TO THE THEORY OF MAPPINGS WITH FINITE AREA DISTORTION

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Abstract

In all dimensions $k = 1, ..., n-1$, we show that mappings f in $\mathbb{R}^n$ with finite distortion of hyperarea satisfy certain modulus inequalities in terms of inner and outer dilatation of the mappings.

1 Introduction

Quasiconformal and quasiregular mappings have been recently generalized to several directions, see e.g. [AIKM], [GI], [HK], [IKO$_1$], [IKO$_2$], [IM], [IR], [IS], [KKM$_1$], [KKM$_2$], [KO], [MRSY$_1$], [MRSY$_2$], [MV$_1$], [MV$_2$], [RSY$_1$] - [RSY$_3$]. In all those generalizations the modulus techniques play a key role. The following concept was proposed in [MRSY$_1$]. Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \to [1, \infty]$ be a measurable function. A homeomorphism $f : D \to \overline{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ is called a **Q−homeomorphism** if

$$M(f\Gamma) \ \leq \ \int\limits_D Q(x) \cdot \rho^n(x) \ dm(x) \tag{1.1}$$

for every family Γ of paths in D and every admissible function ρ for Γ.

Recall that, given a family of paths Γ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called **admissible** for Γ, abbr. $\rho \in adm\,\Gamma$, if

$$\int\limits_\gamma \rho\, ds \ \geq \ 1 \tag{1.2}$$

for each $\gamma \in \Gamma$. The (conformal) **modulus** of Γ is the quantity

$$M(\Gamma) = \inf_{\rho \in adm\,\Gamma} \int\limits_D \rho^n(x)\, dm(x) \tag{1.3}$$

with the measure and the integral by Lebesgue.

In the work [MRSY$_2$], the concept has been extended to mappings with branching. Note that the modulus inequality (1.1) in the definition of a $Q-$homeomorphism has first appeared for $n = 2$ in connection with the so-called $BMO-$quasiconformal mappings, see [RSY$_1$] - [RSY$_3$], cf. also $V(6.6)$ in [LV] in the theory of quasiconformal mappings. In this paper, we consider the modulus of families of surfaces of various dimensions in $\mathbb{R}^n$ and introduce the notation of $(k, Q)-$mappings.

Below we assume that Ω is an open set in $\mathbb{R}^n$, $n \geq 2$, and that all mappings $f : \Omega \to \mathbb{R}^n$ are continuous. Similarly [MRSY$_2$], given a pair $Q(x,y) = (Q_1(x), Q_2(y))$ of measurable functions $Q_1 : \Omega \to [1, \infty]$ and $Q_2 : \Omega_* \to [1, \infty]$ and $k = 1, ..., n-1$, we say that a mapping $f : \Omega \to \mathbb{R}^n$, $f(\Omega) = \Omega_*$, is a $(\mathbf{k}, \mathbf{Q})-$**mapping** if

$$M(f\Gamma) \ \leq \ \int\limits_{\Omega} Q_1(x) \cdot \rho^n(x)\ dm(x) \tag{1.4}$$

and

$$M(\Gamma) \ \leq \ \int\limits_{\Omega_*} Q_2(y) \cdot \rho_*^n(y)\ dm(y) \tag{1.5}$$

for every family Γ of $k-$dimensional surfaces S in Ω and all $\rho \in adm\, \Gamma$ and $\rho_* \in adm\, f\Gamma$.

Given a mapping $\varphi : E \to \mathbb{R}^n$ and a point $x \in E \subseteq \mathbb{R}^n$, let

$$L(x, \varphi) \ = \ \limsup_{y \to x\ y \in E} \frac{|\varphi(y) - \varphi(x)|}{|y - x|}, \tag{1.6}$$

and

$$l(x, \varphi) \ = \ \liminf_{y \to x\ y \in E} \frac{|\varphi(y) - \varphi(x)|}{|y - x|}. \tag{1.7}$$

A mapping $f : \Omega \to \mathbb{R}^n$ is said to be of **finite metric distortion**, abbr. $f \in FMD$, if f has $(N)-$property and

$$0 \ < \ l(x, f) \ \leq \ L(x, f) \ < \ \infty \qquad a.e. \tag{1.8}$$

Note that a mapping $f : \Omega \to \mathbb{R}^n$ is of FMD if and only if f is differentiable a.e. and has $(N)-$ and $(N^{-1})-$properties, see Corollary 3.4 in [MRSY$_2$]. Recall that a mapping $f : X \to Y$ between measurable spaces (X, Σ, μ) and (X', Σ', μ') is said to have $(\mathbf{N})-$**property** if $\mu'(f(E)) = 0$ whenever $\mu(E) = 0$. Similarly, f has the $(\mathbf{N^{-1}})-$**property** if $\mu(E) = 0$ whenever $\mu'(f(E)) = 0$.

We say that a mapping $f : \Omega \to \mathbb{R}^n$ has $(\mathbf{A_k})-$**property** if the two conditions hold:

$(A_k^{(1)})$: for a.e. $k-$dimensional surface S in Ω the restriction $f|_S$ has $(N)-$property;

$(A_k^{(2)})$: for a.e. $k-$dimensional surface S_* in $\Omega_* = f(\Omega)$ the restriction $f|_S$ has $(N^{-1})-$property for each lifting S of S_*.

Here a surface S in Ω is a **lifting** of a surface S_* in $\mathbb{R}^n$ under a mapping $f : \Omega \to \mathbb{R}^n$ if $S_* = f \circ S$. We also say that a mapping $f : \Omega \to \mathbb{R}^n$ is **of finite distortion of area in dimension** $k = 1, ..., n-1$, abbr. $f \in FAD_k$, if $f \in FMD$ and has the $(A_k)-$property. Note that analogues of $(A_k)-$properties and the classes FAD_k have been first formulated in the mentioned work [MRSY$_2$] for $k = 1$ where it is additionally requested local rectifiability of S_* and S in $(A_k^{(1)})-$

and $(A_k^{(2)})-$properties, respectively. Thus, the mapping class FLD (finite length distortion) in [MRSY$_2$] is a subclass of FAD_1. Finally, we say that a mapping $f : \Omega \to \mathbb{R}^n$ is of **finite area distortion**, abbr. $f \in FAD$, if $f \in FAD_k$ for every $k = 1, ..., n-1$.

We show that every mapping f with finite area distortion is a $(k, Q)-$mapping for every $k = 1, ..., n-1$ with

$$Q(x, y) \;=\; \left(K_I(x), \sum_{z \in f^{-1}(y)} K_O(z)\right) \tag{1.9}$$

where

$$K_I(x) \;=\; K_I(x, f) \;=\; \begin{cases} \frac{|J(x,f)|}{l(f'(x))^n}, & \text{if } J(x, f) \neq 0 \\ 1, & \text{if } f'(x) = 0 \end{cases} \tag{1.10}$$

$$K_O(x) \;=\; K_O(x, f) \;=\; \begin{cases} \frac{||f'(x)||^n}{|J(x,f)|}, & \text{if } J(x, f) \neq 0 \\ 1, & \text{if } f'(x) = 0 \end{cases} \tag{1.11}$$

and $K_I(x, f) = \infty = K_O(x, f)$ otherwise. As usual, here $f'(x)$ denotes the Jacobian matrix of f at the point of differentiability x, $J(x, f) = \det f'(x)$ is its determinant and

$$l(f'(x)) \;=\; \min \{|f'(x)h| : \; h \in \mathbb{R}^n, \; |h| = 1\} \tag{1.12}$$

and

$$||f'(x)|| \;=\; \max \{|f'(x)h| : \; h \in \mathbb{R}^n, |h| = 1\} \; . \tag{1.13}$$

The quantity $K_I(x, f)$ is called the **inner dilatation** and $K_O(x, f)$ the **outer dilatation** of the mapping f.

2 Preliminaries

Below H^k, $k = 1, ..., n-1$ denotes the **k−dimensional Hausdorff measure** in $\mathbb{R}^n$, $n \geq 2$. More precisely, if A is a set in $\mathbb{R}^n$, then

$$H^k(A) \;=\; \sup_{\varepsilon > 0} \; H^k_\varepsilon(A), \tag{2.1}$$

$$H^k_\varepsilon(A) \;=\; V_k \inf \; \sum_{i=1}^{\infty} \left(\frac{\delta_i}{2}\right)^k \tag{2.2}$$

where the infimum is taken over all countable collections of numbers $\delta_i \in (0, \varepsilon)$ such that some sets A_i in $\mathbb{R}^n$ with diameters δ_i cover A. Here V_k denotes the volume of the unit ball in $\mathbb{R}^k$. H^k is an outer measure in the sense of **Caratheodory**, i.e.,

1) $H^k(X) \;\leq\; H^k(Y)$ whenever $X \subseteq Y$;

2) $H^k(\Sigma X_i) \;\leq\; \Sigma H^k(X_i)$ for each sequence X_i of sets;

3) $H^k(X \cup Y) \;=\; H^k(X) + H^k(Y)$ whenever $dist(X, Y) > 0$.

A set $E \subset \mathbb{R}^n$ is called **measurable** with respect to H^k if $H^k(X) = H^k(X \cap E) + H^k(X \setminus E)$ for every set $X \subset \mathbb{R}^n$. It is well-known that every Borel set is measurable with respect to any outer measure in the sense of Caratheodory, see e.g. [Sa], p. 52. Moreover, H^k is Borel regular, i.e., for every set $X \subset \mathbb{R}^n$ there is a Borel set $B \subset \mathbb{R}^n$ such that $X \subset B$ and $H^k(X) = H^k(B)$, see e.g. [Sa], p. 53, and 2.10.1 in [Fe]. The latter implies that, for every measurable set $E \subset \mathbb{R}^n$, there exist Borel sets B_* and $B^* \subset \mathbb{R}^n$ such that $B_* \subset E \subset B^*$ and $H^k(B^* \setminus B_*) = 0$, see e.g. 2.2.3 in [Fe]. In particular, $H^k(B^*) = H^k(E) = H^k(B_*)$.

Let ω be an open set in $\mathbb{R}^k$, $k = 1, ..., n-1$. A (continuous) mapping $S : \omega \to \mathbb{R}^n$ is called a $k-$dimensional surface S in $\mathbb{R}^n$. Sometimes we call the image $S(\omega) \subseteq \mathbb{R}^n$ by the surface S, too. The number of preimages

$$(2.3) \quad N(S, y) \ = \ N(S, y, \omega) \ = \ card\, S^{-1}(y) \ = \ card\, \{x \in \omega : \ S(x) = y\}$$

is said to be a **multiplicity function** of the surface S at a point $y \in \mathbb{R}^n$. In the other words, $N(S, y)$ means the multiplicity of covering of the point y by the surface S. It is known that multiplicity function is lower semi-continuous, i.e.,

$$N(S, y) \ \geq \ \liminf_{m\to\infty} \ N(S, y_m)$$

for every sequence $y_m \in \mathbb{R}^n$, $m = 1, 2, ...$ such that $y_m \to y \in \mathbb{R}^n$ as $m \to \infty$, see e.g. [RR], p. 160. Thus, the function $N(S, y)$ is Borel measurable and hence measurable with respect to every Hausdorff measure H^k, see e.g. [Sa], p. 52.

The $k-$dimensional Hausdorff area in $\mathbb{R}^n$ (or simply **area**) associated with a surface $S : \omega \to \mathbb{R}^n$ is given by

$$(2.4) \qquad S(B) \ = \ \int\limits_B N(S, y)\ dH^k y$$

for every Borel set B and, more generally, for an arbitrary set which is measurable with respect to H^k in $\mathbb{R}^n$. The surface S is **rectifiable** if $S(\mathbb{R}^n) < \infty$.

If $\rho : \mathbb{R}^n \to [0, \infty]$ is a Borel function, then its **integral over** S is defined by the equality

$$(2.5) \qquad \int\limits_S \rho\ dS \ = \ \int\limits_{\mathbb{R}^n} \rho(y)\, N(S, y)\ dH^k y\ .$$

Given a family Γ of $k-$dimensional surfaces S, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called **admissible** for Γ, abbr. $\rho \in adm\, \Gamma$, if

$$(2.6) \qquad \int\limits_S \rho^k\ dS \ \geq \ 1$$

for every $S \in \Gamma$. Given $p \in (0, \infty)$, the **p−modulus** of Γ is the quantity

$$(2.7) \qquad M_p(\Gamma) \ = \ \inf_{\rho \in adm \Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)\ dm(x)\ .$$

The modulus is itself an outer measure on the set of families of surfaces.

We say that Γ_2 is **minorized** by Γ_1 and write $\Gamma_2 > \Gamma_1$ if every $S \subset \Gamma_2$ has a subsurface which belongs to Γ_1. It is known that $M_p(\Gamma_1) \geq M_p(\Gamma_2)$, see [Fu], p. 176-178. We also say that a property P holds for **p−a.e.** (almost every) $k-$dimensional surface S in a family Γ if a subfamily of all surfaces of Γ for which P fails has the $p-$modulus zero. If $0 < q < p$, then P also holds for $q-$a.e. S, see Theorem 3 in [Fu]. In the case $p = n$, we write simply a.e.

2.8. Remark. The definition of the modulus immediately implies that, for every $p \in (0, \infty)$ and $k = 1, ..., n-1$

1) $p-$a.e. $k-$dimensional surface in $\mathbb{R}^n$ is rectifiable;

2) given a Borel set B in $\mathbb{R}^n$ of (Lebesgue) measure zero,

$$S(B) \; = \; 0 \tag{2.9}$$

for $p-$a.e. $k-$dimensional surface S in $\mathbb{R}^n$.

2.10. Lemma. *Let $k = 1, ..., n-1$, $p \in [k, \infty)$ and let C be an open cube in $\mathbb{R}^n$, $n \geq 2$, whose edges are parallel to coordinate axes. If a property P holds for $p-$a.e. $k-$dimensional surface S in C, then P also holds for a.e. $k-$dimensional plane in C which is parallel to a $k-$dimensional coordinate plane H.*

The latter a.e. is related to the Lebesgue measure in the corresponding $(n-k)$-dimensional coordinate plane $H^{\perp}$ which is perpendicular to H.

Proof. Let us assume that the conclusion is not true. Then by regularity of the Lebesgue measure m_{n-k} in $H^{\perp}$ there is a Borel set B such that $m_{n-k}(B) > 0$ and P fails for a.e. $k-$dimensional plane S in C which is parallel to H and intersects B. If a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is admissible for the given family Γ of surfaces S such that $\rho \equiv 0$ outside of $C_0 \times B$ where C_0 is the projection of C on H, then by the Hölder inequality

$$\int\limits_{C_0 \times B} \rho^k(x) \; dm(x) \; \leq \; \left(\int\limits_{C_0 \times B} \rho^p(x) \; dm(x) \right)^{\frac{k}{p}} \left(\int\limits_{C_0 \times B} dm(x) \right)^{\frac{p-k}{p}}$$

and hence by the Fubini theorem

$$\int\limits_{\mathbb{R}^n} \rho^p(x) \; dm(x) \; \geq \; \frac{\left(\int\limits_{C_0 \times B} \rho^k(x) \; dm(x) \right)^{\frac{p}{k}}}{\left(\int\limits_{C_0 \times B} dm(x) \right)^{\frac{p-k}{k}}} \; \geq \; \frac{(m_{n-k}(B))^{\frac{p}{k}}}{(h^k \cdot m_{n-k}(B))^{\frac{p-k}{k}}} \; ,$$

i.e.,

$$M_p(\Gamma) \; \geq \; \frac{m_{n-k}(B)}{h^{p-k}}$$

where h is the length of the edge of the cube C. Thus, $M_p(\Gamma) > 0$ that contradicts the hypothesis of the lemma.

The following statement is an analogue of the Fubini theorem, cf. e.g. [Sa], p. 77. It extends Theorem 33.1 in [Va], cf. also Theorem 3 in [Fu] and Lemma 2.13 in [MRSY$_2$].

2.11. Theorem. *Let $k = 1, ..., n-1$, $p \in [k, \infty)$ and let E be a subset in open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Then E is measurable by Lebesgue in $\mathbb{R}^n$ if and only if E is measurable with respect to area on $p-$a.e. $k-$dimensional surface S in Ω. Moreover, $|E| = 0$ if and only if*

$$S(E) = 0 \tag{2.12}$$

on $p-$a.e. $k-$dimensional surface S in Ω.

Proof. By the Lindelöf property in $\mathbb{R}^n$ and the minorant property of M_p, we may assume without loss of generality that Ω is an open cube C in $\mathbb{R}^n$ whose edges are parallel to the coordinate axes.

Suppose first that E is Lebesgue measurable in $\mathbb{R}^n$. Then by the regularity of the Lebesgue measure there exist Borel sets B_* and B^* in $\mathbb{R}^n$ such that $B_* \subset E \subset B^*$ and $|B^* \backslash B_*| = 0$. Thus, by 2) in Remark 2.8 $S(B^* \backslash B_*) = 0$ and hence E is measurable by area on $p-$a.e. $k-$dimensional surface S in C. Conversely, if the latter is true, then E is measurable by area on a.e. $k-$dimensional plane H in C which is parallel to a $k-$dimensional coordinate plane, see Lemma 2.10. Thus, E is measurable by the Fubini theorem.

Now, suppose that $|E| = 0$. Then there is a Borel set B such that $|B| = 0$ and $E \subset B$. Then by 2) in Remark 2.8 the relation (2.12) holds for $p-$a.e. $k-$dimensional surface S in C. Conversely, if the latter is true, then, in particular, $S(E) = 0$ on a.e. $k-$dimensional plane H in C which is parallel to a $k-$dimensional coordinate plane, see Lemma 2.10. Thus, $|E| = 0$ again by the Fubini theorem.

2.13. Remark. Say by the Lusin theorem, see e.g. 2.3.5 in [Fe], for every measurable function $\rho : \mathbb{R}^n \to [0, \infty]$, there is a Borel function $\rho^* : \mathbb{R}^n \to [0, \infty]$ such that $\rho^* = \rho$ a.e. in $\mathbb{R}^n$. Thus, by Theorem 2.11 ρ is measurable on $p-$a.e. $k-$dimensional surface S in $\mathbb{R}^n$ for every $p \in (0, \infty)$ and $k = 1, ..., n-1$.

A Lebesgue measurable function $\rho : \mathbb{R}^n \to [0, \infty]$ is said to be **p−extensively admissible** for a family Γ of $k-$dimensional surfaces S in $\mathbb{R}^n$, abbr. $\rho \in ext_p\, adm \Gamma$, if

$$\int\limits_S \rho^k \, dS \; \geq \; 1 \tag{2.14}$$

for $p-$a.e. $S \in \Gamma$. The **p−extensive modulus** $\overline{M}_p(\Gamma)$ of Γ is the quantity

$$\overline{M}_p(\Gamma) \; = \; \inf \int\limits_{\mathbb{R}^n} \rho^p(x) \; dm(x) \tag{2.15}$$

where the infimum is taken over all $\rho \in ext_p\, adm\, \Gamma$. In the case $p = n$, we use notations $\overline{M}(\Gamma)$ and $\rho \in ext\, adm\, \Gamma$, respectively.

2.16. Corollary. *For every $p \in (0,\infty)$, $k = 1, ..., n-1$, and every family Γ of $k-$dimensional surfaces in $\mathbb{R}^n$,*

$$\overline{M}_p(\Gamma) \ = \ M_p(\Gamma). \tag{2.17}$$

Indeed, $\overline{M}_p(\Gamma) \le M_p(\Gamma)$ by definition and $\overline{M}_p(\Gamma) \ge M_p(\Gamma)$ by Remark 2.13.

3 Modulus inequalities

The following lemma makes possible to extend the so–called K_0–inequality from the theory of quasiregular mappings to FAD mappings, see e.g. [MRV], p. 16, [Ri], p. 31, [Vu], p. 130, cf. also [KO] and [MRSY$_2$].

3.1. Lemma. *Let a mapping $f : \Omega \to \mathbb{R}^n$ be of finite metric distortion with $(A_k^{(1)}-)$property for some $k = 1, ..., n-1$ and let a set $E \subset \Omega$ be measurable by Lebesgue. Then*

$$M(\Gamma) \ \le \int\limits_{f(E)} K_I(y, f^{-1}, E) \cdot \rho_*^n(y) \ dm(y) \tag{3.2}$$

for every family Γ of $k-$dimensional surfaces S in E and $\rho_* \in ext\ adm\ f\Gamma$ where

$$K_I(y, f^{-1}, E) \ = \sum_{x \in E \cap f^{-1}(y)} K_O(x, f) \ . \tag{3.3}$$

In particular, here we have in the case $E = \Omega$

$$K_I(y, f^{-1}, D) \ = \ K_I(y, f^{-1}) \ := \sum_{x \in f^{-1}(y)} K_O(x, f) \ . \tag{3.4}$$

Proof. Let B be a (Borel) set of all points x in Ω where f has a differential $f'(x)$ and $J(x,f) = \det f'(x) \neq 0$. Then $B_0 = \Omega \setminus B$ has the Lebesgue measure zero in $\mathbb{R}^n$ because $f \in FMD$. It is known that B is the union of a countable collection of Borel sets B_l, $l = 1, 2, ...$ such that $f_l = f|_{B_l}$ is a homeomorphism which is bi–Lipschitz, see e.g. 3.2.2 in [Fe]. Setting $B_1^* = B_1$, $B_2^* = B_2 \setminus B_1$ and

$$B_l^* \ = \ B_l \setminus \bigcup_{m=1}^{l-1} B_m$$

we may assume that B_l are mutually disjoint. Note that by 2) in Remark 2.8 $S(B_0) = 0$ for a.e. $k-$dimensional surface S in Ω and by $(A_k^{(1)})-$property $S_*(f(B_0)) = 0$ where $S_* = f \circ S$ also for a.e. $k-$dimensional surface S.

Given $\rho_* \in ext\ adm\ f\Gamma$, set

$$\rho(x) \ = \ \begin{cases} \rho_*(f(x))\|f'(x)\| \ , & \text{for } x \in \Omega \setminus B_0, \\ 0 \ , & \text{otherwise.} \end{cases} \tag{3.5}$$

We may assume without loss of generality that $\rho_* \equiv 0$ outside of $f(E)$. Arguing piecewise on B_l, we have by 3.2.20 and 1.7.6 in [Fe] and Theorem 2.11, see also Remark 2.13, that

$$\int\limits_S \rho^k \, dS \ \geq \ \int\limits_{S_*} \rho_*^k \, dS \ \geq \ 1 \tag{3.6}$$

for a.e. $S \in \Gamma$, i.e., $\rho \in ext\, adm\, \Gamma$. Hence by (2.17)

$$M(\Gamma) \leq \int\limits_\Omega \rho^n(x)\, dm(x)\,. \tag{3.7}$$

Now, the change of variables, see e.g. [Mu], p. 31, we obtain that

$$\int\limits_{f(B_l \cap E)} K_O(f_l^{-1}(y), f) \cdot \rho_*^n(y) \ dm(y) \ = \ \int\limits_\Omega \rho_l^n(x) \ dm(x) \tag{3.8}$$

where $\rho_l = \rho \cdot \chi_{B_l}$ and every $f_l = f|_{B_l}$, $l = 1, 2, \ldots$ is injective by the construction.

Thus, by the Lebesgue monotone convergence theorem, see e.g. [Sa], p. 27,

$$\int\limits_{f(E)} K_I(y, f^{-1}, E) \cdot \rho_*^n(y) \ dm(y) \ = \ \int\limits_\Omega \sum_{l=1}^\infty \rho_l^n(x) \ dm(x) \ \geq \ M(\Gamma)\ . \tag{3.9}$$

The next inequality is a generalized form of the K_I–inequality which is also known as Poletskii's inequality, see [Pol], [Ri], p. 49–51, and [Vu], p. 131, cf. [MRSY$_2$].

3.10. Lemma. *Let $f : \Omega \to \mathbb{R}^n$ be an FMD mapping with $(A_k^{(2)})$–property for some $k = 1, ..., n-1$. Then*

$$M(f\Gamma) \ \leq \ \int\limits_\Omega K_I(x, f) \cdot \rho^n(x) \ dm(x) \tag{3.11}$$

for every family Γ of k–dimensional surface S in Ω and $\rho \in ext\, adm\, \Gamma$.

Proof. Let B_l, $l = 0, 1, 2, \ldots$, be given as above in the proof of Lemma 3.1. By the construction and (N)–property $|f(B_0)| = 0$. Thus, by Theorem 2.11 $S_*(f(B_0)) = 0$ for a.e. $S_* \in f\Gamma$ and hence by $(A_k^{(2)})$–property $S(B_0) = 0$ for a.e. $S_* \in f\Gamma$ where S is an arbitrary lifting of S_* under the mapping f, i.e., $S_* = f \circ S$.

Let $\rho \in ext\, adm\, \Gamma$ and

$$\tilde{\rho}(y) \ = \sup_{x \in f^{-1}(y) \cap \Omega \setminus B_0} \rho_*(x) \tag{3.12}$$

where

$$\rho_*(x) \ = \ \begin{cases} \rho(x)/l(f'(x)), & \text{for } x \in \Omega \setminus B_0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.13}$$

Note that $\tilde{\rho} = \sup \rho_l$ where

$$\rho_l(y) \ = \ \begin{cases} \rho_*(f_l^{-1}(y)), & \text{for } y \in f(B_l), \\ 0, & \text{otherwise}, \end{cases} \tag{3.14}$$

and every $f_l = f|_{B_l}$, $l = 1, 2, \ldots$ is injective. Thus, the function $\tilde{\rho}$ is measurable, see e.g. [Sa], p. 15.

Arguing as in (3.6) we obtain that

$$\int\limits_{S_*} \tilde{\rho}^{\,k} \ dS_* \ \geq \int\limits_{S} \rho^k \ dS \ \geq \ 1 \tag{3.15}$$

for a.e. $S_* = f \circ S \in f\Gamma$ and, thus, $\tilde{\rho} \in ext\ adm\ f\Gamma$. Hence (2.17) yields

$$M(f\Gamma) \ \leq \int\limits_{f(\Omega)} \tilde{\rho}^{\,n}(y) \ dm(y) \ . \tag{3.16}$$

Further, by the change of variables we have that

$$\int\limits_{B_l} K_I(x, f) \cdot \rho^n(x) \ dm(x) \ = \int\limits_{f(\Omega)} \rho_l(y) \ dm(y) \ . \tag{3.17}$$

Finally, by Lebesgue's theorem we obtain the desired inequality

$$\int\limits_{\Omega} K_I(x,f) \cdot \rho^n(x)\, dm(x) \ = \sum_{l=1}^{\infty} \int\limits_{f(\Omega)} \rho_l(y)\, dm(y) \ = \int\limits_{f(\Omega)} \sum_{l=1}^{\infty} \rho_l(y)\, dm(y) \ \geq \ M(f\Gamma) \, .$$

Combining Lemmas 3.1 and 3.10 we come to the main result.

3.18. Theorem. *Let a mapping $f : \Omega \to \mathbb{R}^n$ belong to the class FAD_k for some $k = 1, ..., n-1$. Then f is a $(k, Q)-$mapping in the dimension k with*

$$Q(x, y) \ = \ (K_I(x, f), K_I(y, f^{-1})) \ . \tag{3.19}$$

3.20. Corollary. *Every FAD mapping f is a $(k, Q)-$mapping for each $k = 1, ..., n-1$ with Q given in (3.19).*

3.21. Remark. If $K_I(f) = \text{ess sup}\, K_I(x, f) < \infty$, then (3.11) for $k = 1$ yields the Poletskii inequality:

$$M(f\Gamma) \ \leq \ K_I(f)\, M(\Gamma) \tag{3.22}$$

for every path family in Ω. If $K_O(f) = \text{ess sup}\, K_O(x, f) < \infty$ and E is a Borel set with $N(f, E) < \infty$, then we have from (3.2) the usual form of the K_O-inequality:

$$M(\Gamma) \ \leq \ N(f, E)\, K_O(f)\, M(f\Gamma) \tag{3.23}$$

for every path family in E.

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