BLOCH–TO–BMOA COMPOSITIONS IN SEVERAL COMPLEX VARIABLES

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Abstract. We study analytic composition operators acting from the Bloch space into $BMOA$. The spaces are defined on the unit ball of $\mathbb{C}^n$. If the symbol satisfies a very mild regularity condition, it is possible to characterize boundedness and compactness.

1. Introduction.

We study analytic mappings $\varphi : \mathbb{B}_n \to \mathbb{B}_m$ and the corresponding analytic composition operators $C_\varphi : f \mapsto f \circ \varphi$. Here $n, m \in \mathbb{N}$ and $\mathbb{B}_n$ is the unit ball of $\mathbb{C}^n$. In the one complex variable case $n = m = 1$, $\mathbb{D} := \mathbb{B}_1$, the investigation of composition operators from the Bloch space $\mathcal{B}(\mathbb{D})$ into $BMOA(\mathbb{D})$ has only recently taken place. Boundedness and compactness of $C_\varphi : \mathcal{B}(\mathbb{D}) \to BMOA(\mathbb{D})$, $C_\varphi : \mathcal{B}_0(\mathbb{D}) \to VMOA(\mathbb{D})$ and $C_\varphi : \mathcal{B}(\mathbb{D}) \to VMOA(\mathbb{D})$ has been studied in [SZ] by Smith and Zhao and by Makhmutov and Tjani in [MT]. Madigan and Matheson [MM] proved that C_φ is always bounded on $\mathcal{B}(\mathbb{D})$. Moreover, [MM] contains a characterization of symbols φ inducing compact composition operators on $\mathcal{B}(\mathbb{D})$ and $\mathcal{B}_0(\mathbb{D})$. The essential norm of a composition operator from $\mathcal{B}(\mathbb{D})$ into $Q_p(\mathbb{D})$ was computed in [LMT].

In the case of several complex variables, Ramey and Ullrich [RU] have studied the case mentioned in the beginning: their result states that if $\varphi : \mathbb{B}_n \to \mathbb{D}$ is Lipschitz, then $C_\varphi : \mathcal{B}(\mathbb{D}) \to BMOA(\mathbb{B}_n)$ is well defined, and consequently bounded by the closed graph theorem. Our results below are, of course, more general. The case of $C_\varphi : \mathcal{B}(\mathbb{B}_n) \to \mathcal{B}(\mathbb{B}_n)$ was considered by Shi and Luo [SL], where they proved that C_φ is always bounded and gave a necessary and sufficient condition for C_φ to be compact.

Our main result states that if $\varphi : \mathbb{B}_n \to \mathbb{B}_m$ satisfies a very mild regularity condition, then the boundedness of $C_\varphi : \mathcal{B}(\mathbb{B}_m) \to BMOA(\mathbb{B}_n)$ is characterized by the fact that $d\mu_\varphi(z) = \frac{(1-|z|^2)|R\varphi(z)|^2}{(1-|\varphi(z)|^2)^2} dA(z)$ is a Carleson measure (see notations below).

Similarly, a corresponding o–growth condition characterizes the compactness.

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$. For $z, w \in \mathbb{C}^n$ let $\langle z, w\rangle = \sum_{i=1}^n z_i \bar{w}_i$ denote the complex inner product on $\mathbb{C}^n$ and $|z| = \langle z, z\rangle^{1/2}$. The radial derivative operator is denoted by R; so, if $f : \mathbb{B}_n \to \mathbb{C}$ is analytic, then

$$Rf(z) := \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z) \qquad \text{for } z \in \mathbb{B}_n.$$

The complex gradient of f is given by $\nabla f(z) = (\frac{\partial f}{\partial z_1}(z), \frac{\partial f}{\partial z_2}(z), ..., \frac{\partial f}{\partial z_n}(z))$. Clearly $Rf(z) = \langle \nabla f(z), \bar{z}\rangle$. Let $\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$ denote the invariant gradient, where φ_a stands for the Möbius transformation of $\mathbb{B}_n$ given by $\varphi_a(z) = \frac{a-z}{1-\langle z,a\rangle}$. Note that on the other hand

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$Rf = \sum_k kF_k$, if $\sum_k F_k$ is the homogeneous expansion of f. If $\varphi : \mathbb{B}_n \to \mathbb{C}^m$ with $\varphi := (\varphi_1, \varphi_2, ..., \varphi_m)$, then $R\varphi := (R\varphi_1, R\varphi_2, ..., R\varphi_m)$.

The Rademacher functions $r_n : [0,1] \to \mathbb{R}$, $n \in \{0\} \cup \mathbb{N}$, are defined by $r_n(t) := \mathrm{sign}(\sin(2^n \pi t))$.

The Bloch space $\mathcal{B}(\mathbb{B}_n)$ is defined to consist of analytic functions $f : \mathbb{B}_n \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{B}} := \sup_{z\in\mathbb{B}_n} |\nabla f(z)|(1-|z|^2) < \infty.$$

Timoney [T] proved that $\|f\|_{\mathcal{B}}$ and $\|f\|_1 := \sup_{z\in\mathbb{B}_n} |Rf(z)|(1-|z|^2)$ are equivalent. The Bloch space $\mathcal{B}(\mathbb{B}_n)$ is a Banach space with the norm $||f|| := |f(0)| + ||f||_{\mathcal{B}}$. The little Bloch space $\mathcal{B}_0(\mathbb{B}_n)$ is the subspace of $\mathcal{B}(\mathbb{B}_n)$ for which $\lim_{|z|\to 1} |Rf(z)|(1-|z|^2) = 0$.

Let g be the invariant Green function defined by

$$g(z) = \int_{|z|}^1 (1-t^2)^{n-1} t^{-2n+1} dt,$$

and let $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^{n+1}}$, where dA is the normalized volume measure in $\mathbb{C}^n$.

The space $BMOA(\mathbb{B}_n)$ can be defined (see [CC] Theorem A, [OYZ1] Prop 1) as the space of analytic functions $f : \mathbb{B}_n \to \mathbb{C}$ with

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^2 g(\varphi_a(z)) d\lambda(z) < \infty.$$

We say that a positive Borel measure on $\mathbb{B}_n$ is a *Carleson measure* if there exists $c > 0$ such that for any $\xi \in \partial\mathbb{B}_n$ and $\delta > 0$ we have

$$\mu(B(\xi, \delta)) \leq c\delta^n,$$

where $B(\xi,\delta) = \{z \in \mathbb{B}_n : 1-\delta < |z| < 1, \frac{z}{|z|} \in S(\xi,\delta)\}$ and $S(\xi,\delta) = \{\nu \in \partial\mathbb{B}_n : |1-\langle \nu, \xi\rangle| < \delta\}$. It is well known that μ is a Carleson measure if and only if

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} d\mu(z) < \infty. \tag{1}$$

We shall write $|||d\mu||| = \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} d\mu(z)$.

There is a lot of bibliography concerning characterizations of BMOA in terms of Carleson measures (see [J1, J2] or see [ASX, OYZ2, Y] for Q_p spaces.) It is known that $f \in BMOA(\mathbb{B}_n)$ (see [OYZ2] Proposition 3.4) if and only if

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^2 (1-|\varphi_a(z)|^2)^n d\lambda(z) < \infty.$$

Now, taking into account that $1-|\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2}$, one obtains, using (1) that $f \in BMOA(\mathbb{B}_n)$ if and only if $\frac{|\tilde{\nabla} f(z)|^2}{1-|z|^2} dA(z)$ is a Carleson measure. Observe now that, a direct computation shows

$$|\tilde{\nabla} f(z)|^2 = (1-|z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2).$$

Therefore, using $|Rf(z)| \leq |\nabla f(z)||z|$, one gets

$$|\tilde{\nabla} f(z)|^2 \geq (1-|z|^2)^2 |\nabla f(z)|^2 \geq (1-|z|^2)^2 |Rf(z)|^2.$$

Thus

$$\frac{|\tilde{\nabla} f(z)|^2}{1-|z|^2}dA(z) \le (1-|z|^2)|\nabla f(z)|^2 dA(z) \le (1-|z|^2)|Rf(z)|^2 dA(z).$$

The following theorem is due to several authors. A complete proof of the equivalences of (i), (ii) and (iii) has been presented by Zhu in [Z]. Further, (iii) and (iv) are equivalent by (1).

Theorem 1. *The following are equivalent.*
(i) $f \in BMOA(\mathbb{B}_n)$.
(ii) $(1-|z|^2)|\nabla f(z)|^2 dA(z)$ *is a Carleson measure.*
(iii) $(1-|z|^2)|Rf(z)|^2 dA(z)$ *is a Carleson measure.*
(iv) $\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |Rf(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z) < \infty$.

Hence we define the space $BMOA(\mathbb{B}_n)$ (or just $BMOA$) to consist of all analytic functions $f : \mathbb{B}_n \to \mathbb{C}$ with

$$\|f\|_{BMOA} := \sup_{a\in\mathbb{B}_n} \Big(\int_{\mathbb{B}_n} |Rf(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z)\Big)^{1/2} < \infty.$$

The space $BMOA$ is a Banach space with the norm $||f|| := |f(0)| + ||f||_{BMOA}$.

Since $C_{\varphi_a} : \mathcal{B}(\mathbb{B}_m) \to \mathcal{B}(\mathbb{B}_m)$ is always bounded and invertible, we can without loss of generality assume that $\varphi(0) = 0$ in our investigation of boundedness and compactness of $C_\varphi : \mathcal{B}(\mathbb{B}_m) \to BMOA(\mathbb{B}_n)$.

2. First results.

Let $\varphi : \mathbb{B}_n \to \mathbb{B}_m$ be an analytic map such that $\varphi(0) = 0$. An application of Schwarz's lemma gives that

$$|\varphi(z)| \le |z|, \tag{2}$$

$$|R\varphi(z)| \le \frac{1-|\varphi(z)|^2}{1-|z|^2}. \tag{3}$$

We define $F_\varphi(z) = \frac{(1-|z|^2)|R\varphi(z)|^2}{(1-|\varphi(z)|^2)^2}$ and write $d\mu_\varphi(z) = F_\varphi(z)dA(z)$.

From (3) it follows that $F_\varphi(z) \le \frac{1}{1-|z|^2}$. Hence, in general $F_\varphi \notin L^1(\mathbb{B}_n, dA)$. Using (1) one has that μ_φ is a Carleson measure if and only if

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|R\varphi(z)|^2}{(1-|\varphi(z)|^2)^2}(1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z) < \infty. \tag{4}$$

We start by showing that this condition is sufficient for the boundedness of the composition operator. The result holds without any additional assumptions.

Theorem 2. *Let* n, $m \in \mathbb{N}$ *and let* $\varphi : \mathbb{B}_n \to \mathbb{B}_m$ *be analytic. If*

$$d\mu_\varphi(z) = \frac{(1-|z|^2)|R\varphi(z)|^2}{(1-|\varphi(z)|^2)^2}dA(z)$$

is a Carleson measure then the operator $C_\varphi : \mathcal{B}(\mathbb{B}_m) \to BMOA(\mathbb{B}_n)$ *is bounded.*

Proof. We have, for every $f \in \mathcal{B}(\mathbb{B}_m)$,

$$R(f \circ \varphi)(z) = \sum_{j=1}^{m} \frac{\partial f}{\partial z_j}(\varphi(z)) R\varphi_j(z),$$

so $|R(f \circ \varphi)(z)| \leq |\nabla f(\varphi(z))| \, |R\varphi(z)|$. Therefore

$$\|C_\varphi f\|^2_{BMOA} = \sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} |R(f \circ \varphi)(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^n d\lambda(z)$$

$$\leq \sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} |\nabla f(\varphi(z))|^2 |R\varphi(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^n d\lambda(z)$$

$$\leq \|f\|^2_{\mathcal{B}} \sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} d\mu_\varphi(z) \leq C \|f\|^2_{\mathcal{B}}.$$

□

This of course contains the case $m = 1$. In that case the reverse direction can also be proven by existing methods, so we get

Theorem 3. *Let $n \in \mathbb{N}$ and let $\varphi : \mathbb{B}_n \to \mathbb{D}$ be analytic. The operator $C_\varphi : \mathcal{B}(\mathbb{D}) \to BMOA(\mathbb{B}_n)$ is bounded, if and only if μ_φ is a Carleson measure.*

To prove the necessity, we take two analytic functions $f_j \in \mathcal{B}(\mathbb{D})$, $j = 1, 2$, such that $|f_1'(z)| + |f_2'(z)| \geq C/(1-|z|)$ for all $z \in \mathbb{D}$ (see [RU]). Since the composition operator is assumed bounded, we get

$$C_1 \geq \sum_{j=1}^{2} \|C_\varphi f_j\|^2_{BMOA} = \sum_{j=1}^{2} \sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} |R(f_j \circ \varphi)(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^n d\lambda(z)$$

$$= \sup_{a \in \mathbb{B}_n} \sum_{j=1}^{2} \int_{\mathbb{B}_n} |f_j'(\varphi(z))|^2 |R\varphi(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^n d\lambda(z)$$

$$\geq C^2/2 \sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} d\mu_\varphi(z).$$

Surprising difficulties arise when trying to generalize the above argument to the case $m \geq 2$. We mention that Choe and Rim generalized in [CR] the construction of the "test functions" of Ramey and Ullrich to higher dimensions. However, this seems not to be enough for a proof of the necessity of the Carleson measure condition of μ_φ. The reason is that as a consequence of the use of the chain rule in the expression $R(f \circ \varphi)$, one will need a lower bound for $|\langle \varphi, R\varphi \rangle|$. This is analyzed in the later sections, see especially (32) and (33) for the derivative of our test functions.

The following necessary conditions for the boundedness of $C_\varphi : \mathcal{B}(\mathbb{B}_m) \to BMOA(\mathbb{B}_n)$ with general n, m, can be derived more easily:

$$\sup_{f:\mathbb{B}_m \to \mathbb{D} \text{ analytic}} \sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|\langle R\varphi(z), \overline{\nabla f(\varphi(z))}\rangle|^2}{(1-|f(\varphi(z))|^2)^2} (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^n d\lambda(z) < \infty, \tag{5}$$

$$\sup_{w\in\mathbb{B}_m}\sup_{a\in\mathbb{B}_n}\int\limits_{\mathbb{B}_n}\frac{|\langle R\varphi(z),\overline{w}\rangle|^2}{(1-|\langle\varphi(z),w\rangle|^2)^2}(1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z)<\infty. \tag{6}$$

Here (5) follows by applying Theorem 3 to the bounded composition operator $C_{f\circ\varphi}$: $\mathcal{B}(\mathbb{D})\to BMOA(\mathbb{B}_m)$. (6) is a special case of (5): $f(z):=\langle z,w\rangle$ for a fixed $w\in\mathbb{B}_m$.

In particular, if $C_\varphi:\mathcal{B}(\mathbb{B}_m)\to BMOA(\mathbb{B}_n)$ is bounded then, for $i=1,...,m$

$$d\mu_{\varphi_i}(z)=\frac{(1-|z|^2)|R\varphi_i(z)|^2}{(1-|\varphi_i(z)|^2)^2}dA(z)$$

are Carleson measures.

3. Basic regularity condition for the symbol.

Let $\varphi:\mathbb{B}_n\to\mathbb{B}_m$ be a holomorphic mapping and $F_\varphi(z)=\frac{(1-|z|^2)|R\varphi(z)|^2}{(1-|\varphi(z)|^2)^2}$. Note that if F_φ is bounded then $d\mu_\varphi\le\|F_\varphi\|_\infty dA(z)$, and hence C_φ is bounded.

For $0<s<1$ we denote

$$\Omega_s:=\{z\in\mathbb{B}_n\mid|\varphi(z)|>s,\ |F_\varphi(z)|>\frac{1}{1-s^2}\}.$$

It follows from (2) that Ω_s is an open subset of $\mathbb{B}_n$ contained into $\{z:|z|>s\}$.

Given $z\in\mathbb{B}_n$ and $0<r<1$, we denote by $I_r(z)\subset\mathbb{B}_n$ the line segment joining rz and z: $I_r(z):=\{\zeta\mid\zeta=sz$ for some $s\in[r,1]\ \}$.

Given $z\in\mathbb{B}_n$ and $0<h<1$, we denote by $J_h(z)\subset\mathbb{B}_n$ the *non–tangential cone*

$$J_h(z):=\Big\{\ \xi\in\mathbb{B}_n\ \Big|\ \Big|\Big\langle\frac{z}{|z|},\frac{z-\xi}{|z-\xi|}\Big\rangle\Big|\ge h\Big\}.$$

Lemma 1. *Assume that the holomorphic mapping $\varphi:\mathbb{B}_n\to\mathbb{B}_m$ satisfies the following condition for some $0<h<1$ and $0<s<1$: For every $z\in\Omega_s$ there exists $0<r<1$ such that the line segment $I_r(z)$ is mapped by φ into the non–tangential cone $J_h(\varphi(z))$. Then*

$$\Big|\Big\langle\frac{\varphi(z)}{|\varphi(z)|},\frac{R\varphi(z)}{|R\varphi(z)|}\Big\rangle\Big|\ge h/2 \tag{7}$$

for all $z\in\Omega_s$.

Proof. Suppose that the contrary of (7) holds for a $z\in\Omega_s$:

$$\Big|\Big\langle\frac{\varphi(z)}{|\varphi(z)|},\frac{R\varphi(z)}{|R\varphi(z)|}\Big\rangle\Big|<\frac{h}{2}. \tag{8}$$

By redefining the corresponding r to be smaller, if necessary, we may assume, by continuity, that

$$\Big|\frac{(R\varphi_1(\zeta_1),R\varphi_2(\zeta_2),...,R\varphi_m(\zeta_m))}{|(R\varphi_1(\zeta_1),R\varphi_2(\zeta_2),...,R\varphi_m(\zeta_m))|}-\frac{R\varphi(z)}{|R\varphi(z)|}\Big|\le\frac{h}{100} \tag{9}$$

for all $\zeta_1,\zeta_2,...,\zeta_m\in I_r(z)$; here $\varphi:=(\varphi_1,\varphi_2,...,\varphi_m)$.

The radial derivative $R\varphi(\xi)$ equals

$$\lim_{\varepsilon\to0}\frac{\varphi(\xi)-\varphi((1-\varepsilon)\xi)}{\varepsilon},$$

hence, by the mean value theorem applied to the function $\psi : s \mapsto \psi(s) := \varphi(sz)$, $s \in [r, 1]$, for $\xi \in I_r(z)$,

$$\varphi(\xi) = \varphi(z) + (R\varphi_1(\zeta_1), R\varphi_2(\zeta_2), ..., R\varphi_m(\zeta_m))\frac{|\xi - z|}{|z|} \tag{10}$$

for some points $\zeta_1, \zeta_2, ..., \zeta_n \in I_r(z)$.

We note that the right hand side of (10) cannot be a point of $J_h(\varphi(z))$: by (8), and (9),

$$\begin{aligned}
&\Big|\Big\langle \frac{\varphi(z)}{|\varphi(z)|}, \frac{\varphi(z) - (\varphi(z) + (R\varphi_1(\zeta_1), R\varphi_2(\zeta_2), ..., R\varphi_n(\zeta_n))|\xi - z|/|z|\ )}{|\varphi(z) - (\varphi(z) + (R\varphi_1(\zeta_1), R\varphi_2(\zeta_2), ..., R\varphi_n(\zeta_n)|\xi - z|/|z|\ )|} \Big\rangle\Big| \\
= \ &\Big|\Big\langle \frac{\varphi(z)}{|\varphi(z)|}, \frac{(R\varphi_1(\zeta_1), R\varphi_2(\zeta_2), ..., R\varphi_m(\zeta_m))|\xi - z|/|z|}{|(R\varphi_1(\zeta_1), R\varphi_2(\zeta_2), ..., R\varphi_m(\zeta_m))|\xi - z|/|z||} \Big\rangle\Big| \\
\le \ &\Big|\Big\langle \frac{\varphi(z)}{|\varphi(z)|}, \frac{R\varphi(z)}{|R\varphi(z)|} \Big\rangle\Big| + \frac{h}{100} \le \frac{3h}{4}. \qquad (11)
\end{aligned}$$

Contradiction: φ does not map $I_r(z)$ into $J_h(\varphi(z))$. Hence, (7) is true. $\square$

4. Properties of lacunary series.

In Sections 4 and 5 the number h, $0 < h < 1$, is fixed to be as in Lemma 1.

We define a pseudometric on the boundary of the unit ball:

$$d(\zeta, \xi) := \Big(1 - |\langle \zeta, \xi\rangle|^2\Big)^{1/2}, \quad \zeta, \xi \in \partial\mathbb{B}_n. \tag{12}$$

Note that d satisfies the triangular inequality. Given $\delta > 0$ and $\zeta \in \partial\mathbb{B}_n$ we denote the d–ball with center ζ and radius δ by

$$E_\delta(\zeta) := \{\xi \in \partial\mathbb{B}_n \mid d(\zeta, \xi) < \delta\}. \tag{13}$$

We say that a set $\Gamma \subset \partial\mathbb{B}_n$ is *d-separated by δ*, if d–balls with radius δ and centers in the points of Γ, are pairwise disjoint.

The following result is proved by Ullrich in [U]. See also Lemma 2.2 of [CR].

Lemma 2. *For every (small) $A > 0$ there exists an $M \in \mathbb{N}$ with the following property: if $\delta > 0$ and $\Gamma \subset \partial\mathbb{B}_n$ is d–separated by $A\delta/2$, then Γ can be decomposed as $\Gamma = \cup_{k=1}^M \Gamma_k$ such that every Γ_k is d–separated by δ.*

Let us fix $0 < A \le 10^{-3}$ such that

$$\sum_{m=1}^{\infty} (m+2)^{2n-2} e^{-m^2/(4A)^2} \le \frac{h}{100 \cdot 3^3}, \tag{14}$$

and let then $M \in \mathbb{N}$ be fixed as in Lemma 2. Further, let us fix $p > 1$ large enough, such that

$$\Big(1 - \frac{1}{p}\Big)^p \ge \frac{1}{3}\ , \text{ and} \tag{15}$$

$$pA^2 \ge \frac{10^6}{h^2}. \tag{16}$$

For every $j = 1, 2, \ldots, M$, choose $\delta_{j,0} > 0$ such that

$$A^2 p^j \delta_{j,0}^2 = 1, \tag{17}$$

and then inductively choose the numbers $\delta_{j,\nu}$ for $\nu = 1, 2, \dots$ such that

$$p^M \delta_{j,\nu}^2 = \delta_{j,\nu-1}^2. \tag{18}$$

Clearly, since $p > 1$, every $(\delta_{j,\nu})_{\nu=1}^{\infty}$ is an exponentially decreasing sequence, and by (16)

$$\delta_{j,\nu}^2 < \frac{h^2}{10^6} \qquad \text{for all } j, \nu. \tag{19}$$

Moreover,

$$A^2 p^{\nu M + j} \delta_{j,\nu}^2 = 1 \tag{20}$$

for every j and ν. For every $j = 1, \dots, M$ and $\nu = 1, 2, \dots$, let $\Gamma^{j,\nu} \subset \partial\mathbb{B}_n$ be a maximal subset which is d–separated by $A\delta_{j,\nu}/2$. (In particular, for every $z \in \partial\mathbb{B}_n$ there exists $\xi \in \Gamma^{j,\nu}$ such that $d(z,\xi) \leq A\delta_{j,\nu}$; otherwise $\Gamma^{j,\nu}$ is not maximal.) Using Lemma 2 we define the sets $\Gamma_{j,\nu M+k}$, which are d–separated by $\delta_{j,\nu}$, such that

$$\Gamma^{j,\nu} = \bigcup_{k=1}^{M} \Gamma_{j,\nu M+k}. \tag{21}$$

Finally we define a set of functions; these depend on some unspecified factors, though we do not display this dependence in the following.

Definition 1. *Let $j, k \in \{1, 2, \dots, M\}$ and $\nu \in \mathbb{N}$ be given. Let $\gamma_{j,k,\nu} : \partial\mathbb{B}_n \times \partial\mathbb{B}_n \to \mathbb{C}$ be an arbitrary function such that*
(i) $|\gamma_{j,k,\nu}(z,\zeta)| \geq h/100$, if z, ζ satisfy $d(z,\zeta) \leq \delta_{j,\nu}$,
(ii) $|\gamma_{j,k,\nu}(z,\zeta)| \leq 1$ for all z and ζ.
Let us define

$$P_{k,\nu M+j}(z) := \sum_{\zeta \in \Gamma_{j,\nu M+[k+j]}} \gamma_{j,k,\nu}(z,\zeta) \langle z, \zeta \rangle^{p^{\nu M+j}-1}, \tag{22}$$

where $[k+j] := k+j$, if $k+j \leq M$, and $[k+j] := k+j-M$, if $M < k+j \leq 2M$.

Lemma 3. *For all ν, the functions of Definition 1 satisfy the bounds*

$$2M^2 \geq \sum_{j,k=1}^{M} |P_{k,\nu M+j}(z)| \geq C := C(h) \qquad \text{for } z \in \partial\mathbb{B}_n \tag{23}$$

Remark. We emphasize that the last $C > 0$ is independent of ν and the choice of the functions $\gamma_{j,k,\nu}$.

Proof. The proof is an improvement of [CR], Theorem 2.1.

Let ν and $z \in \partial\mathbb{B}_n$ be given. By the constructions above we can pick j and k such that for some $\xi \in \Gamma_{j,\nu M+[k+j]}$ we have $d(z,\xi) \leq A\delta_{j,\nu} \leq \delta_{j,\nu}$. We have, by (20), Definition 1 (i) and (15),

$$\begin{aligned} |\gamma_{j,k,\nu}(z,\xi)\langle z,\xi\rangle^{p^{\nu M+j}-1}| &\geq \frac{h}{100}\Big(1 - A^2\delta_{j,\nu}^2\Big)^{p^{\nu M+j}/2} \\ &= \frac{h}{100}\Big(1 - \frac{1}{p^{\nu M+j}}\Big)^{p^{\nu M+j}/2} \geq \frac{h}{300}. \end{aligned} \tag{24}$$

We aim to show that the contribution of the other terms in (22) is negligible in comparison with this term. Since we are proving a lower bound, it suffices to consider just the indices j and k fixed above.

For $0 < r < 1$ and $\zeta \in \partial\mathbb{B}_n$, the normalized surface area measure σ of $E_r(\zeta)$ can be calculated:

$$\sigma(E_r(\zeta)) = r^{2n-2}. \tag{25}$$

Let us define for every $m = 0, 1, 2, \ldots$, the set

$$H_m(z) := \{\zeta \in \Gamma_{j,\nu M+[k+j]} \mid m\delta_{j,\nu} \le d(z,\zeta) \le (m+1)\delta_{j,\nu}\}. \tag{26}$$

The number $\#(H_0(z))$, i.e. the cardinality of $H_0(z)$, equals 1, by the construction of the sets Γ. To count $\#(H_m(z))$ for $m > 0$, we have

$$\bigcup_{\zeta\in H_m(z)} E_{\delta_{j,\nu}}(\zeta) \subset E_{(m+2)\delta_{j,\nu}}(z),$$

hence, by (25),

$$\delta_{j,\nu}^{2n-2}\#(H_m(z)) = \sigma(E_{\delta_{j,\nu}}(\zeta))\#(H_m(z)) \le \sigma(E_{(m+2)\delta_{j,\nu}}(z)).$$

We thus get

$$\#(H_m(z)) \le (m+2)^{2n-2}. \tag{27}$$

By (26) and (12),

$$1 - (m+1)^2\delta_{j,\nu}^2 \le |\langle z,\zeta\rangle|^2 \le 1 - m^2\delta_{j,\nu}^2,$$

if $\zeta \in H_m(z)$. Using this and (27),

$$\begin{aligned}
&\sum_{\substack{\zeta\in\Gamma_{j,\nu M+[k+j]}\\ \zeta\neq\xi}} |\gamma_{j,k,\nu}(z,\zeta)|\, |\langle z,\zeta\rangle|^{p^{\nu M+j}-1} \\
\le\ & \sum_{\substack{\zeta\in\Gamma_{j,\nu M+[k+j]}\\ \zeta\neq\xi}} |\langle z,\zeta\rangle|^{p^{\nu M+j}-1} = \sum_{m=1}^{\infty}\sum_{\zeta\in H_m(z)} |\langle z,\zeta\rangle|^{p^{\nu M+j}-1} \\
\le\ & \sum_{m=1}^{\infty}(1-m^2\delta_{j,\nu}^2)^{\frac12 p^{\nu M+j}-\frac12}\#(H_m(z)) \\
\le\ & \sum_{m=1}^{\infty}(1-m^2\delta_{j,\nu}^2)^{\frac12 p^{\nu M+j}-\frac12}(m+2)^{2n-2} \\
\le\ & \sum_{m=1}^{\infty} e^{-\frac12 m^2\delta_{j,\nu}^2(p^{\nu M+j}-1)}(m+2)^{2n-2} \\
\le\ & \sum_{m=1}^{\infty} e^{-m^2(\frac{1}{2A^2}-\frac12)}(m+2)^{2n-2} \\
\le\ & \sum_{m=1}^{\infty} e^{-m^2/(4A)^2}(m+2)^{2n-2} \le \frac{h}{100\cdot 3^3},
\end{aligned}$$

by (20) and (14). Combining with (24), the lower bound in (23) follows. Finally, we see that $|P_{k,\nu M+j}(z)| \le 2$ for all $z \in \mathbb{B}_n$. $\square$

Lemma 4. *For every $\nu \in \mathbb{N}$ and $j, k = 1, \ldots, M$, let the set $\Gamma_{j,\nu M+[k+j]} \subset \mathbb{B}_n$ be as above, and let $(\alpha_\nu)_{\nu\in\mathbb{N}}$ be a complex valued sequence with $|\alpha_\nu| \le 1$ for every ν. Then every analytic function*

$$f(z) := \sum_{\nu\in\mathbb{N}} \alpha_\nu Q_{k,\nu M+j}(z) := \sum_{\nu\in\mathbb{N}} \alpha_\nu \sum_{\zeta\in\Gamma_{j,\nu M+[k+j]}} \langle z,\zeta\rangle^{p^{\nu M+j}} \quad , \qquad z \in \mathbb{B}_n, \tag{28}$$

belongs to $\mathcal{B}(\mathbb{B}_n)$, and $\|f\|_{\mathcal{B}} \le C$ (C is independent of ν, j and k). If the sequence $(\alpha_\nu)_{\nu\in\mathbb{N}}$ tends to zero, then $f \in \mathcal{B}_0(\mathbb{B}_n)$.

Proof. It is elementary to show that $R(Q_{k,\nu M+j}) = p^{\nu M+j} Q_{k,\nu M+j}$. Then we obtain

$$|R(Q_{k,\nu M+j})(z)| \le p^{\nu M+j} |z|^{p^{\nu M+j}} Q_{k,\nu M+j}(\frac{z}{|z|}),$$

and moreover

$$|R(Q_{k,\nu M+j})(z)| \le C p^{\nu M+j} |z|^{p^{\nu M+j}} \le C \frac{p^M}{p^M-1}(p^{\nu M+j} - p^{(\nu-1)M+j})|z|^{p^{\nu M+j}}.$$

This gives

$$\begin{aligned} |Rf(z)| &\le C(\sum_{\nu\in\mathbb{N}} |\alpha_\nu||R(Q_{k,\nu M+j})(z)| \\ &\le C\frac{p^M}{p^M-1}\sum_{\nu\in\mathbb{N}}(p^{\nu M+j} - p^{(\nu-1)M+j})|z|^{p^{\nu M+j}} \\ &\le C\frac{p^M}{p^M-1}(\sum_{\nu\in\mathbb{N}} \sum_{p^{(\nu-1)M+j} \le n < p^{\nu M+j}} |z|^n) \le \frac{C_p}{1-|z|}. \end{aligned}$$

If $\alpha_\nu \to 0$, then we can choose N so big that $|\alpha_\nu| < \varepsilon$ for $\nu \ge N$. With

$$f(z) := \sum_{\nu=0}^{N-1} \alpha_\nu Q_{k,\nu M+j}(z) + \sum_{\nu=N}^{\infty} \alpha_\nu Q_{k,\nu M+j}(z)$$

we see that

$$\lim_{|z|\to 1} (1-|z|^2)|Rf(z)| \le 2C_p\, \varepsilon$$

for all $\varepsilon > 0$. $\square$

5. Main results.

Recall that for $\varphi : \mathbb{B}_n \to \mathbb{B}_m$ is holomorphic, $F_\varphi(z) = \frac{|R\varphi(z)|^2(1-|z|^2)}{(1-|\varphi(z)|^2)^2}$ and $\Omega_r := \{z \in \mathbb{B}_n \mid |\varphi(z)| > r,\ |F_\varphi(z)| > \frac{1}{1-r^2}\}$, which is an open subset of $\mathbb{B}_n$ for $0 < r < 1$.

Let us use the notation $d\mu_{\varphi,s}(z) = \chi_{\Omega_s}(z) F_\varphi(z) dA(z)$. Clearly $|||d\mu_{\varphi,s}||| \le |||d\mu_\varphi|||$.

Proposition 1. *Let $n \in \mathbb{N}$. Then $d\mu_\varphi(z) = F_\varphi(z)dA(z)$ is a Carleson measure if and only if $d\mu_{\varphi,s}(z) = \chi_{\Omega_s}(z)F_\varphi(z)dA(z)$ is a Carleson measure for some $0 < s < 1$.*

Proof. It suffices to show that

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n\setminus\Omega_s} F_\varphi(z) \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} dA(z) \le C. \tag{29}$$

If $z \in \mathbb{B}_n \setminus \Omega_s$ then either $F_\varphi(z) \le \frac{1}{1-s^2}$ or $|\varphi(z)| \le s$.

If $|\varphi(z)| \leq s$ and $a \in \mathbb{B}_n$ then

$$\int\limits_{\mathbb{B}_n \setminus \Omega_s} F_\varphi(z) \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} dA(z) \leq \frac{1}{1-s^2} \int\limits_{\mathbb{B}_n} (1-|z|^2)|R\varphi(z)|^2 \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} dA(z) \leq \frac{C}{1-s^2}$$

where the last estimate follows from the embedding $H^\infty(\mathbb{B}_n) \subset BMOA(\mathbb{B}_n)$ and $\varphi_i \in H^\infty(\mathbb{B}_n)$ for $i = 1, ..., m$.

If $F_\varphi(z) \leq \frac{1}{1-s^2}$ and $a \in \mathbb{B}_n$ then

$$\int\limits_{\mathbb{B}_n \setminus \Omega_s} F_\varphi(z) \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} dA(z) \leq \frac{1}{1-s^2} \int\limits_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} dA(z)$$

where the last integral is bounded by $1-|a|^2$ if $n > 1$ and by $(1-|a|^2)\log \frac{1}{1-|a|^2}$ if $n = 1$ (see Rudin [R], p.17 for this estimate). Hence (29) is shown.

■

Theorem 4. *Assume that φ satisfies the non–tangentiality condition of Lemma 1. Then the composition operator $C_\varphi : f \mapsto f \circ \varphi$ is bounded from $\mathcal{B}(\mathbb{B}_m)$ into $BMOA(\mathbb{B}_n)$ if and only if $d\mu_\varphi(z) = F_\varphi(z)dA(z)$ is a Carleson measure.*

Proof. The "if"-statement is Theorem 2.

We turn to the "only if"–statement. Let $h, s \in (0,1)$ be fixed as in Lemma 1.

From Proposition 1 it suffices to show that

$$\sup_{a \in \mathbb{B}_n} \int\limits_{\Omega_s} F_\varphi(z) \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} dA(z) \leq C. \tag{30}$$

For every $j, k = 1, 2, \ldots, M$ and $t \in [0,1]$ we define the analytic function

$$f_{j,k,t}(z) := \sum_{\nu \in \mathbb{N}} r_\nu(t) Q_{k,\nu M+j}(z), \qquad z \in \mathbb{B}_m,$$

where $r_\nu : [0,1] \to \mathbb{R}$ is the νth Rademacher function and $Q_{k,\nu M+j} := \sum_{\zeta \in \Gamma_{j,\nu M+[k+j]}} \langle z, \zeta\rangle^{p^{\nu M+j}}$. Lemma 4 states that every $f_{j,k,t}$ belongs to $\mathcal{B}(\mathbb{B}_m)$ and that $\|f_{j,k,t}\|_{\mathcal{B}} \leq C_1$.

We are assuming that the composition operator $C_\varphi : \mathcal{B}(\mathbb{B}_m) \to BMOA(\mathbb{B}_n)$ is bounded. Defining the measure $d\mu_a(z) := (1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z)$ on $\mathbb{B}_n$, this means that the operator family

$$T_a : \mathcal{B}(\mathbb{B}_m) \to L^2(d\mu_a) \quad , \quad f \mapsto R(f \circ \varphi) \ ,$$

is bounded uniformly with respect to a. (Denote the norm of $L^2(d\mu_a)$ by $\|\cdot\|_{2,a}$.) We thus find a constant $C_2 > 0$ such that

$$\sup_{a \in \mathbb{B}_n} \|R(f_{j,k,t} \circ \varphi)\|_{2,a}^2 \leq C_2$$

for all j, k and t. Integrating with respect to t, using Fubini's theorem and the orthogonality property of the Rademacher functions we get

$$\int_{\mathbb{B}_n} \sum_{\nu \in \mathbb{N}} |R(Q_{k,\nu M+j} \circ \varphi)(z)|^2 d\mu_a(z) = \int\limits_0^1 \|R(f_{j,k,t} \circ \varphi)\|_{2,a}^2 dt \leq C_2. \tag{31}$$

This inequality still holds with a different C_2, if a summation over all indices j and k is added to the left hand side; for each ν there exist M indices j and k.

Let us fix ν for a moment and bound $R(Q_{k,\nu M+j}\circ\varphi)$ from below. For all $z\in\Omega_s$ we have

$$\begin{aligned} & R(Q_{k,\nu M+j}\circ\varphi)(z) \\ &= p^{\nu M+j}\sum_{\zeta\in\Gamma_{j,\nu M+[k+j]}}\langle\varphi(z),\zeta\rangle^{p^{\nu M+j}-1}\langle R\varphi(z),\zeta\rangle \\ &= p^{\nu M+j}|\varphi(z)|^{p^{\nu M+j}-1}\,|R\varphi(z)|\sum_{\zeta\in\Gamma_{j,\nu M+[k+j]}}\langle\eta(z),\zeta\rangle^{p^{\nu M+j}-1}\langle\eta'(z),\zeta\rangle, \end{aligned}\tag{32}$$

where we denoted $\eta:=\varphi/|\varphi|$ and $\eta':=R\varphi/|R\varphi|$.

We claim that

$$\sum_{j,k=1}^{M}\Big|\sum_{\zeta\in\Gamma_{j,\nu M+[k+j]}}\langle\eta(z),\zeta\rangle^{p^{\nu M+j}-1}\langle\eta'(z),\zeta\rangle\Big|\geq C(h)\tag{33}$$

for every $z\in\Omega_s$. To prove this we use Lemma 3. Given z we find j and k such that $d(\eta(z),\xi)\leq A\delta_{j,\nu}\leq h10^{-6}$ for some $\xi\in\Gamma_{j,\nu M+[k+j]}$. Let $\xi_1:=\xi-\langle\xi,\eta(z)\rangle\eta(z)$. Use the definition of d to obtain that $|\xi_1|\leq\sqrt{2}\,h10^{-6}$ and $|\langle\xi,\eta(z)\rangle|\geq\frac{1}{2}$. By Lemma 1,

$$|\langle\eta'(z),\xi\rangle|\geq|\langle\xi,\eta(z)\rangle|\,|\langle\eta'(z),\eta(z)\rangle|-|\langle\eta'(z),\xi_1\rangle|\geq\frac{h}{10}.\tag{34}$$

In Lemma 3 we choose $w\in\partial\mathbb{B}_n$ such that $w=\eta(z)$, and then $\gamma_{j,k,\nu}(w,\zeta):=\langle\eta'(z),\zeta\rangle$ for all j,k. For other values w, the numbers $\gamma_{j,k,\nu}(w,\zeta)$ are set equal 1. In Lemma 3, $P_{k,\nu M+j}(w)$ coincides with

$$\sum_{\zeta\in\Gamma_{j,\nu M+[k+j]}}\langle\eta(z),\zeta\rangle^{p^{\nu M+j}-1}\langle\eta'(z),\zeta\rangle$$

for all j,k, and because of (34), Lemma 3 applies. Hence, (33) follows. The result is just for this z, but the estimate is z–independent.

Returning to (32) and observing that

$$\Big(\sum_{j,k=1}^{M}|R(Q_{k,\nu M+j}\circ\varphi)(z)|\Big)^2\leq M^2\sum_{j,k=1}^{M}|R(Q_{k,\nu M+j}\circ\varphi)(z)|^2$$

it follows

$$M^2\sum_{j,k=1}^{M}|R(Q_{k,\nu M+j}\circ\varphi)(z)|^2\geq C^2(h)p^{2\nu M}|\varphi(z)|^{2(p^{(\nu+1)M}-1)}|R\varphi(z)|^2$$

for every $z\in\Omega_s$. Hence by (31),

$$\begin{aligned} M^2C_2 &\geq C^2(h)\sup_{a\in\mathbb{B}_n}\int_{\Omega_s}\sum_{\nu\in\mathbb{N}}p^{2\nu M}|\varphi(z)|^{2(p^{(\nu+1)M}-1)}|R\varphi(z)|^2d\mu_a(z) \\ &\geq C_4\sup_{a\in\mathbb{B}_n}\int_{\Omega_s}\frac{|R\varphi(z)|^2}{(1-|\varphi(z)|^2)^2}(1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z), \end{aligned}\tag{35}$$

for some constant C_4. In the last inequality we used ($0 < b < 1$)

$$\begin{aligned}\frac{1}{(1-b)^2} &= \sum_{n=0}^{\infty}(n+1)b^n \\ &\leq C_1 \sum_{\nu=0}^{\infty} \sum_{n=p^{(\nu+1)M}}^{p^{(\nu+1)M+M}} n b^{n-1} \\ &\leq C_1 \sum_{\nu=0}^{\infty} \sum_{n=p^{(\nu+1)M}}^{p^{(\nu+1)M+M}} p^{(\nu+1)M+M} b^{p^{(\nu+1)M}-1} \\ &\leq C_2 \sum_{\nu=0}^{\infty} p^{2(\nu+1)M} b^{p^{(\nu+1)M}-1} \\ &\leq C_3 \sum_{\nu=0}^{\infty} p^{2\nu M} b^{p^{(\nu+1)M}-1}.\end{aligned}$$

Thus (30) is shown and the proof is finished. □

Proposition 2. *If* $\lim_{r\to 1} |||d\mu_{\varphi,r}||| = 0$, *i.e.*

$$\lim_{r\to 1} \sup_{a\in\mathbb{B}_n} \int_{\Omega_r} F_\varphi(z) \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} dA(z) = 0, \tag{36}$$

then $C_\varphi : \mathcal{B}(\mathbb{B}_m) \to BMOA(\mathbb{B}_n)$ *is compact.*

Proof. For every $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that as $r \in [\delta, 1)$ we have

$$\sup_{a\in\mathbb{B}_n} \int_{\Omega_r} F_\varphi(z) \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} dA(z) < \varepsilon.$$

This estimate and (29) show that μ_φ is a Carleson measure, and hence C_φ is bounded. Let us now show that it is compact.

Let (f_i) be a sequence in $\mathcal{B}(\mathbb{B}_m)$, $||f_i|| \leq 1$, which converges to zero uniformly on compact subsets of $\mathbb{B}_m$. We show that $f_i \circ \varphi \to 0$ in the norm of $BMOA(\mathbb{B}_n)$. Since $||f_i|| \leq 1$ and $|R(f_i \circ \varphi)(z)| \leq |\nabla f_i(\varphi(z))| \, |R\varphi(z)| \leq \frac{|R\varphi(z)|}{1-|\varphi(z)|^2}$, we have for all i,

$$\sup_{a\in\mathbb{B}_n} \int_{\Omega_\delta} |R(f_i \circ \varphi)(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^n d\lambda(z) < \varepsilon.$$

Now $f_i \to 0$ on compact subsets of $\mathbb{B}_m$, so we get that there exists $i_0 \in \mathbb{N}$ such that $\sup_{z\in\mathbb{B}_n\setminus\Omega_\delta} |\nabla f_i(\varphi(z))|^2 < \varepsilon$ for all $i \geq i_0$. Thus, if $i \geq i_0$,

$$\begin{aligned}&\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n\setminus\Omega_\delta} |R(f_i \circ \varphi)(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^n d\lambda(z) \\ \leq\; &\sup_{z\in\mathbb{B}_n\setminus\Omega_\delta} |\nabla f_i(\varphi(z))|^2 \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |R\varphi(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^n d\lambda(z) < C\,\varepsilon.\end{aligned}$$

Hence it follows that $|f(\varphi(0))| + ||f_i \circ \varphi||_{BMOA(\mathbb{B}_n)} \to 0$. □

Lemma 5. *Suppose that μ_φ is a Carleson measure. If $C_\varphi : \mathcal{B}(\mathbb{B}_m) \to BMOA(\mathbb{B}_n)$ is compact, then*

$$\lim_{r\to 1} \sup_{\substack{f\in\mathcal{B}_0(\mathbb{B}_m),\\ ||f||\le 1}} \sup_{a\in\mathbb{B}_n} \int_{\Omega_r} |R(f\circ\varphi)(z)|^2 (1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z) = 0. \tag{37}$$

Proof. Since $C_\varphi(\{f \in \mathcal{B}_0(\mathbb{B}_m) : ||f|| \le 1\})$ is relatively compact in $BMOA(\mathbb{B}_n)$, there are, for each $\varepsilon > 0$, functions $f_i \in \mathcal{B}_0(\mathbb{B}_m)$, $||f_i|| \le 1$, $i = 1, ...N$, such that for each $f \in \mathcal{B}_0(\mathbb{B}_m)$, $||f|| \le 1$, there exists $j \in \{1, ..., N\}$ with

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |R(f\circ\varphi)(z) - R(f_j\circ\varphi)(z)|^2 (1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z) < \varepsilon.$$

For every $f_i \in \mathcal{B}_0(\mathbb{B}_m)$, $i = 1, ..., N$, there is $\delta_i \in (0, 1)$ and $\delta := \max_{1\le i\le N} \delta_i$ such that as $r \in [\delta, 1)$ we have

$$|\nabla f_i(w)|(1-|w|^2) < \sqrt{\varepsilon}$$

for all $r < |w| < 1$. Observe that $r < |\varphi(z)| < 1$ for $z \in \Omega_r$. Therefore, for given $a \in \mathbb{B}_n$ and $f \in \mathcal{B}_0(\mathbb{B}_m)$, $||f|| \le 1$, one obtains

$$\begin{aligned}
&\int_{\Omega_r} |R(f\circ\varphi)(z)|^2 (1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z)\\
\le\ & 2\int_{\Omega_r} (|R(f\circ\varphi)(z) - R(f_j\circ\varphi)(z)|^2) (1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z)\\
+\ & 2\int_{\Omega_r} |R(f_j\circ\varphi)(z)|^2 (1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z)\\
\le\ & \varepsilon\Big(2 + 2\sup_{a\in\mathbb{B}_n}\int_{\mathbb{B}_n} F_\varphi(z)\frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} dA(z)\Big).
\end{aligned}$$

This proves the lemma. □

Theorem 5. *Suppose that φ satisfies the non–tangentiality condition of Lemma 1. Then $C_\varphi : \mathcal{B}(\mathbb{B}_m) \to BMOA(\mathbb{B}_n)$ is compact if and only if*

$$\lim_{r\to 1} |||d\mu_{\varphi,r}||| = 0.$$

Proof. The "if"–statement is Proposition 2.

Suppose conversely that $C_\varphi : \mathcal{B} \to BMOA$ is compact. Let $(\alpha_m)_{m\in\mathbb{N}} \in (\frac{1}{2}, 1)$ be such that $|\alpha_m| \to 1$. For every $j, k = 1, 2, \ldots, M$, $m \in \mathbb{N}$ and $t \in [0, 1]$ we define

$$g_{j,k,m,t}(z) := \sum_{\nu\in\mathbb{N}} r_\nu(t)(\alpha_m)^{p^{\nu M+j}-1} Q_{k,\nu M+j}(z), \quad z \in \mathbb{B}_m,$$

where r_ν is the νth Rademacher function and $Q_{k,\nu M+j} := \sum_{\zeta\in\Gamma_{j,\nu M+[k+j]}} \langle z,\zeta\rangle^{p^{\nu M+j}}$. It follows from Lemma 4 that every $g_{j,k,m,t} \in \mathcal{B}_0(\mathbb{B}_m)$ and that $\|g_{j,k,m,t}\|_{\mathcal{B}} \le C_1$. Let $h \in (0, 1)$ and $s \in (\frac{1}{2}, 1)$ be fixed as in Lemma 1.

Let $\varepsilon > 0$ be given. By Lemma 5 there exists $\delta \in (s, 1)$ such that as $r \in [\delta, 1)$ we have

$$\sup_{a\in\mathbb{B}_n} \int_{\Omega_r} |R(g_{j,k,m,t}\circ\varphi)(z)|^2 (1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z) < C_1^2\varepsilon,$$

for all j, k, m, t.

Let $a \in \mathbb{B}_n$ be fixed. Integrating with respect to t, using Fubini's theorem and the orthogonality property of the Rademacher functions we obtain that

$$\begin{aligned} C_1^2\varepsilon &\geq \int_0^1 \int_{\Omega_r} |R(g_{j,k,m,t}\circ\varphi)(z)|^2\,(1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z)dt \\ &= \int_{\Omega_r} \sum_{\nu\in\mathbb{N}} |\alpha_m|^{2p^{\nu M+j}-2} |R(Q_{k,\nu M+j}\circ\varphi)(z)|^2\,(1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z). \end{aligned} \tag{38}$$

Let us bound $R(Q_{k,\nu M+j}\circ\varphi)$ from below as $z\in\Omega_r$. For all $z\in\Omega_r$ we have

$$\begin{aligned} R(Q_{k,\nu M+j}\circ\varphi)(z) &= p^{\nu M+j} \sum_{\zeta\in\Gamma_{j,\nu M+[k+j]}} \langle\varphi(z),\zeta\rangle^{p^{\nu M+j}-1}\langle R\varphi(z),\zeta\rangle \\ &= p^{\nu M+j}|\varphi(z)|^{p^{\nu M+j}-1}\,|R\varphi(z)| \sum_{\zeta\in\Gamma_{j,\nu M+[k+j]}} \langle\eta(z),\zeta\rangle^{p^{\nu M+j}-1}\langle\eta'(z),\zeta\rangle, \end{aligned}$$

where we denoted $\eta := \varphi/|\varphi|$ and $\eta' := R\varphi/|R\varphi|$. As in the proof of Theorem 4 we have that

$$\sum_{j,k=1}^{M} \Big| \sum_{\zeta\in\Gamma_{j,\nu M+[k+j]}} \langle\eta(z),\zeta\rangle^{p^{\nu M+j}-1}\langle\eta'(z),\zeta\rangle \Big| \geq C(h)$$

for every $z\in\Omega_r$. For each $r\in[\delta,1)$ and $z\in\Omega_r$, we thus obtain

$$\sum_{j,k=1}^{M} |R(Q_{k,\nu M+j}\circ\varphi)(z)| \geq C(h)p^{\nu M}|\varphi(z)|^{p^{\nu M}-1}2^{-p^M}|R\varphi(z)|.$$

Since

$$\Big(\sum_{j,k=1}^{M} |R(Q_{k,\nu M+j}\circ\varphi)(z)|\Big)^2 \leq M^2 \sum_{j,k=1}^{M} |R(Q_{k,\nu M+j}\circ\varphi)(z)|^2$$

it follows from (38) that

$$M^4C_1^2\varepsilon \geq 2\ C^2(h)2^{-2p^M} \int_{\Omega_r} \sum_{\nu\in\mathbb{N}} p^{2\nu M}|\alpha_m\varphi(z)|^{2p^{\nu M}-2}|R\varphi(z)|^2\,(1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z).$$

Using that

$$\sum_{\nu\in\mathbb{N}} p^{2\nu M}|\alpha_m\varphi(z)|^{2p^{\nu M}-2} \geq \frac{C_2}{(1-|\alpha_m\varphi(z)|^2)^2}$$

for some constant C_2, we get that there is a constant C_3 such that

$$C_3\varepsilon \geq \int_{\Omega_r} \frac{|R\varphi(z)|^2}{(1-|\alpha_m\varphi(z)|^2)^2}\,(1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z).$$

By Fatou's lemma, we have for each $r\in[\delta,1)$ that

$$\begin{aligned} &\int_{\Omega_r} \frac{|R\varphi(z)|^2}{(1-|\varphi(z)|^2)^2}\,(1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z) \\ \leq\ &\liminf_{m\to\infty} \int_{\Omega_r} \frac{|R\varphi(z)|^2}{(1-|\alpha_m\varphi(z)|^2)^2}\,(1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z) \leq C_3\varepsilon. \end{aligned}$$

Hence, as $a \in \mathbb{B}_n$ was picked arbitrary, we get

$$\sup_{a\in\mathbb{B}_n} \int_{\Omega_r} \frac{|R\varphi(z)|^2}{(1-|\varphi(z)|^2)^2} (1-|z|^2)^2(1-|\varphi_a(z)|^2)^n d\lambda(z) \leq C_3\varepsilon.$$

This proves the statement. $\square$

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