Some elementary inequalities in gas dynamics equation

V.A. Klyachin, A.V. Kochetov, V.M. Miklyukov

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Abstract

We describe sets on which differences of solutions of the gas dynamics equation satisfies some special conditions.

1 Main Results

Consider the gas dynamics equation

$$\text{div } (\sigma(|\bigtriangledown f|) \bigtriangledown f(x)) = 0, \tag{1.1}$$

where

$$\sigma(t) = \left(1 - \frac{\gamma - 1}{2} t^2\right)^{\frac{1}{\gamma - 1}}.$$

Here γ is a constant, $-\infty < \gamma < +\infty$, characterizing the flow of substance. For different values γ it can be a flow of gas, fluid, plastic, electric or chemical field in different mediums, etc. (see, for example, [1, §2], [2, §15, Chapter IV]).

For $\gamma = -1$ the equation (1.1) is known as the minimal surfaces equation

$$\text{div} \left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right) = 0$$

(Chaplygin's gas).

For $\gamma = 1 \pm 0$ we have

$$\text{div} \left(\exp\left\{-\frac{1}{2} |\nabla f|^2\right\} \nabla f\right) = 0.$$

For $\gamma = -\infty$ the equation (1.1) becomes the Laplace equation.

The solution of the equation (1.1), in which the weight function σ is a function of the variable $(x_1, \ldots, x_n)$, is called *σ-harmonic functions.* To learning this kind of functions devoted a large quantity of works (see., e.g., [3], [4] and quoted there literature).

Let $n \geq 2$. We set $\Omega_\gamma = \mathbf{R}^n$ for $\gamma \leq 1$ and

$$\Omega_\gamma = \left\{\xi \in \mathbf{R}^n : |\xi| < \sqrt{\frac{2}{\gamma - 1}}\right\}$$

for $\gamma > 1$.

Let $\xi,\ \eta \in \mathbf{R}^n$. The following inequalities are very important in work with the equation (1.1):

$$c_1 \sum_{i=1}^{n} (\xi_i - \eta_i)^2 \quad \le \quad \sum_{i=1}^{n} \left(\sigma(|\xi|)\xi_i - \sigma(|\eta|)\eta_i\right)(\xi_i - \eta_i), \tag{1.2}$$

$$\sum_{i=1}^{n} \left(\sigma(|\xi|)\xi_i - \sigma(|\eta|)\eta_i\right)^2 \quad \le \quad c_2 \sum_{i=1}^{n} \left(\sigma(|\xi|)\xi_i - \sigma(|\eta|)\eta_i\right)(\xi_i - \eta_i), \tag{1.3}$$

where $c_1,\ c_2 > 0$ are some constants.

In the general case the inequalities (1.2) and (1.3) are valid only for the subsets of the set $\Omega_\gamma \times \Omega_\gamma$ with constants c_1 and c_2 depending on these subsets. The purpose of the given paper is a description of such dependence.

We fix $c_1 > 0, c_2 > 0$ and γ. Introduce the sets

$$\mathcal{A}_\gamma(c_1) = \{(\xi,\eta) : \xi,\eta \in \Omega_\gamma \quad \text{satisfy} \quad (1.2)\},$$

$$\mathcal{B}_\gamma(c_2) = \{(\xi,\eta) : \xi,\eta \in \Omega_\gamma \quad \text{satisfy} \quad (1.3)\}.$$

We set $\Sigma_\gamma = \{x \in \mathbf{R} : x \ge 0\}$ for $\gamma \le 1$ and

$$\Sigma_\gamma = \left\{ x \in \mathbf{R} : 0 \le x < \sqrt{\frac{2}{\gamma - 1}} \right\}$$

for $\gamma > 1$.

Further, we will need the functions defined on the set $\Sigma_\gamma \times \Sigma_\gamma$ and prescribed by the relations

$$I_\gamma^-(x,y) = \frac{x\,\sigma(x) - y\,\sigma(y)}{x - y} \qquad \text{for } x \ne y,$$

$$I_\gamma^-(x,y) = \sigma(x) + \sigma'(x)x \qquad \text{for } x = y$$

and

$$I_\gamma^+(x,y) = \frac{x\,\sigma(x) + y\,\sigma(y)}{x + y} \quad \text{for } x^2 + y^2 > 0,$$

$$I_\gamma^+(0,0) = 1.$$

Note that the functions $I_\gamma^-(x,y)$ and $I_\gamma^+(x,y)$ are continuous in the closing of the set $\Sigma_\gamma \times \Sigma_\gamma$ and they are C^∞-differentiable in the each inner points of this set.

Generally, the sets $\mathcal{A}_\gamma(c_1)$ and $\mathcal{B}_\gamma(c_2)$ have a complicated structure. We shall describe them by comparing with canonical sets of the "simplest form". For arbitrary $\varepsilon \ge 0$ we put

$$W_\gamma^-(\varepsilon) = \{(\xi,\eta) :\ \xi,\eta \in \Omega_\gamma, \quad I_\gamma^-(|\xi|,|\eta|) \ge \varepsilon\},$$

$$W_\gamma^+(\varepsilon) = \{(\xi,\eta) :\ \xi,\eta \in \Omega_\gamma, \quad I_\gamma^+(|\xi|,|\eta|) \ge \varepsilon\},$$

$$V_\gamma^-(\varepsilon) = \{(\xi,\eta) :\ \xi,\eta \in \Omega_\gamma, \quad I_\gamma^-(|\xi|,|\eta|) \le \varepsilon\},$$

$$V_\gamma^+(\varepsilon) = \{(\xi,\eta) :\ \xi,\eta \in \Omega_\gamma, \quad I_\gamma^+(|\xi|,|\eta|) \le \varepsilon\}.$$

Also we will need the sets

$$D_\gamma = \{(\xi,\xi) : \xi \in \Omega_\gamma\},$$

$$Q_\gamma = \{(\xi,\eta) : \ \xi,\eta \in \Omega_\gamma, \quad \xi\sigma(|\xi|) = \eta\sigma(|\eta|)\}.$$

The following assertions are the main result of this paper.

1.4. Theorem. *Let $\gamma \in \mathbf{R}$. Then the following relations are true*

$$\left(W_\gamma^-(\varepsilon) \cup D_\gamma\right) \subset \mathcal{A}_\gamma(\varepsilon) \subset \left(W_\gamma^+(\varepsilon) \cup D_\gamma\right) \quad \text{for all} \quad \varepsilon \in (0,\ 1); \tag{1.5}$$

$$\mathcal{A}_\gamma(\varepsilon) = D_\gamma \quad \text{for all} \quad \varepsilon \in [1,\ +\infty). \tag{1.6}$$

1.7. Theorem. *a) If $\gamma \in (-\infty,-1]$ then*

$$\left(V_\gamma^+(\varepsilon) \cup D_\gamma\right) \subset \mathcal{B}_\gamma(\varepsilon) \subset \left(V_\gamma^-(\varepsilon) \cup D_\gamma\right) \quad \text{for all} \quad \epsilon \in (0,\ 1); \tag{1.8}$$

$$\mathcal{B}_\gamma(\varepsilon) = \mathbf{R}^{2n} \quad \text{for all} \quad \varepsilon \in [1,\ +\infty). \tag{1.9}$$

b) If $\gamma \in (-1,+\infty)$ then

$$\left(V_\gamma^+(\varepsilon) \cap W_\gamma^-(0)\right) \subset \mathcal{B}_\gamma(\varepsilon) \subset \left(V_\gamma^-(\varepsilon) \cup Q_\gamma\right) \quad \text{for all} \quad \varepsilon \in (0,\ 1); \tag{1.10}$$

$$W_\gamma^-(0) \subset \mathcal{B}_\gamma(\varepsilon) \quad \text{for all} \quad \varepsilon \in [1,\ +\infty). \tag{1.11}$$

First the relation (1.9) was proved for $\gamma = -1$ and $\varepsilon = 1$ in [5]. Later it was repeatedly proved with these γ and ε in [6], [7], [8] and [9].

2 Proofs of main theorems

We will need the following elementary assertion.

2.12. Lemma. *The function σ has the following properties:*

1) the domain of σ is the set Σ_γ, moreover, $\sigma(0) = 1$, $\sigma(+\infty) = 0$ for $\gamma \le 1$ and $\sigma\left(\sqrt{\frac{2}{\gamma-1}}\right) = 0$ for $\gamma > 1$;

2) for all $t \in \Sigma_\gamma$ we have

$$0 \le \sigma(t) < 1;$$

3) the function σ is decreasing on Σ_γ moreover

$$\sigma'(t) = -t(1 - \frac{\gamma-1}{2}t^2)^{\frac{2-\gamma}{\gamma-1}} < 0$$

for all $t > 0, t \in \Sigma_\gamma$;

4) the function $\theta(t) = t\sigma(t)$ is increasing on $[0,+\infty)$ for all $\gamma \in (-\infty,-1]$;

5) for every $\gamma \in (-1,+\infty)$, the function θ is increasing on $[0,\sqrt{\frac{2}{\gamma+1}}]$ and decreasing on $[\sqrt{\frac{2}{\gamma+1}},+\infty) \cap \Sigma_\gamma$;

6) *for every* $\gamma \in (-\infty,-1] \cup [2,+\infty)$, *the derivative* θ' *is decreasing on* Σ_γ *;*

7) *for every* $\gamma \in (-1,2)$, *the derivative* θ' *is decreasing on* $[0,\sqrt{\frac{6}{\gamma+1}}]$ *and increasing on* $[\sqrt{\frac{6}{\gamma+1}},+\infty) \cap \Sigma_\gamma$ *.*

The **proof** follows from the equalities:

$$\sigma'(t) = -t\left(1-\frac{\gamma-1}{2}t^2\right)^{\frac{2-\gamma}{\gamma-1}} \qquad \text{for } \gamma \neq 1,$$

$$\sigma'(t) = -t\exp\{-\frac{1}{2}t^2\} \qquad \text{for } \gamma = 1,$$

$$\theta'(t) = \left(1-\frac{\gamma+1}{2}t^2\right)\left(1-\frac{\gamma-1}{2}t^2\right)^{\frac{2-\gamma}{\gamma-1}} \qquad \text{for } \gamma \neq 1,$$

$$\theta'(t) = (1-t^2)\exp\{-\frac{1}{2}t^2\} \qquad \text{for } \gamma = 1,$$

$$\theta''(t) = -t\left(3-\frac{\gamma+1}{2}t^2\right)\left(1-\frac{\gamma-1}{2}t^2\right)^{\frac{3-2\gamma}{\gamma-1}} \qquad \text{for } \gamma \neq 1,$$

$$\theta''(t) = t(t^2-3)\exp\{-\frac{1}{2}t^2\} \qquad \text{for } \gamma = 1.$$

□

2.13. Lemma. *Let* $\gamma \in \mathbf{R}$. *Then for all* $x, y \in \Sigma_\gamma$, $x^2+y^2 \neq 0$ *we have*

$$I_\gamma^-(x,y) \leq I_\gamma^+(x,y) < 1.$$

Proof. Let x, y satisfy the assumptions of Lemma. If $x = y$ then

$$I_\gamma^-(x,y) = \sigma(x) + x\sigma'(x) < \sigma(x) = I_\gamma^+(x,y) < 1.$$

Suppose that $x > y$. Since

$$\sigma(x) < \sigma(y),$$

we obtain

$$\begin{aligned} I_\gamma^-(x,y) &= \frac{x\,\sigma(x) - y\,\sigma(y)}{x-y} \leq \frac{x\,\sigma(x) - y\,\sigma(x)}{x-y} = \sigma(x) \\ &= \frac{x\,\sigma(x) + y\,\sigma(x)}{x+y} \leq \frac{x\,\sigma(x) + y\,\sigma(y)}{x+y} = I_\gamma^+(x,y) \\ &< \frac{x\,\sigma(y) + y\,\sigma(y)}{x+y} = \sigma(y) < 1. \end{aligned}$$

The case $x < y$ is analogous. □

2.14. Lemma. *Let* $\gamma \in \mathbf{R}$. *The sets* $W_\gamma^-(\varepsilon)$, $W_\gamma^+(\varepsilon)$, $V_\gamma^-(\varepsilon)$ *and* $V_\gamma^+(\varepsilon)$ *have the following properties:*

1) $W_\gamma^-(\varepsilon) = W_\gamma^+(\varepsilon) = \emptyset$ *for all* $\varepsilon > 1$;

2) $W_\gamma^-(1) = W_\gamma^+(1) = \{0\}$;

3) $W_\gamma^-(\varepsilon) \subset W_\gamma^+(\varepsilon)$ *for all* $\varepsilon \in (0,\ 1)$;

4) $V_\gamma^-(\varepsilon) = V_\gamma^+(\varepsilon) = \Omega_\gamma \times \Omega_\gamma$ *for all* $\varepsilon \geq 1$;

5) $V_\gamma^+(\varepsilon) \subset V_\gamma^-(\varepsilon)$ *for all* $\varepsilon \in (0,\ 1)$;

6) $W_\gamma^+(0) = \Omega_\gamma \times \Omega_\gamma, \quad V_\gamma^+(0) = \emptyset$;

7) $W_\gamma^-(0) = \mathbf{R}^{2n}, \quad V_\gamma^-(0) = \emptyset$ *for all* $\gamma \leq -1$.

The **proof** follows from Lemma 2.12 and Lemma 2.13.

Further, we set

$$H_\gamma = \{(\xi,\eta) :\ \xi,\eta \in \Omega_\gamma,\ |\xi| = |\eta|,\ \xi \neq \eta\},$$

$$G_\gamma = \{(\xi,\eta) :\ \xi,\eta \in \Omega_\gamma,\ |\xi| \neq |\eta|\},$$

$$U_\gamma^- = \{(\xi,\eta) :\ \xi,\eta \in \Omega_\gamma,\ I_\gamma^-(|\xi|,|\eta|) < 0\},$$

$$U_\gamma^+ = \{(\xi,\eta) :\ \xi,\eta \in \Omega_\gamma,\ I_\gamma^-(|\xi|,|\eta|) > 0\},$$

$$P_\gamma = \{(\xi,\eta) :\ \xi,\eta \in \Omega_\gamma,\ |\xi|\sigma(|\xi|) = |\eta|\sigma(|\eta|), \quad \xi\,\sigma(|\xi|) \neq \eta\,\sigma(|\eta|)\},$$

$$F_\gamma^+(\varepsilon) = \left(V_\gamma^+(\varepsilon) \cap U_\gamma^+\right) \cup Q_\gamma \cup \left(V_\gamma^+(\varepsilon) \cap P_\gamma\right),$$

$$F_\gamma^-(\varepsilon) = \left(V_\gamma^-(\varepsilon) \cap U_\gamma^+\right) \cup Q_\gamma \cup \left(V_\gamma^+(\varepsilon) \cap P_\gamma\right) \cup \left(V_\gamma^+(\varepsilon) \cap U_\gamma^-\right).$$

For every $\xi,\eta \in \mathbf{R}^n$, their inner product is denoted by $\langle \xi,\eta\rangle$. Obviously, the inequalities (1.2), (1.3) with some constant $\varepsilon > 0$ can be written as

$$\varepsilon|\xi-\eta|^2 \leq \langle \sigma(|\xi|)\xi - \sigma(|\eta|)\eta, \xi-\eta\rangle, \tag{2.15}$$

$$|\sigma(|\xi|)\xi - \sigma(|\eta|)\eta|^2 \leq \varepsilon\langle \sigma(|\xi|)\xi - \sigma(|\eta|)\eta, \xi-\eta\rangle, \tag{2.16}$$

respectively.

Let φ be the angle between the vectors ξ and η. We have

$$|\xi-\eta|^2 = |\xi|^2 + |\eta|^2 - 2|\xi||\eta|\cos\varphi,$$

$$\langle \sigma(|\xi|)\xi - \sigma(|\eta|)\eta, \xi-\eta\rangle = \sigma(|\xi|)|\xi|^2 + \sigma(|\eta|)|\eta|^2 - (\sigma(|\xi|)+\sigma(|\eta|))|\xi||\eta|\cos\varphi,$$

$$|\sigma(|\xi|)\xi - \sigma(|\eta|)\eta|^2 = \sigma^2(|\xi|)|\xi|^2 + \sigma^2(|\eta|)|\eta|^2 - 2\sigma(|\xi|)\sigma(|\eta|)|\xi||\eta|\cos\varphi.$$

We set

$$\begin{aligned}
\Upsilon(\varphi) &= |\xi|^2 + |\eta|^2 - 2|\xi||\eta|\cos\varphi,\\
\Phi(\varphi) &= \sigma(|\xi|)|\xi|^2 + \sigma(|\eta|)|\eta|^2 - (\sigma(|\xi|)+\sigma(|\eta|))|\xi||\eta|\cos\varphi,\\
\Psi(\varphi) &= \sigma^2(|\xi|)|\xi|^2 + \sigma^2(|\eta|)|\eta|^2 - 2\sigma(|\xi|)\sigma(|\eta|)|\xi||\eta|\cos\varphi.
\end{aligned}$$

Proof of Theorem 1.4. It is clear that the inequality (2.15) holds for all $(\xi,\eta)\in D_\gamma$. Let $(\xi,\eta)\in\mathcal{A}_\gamma(\varepsilon)\cap H_\gamma$. In this case the inequality (2.15) is rewritten in the form

$$\varepsilon\le\sigma(|\xi|)=\sigma(|\eta|).$$

Hence,

$$\mathcal{A}_\gamma(\epsilon)\cap H_\gamma=W_\gamma^+(\epsilon)\cap H_\gamma.$$

Using Lemma 2.14, we see that

$$\left(W_\gamma^-(\varepsilon)\cap H_\gamma\right)\subset\left(\mathcal{A}_\gamma(\varepsilon)\cap H_\gamma\right)\subset\left(W_\gamma^+(\varepsilon)\cap H_\gamma\right). \tag{2.17}$$

Now we assume that $(\xi,\eta)\in G_\gamma$. Then $\Upsilon(\varphi)>0$ and after simple calculations we find

$$\frac{\partial}{\partial\varphi}\left(\frac{\Phi(\varphi)}{\Upsilon(\varphi)}\right)=\frac{(\sigma(|\eta|)-\sigma(|\xi|))(|\xi|^2-|\eta|^2)|\xi||\eta|\sin\varphi}{\Upsilon^2(\varphi)}.$$

By the property 3) of Lemma 2.12 we have

$$(\sigma(|\eta|)-\sigma(|\xi|))(|\xi|^2-|\eta|^2)>0.$$

Therefore,

$$\begin{aligned}\min_{\varphi\in[0,\pi]}\left(\frac{\Phi(\varphi)}{\Upsilon(\varphi)}\right)&=\frac{\Phi(0)}{\Upsilon(0)}\\&=\frac{\sigma(|\xi|)|\xi|^2+\sigma(|\eta|)|\eta|^2-(\sigma(|\xi|)+\sigma(|\eta|))|\xi||\eta|}{(|\xi|-|\eta|)^2}=I_\gamma^-(|\xi|,|\eta|)\end{aligned}$$

and

$$\begin{aligned}\max_{\varphi\in[0,\pi]}\left(\frac{\Phi(\varphi)}{\Upsilon(\varphi)}\right)&=\frac{\Phi(\pi)}{\Upsilon(\pi)}\\&=\frac{\sigma(|\xi|)|\xi|^2+\sigma(|\eta|)|\eta|^2+(\sigma(|\xi|)+\sigma(|\eta|))|\xi||\eta|}{(|\xi|+|\eta|)^2}=I_\gamma^+(|\xi|,|\eta|).\end{aligned}$$

Then for all $(\xi,\eta)\in G_\gamma$ the following inequalities are valid

$$I_\gamma^-(|\xi|,|\eta|)\le\frac{\langle\sigma(|\xi|)\xi-\sigma(|\eta|)\eta,\xi-\eta\rangle}{|\xi-\eta|^2}\le I_\gamma^+(|\xi|,|\eta|).$$

This implies

$$\left(W_\gamma^-(\varepsilon)\cap G_\gamma\right)\subset\left(\mathcal{A}_\gamma(\varepsilon)\cap G_\gamma\right)\subset\left(W_\gamma^+(\varepsilon)\cap G_\gamma\right).$$

From this, by (2.17) and Lemma 2.14 we obtain (1.5) and (1.6). $\square$

Proof of Theorem 1.7. *a*) It is clear that (2.16) holds for all $(\xi,\eta)\in D_\gamma$. Let $(\xi,\eta)\in\mathcal{B}_\gamma(\varepsilon)\cap H_\gamma$. In this case the inequality (2.16) becomes

$$\sigma(|\xi|)=\sigma(|\eta|)\le\varepsilon.$$

Then

$$\mathcal{B}_\gamma(\varepsilon) \cap H_\gamma = V_\gamma^+(\varepsilon) \cap H_\gamma.$$

Using Lemma 2.14, we see that

$$\left(V_\gamma^+(\varepsilon) \cap H_\gamma\right) \subset (\mathcal{B}_\gamma(\varepsilon) \cap H_\gamma) \subset \left(V_\gamma^-(\varepsilon) \cap H_\gamma\right). \tag{2.18}$$

Now we assume that $(\xi, \eta) \in G_\gamma$. Then by the inequality

$$\Psi(\varphi) \geq (\sigma(|\xi|)|\xi| - \sigma(|\eta|)|\eta|)^2$$

and by the property 4) of Lemma 2.12 we can conclude that $\Psi(\varphi) > 0$ for all $\varphi \in [0, \pi]$. Next after simple calculations, we obtain

$$\frac{\partial}{\partial \varphi}\left(\frac{\Phi(\varphi)}{\Psi(\varphi)}\right) = \frac{(\sigma(|\xi|) - \sigma(|\eta|))(|\xi|^2\sigma^2(|\xi|) - |\eta|^2\sigma^2(|\eta|))|\xi||\eta| \sin\varphi}{\Psi^2(\varphi)}.$$

By the properties 3) and 4) of Lemma 2.12 it follows that

$$(\sigma(|\xi|) - \sigma(|\eta|))(|\xi|^2\sigma^2(|\xi|) - |\eta|^2\sigma^2(|\eta|)) < 0. \tag{2.19}$$

Therefore

$$\min_{\varphi\in[0,\pi]}\left(\frac{\Phi(\varphi)}{\Psi(\varphi)}\right) = \frac{\Phi(\pi)}{\Psi(\pi)}$$
$$= \frac{\sigma(|\xi|)|\xi|^2 + \sigma(|\eta|)|\eta|^2 + (\sigma(|\xi|) + \sigma(|\eta|))|\xi||\eta|}{\sigma^2(|\xi|)|\xi|^2 + \sigma^2(|\eta|)|\eta|^2 + 2\sigma(|\xi|)\sigma(|\eta|)|\xi||\eta|} = \frac{1}{I_\gamma^+(|\xi|, |\eta|)}.$$

and

$$\max_{\varphi\in[0,\pi]}\left(\frac{\Phi(\varphi)}{\Psi(\varphi)}\right) = \frac{\Phi(0)}{\Psi(0)}$$
$$= \frac{\sigma(|\xi|)|\xi|^2 + \sigma(|\eta|)|\eta|^2 - (\sigma(|\xi|) + \sigma(|\eta|))|\xi||\eta|}{\sigma^2(|\xi|)|\xi|^2 + \sigma^2(|\eta|)|\eta|^2 - 2\sigma(|\xi|)\sigma(|\eta|)|\xi||\eta|} = \frac{1}{I_\gamma^-(|\xi|, |\eta|)}.$$

Thus for all $(\xi, \eta) \in G_\gamma$, the following inequalities are true

$$\frac{1}{I_\gamma^+(|\xi|, |\eta|)} \leq \frac{\langle \sigma(|\xi|)\xi - \sigma(|\eta|)\eta, \xi - \eta\rangle}{|\sigma(|\xi|)\xi - \sigma(|\eta|)\eta|^2} \leq \frac{1}{I_\gamma^-(|\xi|, |\eta|)}. \tag{2.20}$$

This implies that

$$\left(V_\gamma^+(\varepsilon) \cap G_\gamma\right) \subset (\mathcal{B}_\gamma(\varepsilon) \cap G_\gamma) \subset \left(V_\gamma^-(\varepsilon) \cap G_\gamma\right).$$

From this, by (2.18) and Lemma 2.14 we obtain the relations (1.8) and (1.9).

b) It is clear that the inequality (2.16) holds for all $(\xi, \eta) \in Q_\gamma$. Moreover, by the property 5 of Lemma 2.12 we have $Q_\gamma \neq D_\gamma$.

Let $(\xi, \eta) \in P_\gamma$. Similarly, we establish that $P_\gamma \neq H_\gamma$. Next, we have

$$\Psi(\varphi) = \sigma^2(|\xi|)|\xi|^2 + \sigma^2(|\eta|)|\eta|^2 - 2\sigma(|\xi|)\sigma(|\eta|)|\xi||\eta| \cos\varphi = 2\sigma^2(|\xi|)|\xi|^2(1 - \cos\varphi)$$

and
$$\begin{aligned}\Phi(\varphi) &= \sigma(|\xi|)|\xi|^2+\sigma(|\eta|)|\eta|^2-(\sigma(|\xi|)+\sigma(|\eta|))|\xi||\eta|\cos\varphi \\ &= \sigma(|\xi|)|\xi|^2+\sigma(|\xi|)|\xi||\eta|-\sigma(|\xi|)|\xi||\eta|\cos\varphi-\sigma(|\xi|)|\xi|^2\cos\varphi \\ &= \sigma(|\xi|)|\xi|(|\xi|+|\eta|)(1-\cos\varphi).\end{aligned}$$

It is easy to see that $\cos\varphi \neq 1$. Indeed, we suppose that $\cos\varphi = 1$. Then the vectors $\xi\sigma(|\xi|)$ and $\eta\sigma(|\eta|)$ are collinear. It implies that $\xi\sigma(|\xi|) = \eta\sigma(|\eta|)$.

We find
$$\frac{\Psi(\varphi)}{\Phi(\varphi)} = \frac{2|\xi|\sigma(|\xi|)}{|\xi|+|\eta|} = I_\gamma^+(|\xi|,|\eta|).$$
Thus, the inequality (2.16) assumes the form
$$I_\gamma^+(|\xi|,|\eta|) \le \varepsilon$$
and we establish that
$$\mathcal{B}_\gamma(\varepsilon)\cap P_\gamma = V_\gamma^+(\varepsilon)\cap P_\gamma. \tag{2.21}$$

Let $(\xi,\eta)\in U_\gamma^+$. By the property 3) of Lemma 2.12 we find that the inequality (2.19) is valid. Therefore the inequalities (2.20) are true and we obtain
$$\left(V_\gamma^+(\varepsilon)\cap U_\gamma^+\right)\subset\left(\mathcal{B}_\gamma(\varepsilon)\cap U_\gamma^+\right)\subset\left(V_\gamma^-(\varepsilon)\cap U_\gamma^+\right). \tag{2.22}$$

Now let $(\xi,\eta)\in U_\gamma^-$. Observe that the set U_γ^- is not empty. It is easy to see that
$$(\sigma(|\xi|)-\sigma(|\eta|))(|\xi|^2\sigma^2(|\xi|)-|\eta|^2\sigma^2(|\eta|))>0.$$
For all $(\xi,\eta)\in U_\gamma^-$ the following inequalities are true
$$\frac{1}{I_\gamma^-(|\xi|,|\eta|)} \le \frac{\langle\sigma(|\xi|)\xi-\sigma(|\eta|)\eta,\xi-\eta\rangle}{|\sigma(|\xi|)\xi-\sigma(|\eta|)\eta|^2} \le \frac{1}{I_\gamma^+(|\xi|,|\eta|)}$$
and we obtain
$$\left(\mathcal{B}_\gamma(\varepsilon)\cap U_\gamma^-\right)\subset\left(V_\gamma^+(\varepsilon)\cap U_\gamma^-\right).$$
From here, by (2.21) and (2.22),
$$F_\gamma^+(\varepsilon)\subset\mathcal{B}_\gamma(\varepsilon)\subset F_\gamma^-(\varepsilon).$$

It is not hard to establish that
$$W_\gamma^-(0)\subset\left(P_\gamma\cup Q_\gamma\cup U_\gamma^+\right), \qquad \left(P_\gamma\cup Q_\gamma\cup U_\gamma^+\cup U_\gamma^-\right)=\Omega_\gamma\times\Omega_\gamma.$$

Then, using Lemma 2.14, we find
$$\left(V_\gamma^+(\varepsilon)\cap W_\gamma^-(0)\right)\subset F_\gamma^+(\varepsilon), \qquad F_\gamma^-(\varepsilon)\subset\left(V_\gamma^-(\varepsilon)\cup Q_\gamma\right).$$

From here we obtain the relations (1.10) and (1.11). □

3 Properties of $W_\gamma^-(\varepsilon)$, $W_\gamma^+(\varepsilon)$, $V_\gamma^-(\varepsilon)$ and $V_\gamma^+(\varepsilon)$

Here we study the sets $W_\gamma^-(\varepsilon)$, $W_\gamma^+(\varepsilon)$, $V_\gamma^-(\varepsilon)$ and $V_\gamma^+(\varepsilon)$. Consider the equation

$$\theta'(t) = \varepsilon, \tag{3.23}$$

where $\theta(t) = t\sigma(t)$ and ε is an arbitrary parameter. It is easy to verify that for $\gamma \neq 1$ the equation (3.23) can be written down in the following form:

$$\frac{2}{\gamma-1}\sigma^{2-\gamma}(t) - \frac{\gamma+1}{\gamma-1}\sigma(t) + \varepsilon = 0.$$

Further, we assume that $\varepsilon \in (0,1)$. We set

$$r = \sqrt{\frac{2(1-\varepsilon^{\gamma-1})}{\gamma-1}} \qquad \text{for } \gamma \neq 1$$

and

$$r = \sqrt{-2\ln\varepsilon} \qquad \text{for } \gamma = 1.$$

Observe that $r \in \Sigma_\gamma$ for all $\gamma \in \mathbf{R}$.

Fix $\varepsilon \in (0,1)$. Assume that $\gamma \leq -1$. It is easy to see that

$$\theta'(0) = 1, \qquad \lim_{t\to+\infty} \theta'(t) = 0.$$

From here and by the property 6) of Lemma 2.12 we deduce that the equation (3.23) has the unique positive solution s and $0 \leq t \leq s$ be the solutions of the inequality $\theta'(t) \geq \varepsilon$ subject to $t \geq 0$.

Further, we have

$$\sigma(r) = \varepsilon = \theta'(s) = \sigma(s) + s\sigma'(s) < \sigma(s).$$

Then the inequality $\sigma(r) < \sigma(s)$ implies $r > s$. Hence, $s \in (0,r)$.

Assume that $\gamma > -1$. By the property 5) of Lemma 2.12 we see that

$$0 \leq t < \sqrt{\frac{2}{\gamma+1}}$$

be the solutions of the inequality $\theta'(t) > 0$ subject to $t \geq 0$. By the properties 6), 7) of Lemma 2.12 we deduce that the function $\theta'(t)$ is decreasing on $\left[0, \sqrt{\frac{2}{\gamma+1}}\right]$. Moreover,

$$\theta'(0) = 1, \qquad \theta'\left(\sqrt{\frac{2}{\gamma+1}}\right) = 0.$$

Therefore the equation (3.23) has the unique positive solution $s < \sqrt{\frac{2}{\gamma+1}}$ and $0 \leq t \leq s$ be the solutions of the inequality $\theta'(t) \geq \varepsilon$ subject to $t \geq 0$. As above, we can show that $s \in (0,r)$.

Thus, we proved the following statement.

3.24. Lemma. *Let $\gamma \in \mathbf{R}$, $\epsilon \in (0,1)$ and $s \in (0,r)$ be a positive solution of (3.23).. Then the following relations hold*

$$\theta'(t) > \varepsilon \quad \text{for all } t \in (0,s), \qquad \theta'(t) < \varepsilon \quad \text{for all } t > s,\ t \in \Sigma_\gamma \tag{3.25}$$

3.26. Remark. It is not hard to establish that for $\gamma > -1$ and $\epsilon = 0$ the relations (3.25) are true with $s = \sqrt{\frac{2}{\gamma+1}}$.

We say that a set $G \subset \mathbf{R}^n$ is an *linearly connected* if any pair of points $x, y \in G$ can be joined on D by an arc.

The sets $W_\gamma^-(\varepsilon)$, $W_\gamma^+(\varepsilon)$, $V_\gamma^-(\varepsilon)$ and $V_\gamma^+(\varepsilon)$ have the following properties.

3.27. Proposition. *a) The set $W_\gamma^-(\varepsilon)$ is linearly connected for $\gamma \in \mathbf{R}$ and $\varepsilon \in (0,1)$.*

b) The set $W_\gamma^-(0)$ is linearly connected for $\gamma > -1$.

c) The set $W_\gamma^+(\varepsilon)$ is linearly connected for $\gamma \in \mathbf{R}$ and $\varepsilon \in (0,1)$.

Proof. *a*) We fix numbers $\gamma \in \mathbf{R}^n$, $\varepsilon \in (0,1)$ and a nonzero point $\zeta = (\xi, \eta) \in W_\gamma^-(\varepsilon)$. To prove the statement, it is sufficient to show that the set $W_\gamma^-(\varepsilon)$ contains the segment $\mathcal{L} = \{(\xi t, \eta t) : 0 \le t \le 1\}$ with the endpoints 0 and ζ.

Indeed, let $\zeta', \zeta'' \in W_\gamma^-(\varepsilon)$ be arbitrary. Let $\mathcal{L}'$, $\mathcal{L}''$ be the segments with the endpoints 0, ζ' and 0, ζ'' respectively. Denote by $\mathcal{L}' \cup \mathcal{L}''$ the double curve which consists of two segments $\mathcal{L}'$ and $\mathcal{L}''$. Then this double curve will join the points ζ', ζ'' and it will lie on $W_\gamma^-(\varepsilon)$.

We prove that the segment $\mathcal{L}$ lies in $W_\gamma^-(\varepsilon)$. Assume that $|\xi| < |\eta|$. For $t \in (0,1)$ we have

$$\begin{aligned}
I_\gamma^-(|\xi t|, |\eta t|) &= \frac{|\xi|\sigma(|\xi t|) - |\eta|\sigma(|\eta t|)}{|\xi| - |\eta|} \ge \frac{|\xi|\sigma(|\xi|) - |\eta|\sigma(|\eta t|)}{|\xi| - |\eta|} \\
&= \frac{|\xi|\sigma(|\xi|) - |\eta|\sigma(|\eta|) + |\eta|\sigma(|\eta|) - |\eta|\sigma(|\eta t|)}{|\xi| - |\eta|} \\
&\ge \varepsilon + \frac{|\eta|(\sigma(|\eta|) - \sigma(|\eta t|))}{|\xi| - |\eta|} \ge \varepsilon.
\end{aligned}$$

The case $|\xi| > |\eta|$ is analogous. Now we assume that $|\xi| = |\eta|$. We write

$$I_\gamma^-(|\xi|, |\eta|) = \theta'(|\xi|) \ge \varepsilon$$

Then by Lemma 3.24 for $t \in (0,1)$ we deduce $|\xi t| \le |\xi| \le s$ and

$$I_\gamma^-(|\xi t|, |\eta t|) = \theta'(|\xi t|) \ge \varepsilon.$$

Hence, the set $W_\gamma^-(\varepsilon)$ contains the segment $\mathcal{L}$.

b) The proof is analogous.

c) We fix numbers $\gamma \in \mathbf{R}^n$, $\varepsilon \in (0,1)$ and a nonzero point $\zeta = (\xi,\eta) \in W_\gamma^+(\varepsilon)$. As above, to prove this statement, it is sufficient to show that the set $W_\gamma^+(\varepsilon)$ contains the segment $\mathcal{L} = \{(\xi t, \eta t) : 0 \le t \le 1\}$. For $t \in (0,1)$ we have

$$I_\gamma^+(|\xi t|, |\eta t|) = \frac{|\xi|\sigma(|\xi t|) + |\eta|\sigma(|\eta t|)}{|\xi| + |\eta|} \ge \frac{|\xi|\sigma(|\xi|) + |\eta|\sigma(|\eta|)}{|\xi| + |\eta|} \ge \varepsilon.$$

Thus, the set $W_\gamma^+(\varepsilon)$ contains the segment $\mathcal{L}$. $\square$

3.28. Proposition. *a) Let $\varepsilon \in (0,1)$ and $\gamma \in \mathbf{R}$. Then*

$$\{(\xi,\eta) : \ \xi,\eta \in \mathbf{R}^n, \ |\xi| \le s, \ |\eta| \le s\} \subset W_\gamma^-(\varepsilon),$$

where $s \in \Sigma_\gamma$ is the unique positive solution of the equation (3.23).

b) If $\gamma > -1$ then

$$\{(\xi,\eta) : \ \xi,\eta \in \mathbf{R}^n, \quad |\xi| \le \sqrt{\frac{2}{\gamma+1}}, \quad |\eta| \le \sqrt{\frac{2}{\gamma+1}}\} \subset W_\gamma^-(0).$$

c) Let $\varepsilon \in (0,1)$ and $\gamma \in \mathbf{R}$. Then

$$V_\gamma^-(\varepsilon) \subset (\Omega_\gamma \times \Omega_\gamma) \setminus \{(\xi,\eta) : \ \xi,\eta \in \mathbf{R}^n, \ |\xi| < s, \ |\eta| < s\},$$

where s is the unique positive solution of the equation (3.23)*.*

d) If $\gamma > -1$ then

$$V_\gamma^-(0) \subset (\Omega_\gamma \times \Omega_\gamma) \setminus \{(\xi,\eta) : \ \xi,\eta \in \mathbf{R}^n, \quad |\xi| < \sqrt{\frac{2}{\gamma+1}}, \quad |\eta| < \sqrt{\frac{2}{\gamma+1}}\}.$$

Proof. *a*) Let $(\xi,\eta) \in \{(\xi,\eta) : \ \xi,\eta \in \mathbf{R}^n, \ |\xi| \le s, \ |\eta| \le s\}$. Using Lemma 3.24, we see that

$$\theta'(|\xi|) \ge \varepsilon, \qquad \theta'(|\eta|) \ge \varepsilon.$$

We assume that $|\xi| = |\eta|$. Then

$$I_\gamma^-(|\xi|,|\eta|) = \theta'(|\xi|) = \theta'(|\eta|) \ge \varepsilon$$

and, hence, $(\xi,\eta) \in W_\gamma^-(\varepsilon)$.

Now we assume that $|\xi| < |\eta|$. Using the well-known Lagrange mean value theorem, we obtain

$$I_\gamma^-(|\xi|,|\eta|) = \theta'(c), \qquad |\xi| \le c \le |\eta|.$$

By Lemma 3.24,

$$\theta'(c) \ge \varepsilon.$$

Hence, $(\xi,\eta) \in W_\gamma^-(\varepsilon)$. The case $|\xi| > |\eta|$ is analogous.

b) The proof is analogous.

c) The proof easy follows from *a*).

d) The proof easy follows from *b*). $\square$

3.29. Proposition. *If $\varepsilon \in (0,1)$ and $\gamma \in \mathbf{R}$, then we have*

a) $\quad W_\gamma^-(\varepsilon) \subset \{(\xi,\eta): \ \xi,\eta \in \mathbf{R}^n, \ |\xi| \le r, \ |\eta| \le r\},$

and

b) $\quad (\Omega_\gamma \times \Omega_\gamma) \setminus \{(\xi,\eta): \ \xi,\eta \in \mathbf{R}^n, \ |\xi| < r, \ |\eta| < r\} \subset V_\gamma^-(\varepsilon).$

Proof. *a)* Let $(\xi,\eta) \in W_\gamma^-(\varepsilon)$. Assume that $|\xi| = |\eta|$. We have

$$\varepsilon \le I_\gamma^-(|\xi|,|\eta|) = \theta'(|\xi|) = \sigma(|\xi|) + |\xi|\sigma'(|\xi|) \le \sigma(|\xi|) = \sigma(|\eta|).$$

Then the inequalities

$$\sigma(|\xi|) \ge \varepsilon, \quad \sigma(|\eta|) \ge \varepsilon$$

imply

$$|\xi| = |\eta| \le r.$$

Hence,

$$(\xi,\eta) \in \{(\xi,\eta): \ \xi,\eta \in \mathbf{R}^n, \ |\xi| \le r, \ |\eta| \le r\}.$$

Now we assume that $|\xi| > |\eta|$. Using the inequality

$$\sigma(|\xi|) < \sigma(|\eta|),$$

we deduce

$$\varepsilon \le I_\gamma^-(|\xi|,|\eta|) = \frac{|\xi|\sigma(|\xi|) - |\eta|\sigma(|\eta|)}{|\xi| - |\eta|} \le \frac{|\xi|\sigma(|\xi|) - |\eta|\sigma(|\xi|)}{|\xi| - |\eta|} = \sigma(|\xi|) < \sigma(|\eta|).$$

From here, $(\xi,\eta) \in \{(\xi,\eta): \ \xi,\eta \in \mathbf{R}^n, \ |\xi| \le r, \ |\eta| \le r\}$. The case $|\xi| < |\eta|$ is analogous.
b) The proof follows from *a)*. □

3.30. Proposition. *If $\varepsilon \in (0,1)$ and $\gamma \in \mathbf{R}$, then*

a) $\quad \{(\xi,\eta): \ \xi,\eta \in \mathbf{R}^n, \ |\xi| \le r, \ |\eta| \le r\} \subset W_\gamma^+(\varepsilon),$

and

b) $\quad V_\gamma^+(\varepsilon) \subset (\Omega_\gamma \times \Omega_\gamma) \setminus \{(\xi,\eta): \ \xi,\eta \in \mathbf{R}^n, \ |\xi| < r, \ |\eta| < r\}.$

Proof. *a)* Let $(\xi,\eta) \in \{(\xi,\eta): \xi,\eta \in \mathbf{R}^n, |\xi| \le r, |\eta| \le r\}$. Then

$$\sigma(|\xi|) \ge \varepsilon, \qquad \sigma(|\eta|) \ge \varepsilon.$$

Assume that $|\xi| = |\eta|$. We have

$$I_\gamma^+(|\xi|,|\eta|) = \sigma(|\xi|) \ge \varepsilon.$$

Hence, $(\xi,\eta) \in W_\gamma^+(\varepsilon)$.
Now we assume that $|\xi| > |\eta|$. We deduce

$$I_\gamma^+(|\xi|,|\eta|) = \frac{|\xi|\sigma(|\xi|) + |\eta|\sigma(|\eta|)}{|\xi| + |\eta|} \ge \frac{|\xi|\sigma(|\xi|) + |\eta|\sigma(|\xi|)}{|\xi| + |\eta|} = \sigma(|\xi|) \ge \varepsilon.$$

From here, $(\xi,\eta) \in W_\gamma^+(\varepsilon)$. The case $|\xi| < |\eta|$ is analogous.
b) The proof follows from *a)*.□

4 Properties of $x_\gamma(\varepsilon)$

For arbitrary $\varepsilon \in (0,1), \gamma \in \mathbf{R}$ we set

$$\begin{aligned} X_\gamma(\varepsilon) &= \Big\{ x \in \Sigma_\gamma :\ \exists y \in \Sigma_\gamma,\ I_\gamma^+(x,y) \geq \varepsilon \Big\}, \\ \bar{X}_\gamma(\varepsilon) &= \Big\{ x \in \Sigma_\gamma :\ \exists y \in \Sigma_\gamma,\ I_\gamma^+(x,y) = \varepsilon \Big\}, \\ x_\gamma(\varepsilon) &= \sup\nolimits_x X_\gamma(\varepsilon). \end{aligned}$$

If $x_\gamma(\varepsilon) < +\infty$ then the following relations are true

$$W_\gamma^+(\varepsilon) \subset \{(\xi,\eta) :\ \xi,\eta \in \mathbf{R}^n,\ |\xi| \leq x_\gamma(\varepsilon),\ |\eta| \leq x_\gamma(\varepsilon)\}$$

and

$$(\Omega_\gamma \times \Omega_\gamma) \setminus \{(\xi,\eta) :\ \xi,\eta \in \mathbf{R}^n,\ |\xi| < x_\gamma(\varepsilon),\ |\eta| < x_\gamma(\varepsilon)\} \subset V_\gamma^+(\varepsilon).$$

We shall study the function $x_\gamma(\varepsilon)$. We have

$$I_\gamma^+(0,r) = \sigma(r) = \varepsilon \qquad \text{for all } \varepsilon \in (0,1), \gamma \in \mathbf{R}.$$

From here, we deduce that $r \in X_\gamma(\varepsilon)$ and $r \in \bar{X}_\gamma(\varepsilon)$. Then the function $x_\gamma(\varepsilon)$ is defined everywhere on (0, 1) and $r \leq x_\gamma(\varepsilon)$. Besides, from the definition of the set Σ_γ we establish

$$x_\gamma(\varepsilon) \leq \sqrt{\frac{2}{\gamma-1}} \qquad \text{for all } \gamma > 1.$$

The function $x_\gamma(\varepsilon)$ has the following properties:

4.31. Proposition. *The function $x_\gamma(\varepsilon)$ is nonincreasing on $(0,1)$.*

The **proof** is evident.

4.32. Proposition. *If $\gamma > 1$ then*

(4.33) $$x_\gamma(\varepsilon) = \sqrt{\frac{2}{\gamma-1}} \qquad \text{for all } \varepsilon \in (0,\varepsilon']$$

and

(4.34) $$x_\gamma(\varepsilon) < \sqrt{\frac{2}{\gamma-1}} \qquad \text{for all } \varepsilon \in (\varepsilon',1),$$

where

(4.35) $$\varepsilon' = \max_{y \in \left[0,\sqrt{\frac{2}{\gamma-1}}\right]} I_\gamma^+\left(\sqrt{\frac{2}{\gamma-1}}, y\right).$$

Proof. Let $\gamma > 1$. We set

$$\alpha(y) \equiv I_\gamma^+\left(\sqrt{\frac{2}{\gamma-1}}, y\right) = \frac{\theta(y)}{y+\sqrt{\frac{2}{\gamma-1}}}.$$

It is easy to see that the function $\alpha(y)$ is positive on $\left(0, \sqrt{\frac{2}{\gamma-1}}\right)$ and it is continuous on $\left[0, \sqrt{\frac{2}{\gamma-1}}\right]$. Therefore there exists

$$\varepsilon' = \max_{y\in[0,\sqrt{\frac{2}{\gamma-1}}]} \alpha(y) > 0.$$

We have

$$\alpha(y) \leq \frac{y}{y+\sqrt{\frac{2}{\gamma-1}}} < 1 \qquad \text{for all } y \in [0, \sqrt{\tfrac{2}{\gamma-1}}].$$

Hence, $\varepsilon' < 1$. Therefore for $\varepsilon \in (0, \varepsilon']$ the equation

$$\alpha(y) = \varepsilon \tag{4.36}$$

has at the minimum one solution $y_0 \in (0, \sqrt{\frac{2}{\gamma-1}})$. Otherwise the equation hasn't solutions.

We fix arbitrary $\varepsilon \in (0, \varepsilon']$, $x \in \Sigma_\gamma$. Let $y_0 \in \Sigma_\gamma$ be a solution of (4.36). We have

$$\varepsilon = \alpha(y_0) = \frac{\theta(y_0)}{y_0+\sqrt{\frac{2}{\gamma-1}}} \leq \frac{\theta(x)+\theta(y_0)}{x+y_0} = I_\gamma^+(x, y_0).$$

From here, we deduce that $x \in X_\gamma(\varepsilon)$. Hence, $X_\gamma(\varepsilon) = \Sigma_\gamma$ for all $\varepsilon \in (0, \varepsilon']$. It proves the relation (4.33).

Now we prove the relation (4.34). Fix $\varepsilon \in (\varepsilon', 1)$. Suppose that

$$x_\gamma(\varepsilon) = \sqrt{\frac{2}{\gamma-1}}.$$

Then for $n \in \mathbf{N}$ there exists a number $x_n \in X_\gamma(\varepsilon)$ such that

$$\sqrt{\frac{2}{\gamma-1}} - \frac{1}{n} < x_n.$$

Moreover,

$$\lim_{n\to\infty} x_n = \sqrt{\frac{2}{\gamma-1}}$$

and for $n \in \mathbf{N}$ there exists $y_n \in \Sigma_\gamma$ satisfying the inequality

$$I_\gamma^+(x_n, y_n) \geq \varepsilon.$$

This inequality implies

$$\theta(x_n) - \varepsilon x_n \geq \varepsilon y_n - \theta(y_n). \tag{4.37}$$

Further, we have

$$\alpha(y_n) = \frac{\theta(y_n)}{y_n + \sqrt{\frac{2}{\gamma-1}}} \le \varepsilon' \qquad \text{for all } n \in \mathbf{N}.$$

From here,

$$\theta(y_n) \le \varepsilon' \left(y_n + \sqrt{\frac{2}{\gamma-1}} \right) \qquad \text{for all } n \in \mathbf{N}. \tag{4.38}$$

Using (4.37) and (4.38), we deduce

$$\theta(x_n) - \varepsilon x_n \ge \varepsilon y_n - \theta(y_n)$$

$$\ge \varepsilon y_n - \varepsilon' \left(y_n + \sqrt{\tfrac{2}{\gamma-1}} \right) \ge -\varepsilon' \sqrt{\tfrac{2}{\gamma-1}}.$$

Letting $n \to \infty$ in the inequality

$$\theta(x_n) - \varepsilon x_n \ge -\varepsilon' \sqrt{\frac{2}{\gamma-1}},$$

we see that $\varepsilon \le \varepsilon'$ and we arrive at a contradiction. □

Prove some auxiliary statements.

4.39. Lemma. *Let*

$$\gamma \in (-\infty, 1], \quad \varepsilon \in (0,1) \tag{4.40}$$

or

$$\gamma \in (1, +\infty), \quad \varepsilon \in (\varepsilon', 1), \tag{4.41}$$

where ε' is defined by (4.35). *Then the set $X_\gamma(\varepsilon)$ is compact.*

Proof. Introduce the set

$$Z_\gamma(\varepsilon) = \{(x,y) \in \Sigma_\gamma \times \Sigma_\gamma : \ I_\gamma^+(x,y) \ge \varepsilon\}.$$

Let $\pi : \mathbf{R^2} \to \mathbf{R}$, $\pi(x,y) = x$ be natural projection. It is clear that $\pi(Z_\gamma(\varepsilon)) = X_\gamma(\varepsilon)$.

Assume that the condition (4.40) holds. The set $Z_\gamma(\varepsilon)$ is closed since the function $I_\gamma^+(x,y)$ is continuous.

The set $Z_\gamma(\varepsilon)$ is bounded. Indeed, we can find a sequence $Z_\gamma(\varepsilon) \ni (x_n, y_n) \to \infty$. Assume that $x_n \to \infty$. Then for the bounded subsequence of $\{y_n\}$ we have

$$\varepsilon \le I_\gamma^+(x_n, y_n) = \frac{x_n\sigma(x_n) + y_n\sigma(y_n)}{x_n + y_n} \le \frac{x_n\sigma(x_n) + y_n}{x_n}.$$

The right part of this inequality tends to zero as $n \to \infty$. Thus we obtain a contradiction to (4.40).

For an unbounded subsequence of $\{y_n\}$ we have

$$\varepsilon \le I_\gamma^+(x_n, y_n) \le \sigma(x_n) + \sigma(y_n).$$

The right part of this inequality tends to zero as $n \to \infty$. Again we obtain a contradiction to (4.40). Hence, the set $Z_\gamma(\varepsilon)$ is bounded. Therefore the set $Z_\gamma(\varepsilon)$ is compact. Because the mapping π is continuous then the set $X_\gamma(\varepsilon) = \pi(Z_\gamma(\varepsilon))$ is compact too.

Assume that the condition (4.41) holds. By (4.34) we have that $\overline{Z_\gamma(\varepsilon)} \subset \Sigma_\gamma \times \Sigma_\gamma$. Here $\overline{Z_\gamma(\varepsilon)}$ denotes the closure of $Z_\gamma(\varepsilon)$. Since the function $I_\gamma^+(x,y)$ is continuous then $Z_\gamma(\varepsilon)$ is compact. Therefore, the set $X_\gamma(\varepsilon)$ is compact too. The lemma is proved. □

4.42. Corollary. *If the condition* (4.40) *or* (4.41) *holds then the set* $\bar{X}_\gamma(\varepsilon)$ *is compact.*

4.43. Lemma. *If the condition* (4.40) *or* (4.41) *holds then*

$$\sup_x X_\gamma(\varepsilon) = \sup_x \bar{X}_\gamma(\varepsilon).$$

Proof. We set

$$a = \sup X_\gamma(\varepsilon), \qquad b = \sup \bar{X}_\gamma(\varepsilon).$$

Obviously, $a \geq b$. Show that $a \leq b$. By Lemma 4.39 we establish that $a \in X_\gamma(\varepsilon)$. Hence, there exists a number $y_0 \in \Sigma_\gamma$ sach that

$$I_\gamma^+(a, y_0) \geq \varepsilon.$$

Assume that

$$I_\gamma^+(a, y_0) = \varepsilon.$$

Then $a \in \bar{X}_\gamma(\varepsilon)$. By Corollary 4.42 we conclude follows that b is the greatest element of the set $\bar{X}_\gamma(\varepsilon)$. Therefore, $a \leq b$.

Now we assume that

$$I_\gamma^+(a, y_0) > \varepsilon.$$

For $\gamma \leq 1$ we have

$$\lim_{x \to +\infty} I_\gamma^+(x, y_0) = 0.$$

Since the function $I_\gamma^+(x,y)$ is continuous then there exists a number $x' > a$ such that

$$I_\gamma^+(x', y_0) = \varepsilon. \tag{4.44}$$

Hence, $x' \in \bar{X}_\gamma(\varepsilon)$. Then $a < x' \leq b$ and we obtain a contradiction.

By (4.35) for $\gamma > 1$, we deduce

$$I_\gamma^+\left(\sqrt{\frac{2}{\gamma - 1}}, y_0\right) \leq \varepsilon' < \varepsilon.$$

Then there exists a number $x' \in \left(a, \sqrt{\frac{2}{\gamma-1}}\right)$ satisfying (4.44). Hence, $x' \in \bar{X}_\gamma(\varepsilon)$. From here, $a < x' \leq b$ and again we obtain a contradiction. The lemma is proved. □

4.45. Lemma. *If the condition* (4.40) *or* (4.41) *holds, then there exists a number* $y_\gamma(\varepsilon) \in \Sigma_\gamma$ *such that*

$$I_\gamma^+(x_\gamma(\varepsilon), y_\gamma(\varepsilon)) = \varepsilon. \tag{4.46}$$

The **proof** follows from Corollary 4.42 and Lemma 4.43. □

Continue to study the function $x_\gamma(\varepsilon)$.

4.47. Proposition. *The function*

$$x_\gamma(\varepsilon) \in C^\infty(0,1) \quad \textit{for all} \quad \gamma \le 1$$

and

$$x_\gamma(\varepsilon) \in C^\infty((0,\varepsilon') \cup (\varepsilon', 1)) \cap C(0,1) \quad \textit{for all} \quad \gamma > 1\,.$$

Proof. Fix ε_0 and γ, satisfying (4.40) or (4.41). Then $x_\gamma(\varepsilon_0) \in \Sigma_\gamma$ and there exists $y_\gamma(\varepsilon_0) \in \Sigma_\gamma$ sach that

$$I^+_\gamma(x_\gamma(\varepsilon_0), y_\gamma(\varepsilon_0)) = \varepsilon_0.$$

We set

$$F(x,y,\varepsilon) = I^+_\gamma(x,y) - \varepsilon.$$

Observe that the function $F(x,y,\varepsilon)$ is C^∞-differentiable in some neighborhood $U \subset \mathbf{R}^3$ of the point $p_0 = (x_\gamma(\varepsilon_0), y_\gamma(\varepsilon_0), \varepsilon_0)$ and $F(p_0) = 0$. We have

$$\frac{\partial F}{\partial x}(p_0) = \frac{\theta'(x_\gamma(\varepsilon_0)) - I^+_\gamma(x_\gamma(\varepsilon_0), y_\gamma(\varepsilon_0))}{x_\gamma(\varepsilon_0) + y_\gamma(\varepsilon_0)} = \frac{\theta'(x_\gamma(\varepsilon_0)) - \varepsilon_0}{x_\gamma(\varepsilon_0) + y_\gamma(\varepsilon_0)}.$$

In Section 3 we proved that $0 < s < r$, where $s \in \Sigma_\gamma$ is the unique positive root of (3.23). Then the inequality $r \le x_\gamma(\varepsilon_0)$ yields

$$\frac{\partial F}{\partial x}(p_0) \ne 0.$$

By the implicit function theorem we deduce that there is an 3-dimensional interval $I = I_x \times I_y \times I_\varepsilon \subset U$ and a function $f \in C^\infty(I_y \times I_\varepsilon)$ such that for all $(x,y,\varepsilon) \in I_x \times I_y \times I_\varepsilon$

$$F(x,y,\varepsilon) = 0 \Leftrightarrow x = f(y,\varepsilon).$$

Here

$$I_x = \{x \in \mathbf{R} : |x - x_\gamma(\varepsilon_0)| < a\}, \qquad I_y = \{y \in \mathbf{R} : |y - y_\gamma(\varepsilon_0)| < b\}$$

and

$$I_\varepsilon = \{\varepsilon \in \mathbf{R} : |\varepsilon - \varepsilon_0| < c\}\,.$$

Moreover,

$$\begin{aligned}
\frac{\partial f}{\partial y}(y_\gamma(\varepsilon_0), \varepsilon_0) &= -[F'_x(p_0)]^{-1}[F'_y(p_0)] = -\frac{\theta'(y_\gamma(\varepsilon_0)) - \varepsilon_0}{\theta'(x_\gamma(\varepsilon_0)) - \varepsilon_0}, \\
\frac{\partial f}{\partial \varepsilon}(y_\gamma(\varepsilon_0), \varepsilon_0) &= -[F'_x(p_0)]^{-1}[F'_\varepsilon(p_0)] = \frac{x_\gamma(\varepsilon_0) + y_\gamma(\varepsilon_0)}{\theta'(x_\gamma(\varepsilon_0)) - \varepsilon_0}.
\end{aligned}$$

It is easy to see that at the point $y_\gamma(\varepsilon_0)$ the function $x = f(y, \varepsilon_0)$ reaches a maximum on I_y. Therefore

$$\frac{\partial f}{\partial y}(y_\gamma(\varepsilon_0), \varepsilon_0) = 0.$$

From this,

$$\theta'(y_\gamma(\varepsilon_0)) = \varepsilon_0$$

and $y_\gamma(\varepsilon_0) = s$.

Further, we set

$$G(y, \varepsilon) = \theta'(y) - \varepsilon.$$

Observe that the function $G(y, \varepsilon)$ is C^∞-differentiable in some neighborhood $V \subset \mathbf{R}^2$ of the point $q_0 = (y_\gamma(\varepsilon_0), \varepsilon_0)$ and $G(q_0) = 0$.

We have

$$\frac{\partial G}{\partial y}(q_0) = \theta''(y_\gamma(\varepsilon_0) = \theta''(s).$$

Suppose that $\gamma \le -1$. By Lemma 2.12 we see that if $\theta''(s) = 0$ then $s = 0$. But, $s > 0$.

Now suppose that $\gamma > -1$. By Lemma 2.12 we see that if $\theta''(s) = 0$ then $s = 0$ or $s = \sqrt{\frac{6}{\gamma+1}}$. But, in Section 3 we showed that

$$0 < s < \sqrt{\frac{2}{\gamma + 1}} \qquad \text{for} \quad \gamma > -1.$$

Therefore

$$\frac{\partial G}{\partial y}(q_0) \neq 0.$$

And by the implicit function theorem the function $y = y_\gamma(\varepsilon)$ is C^∞-differentiable in the point ε_0. Then there is an interval

$$I'_\varepsilon = \{\varepsilon \in \mathbf{R} : |\varepsilon - \varepsilon_0| < c'\} \subset I_\varepsilon$$

such that

$$y_\gamma(\varepsilon) \in I_y \qquad \text{for all} \quad \varepsilon \in I'_\varepsilon.$$

Hence, for all $(x, \varepsilon) \in I_x \times I'_\varepsilon$

$$F(x, y_\gamma(\varepsilon), \varepsilon) = 0 \Leftrightarrow x = f(y_\gamma(\varepsilon), \varepsilon).$$

Fix arbitrary $\varepsilon \in I'_\varepsilon$. Next,

$$x = f(y_\gamma(\varepsilon), \varepsilon) = f(s, \varepsilon).$$

From here,

$$F(x, s, \varepsilon) = 0.$$

Rewrite this equality in the form

$$\varphi(x) = -\varphi(s),$$

where

$$\varphi(t) = \varphi(t, \varepsilon) = \theta(t) - t\varepsilon.$$

We have

$$\varphi'(t) = \theta'(t) - \varepsilon.$$

By Lemma 3.24 we conclude that the function $\varphi(t)$ is strictly increasing on $(0,s)$ and strictly decreasing on $(s,+\infty)\cap\Sigma_\gamma$. Moreover, $\varphi(0)=\varphi(r)=0$ and by (4.46), $\varphi(x_\gamma(\varepsilon))=-\varphi(s)$. Then it is not hard to check that $x=x_\gamma(\varepsilon)$.

Thus, we proved that

$$x_\gamma(\varepsilon)=f(y_\gamma(\varepsilon),\varepsilon)\qquad\text{for all } \varepsilon\in I'_\varepsilon.$$

Therefore, the function $x_\gamma(\varepsilon)$ is C^∞-differentiable in the point ε_0 and, using (4.47), we deduce

$$x'_\gamma(\varepsilon_0)=\frac{\partial f}{\partial y}(y_\gamma(\varepsilon_0),\varepsilon_0)y'_\gamma(\varepsilon_0)+\frac{\partial f}{\partial\varepsilon}(y_\gamma(\varepsilon_0),\varepsilon_0)=\frac{\partial f}{\partial\varepsilon}(y_\gamma(\varepsilon_0),\varepsilon_0).$$

Fix $\gamma>1$. By (4.33) we conclude that the function $x_\gamma(\varepsilon)$ is C^∞-differentiable on $(0,\varepsilon')$. Show that the function $x_\gamma(\varepsilon)$ is not differentiable in the point ε'. Clearly,

$$\lim_{\varepsilon\to\varepsilon'-0}x'_\gamma(\varepsilon)=0.$$

For arbitrary $\varepsilon\in(\varepsilon',1)$ we have

$$|x'_\gamma(\varepsilon)|=\left|\frac{\partial f}{\partial\varepsilon}(y_\gamma(\varepsilon),\varepsilon)\right|=\left|\frac{x_\gamma(\varepsilon)+y_\gamma(\varepsilon)}{\theta'(x_\gamma(\varepsilon))-\varepsilon}\right|\geq\frac{x_\gamma(\varepsilon)}{1+\varepsilon}\geq\frac{r}{1+\varepsilon}.$$

Hence, the function $x'_\gamma(\varepsilon)$ does not tend to 0 as $\varepsilon\to\varepsilon'+0$. Therefore the function $x_\gamma(\varepsilon)$ is not differentiable in the point ε'.

Prove that function $x_\gamma(\varepsilon)$ is continuous in the point ε'. By (4.33), we have

$$\lim_{\varepsilon\to\varepsilon'-0}x_\gamma(\varepsilon)=\sqrt{\frac{2}{\gamma-1}}.$$

Show that

$$\lim_{\varepsilon\to\varepsilon'+0}x_\gamma(\varepsilon)=\sqrt{\frac{2}{\gamma-1}}.\tag{4.48}$$

Let $y_\gamma(\varepsilon')\in\Sigma_\gamma$ is a solution of the equation

$$\alpha(y)=\varepsilon',$$

Here, as above,

$$\alpha(y)=I^+_\gamma\left(\sqrt{\frac{2}{\gamma-1}},y\right).$$

Then

$$\frac{\theta(y_\gamma(\varepsilon'))}{y_\gamma(\varepsilon')+\sqrt{\frac{2}{\gamma-1}}}=\varepsilon'\tag{4.49}$$

and

$$\alpha'(y_\gamma(\varepsilon'))=\frac{\theta'(y_\gamma(\varepsilon'))\left(y_\gamma(\varepsilon')+\sqrt{\frac{2}{\gamma-1}}\right)-\theta(y_\gamma(\varepsilon'))}{\left(y_\gamma(\varepsilon')+\sqrt{\frac{2}{\gamma-1}}\right)^2}=0.$$

From this,

$$\theta(y_\gamma(\varepsilon'))=\theta'(y_\gamma(\varepsilon'))\left(y_\gamma(\varepsilon')+\sqrt{\frac{2}{\gamma-1}}\right),$$

and, using (4.49), we conclude that

$$\theta'(y_\gamma(\varepsilon')) = \varepsilon'. \tag{4.50}$$

Since

$$\theta'(y_\gamma(\varepsilon)) = \varepsilon \qquad \text{for all } \varepsilon \in (\varepsilon', 1),$$

then

$$\lim_{\varepsilon \to \varepsilon'+0} \theta'(y_\gamma(\varepsilon)) = \varepsilon' = \theta'(y_\gamma(\varepsilon')). \tag{4.51}$$

By Lemma 2.12, the function $\varepsilon = \theta'(y)$ is continuous and strictly decreasing on $\left(0, \sqrt{\frac{2}{\gamma+1}}\right)$. Moreover, $y_\gamma(\varepsilon) \in \left(0, \sqrt{\frac{2}{\gamma+1}}\right)$ for all $\varepsilon \in (\varepsilon', 1)$. Then by (4.51), we establish

$$\lim_{\varepsilon \to \varepsilon'+0} y_\gamma(\varepsilon) = y_\gamma(\varepsilon').$$

We can rewrite the equality (4.46) in the form

$$\theta(x_\gamma(\varepsilon)) - x_\gamma(\varepsilon)\varepsilon = -(\theta(y_\gamma(\varepsilon)) - y_\gamma(\varepsilon)\varepsilon).$$

Using (4.49), we obtain

$$\lim_{\varepsilon \to \varepsilon'+0} (\theta(x_\gamma(\varepsilon)) - x_\gamma(\varepsilon)\varepsilon) = -(\theta(y_\gamma(\varepsilon')) - y_\gamma(\varepsilon')\varepsilon') = -\varepsilon' \sqrt{\frac{2}{\gamma-1}}.$$

From here,

$$\lim_{\varepsilon \to \varepsilon'+0} \varphi(x_\gamma(\varepsilon), \varepsilon) = -\varepsilon' \sqrt{\frac{2}{\gamma-1}}. \tag{4.52}$$

Here, as above,

$$\varphi(t) = \varphi(t, \varepsilon) = \theta(t) - t\varepsilon.$$

Suppose that (4.48) is not true. That is, for some sequence $\varepsilon_i \to \varepsilon' + 0$ of numbers, the inequality

$$x_\gamma(\varepsilon_i) \le \sqrt{\frac{2}{\gamma-1}} - m$$

holds with some constant $m > 0$. By Lemma 3.24, we see that the function $\varphi(t)$ is continuous and strictly decreasing on $\left[r, \sqrt{\frac{2}{\gamma-1}}\right]$. Moreover, $x_\gamma(\varepsilon) \in \left[r, \sqrt{\frac{2}{\gamma-1}}\right]$ for all $\varepsilon \in (\varepsilon', 1)$. Then

$$\varphi(x_\gamma(\varepsilon_i), \varepsilon_i) > \varphi\left(\sqrt{\frac{2}{\gamma-1}} - m, \varepsilon_i\right) > \varphi\left(\sqrt{\frac{2}{\gamma-1}}, \varepsilon_i\right) = -\varepsilon_i \sqrt{\frac{2}{\gamma-1}} > -\varepsilon' \sqrt{\frac{2}{\gamma-1}}.$$

Letting $\varepsilon_i \to \varepsilon' + 0$, we obtain a contradiction to (4.52).

Thus, the function $x_\gamma(\varepsilon)$ is continuous in the point ε'. □

Proving of Proposition 4.47, we established the following statements.

4.53. Proposition. *For all $\gamma > 1$, we have*

$$\lim_{\varepsilon \to \varepsilon'+0} x_\gamma(\varepsilon) = \sqrt{\frac{2}{\gamma-1}}.$$

4.54. Proposition. *The function $x_\gamma(\varepsilon)$ is strictly decreasing on $(0,1)$ for $\gamma \le 1$ and strictly decreasing on $(\varepsilon', 1)$ for $\gamma > 1$. Moreover,*

$$x'_\gamma(\varepsilon) = \frac{x_\gamma(\varepsilon) + y_\gamma(\varepsilon)}{\theta'(x_\gamma(\varepsilon)) - \varepsilon} < 0$$

for all γ and ε, satisfying (4.40) or (4.41).

4.55. Proposition. *For $\gamma \in \mathbf{R}$ we have*

$$\lim_{\varepsilon \to 1-0} x_\gamma(\varepsilon) = 0.$$

Proof. Let ε and γ satisfy (4.40) or (4.41). Then

$$0 < y_\gamma(\varepsilon) = s \le r.$$

Letting $\varepsilon \to 1-0$ we obtain

$$\lim_{\varepsilon \to 1-0} y_\gamma(\varepsilon) = 0.$$

Show that

$$\lim_{\varepsilon \to 1-0} x_\gamma(\varepsilon) = 0.$$

Indeed, suppose that this is not true, that is, there is a number $\varepsilon_0 \in (0,1)$ and a sequence $\varepsilon_i \to 1$ ($\varepsilon_0 < \varepsilon_i < 1$) such that the inequality

$$c < x_\gamma(\varepsilon_i) \le x_\gamma(\varepsilon_0).$$

holds with some constant $c > 0$. We can consider that

$$\lim_{\varepsilon_i \to 1} x_\gamma(\varepsilon_i) = a \in [c, x_\gamma(\varepsilon_0)].$$

We have

$$1 = \lim_{\varepsilon_i \to 1} \varepsilon_i = \lim_{\varepsilon_i \to 1} I^+_\gamma(x_\gamma(\varepsilon_i), y_\gamma(\varepsilon_i)) = I^+_\gamma(a, 0) = \sigma(a).$$

From here, $a = 0 < c$ and we obtain a contradiction. □

4.56. Proposition. *For all $\gamma \le 1$ we have*

$$\lim_{\varepsilon \to 0+} x_\gamma(\varepsilon) = +\infty.$$

Proof. Letting $\varepsilon \to 0+$ in the inequality $x_\gamma(\varepsilon) \ge r$, we obtain required . □

4.57. Proposition. *a) If $\gamma \in (-\infty, -1]$, then*

$$\lim_{\varepsilon \to 0+} x_\gamma(\varepsilon)\varepsilon^{-\alpha} = 0 \qquad \textit{for every} \quad \alpha < \tfrac{\gamma-1}{2}.$$

b) If $\gamma \in (-1,1)$*, then*

$$\lim_{\varepsilon\to 0+} x_\gamma(\varepsilon)\varepsilon = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2-2\gamma}}.$$

c) If $\gamma = 1$*, then*

$$\lim_{\varepsilon\to 0+} x_\gamma(\varepsilon)\varepsilon = \exp\{-\frac{1}{2}\}.$$

Proof. a) Let $\gamma < -1$. Using the inequalities $0 < y_\gamma(\varepsilon) \le r$, we obtain

$$\lim_{\varepsilon\to 0+} y_\gamma(\varepsilon)\varepsilon^{-\alpha} = 0 \qquad \text{for every} \quad \alpha < \tfrac{\gamma-1}{2} \tag{4.58}$$

We set

$$\mu(t) = \left(1 - \frac{\gamma+1}{2}t^2\right)\left(1 - \frac{\gamma-1}{2}t^2\right)^{-1}.$$

Obviously,

$$\lim_{t\to+\infty} \mu(t) = \frac{\gamma+1}{\gamma-1}.$$

It is easy to see the function $\mu(t)$ is strictly decreasing on $(0,+\infty)$. Therefore

$$\mu(t) > \frac{\gamma+1}{\gamma-1} \qquad \text{for all } t \ge 0.$$

Next,

$$\varepsilon = \theta'(y_\gamma(\varepsilon)) = \left(1 - \frac{\gamma-1}{2}y_\gamma^2(\varepsilon)\right)^{\frac{1}{\gamma-1}-1}\left(1 - \frac{\gamma+1}{2}y_\gamma^2(\varepsilon)\right) > \sigma(y_\gamma(\varepsilon))\frac{\gamma+1}{\gamma-1}.$$

From here,

$$1 \le \frac{\sigma(y_\gamma(\varepsilon))}{\varepsilon} \le \frac{\gamma-1}{\gamma+1}. \tag{4.59}$$

We notice that the equation $I_\gamma^+(x,y) = \varepsilon$ we can write as

$$x\left(\frac{\sigma(x)}{\varepsilon} - 1\right) = y\left(\frac{\sigma(y)}{\varepsilon} - 1\right).$$

Then by (4.58), (4.59) for all $\alpha < \frac{\gamma-1}{2}$ we have

$$0 = \lim_{\varepsilon\to 0+} y_\gamma(\varepsilon)\varepsilon^{-\alpha}\left(\frac{\sigma(y_\gamma(\varepsilon))}{\varepsilon} - 1\right) = \lim_{\varepsilon\to 0+} x_\gamma(\varepsilon)\varepsilon^{-\alpha}\left(1 - \frac{\sigma(x_\gamma(\varepsilon))}{\varepsilon}\right). \tag{4.60}$$

Assume that there is $\alpha < \frac{\gamma-1}{2}$ such that

$$\lim_{\varepsilon\to 0+} x_\gamma(\varepsilon)\varepsilon^{-\alpha} \ne 0.$$

Then for some sequence $\varepsilon_i \to 0$ of positive numbers the inequality

$$x_\gamma(\varepsilon_i)\varepsilon_i^{-\alpha} \ge m \tag{4.61}$$

holds with some constant $m > 0$.

By (4.60) we obtain

$$\lim_{\varepsilon_i \to 0+} \frac{\sigma(x_\gamma(\varepsilon_i))}{\varepsilon_i} = 1.$$

By (4.61),

$$\lim_{\varepsilon_i \to 0+} \frac{\sigma(x_\gamma(\varepsilon_i))}{\varepsilon_i} \le \lim_{\varepsilon_i \to 0+} \frac{\sigma(m\varepsilon_i^\alpha)}{\varepsilon_i} = 0.$$

and we obtain a contradiction.

Let $\gamma = -1$. We have

$$\varepsilon = \theta'(y_\gamma(\varepsilon)) = \left(1 + y_\gamma^2(\varepsilon)\right)^{-\frac{3}{2}}.$$

From here

$$y_\gamma(\varepsilon) = \sqrt{\varepsilon^{-\frac{2}{3}} - 1}$$

and

$$\sigma(y_\gamma(\varepsilon)) = \varepsilon^{\frac{1}{3}}.$$

For $\alpha < -1$ we have

$$\begin{aligned} \lim_{\varepsilon \to 0+} x_\gamma(\varepsilon)\varepsilon^{-\alpha}\left(1 - \frac{\sigma(x_\gamma(\varepsilon))}{\varepsilon}\right) &= \lim_{\varepsilon \to 0+} y_\gamma(\varepsilon)\varepsilon^{-\alpha}\left(\frac{\sigma(y_\gamma(\varepsilon))}{\varepsilon} - 1\right) \\ &= \lim_{\varepsilon \to 0+} \varepsilon^{-\alpha-1}(1 - \varepsilon^{2/3})^{3/2} = 0. \end{aligned}$$

Assume that there exits $\alpha < -1$ such that

$$\lim_{\varepsilon \to 0+} x_\gamma(\varepsilon)\varepsilon^{-\alpha} \neq 0.$$

Then for some sequence $\varepsilon_i \to 0$ of positive numbers the inequality (4.61) holds with some constant $m > 0$. As above we obtain a contradiction.

b) By Proposition 4.56 we deduce

$$\lim_{\varepsilon \to 0+} \theta(x_\gamma(\varepsilon)) = 0. \tag{4.62}$$

Notice that the function $\theta'(t)$ is continuous and the equation

$$\theta'(t) = 0$$

has the unique solution $s = \sqrt{\frac{2}{\gamma+1}}$. Then the equality

$$\theta'(y_\gamma(\varepsilon)) = \varepsilon$$

yields

$$\lim_{\varepsilon \to 0+} y_\gamma(\varepsilon) = \sqrt{\frac{2}{\gamma + 1}}. \tag{4.63}$$

By (4.62), (4.63) we obtain

$$\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2-2\gamma}} = \theta\left(\sqrt{\frac{2}{\gamma+1}}\right) = \lim_{\varepsilon\to 0+}(\theta(y_\gamma(\varepsilon)) - y_\gamma(\varepsilon)\varepsilon) =$$

$$= \lim_{\varepsilon\to 0+}(x_\gamma(\varepsilon)\varepsilon - \theta(x_\gamma(\varepsilon))) = \lim_{\epsilon\to 0+} x_\gamma(\varepsilon)\varepsilon.$$

c) The proof is analogous. □

4.64. Proposition. *a) If $\gamma \neq 1$, then*

$$\lim_{\varepsilon\to 1-0} \frac{x_\gamma(\varepsilon)}{(1-\varepsilon)^\alpha} = +\infty \qquad \text{for all } \alpha > \tfrac{1}{2}. \tag{4.65}$$

b) If $\gamma = 1$, then

$$\lim_{\varepsilon\to 1-0} \frac{x_\gamma(\varepsilon)}{\ln^\alpha \varepsilon} = +\infty \qquad \text{for all } \alpha > \tfrac{1}{2}.$$

Proof. a) Assume that $\gamma > 1$. Then

$$x_\gamma(\varepsilon) \geq r = \sqrt{\frac{2(1-\varepsilon^{\gamma-1})}{\gamma-1}}$$

Using L'Hospital rule, we find

$$\lim_{\varepsilon\to 1-0} \frac{1-\varepsilon^{\gamma-1}}{(1-\varepsilon)^{2\alpha}} = \frac{\gamma-1}{2\alpha} \lim_{\varepsilon\to 1-0} \frac{\varepsilon^{\gamma-2}}{(1-\varepsilon)^{2\alpha-1}} = +\infty \qquad \text{for all } \alpha > \tfrac{1}{2}.$$

From thies we obtain (4.65). The case $\gamma < 1$ is analogous.
b) The proof is analogous. □

Address :
Department of Mathematics
2 Prodolnaya 30, 400062 Volgograd
Volgograd State University
RUSSIA

Vladimir Klyachin : E-mail: `klchnv@mail.ru`

Alexey Kochetov : E-mail: `kochetov.alexey@mail.ru`

Vladimir Miklyukov : E-mail: `miklyuk@mail.ru`

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