}{b_1} - \frac{a_0}{b_0} \right) > 0.$$

Let $n \geq 2$. First let $n = 2m$ be even. Then

$$\begin{aligned}
T_n = T_{2m} &= \sum_{k=0}^{2m} (2m-k)(a_{2m-k}b_k - a_k b_{2m-k}) \\
&= \sum_{k=0}^{2m-1} (2m-k)(a_{2m-k}b_k - a_k b_{2m-k}) \\
&= \sum_{k=0}^{m-1} (2m-k)(a_{2m-k}b_k - a_k b_{2m-k}) + 0 \\
&\quad + \sum_{k=m+1}^{2m-1} (2m-k)(a_{2m-k}b_k - a_k b_{2m-k}) \\
&= \sum_{k=0}^{m-1} (2m-k)(a_{2m-k}b_k - a_k b_{2m-k}) \\
&\quad + \sum_{k=1}^{m-1} k(a_k b_{2m-k} - a_{2m-k}b_k) \\
&= 2m(a_{2m}b_0 - a_0 b_{2m}) + \sum_{k=1}^{m-1} (2m-2k)(a_{2m-k}b_k - a_k b_{2m-k}) \\
&= 2m b_0 b_{2m} \left(\frac{a_{2m}}{b_{2m}} - \frac{a_0}{b_0} \right) + \sum_{k=1}^{m-1} (2m-2k) b_k b_{2m-k} \left(\frac{a_{2m-k}}{b_{2m-k}} - \frac{a_k}{b_k} \right) > 0.
\end{aligned}$$

Next, let $n = 2m+1$ be odd. Then

$$\begin{aligned}
T_n &= \sum_{k=0}^{2m+1} (2m+1-k)(a_{2m+1-k}b_k - a_k b_{2m+1-k}) \\
&= \sum_{k=0}^{2m}(2m+1-k)(a_{2m+1-k}b_k - a_k b_{2m+1-k}) \\
&= (2m+1)(a_{2m+1}b_0 - a_0 b_{2m+1}) + \sum_{k=1}^{m}(2m+1-k)(a_{2m+1-k}b_k - a_k b_{2m+1-k}) \\
&\quad + \sum_{k=m+1}^{2m} (2m+1-k)(a_{2m+1-k}b_k - a_k b_{2m+1-k}) \\
&= (2m+1)(a_{2m+1}b_0 - a_0 b_{2m+1}) + \sum_{k=1}^{m}(2m+1-k)(a_{2m+1-k}b_k - a_k b_{2m+1-k}) \\
&\quad + \sum_{k=1}^{m} k(a_k b_{2m+1-k} - a_{2m+1-k}b_k) \\
&= (2m+1)(a_{2m+1}b_0 - a_0 b_{2m+1}) + \sum_{k=1}^{m}(2m+1-2k)(a_{2m+1-k}b_k - a_k b_{2m+1-k}) \\
&= (2m+1)(b_0 b_{2m+1}) \left(\frac{a_{2m+1}}{b_{2m+1}} - \frac{a_0}{b_0}\right) \\
&\quad + \sum_{k=1}^{m}(2m+1-2k) b_k b_{2m+1-k} \left(\frac{a_{2m+1-k}}{b_{2m+1-k}} - \frac{a_k}{b_k}\right) > 0. \qquad \square
\end{aligned}$$

4.3. **Theorem.** *Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two real power series converging on the interval $(-R, R)$. If the sequence $\{a_n/b_n\}$ is increasing (decreasing), and $b_n > 0$ for all n, then the function*

$$f(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n}$$

is also increasing (decreasing) on $(0, R)$. In fact, the function $f'(x)\left(\sum_{n=0}^{\infty} b_n x^n\right)^2$ has positive Maclaurin coefficients.

Proof.

$$\begin{aligned}
f'(x)\left(\sum_{n=0}^{\infty} b_n x^n\right)^2 &= \sum_{n=0}^{\infty} b_n x^n \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} n b_n x^{n-1} \\
&= (1/x) \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (n-k)(a_{n-k}b_k - a_k b_{n-k})\right) x^n.
\end{aligned}$$

The result follows from Lemma 4.2. $\square$

The following theorem solves the corresponding problem in the case where we have a quotient of two polynomials instead of two power series.

4.4. **Theorem.** *Let $f_n(x) = \sum_{k=0}^n a_k x^k$ and $g_n(x) = \sum_{k=0}^n b_k x^k$ be real polynomials, with $b_k > 0$ for all k. If the sequence $\{a_k/b_k\}$ is increasing (decreasing), then so is the function $f_n(x)/g_n(x)$ for all $x > 0$. In fact, $g_n f_n' - f_n g_n'$ has positive (negative) coefficients.*

Proof. We prove the increasing case by induction on n. The proof of the decreasing case is similar. Let first $n = 1$. Then

$$\frac{f_1(x)}{g_1(x)} = \frac{a_0 + a_1 x}{b_0 + b_1 x}.$$

Hence

$$\begin{aligned} g_1 f_1' - f_1 g_1' &= (b_0 + b_1 x)a_1 - (a_0 + a_1 x)b_1 \\ &= a_1 b_0 - a_0 b_1 \\ &= b_0 b_1 \left(\frac{a_1}{b_1} - \frac{a_0}{b_0} \right) > 0. \end{aligned}$$

Next, assume that the claim holds for all $k \le n$. Now

$$\frac{f_{n+1}(x)}{g_{n+1}(x)} = \frac{f_n(x) + a_{n+1}x^{n+1}}{g_n(x) + b_{n+1}x^{n+1}}.$$

We get

$$\begin{aligned} g_{n+1}f_{n+1}' - f_{n+1}g_{n+1}' &= (g_n + b_{n+1}x^{n+1})(f_n' + (n+1)a_{n+1}x^n) \\ &\quad -(f_n + a_{n+1}x^{n+1})(g_n' + (n+1)b_{n+1}x^n) \\ &= (g_n f_n' - f_n g_n') + (n+1)x^n(g_n a_{n+1} - f_n b_{n+1}) \\ &\quad +x^{n+1}(b_{n+1}f_n' - a_{n+1}g_n') \\ &= (g_n f_n' - f_n g_n') + (n+1)x^n \sum_{k=0}^{n} (a_{n+1}b_k - b_{n+1}a_k)x^k \\ &\quad +x^{n+1} \sum_{k=1}^{n} k(a_k b_{n+1} - b_k a_{n+1})x^{k-1} \\ &= (g_n f_n' - f_n g_n') + \sum_{k=0}^{n} (n+1-k)(a_{n+1}b_k - b_{n+1}a_k)x^{n+k} \\ &= (g_n f_n' - f_n g_n') + \sum_{k=0}^{n} (n+1-k)b_k b_{n+1} \left(\frac{a_{n+1}}{b_{n+1}} - \frac{a_k}{b_k} \right) x^{n+k}. \end{aligned}$$

Hence each coefficient is positive. $\square$

4.5. **Lemma.** *Let $a, b, c, K > 0$.*

(1) If $a + b \ge c$, then $\mu_{a,b,c} : (0,1) \to (0,\infty)$ and $\varphi_K^{a,b,c} : (0,1) \to (0,1)$ are decreasing and increasing homeomorphisms, respectively.

(2) If $a + b > c$, then the function $f(r) = (r/r')^{2(a+b-c)}\mu_{a,b,c}(r)$ is decreasing from $(0,1)$ onto $((B(a,b)^2)/(2B(c,a+b-c)), B(c,a+b-c)/2)$.

(3) If $a+b>c$ and $K>1$, then

$$r<\varphi_K^{a,b,c}(r)<K^{1/(2(a+b-c))}r\,,$$

for all $r\in(0,1)$.

(4) If $a+b<c$, then $\mu_{a,b,c}:[0,1]\to[B(a,b)/(2d),B(a,b)d/2]$, is a decreasing homeomorphism, where $d=F(a,b;c;1)$ is given by (3.9). In this case, $\varphi_K^{a,b,c}$ is defined if and only if $K\geq 1$. If $K>1$, then $\varphi_K^{a,b,c}$ maps $[0,1]$ onto a proper subset of $[0,1]$.

Proof. (1) Since $a,b,c>0$ and $a+b\geq c$, it follows from (3.9) that the function $F(a,b;c;r)$ is an increasing homeomorphism of $[0,1)$ onto $[1,\infty)$. Hence the function $\mu_{a,b,c}$ is a strictly decreasing homeomorphism $(0,1)\to(0,\infty)$. The assertion about $\varphi_K^{a,b,c}$ follows from these facts.

(2) From (3.9),

$$f(r)=\frac{B(a,b)}{2}\frac{F(c-a,c-b;c;r'^2)}{F(c-a,c-b;c;r^2)},,$$

which is decreasing with required limiting values at 0 and 1.

(3) With $s=\varphi_K^{a,b,c}(r)>r$, from (2) we get

$$\begin{aligned} f(s) &= (s/s')^{2(a+b-c)}\mu_{a,b,c}(s)=(s/s')^{2(a+b-c)}\mu_{a,b,c}(r)/K \\ &< f(r)=(r/r')^{2(a+b-c)}\mu_{a,b,c}(r)\,, \end{aligned}$$

so that

$$(s/s')^{2(a+b-c)}<K(r/r')^{2(a+b-c)}\,,$$

which gives $s/s'<K^{1/(2(a+b-c))}r/r'$. Hence

$$r<s<K^{1/(2(a+b-c))}r(s'/r')<K^{1/(2(a+b-c))}r\,.$$

(4) For $a,b,c>0$ and $a+b<c$, (3.9) implies that $F(a,b;c;r)$ is an increasing homeomorphism $[0,1]\to[1,F(a,b;c;1)]$. Now the claim follows from (1.3) and (1.5). $\square$

The function $\mu_{a,b,c}$ is a natural generalization for the function μ_a in [AQVV, (1.3)]. Namely,

$$\mu_{a,1-a,1}(r)=\frac{B(a,1-a)}{2}\frac{F(a,1-a;1;r'^2)}{F(a,1-a;1;r^2)}=\mu_a(r)\,, \tag{4.6}$$

since by (3.4) and (3.5),

$$B(a,1-a)=\frac{\Gamma(a)\Gamma(1-a)}{\Gamma(1)}=\frac{\pi}{\sin(\pi a)}\,.$$

Clearly (1.3) and (1.5) imply the identities (cf. [AQVV, (4.8), (4.9)]),

$$\mu_{a,b,c}(r)\mu_{a,b,c}(r')=\left(\frac{B(a,b)}{2}\right)^2\,,\;\; r\in(0,1)\,, \tag{4.7}$$

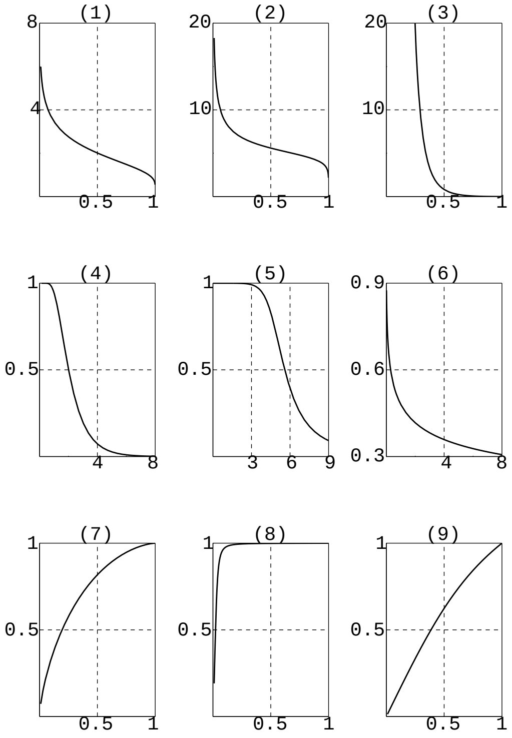

Figure 4: (1) $\mu_{0.5,0.5,1}$, (2) $\mu_{0.1,1.1,1}$, (3) $\mu_{2.5,1.5,2}$, (4) $\mu^{-1}_{0.5,0.5,1}$, (5) $\mu^{-1}_{0.1,1.1,1}$, (6) $\mu^{-1}_{2.5,1.5,2}$, (7) $\varphi^{0.5,0.5,1}_{1.5}$, (8) $\varphi^{0.1,1.1,1}_{2.5}$, (9) $\varphi^{2.5,1.5,2}_{3.5}$.

$$\mu^{-1}_{a,b,c}(x)^2 + \mu^{-1}_{a,b,c}(y)^2 = 1\,, \tag{4.8}$$

where $x, y > 0$ with $xy = (B(a,b)/2)^2$, and

$$\varphi^{a,b,c}_K(r)^2 + \varphi^{a,b,c}_{1/K}(r')^2 = 1\,. \tag{4.9}$$

Moreover, from (1.3) and (3.9) we get, for $c < a+b$,

$$r'^{2(c-a-b)}\mu_{a,b,c}(r) = r^{2(c-a-b)}\mu_{c-a,c-b,c}(r)\,,\;\; r \in (0,1)\,. \tag{4.10}$$

4.11. **Lemma.** *Let f be a bijection from a real interval I onto $(0,\infty)$ and let g be defined on I by $g(x) = af(x)$, where $a > 0$ is a constant. Then for each constant $K > 0$, we have*

$$f^{-1}(Kf(x)) = g^{-1}(Kg(x))\,.$$

Proof. Let $u = f^{-1}(Kf(x))$. Then $f(u) = Kf(x)$, so that $af(u) = aKf(x)$, that is, $g(u) = Kg(x)$. Hence $u = g^{-1}(Kg(x))$. $\square$

4.12. **Remark.** For $a,b,c>0$ with $a+b\ge c$, $K>0$, and $r\in(0,1)$ denote

$$\tilde{\mu}_{a,b,c}(r)=\frac{F(a,b;c;r'^2)}{F(a,b;c;r^2)}$$

and

$$\tilde{\varphi}_K^{a,b,c}(r)=\tilde{\mu}_{a,b,c}^{-1}(\tilde{\mu}_{a,b,c}(r)/K)\,.$$

By Lemma 4.11 we see that

$$\tilde{\varphi}_K^{a,b,c}(r)=\varphi_K^{a,b,c}(r)\,.$$

4.13. **Remark.** Observe first that $\lim_{c\to a+}B(a,c-a)=\infty$ and $\Gamma(c-a)(c-a,n)=\Gamma(c-a+n)$, which tends to $\Gamma(n)=(n-1)!$, as $c\to a+$.

(1) $\lim_{c\to a+}(\mathcal{K}_{a,c}(r)-(B(a,c-a)/2))=\log(1/r')$.

(2) $\lim_{c\to a+}(\mathcal{E}_{a,c}(r)-(B(a,c-a)/2))=\log(1/r')-\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{r^{2n}}{a+n-1}\right)$.

In particular, for each fixed $r\in(0,1)$ all the three functions $\mathcal{K}_{a,c}(r),\mathcal{E}_{a,c}(r)$, and $\mathcal{E}_{a,c}(r)-r'^2\mathcal{K}_{a,c}(r)$ tend to ∞ as $c\to a+$.

4.14. **Theorem.** *The following differentiation formulae hold:*

$$\frac{d\mathcal{K}_{a,b,c}}{dr}=\frac{2}{rr'^2}\left((c-a)\mathcal{E}_{a,b,c}+(br^2+a-c)\mathcal{K}_{a,b,c}\right)\,, \tag{4.15}$$

$$\frac{d\mathcal{E}_{a,b,c}}{dr}=\frac{2(a-1)}{r}\left(\mathcal{K}_{a,b,c}-\mathcal{E}_{a,b,c}\right)\,, \tag{4.16}$$

$$\begin{aligned}\frac{d}{dr}(\mathcal{K}_{a,b,c}-\mathcal{E}_{a,b,c}) &= \frac{2}{rr'^2}\Big(((c-a)-(1-a)r'^2)\mathcal{E}_{a,b,c}\\ &\quad -((a+b)r^2-c+r'^2)\mathcal{K}_{a,b,c}\Big)\,,\end{aligned} \tag{4.17}$$

$$\frac{d}{dr}(\mathcal{E}_{a,b,c}-r'^2\mathcal{K}_{a,b,c})=\frac{2}{r}((1-c)\mathcal{E}_{a,b,c}+(c-1-(b-1)r^2)\mathcal{K}_{a,b,c})\,, \tag{4.18}$$

$$\frac{d}{dr}\mu_{a,b,c}(r)=-\frac{B(a,b)\mathcal{M}(a,b,c,r^2)}{rr'^2v(r^2)^2}=-\frac{B(a,b)^3\mathcal{M}(r^2)}{4rr'^2\mathcal{K}^2}\,, \tag{4.19}$$

$$\frac{\mathcal{M}(a,b,c,s^2)}{\mathcal{M}(a,b,c,r^2)}\frac{ds}{dr}=\frac{1}{K}\frac{ss'^2v(s^2)^2}{rr'^2v(r^2)^2}=\frac{1}{K}\frac{ss'^2\mathcal{K}(s)^2}{rr'^2\mathcal{K}(r)^2}\,,\quad s=\varphi_K^{a,b,c}(r)\,. \tag{4.20}$$

Proof. From (3.6)

$$\frac{dF}{dz}=\frac{(c-a)u+(a-c+bz)v}{z(1-z)}\,.$$

Put $z=r^2$ and multiply both sides by $B(a,b)/2$. This gives (4.15).

For (4.16) recall that by (3.7)

$$\frac{du}{dz} = \frac{(a-1)}{z}(v-u)\,.$$

Put $z = r^2$ and multiply both sides by $B(a,b)/2$. This gives (4.16).

The formulae (4.17) and (4.18) follow from (4.15) and (4.16).

From (3.10), putting $z = r^2$, we get

$$\mathcal{M}(r^2) = -r^2 r'^2 \frac{d\mu}{dr}\left(\frac{2}{B(a,b)}\right)^3 \mathcal{K}^2\left(\frac{1}{2r}\right),$$

so that

$$\frac{d\mu}{dr} = -\frac{B(a,b)^3\mathcal{M}(r^2)}{4rr'^2\mathcal{K}^2}.$$

Denote $s = \varphi_K^{a,b,c}(r)$. Then by (1.5)

$$\mu_{a,b,c}(s) = \frac{1}{K}\mu_{a,b,c}(r)$$

so that

$$\frac{d}{ds}\mu_{a,b,c}(s)\frac{ds}{dr} = \frac{1}{K}\frac{d}{dr}\mu_{a,b,c}(r)\,.$$

Now (4.20) follows from (4.19). $\square$

4.21. **Lemma.** *(cf. [AQVV, Lemmas 5.2 and 5.4]) For $0 < a,b < \min\{c,1\}$ and $c \le a+b$, denote $\mathcal{K} = \mathcal{K}_{a,b,c}$ and $\mathcal{E} = \mathcal{E}_{a,b,c}$. Then the function*

(1) $f_1(r) = (\mathcal{K}-\mathcal{E})/(r^2\mathcal{K})$ is strictly increasing from $(0,1)$ onto $(b/c,1)$. In particular, we have the sharp inequality,

$$\frac{b}{c} < \frac{\mathcal{K}-\mathcal{E}}{r^2\mathcal{K}} < 1$$

for all $r \in (0,1)$.

(2) $f_2(r) = (\mathcal{E} - r'^2\mathcal{K})/r^2$ has positive Maclaurin coefficients and maps $(0,1)$ onto $(B(a,b)(c-b)/(2c), C)$, where

$$C = \frac{B(a,b)B(c,c+1-a-b)}{2B(c+1-a,c-b)}\,.$$

(3) $f_3(r) = r'^{-2(c+1-a-b)}\mathcal{E}$ has positive Maclaurin coefficients and maps $[0,1)$ onto $(B(a,b)/2,\infty)$.

(4) $f_4(r) = r'^{2(a+b-c)}\mathcal{K}$ has positive Maclaurin coefficients and maps $[0,1)$ onto $[B(a,b)/2, B(c,a+b-c)/2)$.

(5) $f_5(r) = r'^{-2}\mathcal{E}$ has positive Maclaurin coefficients and maps $[0,1)$ onto $[B(a,b)/2,\infty)$.

(6) $f_6(r) = r'^2\mathcal{K}$ has negative Maclaurin coefficients, except for the constant term, and maps $[0,1)$ onto $(0, B(a,b)/2]$.

(7) $f_7(r) = \mathcal{K}$ has positive Maclaurin coefficients and is log-convex from $[0,1)$ onto $[B(a,b)/2,\infty)$. In fact, $(d/dr)(\log\mathcal{K})$ also has positive Maclaurin coefficients.

(8) $f_8(r) = (\mathcal{E} - r'^2\mathcal{K})/(r^2\mathcal{K})$ *is strictly decreasing from* $(0,1)$ *onto* $(0, 1-(b/c))$.

(9) $f_9(r) = (\mathcal{K}-\mathcal{E})/(\mathcal{E} - r'^2\mathcal{K})$ *is strictly increasing from* $(0,1)$ *onto* $(b/(c-b), \infty)$.

(10) $f_{10}(r) = (\mathcal{K} - B(a,b)/2)/\log(1/r')$ *is strictly increasing from* $(0,1)$ *onto* $(abB(a,b)/c, D)$*, where* $D = 1$ *if* $c = a+b$ *and* $D = \infty$ *if* $c < a+b$.

(11) $f_{11}(r) = (B(a,b)/2 - r'^2\mathcal{K})/r^2$ *has positive Maclaurin coefficients and maps* $(0,1)$ *onto* $(B(a,b)(c-ab)/(2c), B(a,b)/2)$.

(12) $f_{12}(r) = (\mathcal{K} - B(a,b)/2)/(r'^{2(c-a-b)} - 1)$*, for* $a+b > c$*, is strictly increasing from* $(0,1)$ *onto* $(abB(a,b)/(2c(a+b-c)), B(c, a+b-c)/2)$.

(13) $f_{13}(r) = ((1-a-(b-c)r^2)\mathcal{E} - (1-a)r'^2\mathcal{K})/r^2$ *has negative Maclaurin coefficients, except for the constant term, with*

$$f_{13}(0+) = (c+1-a)(c-b)B(a,b)/(2c)$$

and $f_{13}(1-) = (c+1-a-b)\mathcal{E}(1)$.

(14) $f_{14}(r) = (c-a)\mathcal{E} - (b-a)r'^2\mathcal{K}$ *has negative Maclaurin coefficients, except for the constant term, and maps* $[0,1]$ *onto* $[C, D]$*, where*

$$C = (c-a)\mathcal{E}(1) \text{ and } D = (c-b)B(a,b)/2.$$

Proof. (1) From (3.6)

$$\frac{dF(a^-)}{dz} = \frac{(a-1)}{z}\big(F - F(a^-)\big) = \frac{(a-1)b}{c}F(a, b+1; c+1; z)\,.$$

Putting $z = r^2$ and multiplying by $B(a,b)/2$, we get

$$\frac{\mathcal{K}-\mathcal{E}}{r^2} = \frac{bB(a,b)}{2c}F(a, b+1; c+1; r^2)\,.$$

Hence

$$f_1(r) = \frac{b}{c}\frac{F(a, b+1; c+1; r^2)}{F(a,b;c;r^2)}\,.$$

Thus $f_1(0) = b/c$. The ratio of the coefficients of the numerator and denominator equals

$$\frac{b}{c}\frac{(a,n)(b+1,n)(c,n)}{(c+1,n)(a,n)(b,n)} = \frac{b}{c}\frac{b+n}{b}\frac{c}{c+n} = \frac{b+n}{c+n} = 1 - \frac{c-b}{c+n},$$

which is increasing in n. Hence the result follows from Theorem 4.3. The limit $f_1(1-) = 1$ follows from (3.9).

(2) From (3.6) and (3.7), we get

$$\frac{dF(a-)}{dz} = \frac{(a-1)}{z}(F - F(a-)) = (a-1)(F + \frac{(b-c)}{c}F(c+))\,.$$

Hence putting $z = r^2$, we get

$$\frac{\mathcal{K}-\mathcal{E}}{r^2} = \mathcal{K} + \frac{(b-c)B(a,b)}{2c}F(a, b; c+1; r^2)\,.$$

Thus,

$$f_2(r) = \frac{(c-b)B(a,b)}{2c}F(a, b; c+1; r^2)\,,$$

which proves the assertion. The limiting values follow from (3.9).

(3) By (3.9),

$$
\begin{aligned}
f_3(r) &= \frac{B(a,b)}{2}\frac{F(a-1,b;c;r^2)}{r'^{2(c+1-a-b)}} \\
&= \frac{B(a,b)}{2}F(c+1-a,c-b;c;r^2),
\end{aligned}
$$

and the result follows.

(4) By (3.9),

$$
\begin{aligned}
f_4(r) &= \frac{B(a,b)}{2}r'^{2(a+b-c)}F(a,b;c;r^2) \\
&= \frac{B(a,b)}{2}F(c-a,c-b;c;r^2),
\end{aligned}
$$

and the result follows.

(5) By (3.9)

$$
f_5(r) = \frac{B(a,b)}{2r'^2}F(a-1,b;c;r^2) = \frac{B(a,b)}{2}(1-r^2)^{-(a+b-c)}F(c+1-a,c-b;c;r^2)\,,
$$

which is a product of two Maclaurin series with positive coefficients, hence the result.

(6) We have that

$$
f_6(r) = \mathcal{E}(r) - (\mathcal{E}(r) - r'^2\mathcal{K}(r))\,.
$$

Hence, by (2), $f_6(r) - B(a,b)/2$ has all Maclaurin coefficients negative.

(7) The positivity of Maclaurin coefficients and the limiting values are clear. Next, by (4.15), after simplification, we get

$$
\frac{d}{dr}\log\mathcal{K} = \frac{2r}{r'^2\mathcal{K}}\left((a+b-c)\mathcal{K} + \frac{(c-a)(\mathcal{E}-r'^2\mathcal{K})}{r^2}\right)\,,
$$

which has positive Maclaurin coefficients by (2) and (6).

(8) $f_8(r) = 1 - f_1(r)$, so that the result follows from (1).

(9) $f_9(r) = f_1(r)/f_8(r)$, hence the result follows from (1) and (8).

(10) The ratio of the coefficients equals

$$
\frac{B(a,b)(a,n)(b,n)}{(c,n)(n-1)!} \equiv T_n\,,
$$

say. Then

$$
\frac{T_{n+1}}{T_n} = \frac{(a+n)(b+n)}{(c+n)n} > 1\,,
$$

so that the monotonicity follows from 4.3. The limiting values follow from (3.9).

(11)

$$
\begin{aligned}
f_{11}(r) &= \mathcal{K} - \frac{\mathcal{K} - B/2}{r^2} \\
&= \frac{B}{2}\left(1 - \frac{ab}{c} + \sum_{n=1}^{\infty}\left(\frac{(a,n)(b,n)}{(c,n)n!}(1 - \frac{(a+n)(b+n)}{(c+n)(n+1)})r^{2n}\right)\right)\,,
\end{aligned}
$$

which has all coefficients positive. The limiting values are obvious.

(12) The ratio has the indeterminate form $0/0$ at $r=0$. The derivative ratio equals

$$\frac{abB(a,b)}{2c(a+b-c)}\frac{F(a+1,b+1;c+1;r^2)}{r'^{2(c-a-b-1)}}=\frac{abB(a,b)}{2c(a+b-c)}F(c-a,c-b;c+1;r^2),$$

by (3.9), and so the result follows by Lemma 4.1.

(13) From (3.6) and (3.7)

$$\begin{aligned}\frac{dF(a-)}{dz} &= \frac{1}{1-z}\left((a-1+b-c)F(a-)+\frac{(c+1-a)(c-b)}{c}F(a-,c+)\right)\\ &= \frac{(a-1)}{z}(F-F(a-))\,.\end{aligned}$$

Multiplying by $z(1-z)B(a,b)/2$, we get

$$z(a-1+b-c)\mathcal{E}+\frac{(c+1-a)(c-b)}{c}zF(a-,c+)\frac{B(a,b)}{2}=(a-1)(1-z)(\mathcal{K}-\mathcal{E})\,.$$

With $z=r^2$ this yields the result.

(14) In the Gauss contiguous relation in [R, Exercise 21(9), p. 71], if we change z to r^2 and multiply by $B(a,b)/2$, then we get $f_{14}(r)=((c-b)B(a,b)/2)F(a,b-1;c;r^2)$, so that the assertions on the coefficients follow. The limiting values are clear. □

4.22. **Remark.** In the classical case, $a=b=1/2$ and $c=1$, the boundary values in the above result 4.21 (13) reduce to $f_{13}(0+)=3\pi/8$ and $f_{13}(1-)=1$, showing that the above result is quite sharp.

4.23. **Theorem.** *Let $0<a<c$, $b=c-a$, $R=R(a,c-a)=-\Psi(a)-\Psi(c-a)-2\gamma$, and $B=B(a,c-a)$. Then*

(1) The function $f(r)=\mathcal{K}_{a,c}(r)+\log r'$ has negative Maclaurin coefficients, except for the constant term, and maps $[0,1)$ onto $(R/2,B/2]$.

(2) The function $g(r)=\mathcal{K}_{a,c}(r)+(1/r^2)\log r'$ has positive Maclaurin coefficients and maps $(0,1)$ onto $((B-1)/2,R/2)$ if $a,b\in(0,1)$, while it has negative Maclaurin coefficients and maps $(0,1)$ onto $(R/2,(B-1)/2)$ if $a,b\in(1,\infty)$.

(3) The function $h(r)=r^2\mathcal{K}_{a,c}(r)/\log(1/r')$ is strictly decreasing (respectively, increasing) from $(0,1)$ onto $(1,B)$ if $a,b\in(0,1)$ (respectively, onto $(B,1)$, if $a,b\in(1,\infty)$).

(4) The function $k(r)=\mathcal{K}_{a,c}(r)/\log((e^{R/2})/r')$ is strictly decreasing from $(0,1)$ onto $(1,B/R)$.

Proof. (1) That $f(0+)=B(a,c-a)/2$ is clear and $f(1-)=R(a,c-a)/2$ follows from [AVV, Theorem 1.52 (2)]. Next,

$$2f(r)=B+B\sum_{n=1}^{\infty}\left(\frac{(a,n)(b,n)}{(c,n)n!}-\frac{1}{n}\right)r^{2n}.$$

Thus, we need to show that $T_n = (a,n)(b,n)/((c,n)(n-1)!) < 1$. Now

$$\frac{T_{n+1}}{T_n} = \frac{(a+n)(b+n)}{(c+n)n} = \frac{(a+b)n+n^2+ab}{(a+b)n+n^2} > 1,$$

and $\lim_{n\to\infty} T_n = 1$ by Stirling's formula. Hence $T_n < 1$ for all $n = 1, 2, 3, \ldots$.

(2)

$$g(0+) = \frac{B}{2} + \lim_{r\to 0} \frac{\log r'}{r^2} = \frac{B}{2} - \frac{1}{2}.$$

Next, $g(r) = f(r) + (1/r^2 - 1)\log r'$, so that $g(1-) = f(1-) = R/2$, from (1). Next,

$$\begin{aligned} 2g(r) &= BF(a,b;c;r^2) - \frac{1}{r^2}\log\frac{1}{1-r^2} \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{B(a,n)(b,n)(n+1)}{(c,n)n!} - 1\right) r^{2n}. \end{aligned}$$

Let

$$T_n = \frac{B(a,n)(b,n)(n+1)}{(c,n)n!}.$$

Then

$$\frac{T_{n+1}}{T_n} = \frac{(a+n)(b+n)(n+2)}{(c+n)(n+1)^2}.$$

Now $(a+n)(b+n)(n+2) - (c+n)(n+1)^2 = -(1-ab)n - (a+b-2ab)$, which is negative (positive) if $a, b \in (0,1)$ ($a, b \in (1,\infty)$). By Stirling's formula, $\lim_{n\to\infty} T_n = 1$. Hence the result follows.

(3)

$$h(r) = \frac{g(r)}{\frac{1}{r^2}\log\frac{1}{r'}} + 1,$$

so that $h(0+) = B$ and $h(1-) = 1$ are clear. Next,

$$h(r) = \frac{B\sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!} r^{2n}}{\sum_{n=0}^{\infty} \frac{1}{n+1} r^{2n}},$$

so that the coefficient ratio equals $(a,n)(b,n)(n+1)/((c,n)n!)$, which is decreasing if $a, b \in (0,1)$ and increasing if $a, b \in (1,\infty)$. Hence the result follows from [PV, Lemma 2.1].

(4) We have $k(r) = 1 + (f(r) - R/2)/\log(e^{R/2}/r')$. Hence the result follows from (1). $\square$

4.24. **Theorem.** *(1) Let $0 < a, b < c$ and $2ab < c \le a+b < c+1/2$. Then the function $f(r) = r'\mathcal{K}(r)$ is strictly decreasing from $[0,1)$ onto $(0, B(a,b)/2]$.*

(2) Let $0 < a, b < c < a+b$. Then the function $g(r) = r'^{2(a+b-c)}(\mathcal{K} - \mathcal{E})/r^2$ has positive Maclaurin coefficients and maps $(0,1)$ onto $(bB(a,b)/(2c), B(c, a+b-c)/2)$.

(3) Let $0 < a, b < 1$, $c = a+b$, and $a(2b+1) < b+1 < 1/a$. Then the function $h(r) = (\mathcal{K} - \mathcal{E})/\log(1/r')$ is decreasing from $(0,1)$ onto $(1, bB(a,b)/c)$.

Proof. (1) Clearly $f(0) = B(a,b)/2$ and by (3.9), $f(1) = 0$. We have $f(r) = g(r)/h(r)$, where

$$g(r) = B(a,b)\sum_0^\infty \frac{(a,n)(b,n)}{(c,n)n!}r^{2n}\,,$$

and

$$h(r) = 2\sum_0^\infty d_n r^{2n}\,,$$

with $d_0 = 1$ and $d_n = (1\cdot 3\cdot \ldots \cdot (2n-1))/(2\cdot 4\cdot \ldots \cdot (2n))$ for $n = 1,2,3,\ldots$. Hence, the coefficient ratio equals

$$T_n = \frac{B(a,b)(a,n)(b,n)2^n}{(c,n)\cdot 1\cdot 3\cdot \ldots \cdot (2n-1)}\,.$$

Then

$$\frac{T_{n+1}}{T_n} = \frac{2(n+1)(n+b)}{(2n+1)(n+c)}\,.$$

Now

$$(2n+1)(n+c) - 2(n+a)(n+b) = n(2c+1-2a-2b) + c - 2ab > 0\,,$$

so that T_n is decreasing and hence by Theorem 4.3, f is also decreasing.

(2) From (3.6), (3.9) and (4.16), we get

$$f(r) = \frac{bB(a,b)}{2c}r'^{2(a+b-c)}F(a,b+1;c+1;r^2) = \frac{bB(a,b)}{2c}F(c+1-a,c-b;c+1;r^2).$$

Hence the assertion follows from (3.9).

(3) As in (2), from (3.6), (3.9) and (4.16), we get

$$h(r) = \frac{bB(a,b)}{c}\frac{r^2F(a,b+1;c+1;r^2)}{2\log(1/r')}.$$

Writing the Maclaurin series expansion of both the numerator and the denominator, the ratio of coefficients equals

$$T_n = \frac{bB(a,b)}{c}\frac{(a,n)(b+1,n)(n+1)}{(c+1,n)n!} = \frac{B(a,b)(a,n)(b,n+1)(n+1)}{(c,n+1)n!}.$$

Hence

$$\frac{T_{n+1}}{T_n} = \frac{(a+n)(b+n+1)(n+2)}{(c+n+1)(n+1)^2}.$$

Then,

$$(c+n+1)(n+1)^2 - (a+n)(b+n+1)(n+2) = n(1-a-ab) + (1+b-2ab-a) > 0.$$

Hence, T_n is decreasing, so that the result follows by (3.9) and Theorem 4.3. $\square$

4.25. **Remark.** Theorem 4.24(3) generalizes [AQVV, Lemma 5.2 (12)]. The latter follows if we put $c = 1$ in Theorem 4.24(3).

4.26. **Differential equations.** The hypergeometric function $w = F(a,b;c;z)$ satisfies the differential equation [R, p. 54]

$$z(1-z)w'' + (c-(a+b+1)z)w' - abw = 0\,.$$

Changing the variable z to z^2, this reduces to

$$z(1-z^2)w'' + (2c-1-(2a+2b+1)z^2)w' - 4abzw = 0\,.$$

In particular, the generalized elliptic integrals, $w = \mathcal{K}_{a,c}(r)$ and $w = \mathcal{K}'_{a,c}(r)$, satisfy, respectively, the differential equations

$$rr'^2w'' + (2c-1-(2c+1)r^2)w' - 4a(c-a)rw = 0\,, \tag{4.27}$$

and

$$rr'^2w'' - (1-(2c+1)r^2)w' - 4a(c-a)rw = 0\,. \tag{4.28}$$

In the special case $c = 1$ the above two equations coincide (cf. [AQVV, (4.3)], [L, 3.8.19, p. 75]). Next, the generalized elliptic integrals $w = \mathcal{E}_{a,c}(r)$ and $w = \mathcal{E}'_{a,c}(r)$ satisfy, respectively, the differential equations

$$rr'^2w'' - (2c-1)r'^2w' + 4(1-a)(c-a)rw = 0\,, \tag{4.29}$$

and

$$rr'^2w'' - (1+(2c-1)r^2)w' + 4(1-a)(c-a)rw = 0\,. \tag{4.30}$$

In the special case $c = 1$ the equations (4.29) and (4.30) are still different unlike in the case of (4.27) and (4.28) (cf. [L, 3.8.17, p. 74 and 3.8.23, p. 75]).

4.31. **Correction.** In [AQVV, (4.3), p. 14] the first differential equation has a symmetry property, namely, it is satisfied both by $\mathcal{K}_a$ and $\mathcal{K}'_a$. However, the second differential equation is satisfied only by $\mathcal{E}_a$, and *not* by $\mathcal{E}'_a$. The differential equation satisfied by $w = \mathcal{E}'_a$ is obtained by putting $c = 1$ in (4.30). Thus it is

$$rr'^2w'' - (1+r^2)w' + 4(1-a)^2rw = 0\,.$$

We use the notation

$$S_w = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2$$

for the *Schwarzian derivative.*

4.32. **Lemma.** *[R2, p. 9] Let w_1, w_2 be linearly independent solutions of the differential equation $w'' + p(z)w' + q(z)w = 0$. Then $W = w_1/w_2$ satisfies the differential equation*

$$S_W(z) = 2q(z) - p'(z) - p(z)^2/2\,.$$

4.33. **Theorem.** *Let $0 < a, b < 1$ and $2c = a + b + 1$. Then the modulus $\mu = \mu_{a,b,c}$ satisfies the differential equation*

$$S_\mu(r) = 2q(r) - p'(r) - p(r)^2/2\,,$$

where

$$p(r) = \frac{2c - 1 - (4c-1)r^2}{rr'^2}\,, \quad q(r) = -\frac{4ab}{r'^2}\,.$$

Proof. Follows immediately from Lemma 4.32 and (4.27). $\square$

4.34. **Theorem.** *Let $0 < a, b < 1$ and $2c = a + b$. Then the function $\nu = \mathcal{E}'/\mathcal{E}$ satisfies the differential equation*

$$S_\nu(r) = 2q(r) - p'(r) - p(r)^2/2\,,$$

where

$$p(r) = \frac{2c - 1 - (4c-1)r^2}{rr'^2}\,, \quad q(r) = -\frac{4(a-1)b}{r'^2}\,.$$

Proof. Follows immediately from Lemma 4.32 and (4.29). $\square$

The classical Landen identities [WW, p. 507], [R, Theorem 23] for the case $a = b = 1/2$, $c = 1$, state that

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r)\,,\ \mathcal{K}\left(\frac{1-r}{1+r}\right) = \frac{1+r}{2}\mathcal{K}'(r) \tag{4.35}$$

for all $r \in (0,1)$. The next pair of inequalities generalize these formulas, see [QV2].

4.36. **Theorem.** *Let $0 < a < \min\{c, 1\}$, $r \in (0,1)$, $s = 2\sqrt{r}/(1+r)$, and $t = (1-r)/(1+r)$. Let $B = B(a, c-a)$ and $R = -\Psi(a) - \Psi(c-a) - 2\gamma$. Then*

(1) $\mathcal{K}_{a,c}(s) \le (1+r)\mathcal{K}_{a,c}(r) \le \mathcal{K}_{a,c}(s) + (R - \log 16)/2$,

(2) $\mathcal{K}_{a,c}(r') \le (2/(1+r))\mathcal{K}_{a,c}(t) \le \mathcal{K}_{a,c}(r') + (R - \log 16)/2$

with equality in each case if and only if $a = 1/2$ and $c = 1$.

4.37. **Theorem.** *For $a > 0$, $r \in [0,1]$, and $t = 2\sqrt{r}/(1+r)$, the following generalizations of Landen identities (4.35) hold:*

(1) $\mathcal{K}_{a,2a}(t) = (1+r)^{2a}\mathcal{K}_{a,a+(1/2)}(r)$,

(2) $\mathcal{K}'_{a,a+(1/2)}(t) = \left(\frac{1+r}{2}\right)^{2a}\mathcal{K}'_{a,2a}(r)$.

Proof. Assertion (1) follows from [R, Theorem 23] by putting $a = b$. Assertion (2) follows from (1), by changing r to r'. $\square$

4.38. **Lemma.** *If $0 < a < \min\{c, 1\}$ and $c \le a + (1/2)$, then the function $f(r) = r\mathcal{K}_{a,c}(r)/\mathrm{arth} r$ is strictly decreasing from $(0,1)$ onto $(1, B/2)$, where $B = B(a, c-a)$.*

Proof. Let $f(r) = g(r)/h(r)$, where $g(r) = r\mathcal{K}_{a,c}(r)$ and $h(r) = \mathrm{arth}\, r$. Then $g(0) = h(0) = 0$ and

$$g'(r)/h'(r) = 2(c-a)\mathcal{E}_{a,c}(r) + (1 - 2(c-a))(r'^2)\mathcal{K}_{a,c}(r)\,,$$

which, by Lemma 4.21 (6), being a sum of two strictly decreasing functions, is also so. Hence, the monotonicity follows from L'Hôpital's Monotone Rule, Lemma 4.1. Finally, $f(0+) = B/2$ by L'Hôpital's Rule and $f(1-) = g'(1-)/h'(1-) = 1$, again by L'Hôpital's Rule. $\square$

4.39. **Conjecture.** *Based on experimental evidence, we make the following conjectures. For the special case $c = 1$, these are proved in [AQVV, Theorems 5.5 and 6.7]. Let $a \in (0,1)$ and let $c \in [1, \infty)$. Then*

(1) The function $f(r) = \mu_{a,c}(r)/\log(1/r)$ is strictly increasing from $(0,1)$ onto $((c + a - 1)B(a,c), \infty)$.

(2) For each $K > 1$, the function $g(r) = \varphi_K^{a,c}(r)/r^{1/K}$ is strictly decreasing from $(0,1]$ onto $[1, e^d)$, $d = (1 - (1/K))R(a, c-a)/2$. In particular,

$$r^{1/K} < \varphi_K^{a,c}(r) < e^d r^{1/K}\,.$$

(3) The function $h(r) = \mu_{a,c}(r) + \log r$ is strictly decreasing from $(0,1]$ onto $[0, R(a, c-a)/2)$.

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