On quasiplanes in $\mathbf{R}^n$ with codimension > 1

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Abstract. Some geometric properties of images of k-dimensional planes under quasiconformal maps of the n-dimensional Euclidean space are proved when $1 \le k \le n-2$.

1 Main Results

Below we use the terminology and notation of [4].

Let $\mathbf{R}^n$ be the $n-$dimensional Euclidean space, $n \ge 2$, let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a mapping of the class $W^1_{n,\mathrm{loc}}(\mathbf{R}^n)$, and let $f' : \mathbf{R}^n \to \mathbf{R}^n$ denote its formal derivative. We put

$$\|f'(x)\| = \max_{|h|=1} |f'(x)\, h| .$$

An injective map $f : \mathbf{R}^n \to \mathbf{R}^n$ is called K-*quasiconformal* [7, p. 250-252] if $f \in W^1_{n,\mathrm{loc}}(\mathbf{R}^n)$ and

$$\|f'(x)\|^n \le K\, J(x,f) \quad \text{a.e. on} \quad \mathbf{R}^n, \tag{1.1}$$

where $J(x,f) = \det(f'(x))$ is the Jacobian of f at the point $x \in \mathbf{R}^n$.

The smallest constant K at (1.1) is called the exterior dilatation of f and denoted by $K_O(f)$. The best constant $K \ge 1$ in the inequality

$$J(x,f) \le K\, \ell(f'(x))^n$$

is called the inner dilatation of $f : \mathbf{R}^n \to \mathbf{R}^n$ and denoted by $K_I(f)$. Here, $\ell(f'(x)) = \min_{|h|=1} |f'(x)\, h|$.

The quantity

$$K(f) = \max\{K_O(f), K_I(f)\}$$

is called the maximal dilatation of f [7, Section **14.1**].

If $f : \mathbf{R}^n \to \mathbf{R}^n$ is quasiconformal, then it is well-known that $f(\mathbf{R}^n) = \mathbf{R}^n$ and the inverse map $f^{-1} : \mathbf{R}^n \to \mathbf{R}^n$ is also quasiconformal in $\mathbf{R}^n$ with $K(f^{-1}) = K(f)$.

Let $d(x', x'')$ be the distance defined by

$$d(x', x'') = \left(\sum_{k=1}^{n} (x'_k - x''_k)^2 \right)^{1/2} .$$

We put

$$B(a,t) = \{x \in \mathbf{R}^n : d(a,x) < t\}, \quad S(a,t) = \{x \in \mathbf{R}^n : d(a,x) = t\}\,.$$

Let $y = (y_1, y_2, ..., y_n)$ be a point in $\mathbf{R}^n$ and $1 \le k \le n-2$. Consider a k-dimensional plane

$$\Pi_0^k = \{y = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n : \quad y_{k+1} = \ldots = y_n = 0\}\,.$$

Let Π^k be a k-dimensional surface in $\mathbf{R}^n$, i.e. a homeomorphic image of a k-dimensional plane. Fix a point $a \in \Pi^k$ and R, $0 < R < \infty$. Denote $\Pi^k(a,R)$ a connected component of $\Pi^k \cap B(a,R)$, $a \in \Pi^k(a,R)$. We say that a k-dimensional surface Π^k is a K-*quasiplane*, if there exists a K-quasiconformal mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ such that $f(\Pi^k) = \Pi_0^k$.

For $n = 2$, the following result is well known [3]. *A curve $\Pi^1 \subset \mathbf{R}^2$ is a K-quasiconformal line if and only if there exists a constant $C(K)$ such that*

$$\left|\frac{\zeta_2 - \zeta_1}{\zeta_3 - \zeta_1}\right| \le C(K)$$

for arbitrary three distinct points on Π^1 such that ζ_2 lies between ζ_1 and ζ_3.

For more information on quasiconformal lines, we refer to [5], [1, Chapter 14], [6], [8], [13], [9]. Quasiplanes Π^{n-1} in $\mathbf{R}^n$ with $\operatorname{codim} \Pi^{n-1} = 1$ were considered in [10] and, in the setup of Riemannian manifolds, in [11].

Here we study the case of k-quasiplanes in $\mathbf{R}^n$, $1 \le k \le n-2$.

For an arbitrary open subset $\Sigma \subset S(a,r) \subset \mathbf{R}^n$ we define the quantity

$$\eta(\Sigma) = \sup_{\alpha} \inf \frac{\left(\int_{\Sigma} |\nabla_S \varphi|^n d\mathcal{H}^{n-1}\right)^{1/n}}{\left(\int\limits_{\Sigma} |\varphi - \alpha|^n d\mathcal{H}^{n-1}\right)^{1/n}}\,. \tag{1.2}$$

Here the infimum is taken over all functions

$$\varphi \in W_n^1(\Sigma), \quad \varphi|_{\partial\Sigma} = 0\,,$$

$\nabla_S \varphi$ is the gradient of φ on $S(a,r)$ and the supremum is taken over all constants α.

1.3. Theorem. *Let Π^k be a K-quasiplane in $\mathbf{R}^n$ with $1 \le k \le n-2$. Then for every point $a \in \Pi^k$ and arbitrary numbers r, R, $0 < r < R < \infty$, the following relation holds*

$$\exp\left\{c(n,K)\int\limits_r^R \eta(\Sigma(a,\tau))\,d\tau\right\} \le D^{2n}(n,K)\left(\frac{R}{r}\right)^{n\beta}, \quad \beta = K^{1/(n-1)}\,, \tag{1.4}$$

Here $c(n,K)$, $D(n,K)$ are some positive constants depending only on the indicated parameters.

2 The language of differential forms

Let $1 \leq k \leq n-2$ and let

$$f = (f_1, \ldots, f_k, f_{k+1}, \ldots, f_n) : \mathbf{R}^n \to \mathbf{R}^n$$

be a K-quasiconformal mapping.

Let Π_0 be a k-dimensional plane of the form

$$\Pi_0 = \{y = (y_1, \ldots, y_k, y_{k+1}, \ldots, y_n) : y_{k+1} = \ldots = y_n = 0\}\,.$$

We will study properties of the k-dimensional surface

$$\Pi^k = f^{-1}(\Pi_0)\,.$$

Fix a point $a \in \Pi$. For an arbitrary R, $0 < R < \infty$ let $\Pi(a,R)$ be the connected component of $\Pi^k \cap B(a,R)$, containing the point a, $\Sigma(R) = S(R) \setminus \overline{\Pi(a,R)}$.

We consider the differential form

$$\omega = \sum_{i=k+1}^{n} (-1)^{i-k-1} f_i\, df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_n\,.$$

It is clear that for a.e. $R \in (0,\infty)$, we have

$$\omega \in W^{1,1}_{\mathrm{loc}}(S)(a,R)) \quad \text{and} \quad \omega|_{\partial\Sigma(R)} = 0\,.$$

We have

$$\begin{aligned} d\omega &= \textstyle\sum_{i=k+1}^{n} (-1)^{i-k-1} df_i\, df_{k+1} \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_n = \\ &= (n-k) df_1 \wedge \ldots \wedge df_n\,. \end{aligned}$$

Thus,

$$*d\omega = (n-k)\, J(x,f) \geq 0\,.$$

We write

$$\varepsilon_f(a,R) = \sup_{\omega_0} \int\limits_{\Sigma(a,R)} (*d\omega)\, d\mathcal{H}^{n-1} \Bigg/ \int\limits_{\Sigma(a,R)} |\omega - \omega_0|\ ,$$

where the supremum is taken over all weakly closed $(n-1)$-forms $\omega_0 \in W^{1,1}_{\mathrm{loc}}(S(a,R))$.

Denote

$$V(a,r) = \int\limits_{B(a,r)} J(x,f)\, d\mathcal{H}^n\,.$$

Because $\mathcal{H}^n(\Pi_0) = 0$ we see that $\mathcal{H}^n(\Pi^k) = 0$ and for a.e. $r > 0$ we have

$$\mathcal{H}^{n-1}(\Pi^k \cap S(a,r)) = 0\,.$$

Thus,

$$\int\limits_{\Sigma(r)} \omega_0 = \int\limits_{S(a,r)\setminus\Pi^k} \omega_0 = \int\limits_{S(a,r)} \omega_0 \,.$$

For an arbitrary form ω_0 and almost all $0 < R < \infty$ the following relations hold

$$\begin{aligned}\int\limits_{\Sigma(R)} |\omega - \omega_0| \ &\geq \int\limits_{S(a,R)} (\omega - \omega_0) = \\ &= \int\limits_{B(a,R)} d(\omega - \omega_0) = \int\limits_{B(a,R)} d\omega = \int\limits_{B(a,R)} (*d\omega) * 1 = \\ &= (n-k) \int\limits_{B(a,R)} J(x,f)\, d\mathcal{H}^n = (n-k) V(a,R)\,.\end{aligned}$$

(It is the standard Gauss formula for functions with generalized derivatives.)

Fix $0 < k < 1$ and choose ω_0 such that

$$k\,\varepsilon_f(a,R) \leq \int\limits_{\Sigma(a,R)} (*d\omega)\, d\mathcal{H}^{n-1} \Bigg/ \int\limits_{\Sigma(a,R)} |\omega - \omega_0| \ .$$

Thus,

$$k\,\varepsilon_f(a,R) \leq \int\limits_{\Sigma(a,R)} J(x,f)\, d\mathcal{H}^n \Bigg/ V(a,R)$$

and setting $k \to 1$ we obtain

$$\varepsilon_f(a,R) \leq \int\limits_{\Sigma(a,R)} J(x,f)\, d\mathcal{H}^n \Bigg/ V(a,R)\,.$$

Since $|\nabla d(a,x)| \equiv 1$ we arrive at the following relation

$$\int\limits_{\Sigma(a,R)} J(x,f)\, d\mathcal{H}^{n-1} = V'(a,R) \geq \varepsilon_f(a,R)\, V(a,R)\,,$$

which is true for a.e. $R \in (0,\infty)$.

The function $t \to V(a,t)$ is absolutely continuous on $[r,R]$ and solving this differential inequality we have

$$\exp\left\{ \int\limits_r^R \varepsilon_f(\tau)\, d\tau \right\} \leq \frac{V(a,R)}{V(a,r)} \tag{2.1}$$

where $0 < r < R < \infty$ are arbitrary.

We put

$$l(a,t) = \min_{x\in S(a,t)} |f(x) - f(a)| \quad L(a,t) = \max_{x\in S(a,t)} |f(x) - f(a)|\,.$$

However,

$$V(a,r) \geq \Omega_n\, l^n(a,r)$$

and

$$V(a,R) \leq \Omega_n\, L^n(a,R)\,,$$

where

$$\Omega_n = \mathcal{H}^n\left(B(0,1)\right)\,.$$

Hence,

$$\frac{V(a,R)}{V(a,r)} \leq \frac{L^n(a,R)}{l^n(a,r)}\,. \tag{2.2}$$

We use the following bound for quasiconformal maps.

2.3. Theorem. *Let $f:\mathbf{R}^n \to \mathbf{R}^n$ be a K-quasiconformal map. Then for every point $a \in \mathbf{R}^n$ and an arbitrary ρ, $0<\rho<\infty$ the following estimate holds*

$$\frac{L(a,\rho)}{l(a,\rho)} \leq D_*\,, \quad D_* = D_*(K) = \exp\left(4K(K+1)\sqrt{K-1}\right). \tag{2.4}$$

For a proof see, [1, Theorem **14.8**]. Results of this type are well-known.

Using (2.4), we see that

$$\frac{L(a,R)}{l(a,r)} = \frac{L(a,R)}{l(a,R)}\,\frac{l(a,R)}{L(a,r)}\,\frac{L(a,r)}{l(a,r)} \leq D_*^2\,\frac{l(a,R)}{L(a,r)}\,, \quad D_* = D_*(K)\,. \tag{2.5}$$

We next show that if $\beta = K^{1/(n-1)}$, then for $0<r<R$

$$\frac{l(a,R)}{L(a,r)} \leq \left(\frac{R}{r}\right)^{\beta}\,. \tag{2.6}$$

Indeed, because the case $l(a,R) \leq L(a,r)$ is clear, we may assume that $l(a,R) > L(a,r)$. Applying the well-known formula for the n-capacity of a spherical condenser in $\mathbf{R}^n$, we obtain

$$\begin{aligned}
\omega_{n-1}\left(\ln\frac{l(a,R)}{L(a,r)}\right)^{1-n} &= \mathrm{cap}\,\left(B(f(a),L(a,r)),B(f(a),l(a,R))\right)\\
&\geq \mathrm{cap}\,\left(f\overline{B}(a,r), fB(a,R)\right)\\
&\geq \tfrac{1}{K}\,\mathrm{cap}\,\left(\overline{B}(a,r),B(a,R)\right) =\\
&= \tfrac{1}{K}\,\omega_{n-1}\left(\ln\tfrac{R}{r}\right)^{1-n}\,,
\end{aligned}$$

where ω_{n-1} is the $(n-1)$-dimensional area of the boundary of $B(0,1)$.

From this inequality we arrive at (2.6).

Thus, by (2.5) and (2.6) we obtain

$$\frac{L(a,R)}{l(a,r)} \leq D_*^2 \left(\frac{R}{r}\right)^{\beta} . \tag{2.7}$$

Using (2.5), we have now

$$\frac{V(a,R)}{V(a,r)} \leq D_*^{2n} \frac{l^n(a,R)}{L^n(a,r)} ,$$

and further by (2.6),

$$\frac{V(a,R)}{V(a,r)} \leq D_*^{2n} \left(\frac{R}{r}\right)^{n\beta} .$$

The estimate (2.1) implies the following statement.

2.8. Theorem. *Let Π^k be a K-quasiplane in $\mathbf{R}^n$ with $1 \leq k \leq n-2$. Then for every point $a \in \Pi^k$ and arbitrary numbers r, R, $0 < r < R < \infty$, we have*

$$\exp\left\{\int_r^R \varepsilon_f(a,\tau)\, d\tau\right\} \leq D_*^{2n}(K) \left(\frac{R}{r}\right)^{n\beta} , \quad \beta = K^{1/(n-1)} . \tag{2.9}$$

3 Proof of Main Theorem

Fix a K-quasiconformal map $f : \mathbf{R}^n \to \mathbf{R}^n$ with $f(\Pi^k) = \Pi_0^k$. We shall estimate $\varepsilon_K(R)$ by $\eta(\Sigma(R))$. Setting

$$\omega_0 = \sum_{i=k+1}^{n} \alpha_i\, df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_n , \quad \varphi = \sum_{i=k+1}^{n} \left| df_1 \wedge \ldots \widehat{df_i} \ldots df_n \right|^2 ,$$

where α_i are arbitrary constants, we have

$$\varepsilon_f(a,R) \geq \int_{\Sigma(R)} *d\omega \Bigg/ \int_{\Sigma(R)} |\omega - \omega_0| . \tag{3.1}$$

Observe that

$$\left|\int\limits_{\Sigma(R)} \omega - \omega_0\right| \leq \int\limits_{\Sigma(R)} |\omega - \omega_0|\, d\mathcal{H}^{n-1}$$

$$\leq \int\limits_{\Sigma(R)} \sqrt{\sum_{i=k+1}^{n} (f_i - \alpha_i)^2} \sqrt{\sum_{i=k+1}^{n} \left|df_1 \wedge \ldots \widehat{df_i} \ldots df_n\right|^2}\, d\mathcal{H}^{n-1}$$

$$\leq \left(\int\limits_{\Sigma(R)} \left(\sum_{i=k+1}^{n} (f_i - \alpha_i)^2\right)^{n/2} d\mathcal{H}^{n-1}\right)^{1/n} \left(\int\limits_{\Sigma(R)} (\varphi)^{n/(2(n-1))}\right)^{(n-1)/n}.$$

It is not difficult to check that

$$\int\limits_{\Sigma(R)} (\varphi)^{n/(2(n-1))}\, d\mathcal{H}^{n-1} \leq c_3(n,K) \int\limits_{\Sigma(R)} |f'(x)|^n \, d\mathcal{H}^{n-1},$$

where $c_3(n,K)$ is a constant.

Next we see that

$$\begin{aligned}\int\limits_{\Sigma(R)} \left(\sum_{i=k+1}^{n} (f_i - \alpha_i)^2\right)^{n/2} d\mathcal{H}^{n-1} &\leq c_4(n) \sum_{i=k+1}^{n} \left(\int\limits_{\Sigma(R)} |f_i - \alpha_i|^2\, d\mathcal{H}^{n-1}\right)^{n/2} \\ &\leq c_4(n)\eta^{-n}(\Sigma(R)) \sum_{i=k+1}^{n} \int\limits_{\Sigma(R)} |\nabla f_i|^n \, d\mathcal{H}^{n-1} \\ &\leq c_5(n,K)\, \eta^{-n}(\Sigma(R)) \int\limits_{\Sigma(R)} |f'(x)|^n\, d\mathcal{H}^{n-1}.\end{aligned}$$

Substituting these estimates into (3.1), we have the inequality

$$\varepsilon_f(a,R) \geq c_5(n,K)\eta(\Sigma(R)) \int\limits_{\Sigma(R)} J(x,f)\, d\mathcal{H}^{n-1} \Bigg/ \int\limits_{\Sigma(R)} |f'(x)|^n\, d\mathcal{H}^{n-1}.$$

Thus, we obtained

$$\varepsilon_f(a,R) \geq c_6(n,K)\, \eta(\Sigma(R)) \tag{3.2}$$

and using Theorem 2.8, we prove Theorem 1.3.

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