QUASIREGULAR MAPPINGS FROM A PUNCTURED BALL INTO COMPACT MANIFOLDS

PEKKA PANKKA

ABSTRACT. We study quasiregular mappings from a punctured unit ball of the Euclidean n-space into compact manifolds. We show that a quasiregular mapping has a limit in the point of punctuation whenever the dimension of the cohomology ring of the compact manifold exceeds a bound given in terms of the dimension and the distortion constant of the mapping.

1. INTRODUCTION

In [2], Bonk and Heinonen prove the following theorem.

Theorem 1 ([2, Theorem 1.1]). *Let N be a closed, connected, and oriented Riemannian n-manifold, $n \geq 2$. If there exists a non-constant K-quasiregular mapping $f\colon \mathbb{R}^n \to N$, then*

$$\dim H^*(N) \leq C(n, K), \tag{1}$$

where $\dim H^*(N)$ *is the dimension of the de Rham cohomology ring $H^*(N)$ of N and $C(n, K)$ is a constant depending on n and K.*

In this paper, we show that Theorem 1 has a local counterpart.

Theorem 2. *Let $n \geq 2$ and $K \geq 1$. There exists a constant $C(n, K)$ such that if N is a closed, connected, and oriented Riemannian n-manifold with* $\dim H^*(N) \geq C(n, K)$ *and $f\colon B^n \setminus \{0\} \to N$ is a K-quasiregular mapping, then the limit* $\lim_{x\to 0} f(x)$ *exists.*

A continuous mapping $f\colon M \to N$ between connected and oriented Riemannian n-manifolds M and N is called *K-quasiregular*, if it belongs to the class $W^{1,n}_{\mathrm{loc}}(M, N)$, and satisfies an inequality

$$\|T_x f\|^n \leq K J(x, f)$$

for almost every $x \in M$. Here $\|T_x f\|$ is the operator norm of the tangent mapping $T_x f\colon T_x M \to T_{f(x)} N$, and $J(x, f)$ is the Jacobian determinant of f at $x \in M$ uniquely defined by the equation $(f^*(\mathrm{vol}_N))_x =$

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$J(x,f)\,(\mathrm{vol}_M)_x$ for almost every $x \in M$. It is well known that quasiregular mappings are almost everywhere differentiable, and that non-constant quasiregular mappings are open and discrete. See e.g. [17] for thorough exposition of properties and history of quasiregular mappings.

The theorem of Bonk and Heinonen can be considered as a Picard type theorem for quasiregular mappings. In that respect Theorem 2 can be viewed as a big Picard type theorem. In order to justify this terminology, let us describe how Theorem 2 implies Theorem 1. Let N be a compact n-manifold with $\dim H^*(N) \ge C(n,K)$, where $C(n,K)$ is as in Theorem 2, and suppose that there exists a non-constant K-quasiregular mapping f from $\mathbb{R}^n$ into N. By another theorem of Bonk and Heinonen, [2, Theorem 1.11], the average of the counting function of f has a lower growth bound of polynomial type, see also Section 5. In particular, the average of the counting function is unbounded. On the other hand, f can be extended to a K-quasiregular mapping from S^n into N by Theorem 2. Hence the average of the counting function of f is in fact bounded. This is a contradiction and Theorem 1 follows.

The proof of Theorem 2 is divided into three parts: the case of Riemannian surfaces, a bound for the first cohomology, and bounds for the higher cohomologies. We consider first the case of Riemannian surfaces. For Riemannian surfaces the claim of Theorem 2 is settled by the following theorem.

Theorem 3. *Let N be a closed Riemannian surface and let $f\colon B^2 \setminus \{0\} \to N$ be a K-quasiregular mapping. If $\dim H^1(N) > 2$, f has a limit at the origin.*

This theorem is an easy consequence of the big Picard theorem for Riemannian surfaces and the Measurable Riemann mapping theorem, see Section 2 for details. Although this result is well known to the experts, we provide a simple proof.

In the same section we show how the method of [2, Corollary 1.6] yields a bound for $H^1(N)$ when the dimension of N is at least three.

Theorem 4. *Let $n \ge 3$ and N be a closed, connected, and oriented Riemannian n-manifold such that $\dim H^1(N) > n$. Then every quasiregular mapping $f\colon B^n \setminus \{0\} \to N$ has a limit at the origin.*

Together with Poincaré duality, Theorem 4 yields the claim of Theorem 2 for quasiregular mappings from punctured ball into compact three manifolds. By theorems 3 and 4, we may take $C(n,K) = 2^n$, if $n = 2$ or $n = 3$, in Theorem 2.

Having Theorems 3 and 4 at our disposal, it is sufficient to consider compact manifolds of dimension $n \ge 4$ with non-trivial cohomology in some of the dimensions $\ell \in \{2,\dots,n-2\}$. We settle this part of the claim of Theorem 2 with the following theorem.

Theorem 5. *Let $n \geq 4$ and $K \geq 1$. There exists a constant $C > 0$ depending only on n and K such that if N is a closed, connected, and oriented Riemannian n-manifold and $f\colon \mathbb{R}^n \setminus \bar{B}^n \to N$ is a K-quasiregular mapping with an essential singularity at infinity, then $\dim H^\ell(N) \leq C$ for every $\ell \in \{2, \ldots, n-2\}$.*

Indeed, we may fix a sense-preserving Möbius mapping $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\sigma(\mathbb{R}^n \setminus \bar{B}^n) = B^n \setminus \{0\}$, and instead of a K-quasiregular mapping $f\colon B^n \setminus \{0\} \to N$ we may consider the mapping $f \circ \sigma$ in Theorem 2.

Although the main idea of the proof of Theorem 5 is the same as in the proof of Theorem 1, we apply the methods of [2] differently. Let us briefly describe the method and then differences. Let N be an n-manifold with $H^\ell(N) \neq 0$ for some $\ell \in \{2, \ldots, n-2\}$, $d = \dim H^\ell(N)$ and $p = n/\ell$. We fix p-harmonic forms $\xi_1, \ldots, \xi_d$ whose cohomology classes span $H^\ell(N)$ and which are uniformly bounded and separated in the L^p-norm. The value distribution theory of quasiregular mappings can now be used to show that there exists a quasiregular mapping ψ from the unit ball B^n of $\mathbb{R}^n$ into $\mathbb{R}^n \setminus \bar{B}^n$ such that forms $\psi^* f^*(\xi_i)$ are uniformly separated in the L^p-norm over $B^n(1/2)$, and uniformly bounded in B^n with constants depending only on n and K. Furthermore, the dilatation of ψ depends only on n. The conclusion that d has a bound depending only on n and K now follows from a Caccioppoli type inequality and compactness of the L^p-Poincaré homotopy operator as in [2].

In the proof of Theorem 1, the use of an auxiliary mapping ψ is not necessary. Since the natural exhaustion of $\mathbb{R}^n$ is with balls centered at the origin, an argument using rescaling and the Mattila-Rickman theorem [11, 5.11] shows that there exist many balls B in $\mathbb{R}^n$ such that forms $f^*(\xi_i)$ are uniformly bounded and separated in the L^p-norm on each ball B. On the other hand, a natural family of domains exhausting $\mathbb{R}^n \setminus \bar{B}^n$, from the point of view of the value distribution, is not a family of Euclidean balls but a family of spherical annuli. In Section 3 we show, using [11, 4.8], that there exists a version of [11, 5.11] for quasiregular mappings from $\mathbb{R}^n \setminus \bar{B}^n$ into a compact manifold with an essential singularity at infinity.

We find a mapping ψ in three steps. First we show that the assumption on the cohomology yields a lower growth bound for the average of the counting function of a quasiregular mapping with an essential singularity at the origin. This theorem corresponds to [2, Theorem 1.11] in our setting, see Section 5. The second step is to replace the use of the rescaling argument with a value distribution lemma from [16]. The last step is to modify a ball decomposition method due to Rickman, see e.g [16] or [17, V.2.14], into an annulus decomposition method. Using the obtained decomposition, we find the mapping ψ.

In the annulus decomposition method we need the average of the counting function of the mapping to have a polynomial type growth. In Section 5 we show that mappings considered in Theorem 5 indeed have this property. Theorem 14 yielding this result is a counterpart of [2, Theorem 1.11] in our setting. We also give an example indicating that the more restrictive assumption on cohomology cannot be relaxed.

The notation used in this paper is standard. Given a Riemannian manifold M, we denote by $B(z,r)$ the open ball centered at $z \in M$ with radius $r > 0$. In $\mathbb{R}^n$ we denote by B^n the open unit ball centered at the origin, and by $B^n(r)$ open ball of radius $r > 0$ centered at the origin.

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2. Bounds for $\dim H^1(N)$

In this section we first consider the proof of Theorem 3 and then the proof of Theorem 4.

In two dimensions we have two well known theorems at our disposal: the big Picard theorem for Riemannian surfaces and the measurable Riemann mapping theorem. Although Theorem 3 is almost a direct consequence of these two theorems, we give a short proof. The big Picard theorem for Riemann surfaces goes back to Picard [14]; see also Ohtsuka ([12] and [13]), Renggli [15], and Royden [18]. For the Measurable Riemann mapping theorem and for notation in the proof, see e.g. [1].

Proof of Theorem 3. Let us first factorize f into a quasiconformal and an analytic mapping. We denote by $\mathbb{D}^*$ the punctured disk $B^2 \setminus \{0\}$. Let $\mu\colon \mathbb{C} \to \mathbb{C}$ be a measurable mapping such that $\mu(z) = \mu_f(z)$ whenever defined in $\mathbb{D}^*$ and $\mu(z) = 0$ otherwise. Since $\|\mu\|_\infty \le (K-1)/(K+1) < 1$, there exists, by the measurable Riemann mapping theorem, a K-quasiconformal mapping $h\colon \mathbb{C} \to \mathbb{C}$ such that $h_{\bar{z}} = \mu h_z$ and $h(0) = 0$. Moreover, $f \circ h^{-1}$ is 1-quasiregular, and hence analytic, in $h(\mathbb{D}^*)$.

Since the Riemann surface N is not a sphere or a torus, the mapping $f \circ h^{-1}|h(\mathbb{D}^*)$ has a limit at the origin by the big Picard theorem for Riemann surfaces. □

In dimensions $n \ge 3$, the measurable Riemann mapping theorem is not at our disposal. In this case, the simply connectedness of $B^n \setminus \{0\}$ allows us to consider a lift of a mapping $f\colon B^n \setminus \{0\} \to N$ into the universal cover of N. The proof of Theorem 4 is essentially a recombination of the arguments given in [2, Corollary 1.6] and [22, Theorem X.1.5]. In the proof we denote by $|E|$ the Lebesgue measure of a measurable set E. We also employ *the conformal capacity of a condenser.* Let F be a compact set and Ω an open set on manifold N such that $F \subset \Omega$. Then

$$\mathsf{cap}_n(\Omega, F) = \inf_u \int_N |\nabla u|^n \,\mathrm{d}x,$$

where the infimum is taken over all functions $u \in C_0^\infty(\Omega)$ such that $u|F \geq 1$, see e.g. [17, II.10]. Furthermore, we say that N is *n-parabolic* if $\mathsf{cap}_n(N, F) = 0$ for every compact set $F \subset N$. Otherwise, we say that N is *n-hyperbolic.*

Proof of Theorem 4. Suppose $d = \dim H^1(N) > n$. Let $\tilde{N}$ be the universal cover of N with the induced Riemannian metric, and let $\tilde{f} \colon B^n \setminus \{0\} \to \tilde{N}$ be the lifting of f to $\tilde{N}$. Then $\tilde{f}$ is K-quasiregular. Fix a point $o \in \tilde{N}$. The proof of [2, Corollary 1.6] shows that there exist a constant $C > 0$ such that for every positive integer R we have

$$C|B(o, CR)| \geq (2R+1)^d.$$

By the proof of [22, Theorem X.1.5], $\tilde{N}$ is n-hyperbolic.

Suppose that f has an essential singularity at the origin. Let us fix a sequence $1 > r_1 > r_2 > \cdots$ such that $r_k \to 0$ as $k \to \infty$ and $|fC_k| \geq |N|/2$ for every k, where $C_k = \bar{B}^n(r_k) \setminus B^n(r_{k+1})$ for every k. Such a sequence can be fixed, since $N \setminus fB^n(r)$ has zero measure for every $r \in (0,1)$, see e.g. [5, Lemma 7].

Since the Riemannian metric of $\tilde{N}$ is inherited from N, $|\tilde{f}C_k| \geq |fC_k|$ for every k. Hence, by [3, Theorem 3.5], there exists $\delta > 0$ such that $\mathsf{cap}_n(\tilde{N}, \tilde{f}C_k) \geq \delta$ for every k. On the other hand, we obtain from the K_I-inequality that

$$\begin{aligned} \mathsf{cap}_n(\tilde{N}, \tilde{f}C_k) &\leq \mathsf{cap}_n(\tilde{f}(B^n \setminus \{0\}), \tilde{f}C_k) \\ &\leq K_I(f)\mathsf{cap}_n(B^n \setminus \{0\}, C_k) \to 0 \end{aligned}$$

as $k \to \infty$, see e.g. [17, II.10.10]. Here $K_I(f)$ is the inner dilatation of f, see e.g. [17, I.2]. Since this is a contradiction, f has a limit at the origin. □

3. A value distribution result of Mattila-Rickman type

In this section, our objective is to prove a version of [11, 5.11] that is suitable for our purposes. Let us first introduce some notation and terminology. Although it is sufficient for us to consider only quasiregular mappings from Euclidean domains into compact manifolds, we follow here [11, Section 2] in full generality.

Let M and N be connected and oriented Riemannian n-manifolds such that M is non-compact and N is compact. Furthermore, let $f \colon M \to N$ be a non-constant quasiregular mapping.

We say that a family of domains $\mathcal{D} = \{D(s) \subset M \colon s \in [a,b)\}$, where $a > 0$ and $b \in (a,\infty]$, is *an exhaustion of M* if every domain $D(s)$ is relatively compact, $\overline{D(s)} \subset D(t)$ for $a \le s < t < b$, and M is a union of the domains in $\mathcal{D}$. We may assume that domains in $\mathcal{D}$ are parameterized by an equation

$$\mathsf{M}_n(\Gamma_{ta}) = \omega_{n-1}\left(\log\frac{t}{a}\right)^{1-n},$$

where Γ_{ts} is the family of paths joining $\partial D(t)$ and $\partial D(s)$, i.e,

$$\Gamma_{ts} = \Delta(\partial D(t), \partial D(s); M)$$

for every $a \le s < t < b$, and ω_{n-1} is the Hausdorff $(n-1)$-measure of the unit sphere in $\mathbb{R}^n$. *The conformal modulus* $\mathsf{M}_n(\Gamma)$ of a path family Γ is defined by

$$\mathsf{M}_n(\Gamma) = \inf_\rho \int_M \rho^n \,\mathrm{d}x,$$

where infimum is taken over all non-negative Borel functions ρ on M such that

$$\int_\gamma \rho \,\mathrm{d}s \ge 1$$

for every locally rectifiable path $\gamma \in \Gamma$, see e.g. [17, II.1].

We say that an exhaustion $\mathcal{D}$ of M is *a $(\kappa,\lambda,\theta_0)$-admissible* if there exist constants $a_0 \in [a,b)$, $\theta_0 > 1$, $\kappa > 0$, and $\lambda \ge n-1$ such that

$$\mathsf{M}_n(\Gamma_{ts}) \le \kappa\omega_{n-1}\left(\log\frac{t}{s}\right)^{-\lambda}$$

for every $a_0 \le s < t < b$ satisfying $t \le \theta_0 s$.

For every relatively compact set $F \subset M$ we define *the counting function of f with respect to F* by

$$n(F,y;f) = \sum_{x \in f^{-1}(y)\cap\overline{F}} i(x;f).$$

Since the image of the branch set of f, i.e. the image of the set where f fails to be a local homeomorphism, has zero Lebesgue measure, $n(F,y;f) = \mathrm{card}\,(f^{-1}(y)\cap\overline{F})$ for almost every $y \in N$ and every relatively compact set $F \subset N$. Moreover, for every relatively compact set F the function $y \mapsto n(F,y;f)$ is upper semicontinuous.

Let μ be a finite non-trivial Borel measure on N. We say that $h\colon [0,\infty) \to [0,\infty)$ is *a calibration function (with constant $p > 2$)* if h is increasing, continuous, $h(0)=0$, $h(r)>0$ for every $r>0$, and

$$\int_0^1 \frac{h(r)^{1/(pn)}}{r}\,\mathrm{d}r < \infty.$$

Furthermore, we say that μ is *h-calibrated (with constant p)* if there exists $p > 2$ and a calibration function h (with constant p) such that

$$\mu(B(x,r)) \leq h(r)$$

for every ball $B(x,r) \subset N$.

For every relatively compact set $F \subset M$ we define *the average of the counting function of f with respect to measure μ* by

$$\nu_\mu(F;f) = \frac{1}{\mu(N)} \int_N n(F,y;f)\, \mathrm{d}\mu(y).$$

For the Lebesgue measure $\mathrm{d}x$ on N we abbreviate $A(F;f) = \nu_{\mathrm{d}x}(F;f)$, and for a fixed exhaustion $\mathcal{D}$ of M we let $A(s) = A(D(s);f)$ and $\nu_\mu(s) = \nu_\mu(D(s);f)$ for $s \in [a,b)$, when there is no possibility of confusion.

Let us now state the main theorem of this section which is a version of a part of [11, 5.11].

Theorem 6. *Let N be a closed, connected, and oriented Riemannian n-manifold and $f\colon \mathbb{R}^n \setminus \bar{B}^n \to N$ a K-quasiregular mapping with an essential singularity at the infinity. Let μ be an h-calibrated measure with $p > 2$ on N. Then there exists a set $E \subset [2,\infty)$ of finite logarithmic measure, i.e.*

$$m_{\log}(E) := \int_E \frac{\mathrm{d}t}{t} < \infty,$$

such that

$$\lim_{\substack{t\to\infty \\ t\notin E}} \frac{\nu_\mu(B^n(t) \setminus \bar{B}^n(2);f)}{A(B^n(t) \setminus \bar{B}^n(2);f)} = 1. \tag{2}$$

Although Theorem 6 is not a direct corollary of [11, 5.11], the main part of the proof is [11, 4.8], as in [11, 5.11]. In order to prove Theorem 6, we use [11, 4.8] to obtain Lemma 10 and then use the proof of [11, 5.11] to obtain Theorem 6. For reader's convenience, let us first formulate [11, 4.8]. For the sharp form of this theorem in the Euclidean setting, see [17, IV.1.7, IV.1.10].

Theorem 7 ([11, 4.8])**.** *Let M be a non-compact, connected, and oriented Riemannian n-manifold admitting a $(\kappa,\lambda,\theta_0)$-admissible exhaustion. Let N be a closed, connected, and oriented Riemannian n-manifold, and let μ be an h-calibrated measure with $p > 2$ on N. Then for every $c > 1$ and $K \geq 1$ there exists $d > 0$ such that the following holds. Let $\mathcal{D} = \{D(s)\colon s \in [a,b)\}$ be a $(\kappa,\lambda,\theta_0)$-admissible exhaustion on M and let $f\colon M \to N$ be a K-quasiregular mapping. Then*

$$cA(\theta s;f) \geq \nu_\mu(s;f) - d(\log\theta)^{-p\lambda} \tag{3}$$

and

$$A(s;f) \leq c\nu_\mu(\theta s;f) + d(\log\theta)^{-p\lambda} \tag{4}$$

whenever $a_0 \leq s < \theta s < b$ and $\theta \leq \theta_0$.

Let us note, that in [11] the formulation of [11, 4.8] does not contain the dependence of d on the other parameters. However, by the proof d depends only on c, K, κ, λ, θ_0, p, μ, and the Riemannian metric of N. In particular, d does not depend on f or $\mathcal{D}$ in any other way than via the parameters K, κ, λ, and θ_0.

We formulate the following corollary of Theorem 7 as a lemma.

Lemma 8. *Let N and μ be as in Theorem 6. Then for every $c > 1$ and $K \geq 1$ there exists $d > 0$ such that for every K-quasiregular mapping $f\colon \mathbb{R}^n \setminus \bar{B}^n \to N$ we have*

$$cA(B^n(\theta r) \setminus \bar{B}^n(2); f) \geq \nu_\mu(B^n(r) \setminus \bar{B}^n(2\theta); f) - d(\log\theta)^{-p(n-1)} \tag{5}$$

and

$$A(B^n(r) \setminus \bar{B}^n(2\theta); f) \leq c\nu_\mu(B^n(\theta r) \setminus \bar{B}^n(2); f) + d(\log\theta)^{-p(n-1)} \tag{6}$$

whenever $\theta \in (1, 2]$ and $r > 2\theta$.

Proof. Let $\theta > 1$ and $r > 2\theta$ be given. Fix $a = \sqrt{2\theta r}$ and $\delta = 2^{1/(1-n)}$ and let

$$D(s) = B^n((s/a)^{1/\delta}a) \setminus \bar{B}^n((s/a)^{-1/\delta}a)$$

for every $s \in (a, a^{1+\delta}]$. Then

$$\begin{aligned} \mathsf{M}_n(\Delta(\partial D(s), \partial D(t))) &= 2\omega_{n-1}\left(\log(t/s)^{1/\delta}\right)^{1-n} \\ &= \omega_{n-1}\left(\log(t/s)\right)^{1-n} \end{aligned}$$

for every $a < s < t < a^{\delta+1}$. Furthermore, $D(a^{\delta+1}) = B^n(2\theta r) \setminus \bar{B}^n$. Hence, for every $a' \in (a, a^{\delta+1})$, domains $D(t)$, $t \in [a', a^{\delta+1})$, form a $(1, n-1, 2)$-admissible exhaustion of $B^n(2\theta r) \setminus \bar{B}^n$.

Since $D(\theta^{-\delta}(a/2)^\delta a) = B^n(r) \setminus \bar{B}^n(2\theta)$ and $D((a/2)^\delta a) = B^n(\theta r) \setminus \bar{B}^n(2)$, the claim now follows from Theorem 7. □

The following lemma, which is a reformulation of [17, III.2.11], connects the essential singularity of f to the growth of averages of the counting function, see also [5, Lemma 3.1].

Lemma 9. *Let N, μ, and f be as in Theorem 6. Then $\nu_\mu(B^n(s) \setminus \bar{B}^n(s_0); f) \to \infty$ as $s \to \infty$ for every $s_0 \geq 2$.*

Proof. Let $s_0 \geq 2$. By [5, Lemma 3.1], the image of any neighborhood of the infinity covers N except of a possibly non-empty set of zero n-capacity. Hence $n(B^n(s) \setminus \bar{B}^n(s_0), y) \to \infty$ as $s \to \infty$ for $y \in N \setminus C$, where C has zero n-capacity. By [11, 4.3], C has zero μ measure. The claim now follows from the monotone convergence theorem. □

Using Lemma 9 we enhance inequalities (5) and (6).

Lemma 10. *Let N, μ, and f be as in Theorem 6. Then for each $c > 1$ there exist $d > 0$ and $s_0 \geq 4$ such that for every $\theta \in (1, 2]$ we have*

$$cA(B^n(\theta s) \setminus \bar{B}^n(2); f) \geq \nu_\mu(B^n(s) \setminus \bar{B}^n(2); f) - d(\log\theta)^{-p(n-1)} \tag{7}$$

and

(8) $\quad A(B^n(s) \setminus \bar{B}^n(2); f) \le c\nu_\mu(B^n(\theta s) \setminus \bar{B}^n(2); f) + d(\log\theta)^{-p(n-1)}$

whenever $s \ge s_0$.

Proof. Let $\theta \in (1, 2]$, and $c' \in (1, c)$. Since, by Lemma 9, $\nu_\mu(B^n(s) \setminus \bar{B}^n(2\theta); f) \to \infty$ and $A(B^n(s) \setminus \bar{B}^n(2\theta); f) \to \infty$ as $s \to \infty$, we may fix $s_0 \ge 4$ such that

$$(c - c')A(B^n(s) \setminus \bar{B}^n(2\theta); f) \ge \nu_\mu(B^n(2\theta) \setminus \bar{B}^n(2); f)$$

and

$$A(B^n(2\theta) \setminus \bar{B}^n(2); f) \le (c - c')\nu_\mu(B^n(s) \setminus \bar{B}^n(2\theta); f).$$

for every $s \ge s_0$. Inequalities (7) and (8) follow by applying Lemma 8 with c'. □

The proof of Theorem 6 now follows from Lemma 10 as the corresponding part of [11, 5.11] from [11, 4.8]. Since we may apply the proof of [11, 5.11] verbatim, we omit the details.

4. Quasiregular mappings and p-harmonic forms

In this section, we recall some parts of the theory of p-harmonic forms and their connection to quasiregular mappings and cohomology of compact manifolds. For details, see e.g. [2], [4], [6], [8], [9], [10], and [19].

Let M be a connected and oriented Riemannian n-manifold with $n \ge 2$. The Riemannian metric of M induces an inner product to the exterior bundle $\bigwedge^\ell T^*M$ for every $\ell \in \{1, \ldots, n\}$, see e.g. [9, 9.6] for details. We denote this inner product by $\langle \cdot, \cdot \rangle$ and the norm given by this inner product by $|\cdot|$. As usual, sections of the bundle $\bigwedge^\ell T^*M$ are called ℓ-forms. The L^p-space of measurable ℓ-forms is denoted by $L^p(\bigwedge^\ell M)$ and the L^p-norm is defined by

$$\|\xi\|_p = \left(\int_M |\xi|^p \,\mathrm{d}x \right)^{1/p}.$$

The local L^p-spaces of ℓ-forms are denoted by $L^p_{\mathrm{loc}}(\bigwedge^\ell M)$. The space of C^∞-smooth ℓ-forms on M is denoted by $C^\infty(\bigwedge^\ell M)$, and the space of compactly supported C^∞-smooth ℓ-forms by $C^\infty_0(\bigwedge^\ell M)$.

In order to define Sobolev spaces, we say that a form $\omega \in L^1_{\mathrm{loc}}(\bigwedge^\ell M)$ has *a weak exterior derivative* $\tau \in L^1_{\mathrm{loc}}(\bigwedge^{\ell+1} M)$ if

$$\int_M \langle \omega, d^*\varphi \rangle \,\mathrm{d}x = \int_M \langle \tau, \varphi \rangle \,\mathrm{d}x$$

for every $\varphi \in C^\infty_0(\bigwedge^{\ell+1} M)$. Here d^* is the adjoint of the exterior derivative $d\colon C^\infty(\bigwedge^\ell M) \to C^\infty(\bigwedge^{\ell+1} M)$, that is, $\langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle$ for every $\alpha \in C^\infty(\bigwedge^\ell M)$ and $\beta \in C^\infty(\bigwedge^{\ell+1} M)$. We denote by $d\omega$ the weak exterior derivative of ω. *The weak exterior coderivative* $d^*\omega$ of ω is defined similarly by saying that a form $\theta \in L^1_{\mathrm{loc}}(\bigwedge^{\ell-1} M)$ is the weak exterior coderivative of $\omega \in L^1_{\mathrm{loc}}(\bigwedge^\ell M)$ if

$$\int_M \langle \omega, d\varphi \rangle \, \mathrm{d}x = \int_M \langle \theta, \varphi \rangle \, \mathrm{d}x$$

for every $\varphi \in C_0^\infty(\bigwedge^{\ell-1} M)$. Weak exterior derivative and coderivative are studied in detail e.g. in [9, Chapter 10], [10, Section 3], and [19].

In what follows, we consider three types of Sobolev spaces of differential forms. For most of our considerations we use *a partial Sobolev space* $W^{d,p}(\bigwedge^\ell M)$ for $p \in (1,\infty)$. We say that $\omega \in W^{d,p}(\bigwedge^\ell M)$ if $\omega \in L^p(\bigwedge^\ell M)$ and $d\omega \in L^p(\bigwedge^{\ell+1} M)$. We equip $W^{d,p}(\bigwedge^\ell M)$ with the norm $\|\omega\|_{d,p} = \|\omega\|_p + \|d\omega\|_p$.

Corresponding to the exterior coderivative we define *a partial Sobolev space* $W^{d^*,p}(\bigwedge^\ell M)$ by saying that $\omega \in W^{d^*,p}(\bigwedge^\ell M)$ if $\omega \in L^p(\bigwedge^\ell M)$ and $d^*\omega \in L^p(\bigwedge^{\ell-1} M)$. We equip this space with the norm $\|\omega\|_{d^*,p} = \|\omega\|_p + \|d^*\omega\|_p$.

We also use the Sobolev space

$$W^{1,p}(\bigwedge^\ell M) = W^{d,p}(\bigwedge^\ell M) \cap W^{d^*,p}(\bigwedge^\ell M)$$

and equip it with the norm $\|\omega\|_{1,p} = \|\omega\|_p + \|d\omega\|_p + \|d^*\omega\|_p$. For a detailed discussion on spaces $W^{1,p}$, $W^{d,p}$, and $W^{d^*,p}$ see e.g. [10] and [19].

We also consider Sobolev spaces $W^{d,p}$, $W^{d^*,p}$, and $W^{1,p}$ on manifolds with boundary. In our applications, all such manifolds are submanifolds of $\mathbb{R}^n$ with smooth boundary and Sobolev spaces $W^{d,p}$, $W^{d^*,p}$, and $W^{1,p}$ are defined as above. See [10, Section 3] for a detailed discussion.

Having Sobolev spaces at our disposal, we may now consider p-harmonic forms and their connection to quasiregular mappings.

Let $\ell \in \{1,\ldots,n-1\}$ and $p > 1$. Let $\mathcal{A}\colon \bigwedge^\ell T^*M \to \bigwedge^\ell T^*M$ be a measurable bundle map such that there exists positive constants α and β satisfying

$$\langle \mathcal{A}(\xi) - \mathcal{A}(\zeta), \xi - \zeta \rangle \geq \alpha(|\xi| + |\zeta|)^{p-2}|\xi - \zeta|^2, \tag{9}$$

$$|\mathcal{A}(\xi) - \mathcal{A}(\zeta)| \leq \beta(|\xi| + |\zeta|)^{p-2}|\xi - \zeta|, \text{ and} \tag{10}$$

$$\mathcal{A}(t\xi) = t|t|^{p-2}\mathcal{A}(\xi) \tag{11}$$

for all $\xi, \zeta \in \bigwedge^\ell T^*_x M$, $t \in \mathbb{R}$, and for almost every $x \in M$. We also assume that $x \mapsto \mathcal{A}_x(\omega)$ is a measurable ℓ-form for every measurable ℓ-form $\omega\colon M \to \bigwedge^\ell T^*M$.

We say that a closed form $\xi \in W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell M)$ is $\mathcal{A}$-*harmonic (of type p)* if it satisfies *an $\mathcal{A}$-harmonic equation*

$$d^*(\mathcal{A}(\xi)) = 0 \tag{12}$$

weakly, i.e.

$$\int_M \langle \mathcal{A}(\xi), d\varphi \rangle \, \mathrm{d}x = 0$$

for every $\varphi \in C_0^\infty(\bigwedge^{\ell-1} M)$. If $\mathcal{A}(\xi) = |\xi|^{p-2}\xi$, we say that equation (12) is *a p-harmonic equation* and its closed weak solutions in $W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell M)$ we call *p-harmonic forms*.

Let $f\colon M \to N$ be a K-quasiregular mapping, $\ell \in \{1, \dots, n\}$, and $p = n/\ell$. Since f is almost everywhere differentiable, we may define the pull-back $f^*(\omega) \in L^p_{\mathrm{loc}}(\bigwedge^\ell M)$ of $\omega \in L^p_{\mathrm{loc}}(\bigwedge^\ell N)$ under f by

$$f^*(\omega)_x = (T_x f)^* \omega_{f(x)}$$

for almost every $x \in M$. The local L^p-integrability of $f^*(\omega)$ follows from quasiregularity of f. Indeed, by quasiregularity, we have for every $\omega \in L^p_{\mathrm{loc}}(\bigwedge^\ell N)$ inequalities

$$\frac{1}{K_I(f)} \int_N N(y,\Omega)|\omega|^p \, \mathrm{d}y \le \int_\Omega |f^*(\omega)|^p \, \mathrm{d}x \le K_O(f) \int_N N(y,\Omega)|\omega|^p \, \mathrm{d}y \tag{13}$$

for every relatively compact domain $\Omega \subset M$. Here $K_O(f)$ is the outer dilatation of f, and $N(\cdot, \Omega)$ is the multiplicity of f defined by $N(y,\Omega) = \mathrm{card}\,(f^{-1}(y) \cap \Omega)$ for every $y \in N$. Note that the conformal exponent $p = n/\ell$ has an essential role in (13).

By [8, Lemma 3.6], $f^*(\omega) \in W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell M)$ and the commutation rule $d(f^*(\omega)) = f^*(d\omega)$ holds whenever $\omega \in W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell N)$.

For a given quasiregular mapping f, we may define a measurable bundle map $\mathcal{A}\colon \bigwedge^\ell T^*M \to \bigwedge^\ell T^*M$ by

$$\mathcal{A}(\xi) = \langle G^*\xi, \xi \rangle^{\frac{p-2}{2}} G^*\xi, \tag{14}$$

where

$$G_x = \begin{cases} J(x,f)^{2/n}(T_x f)^{-1}\left((T_x f)^{-1}\right)^t, & J(x,f) > 0 \\ Id, & \text{otherwise.} \end{cases} \tag{15}$$

Here $\left((T_x f)^{-1}\right)^t : T_x M \to T_{f(x)}N$ is the adjoint of the tangent map $(T_x f)^{-1}\colon T_{f(x)}N \to T_x M$.

Let ξ be a p-harmonic ℓ-form on N with $p = n/\ell$. Then $f^*(\xi)$ is a closed form and satisfies an equation

$$d^*(\mathcal{A}(f^*(\xi)) = 0 \tag{16}$$

weakly. For a detailed discussion, see e.g. [10, Section 7] and [8, Section 4]. For reader's convenience, let us indicate the steps. Direct computation shows that

$$\langle G^* f^*(\xi), f^*(\xi) \rangle^{\frac{p-2}{2}} = J(\cdot, f)^{(p-2)/p} |\xi|^{p-2} \circ f$$

almost everywhere in M. Furthermore, by [8, pp. 49],

$$\begin{aligned} G^*f^*(\xi) &= (-1)^{\ell(n-\ell)} G^*f^*(\star\star\xi) \\ &= (-1)^{\ell(n-\ell)} G^*\left(J(\cdot,f)^{(n-2(n-\ell))/n}(G^{-1})^*\star f^*(\star\xi)\right) \\ &= (-1)^{\ell(n-\ell)} J(\cdot,f)^{(2\ell-n)/n}\left(\star f^*(\star\xi)\right) \end{aligned}$$

almost everywhere in M. Hence

$$\begin{aligned} \langle G^*f^*(\xi), f^*(\xi)\rangle^{\frac{p-2}{2}} G^*f^*(\xi) &= (-1)^{\ell(n-\ell)}(|\xi|^{p-2}\circ f)\star f^*(\star\xi) \\ &= (-1)^{\ell(n-\ell)}\star f^*(|\xi|^{p-2}(\star\xi)), \end{aligned}$$

and

$$\begin{aligned} d^*\left(\star f^*((-1)^{\ell(n-\ell)}|\xi|^{p-2}(\star\xi))\right) &= (-1)^{n\ell+1}\star f^*(d(\star(|\xi|^{p-2}\xi))) \\ &= (-1)^{\ell(n-\ell)}(\star f^*\star)d^*(|\xi|^{p-2}\xi)) = 0 \end{aligned}$$

weakly.

The following connection between the de Rham cohomology and p-harmonic forms is crucial for our forthcoming considerations. Let N be a closed, connected, and oriented Riemannian n-manifold. By [19, Section 7], for every $p \in (1,\infty)$ every cohomology class of N weakly contains a p-harmonic representative, that is, for every closed form $\omega \in C^\infty(\bigwedge^\ell N)$ there exists a p-harmonic form $\xi \in W^{1,p}(\bigwedge^\ell N)$ such that $\xi - \omega = d\gamma$ for some $\gamma \in W^{1,p}(\bigwedge^{\ell-1} N)$. Since the p-harmonic equation is the Euler equation of the variational integral

$$\xi \mapsto \int_N |\xi|^p \,\mathrm{d}x, \tag{17}$$

a p-harmonic form minimizes the p-energy within its cohomology class.

The following lemma is a Caccioppoli type inequality for $\mathcal{A}$-harmonic forms on B^n.

Lemma 11 ([2, Lemma 5.8])**.** *Let $\ell \in \{1,\ldots,n\}$, $p \ge 2$, $r \in (0,1)$, and let $\mathcal{A}$ be a measurable bundle map $\bigwedge^\ell T^*B^n \to \bigwedge^\ell T^*B^n$ satisfying* (9)-(11)*. Let $\xi,\zeta \in W^{d,p}_{\mathrm{loc}}(\bigwedge^{\ell-1} B^n)$ be such that $d\xi$ and $d\zeta$ are $\mathcal{A}$-harmonic ℓ-forms. Then*

$$\|(d\xi - d\zeta)|B^n(r)\|_p^p \le \frac{C}{1-r}\left(\|d\xi|D\|_p^{p-1} + \|d\zeta|D\|_p^{p-1}\right)\|(\xi-\zeta)|D\|_p, \tag{18}$$

where the constant C depends only on n, p, and constants α and β of $\mathcal{A}$. Here $D = B^n \setminus \bar{B}^n(r)$.

In [7] Iwaniec and Lutoborski introduce a L^p-version of the Poincaré homotopy operator $T\colon L^p(\bigwedge^\ell B^n) \to L^p(\bigwedge^{\ell+1} B^n)$, $p \in (1,\infty)$, satisfying

$$\omega = dT\omega + Td\omega \tag{19}$$

for every $\omega \in W^{d,p}(\bigwedge^\ell B^n)$. By [7, Proposition 4.1], the operator T is compact. We state as a lemma the punch line of the proof of [2, Theorem 1.1], where the compactness of T is combined with the Caccioppoli type estimate of Lemma 11. For reader's convenience, we give a sketch of a proof.

Lemma 12. *Let $c \in (0,1)$, $p \geq 2$, and let $\mathcal{A} \colon \bigwedge^\ell T^*B^n \to \bigwedge^\ell T^*B^n$ be a bundle map satisfying (9) - (11). Let $\xi_1, \ldots, \xi_k \in W^{d,p}(\bigwedge^\ell B^n)$ be $\mathcal{A}$-harmonic forms such that $\|\xi_i\|_p \leq 1$ for every i and*

$$\|(\xi_i - \xi_j)|B^n(r)\|_p \geq c > 0$$

for some $r \in (0,1)$ and for every $j \neq i$. Then k is bounded from above by a constant depending only on n, p, r, c, and constants α and β of $\mathcal{A}$.

Sketch of a proof. Since forms ξ_i are closed, (19) yields $\xi_i = dT\xi_i$ for every i. By Lemma 11,

$$\|T\xi_i - T\xi_j\|_p \geq \frac{1-r}{2C}\|\xi_i - \xi_j\|_p \geq c'$$

for every i and j, where $c' > 0$ depends only on n, p, $\mathcal{A}$, r, and c. Since forms $T\xi_i$ are contained in the image of the unit ball of $W^{d,p}(\bigwedge^\ell B^n)$ under T, and therefore in a relatively compact set, the number of forms ξ_i is bounded from above by a constant depending on $\|T\|$ and c. By [7, Proposition 4.1], $\|T\|$ depends only on n and p. The claim now follows. $\square$

Let us end this section with a note on p-harmonic forms and h-calibrated measures. In sections to come, we use p-harmonic forms to produce weighted Lebesgue measures. By Ural'tseva's theorem p-harmonic forms are locally Hölder continuous for every $p \in (1,\infty)$ [21, Theorem 1], see also [20]. Therefore for every compact manifold N and every p-harmonic form ξ on N the measure $\mathrm{d}\mu = |\xi|^p\,\mathrm{d}x$ satisfies $\mu(B(z,r)) \leq Cr^n$ for every ball $B(z,r)$ in N, where C is independent of the ball $B(z,r)$. If, in addition, $\xi \neq 0$, the measure μ is h-calibrated with any exponent $q > 2$.

5. A lower growth bound for the average of the counting function

In [2] the following lower growth bound for the average of the counting function of a quasiregular mapping from $\mathbb{R}^n$ into a compact manifold is established.

Theorem 13 ([2, Theorem 1.11])**.** *Let $f \colon \mathbb{R}^n \to N$ be a non-constant K-quasiregular mapping into a closed, connected, and oriented Riemannian n-manifold N, $n \geq 2$. If the ℓ-th cohomology group $H^\ell(N)$ of*

N is nontrivial for some $\ell = 1, \dots, n-1$, then there exists a positive constant $\alpha = \alpha(n, K)$ such that

$$\liminf_{r\to\infty} \frac{A(B^n(r); f)}{r^\alpha} > 0.$$

In this section we show that the average of the counting function of a quasiregular mapping from $B^n \setminus \{0\}$ into a compact manifold has a similar type of growth if the mapping has an essential singularity at the origin. However, in the case of a punctured ball we have to assume more on the cohomology of the target manifold as the following example reveals.

Let $N = S^{n-1} \times S^1$, and define $f\colon \mathbb{R}^n \setminus \bar{B}^n \to N$ by $x \mapsto \varphi(x/2^k)$ for every $x \in \bar{B}^n(2^{k+1}) \setminus \bar{B}^n(2^k)$ and every $k \in \mathbb{N}$, where $\varphi\colon \bar{B}^n(2) \setminus \bar{B}^n \to N$ is defined by $x \mapsto (x/|x|, e^{i2\pi|x|})$. Then every point in N has exactly one preimage point in $\bar{B}^n(2^{k+1}) \setminus \bar{B}^n(2^k)$ for every $k \in \mathbb{N}$. Hence $A(B^n(r) \setminus \bar{B}^n(2); f) \sim \log r$ for large r. In particular,

$$\frac{A(B^n(r) \setminus \bar{B}^n(2); f)}{r^\alpha} \to 0$$

as $r \to \infty$ for every $\alpha > 0$.

Having this example in mind, we formulate a counterpart of Theorem 13 as follows. Here and in the sections to come we abuse the notation by denoting $A(t; f) = A(B^n(t) \setminus \bar{B}^n(2); f)$ and $\nu_\mu(t; f) = \nu_\mu(B^n(t) \setminus \bar{B}^n(2); f)$ for every h-calibrated measure μ and every quasiregular mapping $f\colon \mathbb{R}^n \setminus \bar{B}^n \to N$ even though $\{B^n(t) \setminus \bar{B}^n(2)\colon t > 2\}$ is not an exhaustion of $\mathbb{R}^n \setminus \bar{B}^n$ in the sense of the definition given in Section 3.

Theorem 14. *Let N be a closed, connected, oriented Riemannian n-manifold such that $H^\ell(N) \neq 0$ for some $\ell \in \{2, \dots, n-2\}$, and let $f\colon \mathbb{R}^n \setminus \bar{B}^n \to N$ be a K-quasiregular mapping having an essential singularity at infinity. Then there exist constants $C_0 > 1$ and $\lambda > 1$ depending only on n, ℓ, and K such that*

$$\liminf_{t\to\infty} \frac{A(\lambda t; f)}{A(t; f)} \geq C_0. \tag{20}$$

Furthermore, there exists $\alpha > 0$ depending only on n, ℓ, and K such that

$$\liminf_{t\to\infty} \frac{A(t; f)}{t^\alpha} > 0. \tag{21}$$

The proof of Theorem 14 is based on another Caccioppoli type inequality and a Poincaré inequality given in [10]. The Poincaré inequality is formulated as follows.

Theorem 15 ([10, Theorem 6.4])**.** *Let M be a compact submanifold with boundary of a closed manifold and let $1 < p < \infty$. For every*

$\omega \in W^{d,p}(\bigwedge^\ell M)$ there exists a closed form $\omega_0 \in L^p(\bigwedge^\ell M)$ such that $\omega - \omega_0 \in W^{1,p}(\bigwedge^\ell M)$ and

$$\|\omega - \omega_0\|_{1,p} \le C_p(M)\|d\omega\|_p,$$

where $C_p(M) > 0$ depends only on p and M.

Before formulating a Caccioppoli type inequality suitable for our purposes, let us introduce some notations. For every $r > 0$ we set $D_r = B^n(r) \setminus \bar{B}^n(4)$, $\Omega_r = B^n(2r) \setminus \bar{B}^n(r)$, and $\Omega'_r = B^n(4r) \setminus \bar{B}^n(r/2)$.

Lemma 16. *Let $1 \le \ell \le n$, $p > 1$, $R > 4$, and let ω be a form in $W^{d,p}(\bigwedge^{\ell-1} \bar{B}^n(2R) \setminus B^n)$ such that $d\omega$ is $\mathcal{A}$-harmonic in $B^n(2R) \setminus \bar{B}^n$. Then*

$$\|d\omega|D_R\|_p^p \le C_1\|d\omega|\Omega_2\|_p^{p-1}\|\omega|\Omega_2\|_p + \frac{C_1}{R}\|d\omega|\Omega_R\|_p^{p-1}\|\omega|\Omega_R\|_p,$$

where $C_1 = 2\beta/\alpha$, and constants α and β are as in (9) *and* (10).

Proof. Let $\varphi \in C_0^\infty(B^n(2R) \setminus \bar{B}^n(2))$ be such that $\varphi|D_R = 1$, $|d\varphi|\Omega_2| \le 2$, and $|d\varphi|\Omega_R| \le 2/R$. Then, by (9),

$$\begin{aligned}\int_{D_R} |d\omega|^p \,\mathrm{d}x &\le \frac{1}{\alpha}\int_{D_R} \langle \mathcal{A}(d\omega), d\omega\rangle \,\mathrm{d}x \\ &\le \frac{1}{\alpha}\int_{B^n(2R)\setminus\bar{B}^n(2)} \varphi\langle \mathcal{A}(d\omega), d\omega\rangle \,\mathrm{d}x,\end{aligned}$$

where α is as in (9). The $\mathcal{A}$-harmonicity of $d\omega$ together with Cauchy-Schwarz and Hadamard-Schwarz inequalities yield

$$\begin{aligned}\int_{B^n(2R)\setminus\bar{B}^n(2)} \varphi\langle \mathcal{A}(d\omega), d\omega\rangle \,\mathrm{d}x &= \int_{B^n(2R)\setminus\bar{B}^n(2)} \langle \mathcal{A}(d\omega), d(\varphi\omega)\rangle \,\mathrm{d}x \\ &\quad - \int_{B^n(2R)\setminus\bar{B}^n(2)} \langle \mathcal{A}(d\omega), d\varphi \wedge \omega\rangle \,\mathrm{d}x \\ &= -\int_{B^n(2R)\setminus\bar{B}^n(2)} \langle \mathcal{A}(d\omega), d\varphi \wedge \omega\rangle \,\mathrm{d}x \\ &\le \int_{\Omega_2\cup\Omega_R} |\mathcal{A}(d\omega)||d\varphi||\omega| \,\mathrm{d}x\end{aligned}$$

By (10) and Hölder's inequality,

$$\begin{aligned}\int_{\Omega_2\cup\Omega_R} |\mathcal{A}(d\omega)||d\varphi||\omega| \,\mathrm{d}x &\le 2\beta\int_{\Omega_2} |d\omega|^{p-1}|\omega| \,\mathrm{d}x \\ &\quad + \frac{2\beta}{R}\int_{\Omega_R} |d\omega|^{p-1}|\omega| \,\mathrm{d}x \\ &\le 2\beta\|d\omega|\Omega_2\|_p^{p-1}\|\omega|\Omega_2\|_p \\ &\quad + \frac{2\beta}{R}\|d\omega|\Omega_R\|_p^{p-1}\|\omega|\Omega_R\|_p,\end{aligned}$$

where β as in (10). The claim now follows from these estimates. $\square$

Lemma 17. *Let $2 \le \ell \le n-1$, $p > 1$, $R > 8$, and let $\tau \in W^{d,p}(\bigwedge^{\ell-1} \bar{B}^n(4R) \setminus B^n)$ be such that $\|d\tau|\Omega_R\|_p > 0$. Then there exists $\omega \in W^{d,p}(\bigwedge^{\ell-1} \bar{B}^n(4R) \setminus B^n)$ such that $d\omega = d\tau$, $\omega|\Omega_2 = \tau|\Omega_2$, and*

$$\|\omega|\Omega_R\|_p \le 2\|\omega|\Omega_R + d\beta|\Omega_R\|_p$$

for every $\beta \in W^{d,p}(\bigwedge^{\ell-2} \overline{\Omega'_R})$.

Proof. Let us first show that

$$\inf_\beta \int_{\Omega_R} |\tau + d\beta|^p \,\mathrm{d}x > 0, \tag{22}$$

where $\beta \in W^{d,p}(\bigwedge^{\ell-2} \overline{\Omega'_R})$. Suppose that this is not the case. Then there exists a sequence (β_k) such that $d\beta_k \to \tau$ in $L^p(\bigwedge^{\ell-1} \overline{\Omega'_R})$. By the Poincaré inequality, we may assume that $\beta_k \to \beta$ in $W^{d,p}(\bigwedge^{\ell-2} \overline{\Omega'_R})$ for some $\beta \in W^{d,p}(\bigwedge^{\ell-2} \overline{\Omega'_R})$. Then $d\beta|\Omega_R = \tau|\Omega_R$, and hence $d\tau|\Omega_R = d^2\beta|\Omega_R = 0$. This contradicts the assumption $\|d\tau|\Omega_R\|_p > 0$.

By (22), there exists $\beta' \in W^{d,p}(\bigwedge^{\ell-2} \overline{\Omega'_R})$ such that

$$\|\tau|\Omega_R + d\beta'|\Omega_R\|_p \le 2 \inf_\beta \|\tau|\Omega_R + d\beta|\Omega_R\|_p,$$

where $\beta \in W^{d,p}(\bigwedge^{\ell-2} \overline{\Omega'_R})$. Let $\psi \in C_0^\infty(\Omega'_R)$ be such that $\psi|\Omega_R = 1$ and let $\omega = \tau + d(\psi\beta')$. Then ω satisfies assumptions of the claim. $\square$

Lemma 18. *Let $2 \le \ell \le n-1$ and $p = n/\ell$. Then for every $R > 0$ and $\omega \in W^{d,p}(\bigwedge^{\ell-1} \overline{\Omega'_R})$ there exists a closed form $\omega_0 \in W^{d,p}(\bigwedge^{\ell-1} \overline{\Omega'_R})$ such that*

$$\|\omega - \omega_0\|_p \le C_p(\overline{\Omega'_2})(R/2)\|d\omega\|_p, \tag{23}$$

where $C_p(\overline{\Omega'_2})$ is as in Theorem 15.

Proof. Let $R > 0$ and $\omega \in W^{d,p}(\bigwedge^{\ell-1} \overline{\Omega'_R})$. Let us define $\psi \colon \mathbb{R}^n \to \mathbb{R}^n$ by $x \mapsto (R/2)x$. Then ψ is conformal. By Theorem 15, we may fix a closed form $\omega_1 \in W^{d,p}(\bigwedge^{\ell-1} \overline{\Omega'_2})$ such that

$$\|\psi^*\omega - \omega_1\|_{1,p} \le C_p(\overline{\Omega'_2})\|d\psi^*\omega\|_p.$$

We show that $\omega_0 = (\psi^{-1})^*\omega_1$ satisfies (23). Since

$$\begin{aligned}
|\psi^*(\omega - \omega_0)|^p &= \left(|\psi^*(\omega - \omega_0)|^{n/(\ell-1)}\right)^{(\ell-1)/\ell} \\
&= \left(|\omega - \omega_0|^{n/(\ell-1)} \circ \psi J(\cdot, \psi)\right)^{(\ell-1)/\ell} \\
&= (|\omega - \omega_0|^p \circ \psi)\, J(\cdot, \psi) J(\cdot, \psi)^{-1/\ell} \\
&= (R/2)^{-p} (|\omega - \omega_0|^p \circ \psi)\, J(\cdot, \psi),
\end{aligned}$$

we have that

$$\begin{aligned}
\left(\int_{\Omega'_R} |\omega-\omega_0|^p \,\mathrm{d}x\right)^{1/p} &= \left(\int_{\Omega'_2} |\omega-\omega_0|^p\circ\psi J(\cdot,\psi)\,\mathrm{d}x\right)^{1/p}\\
&= (R/2)\left(\int_{\Omega'_2} |\psi^*\omega-\omega_1|^p\,\mathrm{d}x\right)^{1/p}\\
&\le (R/2)C_p(\overline{\Omega'_2})\|d\psi^*\omega\|_p\\
&= C_p(\overline{\Omega'_2})(R/2)\|\psi^* d\omega\|_p\\
&= C_p(\overline{\Omega'_2})(R/2)\|d\omega\|_p.
\end{aligned}$$

□

Proof of Theorem 14. By replacing f with the mapping $x\mapsto f(2x)$, we may assume that $f\colon \mathbb{R}^n\setminus \bar{B}^n(1/2)\to N$. Let $2\le \ell\le n-2$ be such that $H^\ell(N)\ne 0$, and let $p=n/\ell$. Fix a p-harmonic ℓ-form ξ on N such that $\|\xi\|_p=1$, and let $\eta=f^*\xi$. Furthermore, let μ be the measure $\mathrm{d}\mu=|\xi|^p\mathrm{d}x$ on N.

Since f has an essential singularity at the infinity,

$$\begin{aligned}
\int_{D_R}|\eta|^p\,\mathrm{d}x &\ge \frac{1}{K_I(f)}\int_N N(y,D_R;f)|\xi|^p\,\mathrm{d}y\\
&= \frac{1}{K_I(f)}\int_N n(y,D_R;f)|\xi|^p\,\mathrm{d}y\\
&= \frac{1}{K_I(f)}\left(\nu_\mu(R;f)-\nu_\mu(4;f)\right)\to\infty
\end{aligned}$$

as $R\to\infty$. Here we used the fact that $n(y,D_R;f)=N(y,D_R;f)$ for all $y\notin f(\partial D_R)\cup fB_f$, where B_f is the branch set of f, and the fact that $f(\partial D_R)\cup fB_f$ has zero measure. Since η is (weakly) closed and $H^\ell(\bar{B}^n(5)\setminus B^n)=0$, we may fix $\tau_0\in W^{d,p}(\bigwedge^{\ell-1}\bar{B}^n(5)\setminus B^n)$ such that $d\tau_0=\eta$, see [10, Theorem 5.7]. We may now choose $R_0>8$ such that

$$\|\eta|D_{R_0}\|_p^p\ge 2C_1\|\eta|\Omega_2\|_p^{p-1}\|\tau_0|\Omega_2\|_p,$$

where C_1 is as in Lemma 16.

Let $R\ge R_0$, and $\tau\in W^{d,p}(\bigwedge^{\ell-1}\bar{B}^n(4R)\setminus B^n)$ such that $d\tau=\eta$ in $\bar{B}^n(4R)\setminus B^n$. Furthermore, we may assume that $\tau|\Omega_2=\tau_0|\Omega_2$. By Lemma 17, we may fix $\omega\in W^{d,p}(\bigwedge^{\ell-1}\bar{B}^n(4R)\setminus B^n)$ such that $d\omega=\eta$, $\omega|\Omega_2=\tau|\Omega_2$, and

$$\|\omega|\Omega_R\|_p\le 2\|\omega|\Omega_R+d\beta|\Omega_R\|_p \tag{24}$$

for every $\beta\in W^{d,p}(\bigwedge^{\ell-2}\overline{\Omega'_R})$. Then, by Lemma 16,

$$\begin{aligned}
\|\eta|D_R\|_p^p &\le C_1\|\eta|\Omega_2\|_p^{p-1}\|\omega|\Omega_2\|_p+\frac{C_1}{R}\|\eta|\Omega_R\|_p^{p-1}\|\omega|\Omega_R\|_p\\
&\le \frac{1}{2}\|\eta|D_R\|_p^p+\frac{C_1}{R}\|\eta|\Omega_R\|_p^{p-1}\|\omega|\Omega_R\|_p.
\end{aligned}$$

Hence

$$\|\eta|D_R\|_p^p \le 2\frac{C_1}{R}\|\eta|\Omega_R\|_p^{p-1}\|\omega|\Omega_R\|_p. \tag{25}$$

Let $\omega_0 \in W^{d,p}(\bigwedge^{\ell-1}\overline{\Omega'_R})$ be a closed form as in Lemma 18, i.e.

$$\|\omega|\Omega'_R - \omega_0\|_p \le C_p(\overline{\Omega'_2})(R/2)\|d\omega|\Omega'_R\|_p.$$

Since $H^{\ell-1}(\overline{\Omega'_R}) = 0$, the form ω_0 is exact and (24) yields

$$\begin{aligned}\|\omega|\Omega_R\|_p &\le 2\|\omega|\Omega_R - \omega_0|\Omega_R\|_p \le 2\|\omega|\Omega'_R - \omega_0\|_p \\ &\le 2C_p(\overline{\Omega'_2})(R/2)\|d\omega|\Omega'_R\|_p = C_p(\overline{\Omega'_2})R\|\eta|\Omega'_R\|_p.\end{aligned} \tag{26}$$

Inequalities (25) and (26) together yield

$$\|\eta|D_R\|_p^p \ \le \ 2C_1C_p(\overline{\Omega'_2})\|\eta|\Omega'_R\|_p^p = C\|\eta|\Omega'_R\|_p^p.$$

Hence

$$\begin{aligned}\|\eta|D_{4R}\|_p^p &\ge \|\eta|D_{R/2}\|_p^p + (1/C)\|\eta|D_R\|_p^p \\ &\ge (1+1/C)\|\eta|D_{R/2}\|_p^p.\end{aligned} \tag{27}$$

for every $R \ge R_0$. Therefore,

$$\liminf_{R\to\infty} \frac{\|\eta|D_{8R}\|_p^p}{\|\eta|D_R\|_p^p} \ge 1 + 1/C > 1.$$

By (13),

$$(1/K_O(f))\|\eta|D_R\|_p^p \le \nu_\mu(R) - \nu_\mu(4) \le K_I(f)\|\eta|D_R\|_p^p$$

for every $R \ge R_0$. Hence for every $k \ge 1$

$$\begin{aligned}\liminf_{R\to\infty}\frac{\nu_\mu(8^kR;f)}{\nu_\mu(R;f)} &\ge \liminf_{R\to\infty}\frac{\nu_\mu(4) + (1/K_O(f))\|\eta|D_{8^kR}\|_p^p}{\nu_\mu(4) + K_I(f)\|\eta|D_R\|_p^p} \\ &= \frac{1}{K_I(f)K_O(f)}\liminf_{R\to\infty}\frac{\|\eta|D_{8^kR}\|_p^p}{\|\eta|D_R\|_p^p} \\ &\ge \frac{(1+1/C)^k}{K_I(f)K_O(f)}.\end{aligned} \tag{28}$$

Let us fix an integer m such that $(1+1/C)^m > 4K_I(f)K_O(f)$. Then

$$\liminf_{R\to\infty}\frac{\nu_\mu(8^mR;f)}{\nu_\mu(R;f)} > 4. \tag{29}$$

By Theorem 6 there exists a set $E \subset (1,\infty)$ of finite logarithmic measure such that

$$\lim_{\substack{t\to\infty \\ t\notin E}} \frac{A(t;f)}{\nu_\mu(t;f)} = 1.$$

Let $R_1 > R_0$ be such that $\nu_\mu(t)/2 \le A(t) \le 2\nu_\mu(t)$ for every $t \in (R_1,\infty)\setminus E$ and $m_{\log}(E\cap[R_1,\infty)) \le (\log R_0)/2$. Then $[R,2R)\setminus E \neq \emptyset$ for every $R > R_1$. By (29),

$$\begin{aligned}\liminf_{R\to\infty} \frac{A(2\cdot 8^m R; f)}{A(R/2;f)} &\ge \liminf_{\substack{R\to\infty\\ R,8^m R\notin E}} \frac{A(8^m R; f)}{A(R;f)}\\ &\ge \frac{1}{4}\liminf_{R\to\infty} \frac{\nu_\mu(8^m R;f)}{\nu_\mu(R;f)} > 1.\end{aligned}$$

Inequality (20) now follows.

To show (21), let us fix $\alpha = \log_8(1+1/C)$. Then, by (28),

$$\begin{aligned}\frac{\nu_\mu(8^k R_0; f)}{(8^k R_0)^\alpha} &\ge \frac{1}{K_I(f)K_O(f)}\left(\frac{1+1/C}{8^\alpha}\right)^k \frac{\nu_\mu(R_0;f)}{R_0^\alpha}\\ &= \frac{1}{K_I(f)K_O(f)}\frac{\nu_\mu(R_0;f)}{R_0^\alpha} > 0\end{aligned}$$

for every $k \ge 1$.

Let $t > 2R_1$, and fix $r \in [t/2,t]\setminus E$ and $k \ge 0$ such that $8^k R_0 \le r < 8^{k+1}R_0$. Then $8^k R_0 \ge r/8 \ge t/16$ and

$$\frac{A(t;f)}{t^\alpha} \ge \frac{\nu_\mu(r;f)}{2t^\alpha} \ge \frac{(8^k R_0)^\alpha}{2t^\alpha}\frac{\nu_\mu(8^k R_0;f)}{(8^k R_0)^\alpha} \ge \frac{1}{32}\frac{\nu_\mu(8^k R_0;f)}{(8^k R_0)^\alpha}.$$

Hence $\liminf_{t\to\infty} A(t;f)/t^\alpha > 0$. □

6. Bounds for $\dim H^\ell(N)$ when $2 \le \ell \le n-2$

In this section we show that there exists a constant C depending only on n and K such that $\dim H^\ell(N) \le C$ for every $\ell \in \{2,\dots,n-2\}$ whenever there exists a quasiregular mapping from $B^n\setminus\{0\}$ into N with an essential singularity at the origin. The proof is based on a modification of a ball decomposition method due to Rickman, see [16]. The following lemma is very common in the value distribution theory. We use here a simplified version of [16, 2.4].

Lemma 19. *Let $n \ge 2$, $\alpha = 2^{-1}(n-1)^{-1}$, N a closed, connected, and oriented Riemannian n-manifold, and let $f\colon \mathbb{R}^n\setminus \bar{B}^n \to N$ be a quasiregular mapping such that $\lim_{s\to\infty} A(s;f) = \infty$. Then there exists a set $E \subset [1,\infty)$ of finite logarithmic measure such that the following holds. For every $\varepsilon > 0$ there exists $s_\epsilon > 0$ such that*

$$1 \le \frac{A(s';f)}{A(s;f)} \le 1+\varepsilon \tag{30}$$

whenever $s' \notin E$ and $s' \ge s_\varepsilon$, where

$$s' = s + \frac{s}{A(s;f)^\alpha}.$$

Proof. Let $m \in \mathbb{Z}_+$, $c_m = 1+1/m$, and $R_0 \geq 1$ such that $A(R_0; f)^{\alpha/2} \geq 3$. Set $\beta(r) = A(r; f)^{\alpha/2}/(2r)$ for $r \geq R_0$. By increasing R_0, if necessary, we may assume that $(r + 1/\beta(r))' \leq r + 2/\beta(r)$ for $r \geq R_0$. Let

$$F_m = \{r > R_0 \colon A(r + 2/\beta(r); f) > c_m^{1/2} A(r; f)\}$$

and

$$E_m = \{(r + 1/\beta(r))' \colon r \in F_m\}.$$

Following [16, 2.4], we obtain $m_{\log}(E_1) < \infty$. We fix an increasing sequence $R_0 \leq d_1 < d_2 < \cdots$ such that $m_{\log}(E_m \cap [d_m, \infty)) < 2^{-m}$.

Let $\varepsilon > 0$ and let $m \in \mathbb{Z}_+$ be such that $(1 + 1/m)^{1/2} \leq 1 + \varepsilon$. Let $s \geq R_0$ be such that $s' \notin E_m$ and $s' \geq d_m$. Fix $r \geq R_0$ to be the least t such that $s' = (t + 1/\beta(t))'$. Then $(r + 1/\beta(r))' \leq r + 2/\beta(r)$ and

$$\begin{aligned} A(s'; f) &= A((r + 1/\beta(r))'; f) \leq A(r + 2/\beta(2); f) \\ &\leq c_m^{1/2} A(r; f) \leq (1 + \varepsilon) A(s; f), \end{aligned}$$

since $s \geq r$. □

Theorem 20. *Let $n \geq 4$, N a closed, connected, and oriented Riemannian n-manifold such that $H^\ell(N) \neq 0$ for some $\ell \in \{2, \ldots, n-2\}$, and let $f \colon \mathbb{R}^n \setminus \bar{B}^n \to N$ be a K-quasiregular mapping with an essential singularity at the infinity. Then there exist a constant $K' \geq 1$ depending only on n, and constants $C_1 > 1$ and $C_2 \in (0, 1)$ depending only on n, ℓ, and K such that for every $R_0 \geq 4$ there exists $R \geq R_0$ and K'-quasiconformal mapping $\psi \colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$A(\psi(B^n); f) \leq C_1 A(\psi(B^n(1/2)); f) \tag{31}$$

and

$$A(\psi(B^n(1/2)); f) \geq C_2 A(R; f)^{1/4}. \tag{32}$$

Moreover, $\psi(B^n) \subset B^n(R') \setminus \bar{B}^n(2)$, where $R' = R + R/A(R; f)^\alpha$ and α as in Lemma 19.

Proof. Let us first consider a decomposition of a cubical annulus into rectangles. Here we follow the idea of the ball decomposition method due to Rickman, see e.g. [16] and [17, V.2.14].

We construct the decomposition in three steps. As the first step, let us consider two concentric closed n-cubes Q_1 and Q_2 with side lengths $0 < r < R$, respectively. Let $Q_1 = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $Q_2 = [c_1, d_1] \times \cdots \times [c_n, d_n]$. For each $i \in \{1, \ldots, n\}$ we divide the interval $[c_i, d_i]$ into closed essentially disjoint subintervals $[c_{ij}, d_{ij}]$, $j \in \{1, \ldots, m_i\}$, in such a way that $a_i = c_{ik}$ and $b_i = d_{im}$ for some k and m for every i, and $r/16 < d_{ij} - c_{ij} < r/8$. Let $\mathcal{C}$ be the collection of rectangles $[c_{1j_1}, d_{1j_1}] \times \cdots \times [c_{nj_n}, d_{nj_n}] \subset Q_2 \setminus \mathrm{int} Q_1$, where $j_i \in \{1, \ldots, m_i\}$ for every i. Then $\mathcal{C}$ is a decomposition of $Q_2 \setminus \mathrm{int} Q_1$

into closed essentially disjoint n-rectangles. Moreover, the number of elements in $\mathcal{C}$ is bounded by

$$\frac{R^n - r^n}{(r/16)^n} = 16^n \frac{R^n - r^n}{r^n}.$$

Furthermore, every rectangle in $\mathcal{C}$ is quasiconformally equivalent to B^n with a uniformly bounded dilatation.

As the second step, let us consider three concentric closed n-cubes Q_1, Q_2, and Q_3 with side lengths $0 < r_1 < r_2 < r_3$, respectively, and assume that $2r_1 > r_2$. We equip both annuli $Q_3 \setminus Q_2$ and $Q_2 \setminus Q_1$ with decompositions as above. Let Q be a closed n-rectangle from the decomposition of $Q_3 \setminus Q_2$, and denote by $2Q$ the concentric closed n-rectangle with double side length. Then the interior of $2Q$ meets at most 3^n rectangles in the decomposition of $Q_3 \setminus Q_2$ and at most 9^n rectangles in the decomposition of $Q_2 \setminus Q_1$. For a closed n-rectangle Q' in the decomposition of $Q_2 \setminus Q_1$ the corresponding numbers are both 3^n. Let us denote by b_2 the maximum number of rectangles in these decompositions any cube $2Q$ meets. Then b_2 has an upper bound depending only on n.

For the third step, let $0 < r_0 < r_1 < \cdots < r_k$ be a sequence and let $Q_0 \subset Q_1 \subset \cdots \subset Q_k$ be a sequence of closed concentric cubes with side lengths r_i, respectively. Suppose that the sequence (r_i) satisfies the following conditions.

(1) $r_i - r_0 = r_k - r_{k-i}$ for every i,
(2) $r_{i+1} - r_i = 2(r_i - r_{i-1})$ for $1 < i < k/2$, and
(3) $r_{i+1} - r_i \leq 2(r_i - r_{i-1})$ if i satisfies $i < k/2 < i + 1$.

Then $r_i = r_0 + (2^i - 1)(r_1 - r_0)$ for $i < k/2$ and $r_i = r_k - (2^{k-i} - 1)(r_1 - r_0)$ for $i > k/2$. Hence $r_i < 2r_{i-1}$ for every $i \in \{1, \ldots, k\}$. Let us fix a decomposition $\mathcal{C}_i$ described above for every cubical annulus $Q_i \setminus \mathrm{int} Q_{i-1}$ where $i \in \{1, \ldots, k\}$. Then $\mathcal{C}_i$ contains at most

$$16^n \frac{r_i^n - r_{i-1}^n}{r_{i-1}^n} \leq 16^n \frac{(2^n - 1) r_{i-1}^n}{r_{i-1}^n} = 16^n (2^n - 1)$$

elements for each $i \in \{1, \ldots, k\}$, and if Q is any of the rectangles in any of the decompositions $\mathcal{C}_i$, rectangle $2Q$ meets at most b_2 rectangles from $\bigcup_i \mathcal{C}_i$. Since $r_k \geq 2(2^{k/2-1} - 1)(r_1 - r_0)$, the total amount of rectangles in $\bigcup_i \mathcal{C}_i$ is at most

$$k 16^n (2^n - 1) \leq \left(4 \log_2 \frac{r_k}{r_1 - r_0} + 1 \right) 16^n (2^n - 1) = b_1 \log_2 \frac{r_k}{r_1 - r_0}, \tag{33}$$

where the constant b_1 depends only on n.

Let us now return to the actual proof. Let $R_0 \geq 4$ be given. We fix a radial bilipschitz mapping $h \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $h((-r, r)^n) = B^n(r)$ for every $r > 0$.

By Theorem 14 and Lemma 19, we may fix $R > \lambda R_0$ such that

$$A(R;f)^\alpha \geq \max\{2\lambda, (2b_1)^4\}, \tag{34}$$
$$A(R;f) \geq C_0 A(R/\lambda; f), \text{ and} \tag{35}$$
$$A(R;f) \geq A(R';f)/2, \tag{36}$$

with b_1 as in (33), C_0 and λ as in Theorem 14, $R' = R + R/A(R;f)^\alpha$, and α as in Lemma 19.

Let $d = R' - R$ and $r_0 < \cdots < r_k$ be a sequence as in the third step of the construction above satisfying additional conditions $r_0 = 2 + d$, $r_1 - r_0 = d$, and $r_k = R$. We denote by $\mathcal{C}$ the decomposition of $(-R,R)^n \setminus [-r_0, r_0]^n$ into closed essentially disjoint n-rectangles with respect to the sequence (r_i) as described above.

By (34), $d \leq R/(2\lambda)$. Hence, by (35),

$$A(R;f) - A(r_0;f) \geq A(R;f) - A(R/\lambda;f) \geq (1 - 1/C_0)A(R;f).$$

Since $h((-R,R)^n \setminus [-r_0,r_0]^n) = B^n(R) \setminus \bar{B}^n(r_0)$, it is sufficient to find $Q \in \mathcal{C}$ such that

$$A(h(2Q);f) \leq C_1 A(h(Q);f) \quad \text{and} \quad A(h(Q);f) \geq C_2 A(R;f)^{1/4},$$

where C_1 and C_2 depend only on n, ℓ, and K. Indeed, given such a cube $Q \in \mathcal{C}$, we may take $\psi\colon \mathbb{R}^n \to \mathbb{R}^n$ to be the mapping $x \mapsto h(x_Q + \ell_Q h^{-1}(x))$, where x_Q is the center and ℓ_Q is the side length of Q. Then $\psi(B^n(1/2)) = h(Q)$, $\psi(B^n) = h(2Q)$, and the dilatation of ψ depends only on n.

Let

$$I_0 = \{Q \in \mathcal{C} \colon A(h(2Q);f) \geq \frac{6b_2}{1-1/C_0} A(h(Q);f)\}$$

and

$$I_1 = \{Q \in \mathcal{C} \colon A(h(Q);f) \leq (1-1/C_0)A(R;f)^{1/4}\}.$$

We show that $\mathcal{C} \setminus (I_0 \cup I_1) \neq \emptyset$.

Since $2Q \subset (-R', R')^n \setminus [-2,2]^n$ for every $Q \in \mathcal{C}$, we obtain

$$\begin{aligned} 2\left(A(R;f) - A(r_0;f)\right) &\geq (1-1/C_0)A(R';f) \\ &\geq \frac{1-1/C_0}{b_2} \sum_{Q \in I_0} A(h(2Q);f) \\ &\geq 6 \sum_{Q \in I_0} A(h(Q);f). \end{aligned}$$

By (33) and (34), $\mathcal{C}$ has at most

$$b_1 \log_2 \frac{R}{d} = b_1 \log_2 A(R;f)^\alpha \leq b_1 A(R;f)^{1/2}.$$

elements. Hence

$$\begin{aligned}\sum_{Q\in I_1} A(h(Q);f) &\le (\mathrm{card}\,\mathcal{C})(1-1/C_0)A(R;f)^{1/4} \\ &\le b_1(1-1/C_0)A(R;f)^{3/4} \le (1-1/C_0)A(R;f)/2.\end{aligned}$$

These estimates yield

$$\sum_{Q\in I_0\cup I_1} A(h(Q);f) \le \frac{5}{6}\left(A(R;f)-A(r_0;f)\right) < \sum_{Q\in\mathcal{C}} A(h(Q);f).$$

Therefore there exists $Q\in\mathcal{C}\setminus(I_0\cup I_1)$. This concludes the proof. $\square$

Proof of Theorem 5. Let $\ell\in\{2,\dots,n-2\}$. We may assume that $\dim H^\ell(N)>0$ and, by the Poincaré duality, $\ell\le n/2$. Let $p=n/\ell$, $d=\dim H^\ell(N)$, and $\xi_1,\dots,\xi_d$ be p-harmonic ℓ-forms such that $\|\xi_i\|_p = 1$ and $1\le\|\xi_i-\xi_j\|_p\le 2$ for every $1\le i\le d$ and $j\ne i$. Furthermore, let μ_i and μ_{ij} be measures such that $d\mu_i=|\xi|^p\,\mathrm{d}x$ and $d\mu_{ij}=|\xi_i-\xi_j|^p\,\mathrm{d}x$ for every $1\le i\le d$ and $j\ne i$.

Let $K'\ge 1$ be as in Theorem 20. By Theorem 7, we may fix $d_1>0$ such that

$$2A(B^n;F)\ge \nu_{\mu_*}(B^n(4/5);F)-d_1(\log 5/4)^{-3(n-1)} \tag{37}$$

and

$$A(B^n(1/2);F)\le 2\nu_{\mu_*}(B^n(5/8);F)+d_1(\log 5/4)^{-3(n-1)}, \tag{38}$$

whenever μ_* is any of the measures μ_i and μ_{ij}, and $F\colon B^n\to N$ is a $K'K$-quasiregular mapping.

Since $A(r;f)\to\infty$ as $r\to\infty$, we may fix $r_0>0$ such that $A(r_0;f)^{1/4}\ge 2d_1(\log 5/4)^{-3(n-1)}$. By Theorem 20, there exist $r>r_0$ and a K'-quasiconformal mapping $\psi\colon\mathbb{R}^n\to\mathbb{R}^n$ such that

$$A(\psi(B^n);f)\le C_1 A(\psi(B^n(1/2));f)$$

and

$$A(\psi(B^n(1/2));f)\ge C_2 A(r_0;f)^{1/4},$$

where $C_1>1$ and $C_2\in(0,1)$ depend only on n, ℓ, and K. Let $F=f\circ\psi\colon B^n\to N$. Then F is $K'K$-quasiregular. Thus, by (37) and (38), we have

$$\begin{aligned}\nu_{\mu_i}(B^n(4/5);F) &\le 2A(B^n;F)+A(r_0;f)^{1/4} \\ &\le 2C_1A(B^n(1/2);F)+A(B^n(1/2);F)/C_2 \\ &= (2C_1+1/C_2)A(B^n(1/2);F)\end{aligned}$$

and

$$\begin{aligned}\nu_{\mu_{ij}}(B^n(5/8);F) &\ge A(B^n(1/2);F)/2-A(r_0;f)^{1/4}/4 \\ &\ge (2-1/C_2)A(B^n(1/2);F)/4\end{aligned}$$

for every i and $j\ne i$.

For every i we fix a form

$$\eta_i = \frac{F^*(\xi_i)}{(K_O(F)(2C_1 + 1/C_0)A(B^n(1/2);F))^{1/p}}.$$

Then forms η_i are $\mathcal{A}$-harmonic. Since

$$\begin{aligned} \int_{B^n(4/5)} |F^*(\xi_i)|^p \, \mathrm{d}x &\leq K_O(F) \int_N n(y;B^n(4/5);F)|\xi_i|^p \, \mathrm{d}y \\ &= K_O(F)\nu_{\mu_i}(B^n(4/5);F) \end{aligned}$$

for every i and

$$\int_{B^n(5/8)} |F^*(\xi_i) - F^*(\xi_j)|^p \, \mathrm{d}x \geq \frac{1}{K_I(F)} \nu_{\mu_{ij}}(B^n(5/8);F)$$

for every $j \neq i$, we obtain

$$\int_{B^n(4/5)} |\eta_i|^p \, \mathrm{d}x \leq 1$$

and

$$\int_{B^n(5/8)} |\eta_i - \eta_j|^p \, \mathrm{d}x \geq \frac{2 - 1/C_2}{4K_I(F)K_O(F)(2C_1 + 1/C_2)}.$$

for every i and $j \neq i$. By Lemma 12, there exists a constant C depending on K, C_1, C_2, and constants α and β of $\mathcal{A}$ such that $d \leq C$. Since constants C_1 and C_2 depend only on n, ℓ, and K, and constants α and β of $\mathcal{A}$ depend only on n and K, this concludes the proof.

□

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Department of Mathematics and Statistics,
P.O. Box 68, FIN-00014 University of Helsinki, Finland.
E-mail: `pekka.pankka@helsinki.fi`