WEAKLY COMPACT COMPOSITION OPERATORS ON VECTOR-VALUED BMOA

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Abstract. The weak compactness of the analytic composition operator $f \mapsto f \circ \varphi$ is studied on $BMOA(X)$, the space of X-valued analytic functions of bounded mean oscillation, where X is a complex Banach space. It is shown that the composition operator is weakly compact on $BMOA(X)$ if X is reflexive and the corresponding composition operator is compact on the scalar-valued $BMOA$. A concrete example is given which shows that $BMOA(X)$ differs from the weak vector-valued $BMOA$ for infinite dimensional Banach spaces X.

1. Introduction

Let φ be an analytic self-map of the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ and X a complex Banach space. The composition operator C_φ induced by φ is the linear map

$$C_\varphi \colon f \mapsto f \circ \varphi$$

defined on the linear space of all analytic functions $f \colon \mathbb{D} \to X$. A fundamental problem concerning composition operators is to relate operator theoretic properties of C_φ to function theoretic properties of φ when restricted to a suitable Banach space of analytic functions. Compactness and weak compactness of C_φ have been studied on many classical Banach spaces such as Hardy spaces (see [27, 12]), Bergman and Bloch spaces, and $BMOA$ [31, 9, 28, 11]. Recently these studies have been extended by considering weak compactness of composition operators on spaces of X-valued analytic functions, where X is an arbitrary complex Banach space. In [24] and [8] results of this type have been obtained e.g. for vector-valued Hardy spaces $H^p(X)$ and vector-valued (weighted) Bergman and Bloch spaces. In this paper we consider composition operators C_φ on $BMOA(X)$, the space of X-valued analytic functions of bounded mean oscillation.

The main goal of this paper is to show that if the map $\varphi \colon \mathbb{D} \to \mathbb{D}$ induces a compact composition operator on $BMOA$ and X is a reflexive complex Banach space, then C_φ is weakly compact on $BMOA(X)$ (see Theorem 4.2). As a consequence we obtain a characterization of the weakly compact composition operators C_φ on $BMOA(X)$ under some restrictions on φ for reflexive Banach spaces X. The idea of the main theorem is to generalize the characterization due to Smith [28] of the compact composition operators on

2000 *Mathematics Subject Classification.* 47B33, 30D50, 46E40.

Key words and phrases. Weakly compact composition operators, bounded mean oscillation, vector-valued analytic functions.

The author was supported by the Finnish Academy of Science and Letters (Väisälä Foundation) and in part by the Academy of Finland, project # 53893.

$BMOA$ to the vector-valued case. For this aim we apply methods developed by Liu, Saksman and Tylli [24].

In the final section we consider a weak version of the vector-valued $BMOA$ denoted by $wBMOA(X)$. By a general result due to Bonet, Domański and Lindström [8] the counterpart for $wBMOA(X)$ of our main theorem holds: If C_φ is compact on $BMOA$ and X is reflexive, then C_φ is weakly compact on $wBMOA(X)$. We provide a concrete example demonstrating that the spaces $BMOA(X)$ and $wBMOA(X)$ are different for any infinite dimensional Banach space X. Thus our main theorem applies to a different setting compared to [8]. An example of this type was earlier given in [21] in the case where X is an infinite dimensional Hilbert space.

2. Preliminaries on vector-valued $BMOA$

In the sequel X will always be a complex Banach space. Let $H^p(X)$ denote the Hardy space of analytic functions $f\colon \mathbb{D} \to X$ such that

$$\|f\|^p_{H^p(X)} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|^p_X \, d\theta < \infty \quad \text{for } 1 \le p < \infty,$$

and $\|f\|_{H^\infty(X)} = \sup_{z\in\mathbb{D}} \|f(z)\|_X < \infty$ for $p = \infty$. One useful way to define the vector-valued $BMOA$ space is to view it as the Möbius invariant version of $H^1(X)$ (cf. [2]): An analytic function $f\colon \mathbb{D} \to X$ belongs to $BMOA(X)$ if and only if

$$\|f\|_{*,X} = \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^1(X)} < \infty,$$

where σ_a is the Möbius transformation $\sigma_a(z) = (a - z)/(1 - \overline{a}z)$ for $a \in \mathbb{D}$. The norm in $BMOA(X)$ is given by $\|f\|_{BMOA(X)} = \|f(0)\|_X + \|f\|_{*,X}$.

An alternative way to consider the vector-valued $BMOA$ is to view it as the space of Poisson extensions of the vector-valued BMO functions on the unit circle $\mathbb{T} = \partial\mathbb{D}$ having vanishing negative Fourier coefficients (cf. [5, 6]). Let $BMOA_{\mathbb{T}}(X)$ denote the space of such functions equipped with the BMO norm on the boundary. By modifying the scalar arguments one sees that $BMOA_{\mathbb{T}}(X) \subset BMOA(X)$, and that the norms of $BMOA(X)$ and $BMOA_{\mathbb{T}}(X)$ are equivalent when restricted to $BMOA_{\mathbb{T}}(X)$. Moreover, $BMOA_{\mathbb{T}}(X)$ can be identified (up to equivalent norms) with the closed subspace of $BMOA(X)$ consisting precisely of the functions $f \in BMOA(X)$ for which the radial limit function $f^*(\zeta) = \lim_{r\to 1} f(r\zeta)$ exists almost everywhere on $\mathbb{T}$ (see e.g. [18, Satz 2.7] for the analogous result for vector-valued Hardy spaces).

For general Banach spaces X the radial limits of $f \in BMOA(X)$ need not exist almost everywhere on $\mathbb{T}$. In fact, the identity $BMOA(X) = BMOA_{\mathbb{T}}(X)$ holds if and only if X has the analytic Radon-Nikodým property (ARNP). Recall that X has the ARNP if and only if the radial limits of every $f \in H^p(X)$ exist almost everywhere on $\mathbb{T}$, and this fact is independent of $p \in [1, \infty]$ [10, 3]. The same fact holds also for $BMOA(X)$ because of the inclusions $H^\infty(X) \subset BMOA(X) \subset H^1(X)$.

We define the space $VMOA(X)$ as the closure in $BMOA(X)$ of the X-valued analytic polynomials, that is, the functions of the form $p(z) = \sum_{k=0}^N x_k z^k$ where $x_k \in X$. Clearly $VMOA(X) \subset BMOA_{\mathbb{T}}(X)$. In fact, $VMOA(X)$ consists of the extensions of the X-valued VMO functions on

$\mathbb{T}$ having vanishing negative Fourier coefficients. By modifying the scalar arguments (see for instance [17]) we see that $f \in VMOA(X)$ if and only if $f \in BMOA_{\mathbb{T}}(X)$ and

$$\lim_{|a|\to 0} \|f \circ \sigma_a - f(a)\|_{H^1(X)} = 0.$$

We denote for simplicity $H^p = H^p(\mathbb{C})$, $BMOA = BMOA(\mathbb{C})$, $VMOA = VMOA(\mathbb{C})$, and $\|f\|_* = \|f\|_{*,\mathbb{C}}$ in the scalar case $X = \mathbb{C}$.

Various questions about vector-valued $BMOA$ functions have been studied earlier by O. Blasco (see for instance [5, 6, 7]). The reader is referred to [2, 16, 17, 32] for the scalar $BMOA$ and $VMOA$ theory.

3. Boundedness of C_φ on $BMOA(X)$

It is well-known that for every analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$ the composition operator $C_\varphi\colon f \mapsto f \circ \varphi$ is bounded on $BMOA$. This fact was first noticed by Stephenson [30, Thm. 3] (see also [1, Thm. 12]). We include here for completeness a proof that C_φ is bounded on $BMOA(X)$ for any complex Banach space X. It is possible to generalize Stephenson's argument to the vector-valued case (this is guaranteed by the boundedness of the composition operator on $H^1(X)$ (see [24, Prop. 1] or [20, Thm. 1])). We give a slightly different argument, in the scalar case due to Smith [28, p. 2716], which motivates our study of weak compactness in the following section. The argument is basically Littlewood's inequality applied to a formula due to Stanton for subharmonic functions.

We first recall some auxiliary concepts. Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be analytic and $0 < r \leq 1$. The partial Nevanlinna counting function $N_r(\varphi, \cdot)\colon \mathbb{D} \to \mathbb{R}$ is defined by

$$N_r(\varphi, z) = \sum_{w\in\varphi^{-1}(z)} \log^+\left(\frac{r}{|w|}\right)$$

for $z \in \mathbb{D} \setminus \{\varphi(0)\}$, each point in the preimage $\varphi^{-1}(z)$ of $z \in \mathbb{D}$ being repeated according to its multiplicity. Moreover, we put $N_r(\varphi, \varphi(0)) = 0$. The standard Nevanlinna counting function is given by $N(\varphi, z) = N_1(\varphi, z) = \sum_{w\in\varphi^{-1}(z)} \log(1/|w|)$. We refer to e.g. [27, Chapter 10] for the properties of the (partial) Nevanlinna counting function. For any complex Banach space X and analytic function $f\colon \mathbb{D} \to X$, the function $z \mapsto \|f(z)\|_X$ is subharmonic on $\mathbb{D}$. Thus we may define the distributional Laplacian $\triangle\|f\|_X$ of $\|f\|_X$, which is a positive measure on $\mathbb{D}$, by setting

$$\int_{\mathbb{D}} \psi(w) d(\triangle\|f\|_X)(w) = \frac{1}{2\pi} \int_{\mathbb{D}} \|f(w)\|_X \triangle\psi(w) dA(w)$$

for every test function $\psi \in C_0^\infty(\mathbb{D})$, where dA denotes the Lebesgue area measure on $\mathbb{D}$. The following lemma states a special case of Stanton's formula [29, Thm. 2], and it will be needed several times in the sequel.

Lemma 3.1 ([24, p. 300-301])**.** *Let $f\colon \mathbb{D} \to X$ and $\varphi\colon \mathbb{D} \to \mathbb{D}$ be analytic functions, $0 < r < 1$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \|(f\circ\varphi)(re^{i\theta})\|_X d\theta = \|f(\varphi(0))\|_X + \int_{\mathbb{D}} N_r(\varphi, w) d(\triangle\|f\|_X)(w),$$

$$\|f\circ\varphi\|_{H^1(X)} = \|f(\varphi(0))\|_X + \int_{\mathbb{D}} N(\varphi, w) d(\triangle\|f\|_X)(w).$$

The special case $\varphi(z) \equiv z$ yields the identities

$$\frac{1}{2\pi}\int_0^{2\pi} \|f(re^{i\theta})\|_X d\theta = \|f(0)\|_X + \frac{1}{2\pi}\int_{\mathbb{D}} \log^+\left(\frac{r}{|w|}\right) d(\triangle\|f\|_X)(w),$$

$$\|f\|_{H^1(X)} = \|f(0)\|_X + \frac{1}{2\pi}\int_{\mathbb{D}} \log\left(\frac{1}{|w|}\right) d(\triangle\|f\|_X)(w).$$

The following estimates are not difficult to obtain by using the Cauchy integral formula (see for instance [17, p. 95]).

Lemma 3.2. *Let $f\colon \mathbb{D} \to X$ be analytic and $z \in \mathbb{D}$. Then*

$$\|f(z) - f(0)\|_X \le \min\left\{\frac{|z|}{1-|z|}\|f\|_{H^1(X)},\ \frac{1}{2}\log\frac{1+|z|}{1-|z|}\|f\|_{*,X}\right\}.$$

We are now ready to prove that every composition operator C_φ is bounded on $BMOA(X)$ for any complex Banach space X.

Proposition 3.3. *Let φ be an analytic self-map of the unit disk. Then $\|f\circ\varphi\|_{*,X} \le \|f\|_{*,X}$ and $C_\varphi\colon BMOA(X) \to BMOA(X)$ is bounded with*

$$\|C_\varphi\|_{\mathcal{L}(BMOA(X))} \le 1 + \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|},$$

where $\|\cdot\|_{\mathcal{L}(BMOA(X))}$ denotes the operator norm on the space of bounded linear operators on $BMOA(X)$.

Proof. For any function $f \in H^1(X)$ and $a \in \mathbb{D}$ one has

$$\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{H^1(X)} = \int_{\mathbb{D}} N(\varphi\circ\sigma_a, w) d(\triangle\|f - f(\varphi(a))\|_X)(w),$$

by Lemma 3.1. By Littlewood's inequality [12, Thm. 2.29], it holds that $N(\varphi\circ\sigma_a, w) \le \log(1/|\sigma_{\varphi(a)}(w)|) = N(\sigma_{\varphi(a)}, w)$ for $w \in \mathbb{D}\setminus\{\varphi(a)\}$. Hence, by applying Lemma 3.1 once more, one obtains

$$\begin{aligned}\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{H^1(X)} &\le \int_{\mathbb{D}} N(\sigma_{\varphi(a)}, w) d(\triangle\|f - f(\varphi(a))\|_X)(w)\\ &= \|f\circ\sigma_{\varphi(a)} - f(\varphi(a))\|_{H^1(X)}\\ &\le \sup_{b\in\mathbb{D}}\|f\circ\sigma_b - f(b)\|_{H^1(X)},\end{aligned}$$

so that the inequality $\|f\circ\varphi\|_{*,X} \le \|f\|_{*,X}$ holds for $f \in BMOA(X)$. Thus

$$\begin{aligned}\|C_\varphi f\|_{BMOA(X)} &= \|f\circ\varphi\|_{*,X} + \|f(\varphi(0))\|_X\\ &\le \|f\|_{*,X} + \|f(0)\|_X + \|f(\varphi(0)) - f(0)\|_X\\ &\le \left(1 + \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)\|f\|_{BMOA(X)},\end{aligned}$$

by Lemma 3.2. $\square$

Remark 3.4. The composition operator C_φ maps the space $BMOA_{\mathbb{T}}(X)$ into itself for any Banach space X. To see this, it is enough to verify that the radial boundary function $(f\circ\varphi)^*$ exists almost everywhere on $\mathbb{T}$ whenever $f \in H^1_{\mathbb{T}}(X)$, where $H^1_{\mathbb{T}}(X)$ is the subspace of $H^1(X)$ consisting of the functions for which the radial limit function exists almost everywhere on $\mathbb{T}$. But this follows from the known facts that $p\circ\varphi \in H^1_{\mathbb{T}}(X)$ for every analytic X-valued polynomial p, and these polynomials form a dense subset of $H^1_{\mathbb{T}}(X)$ (see for instance [18, p. 57]).

It is well-known that $C_\varphi(VMOA) \subset VMOA$ if and only if $\varphi \in VMOA$ [1, Thm. 12]. We include the vector-valued argument for completeness.

Corollary 3.5. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic self-map of the unit disk. Then $C_\varphi(VMOA(X)) \subset VMOA(X)$ if and only if $\varphi \in VMOA$.*

Proof. Suppose that C_φ maps $VMOA(X)$ into itself. In particular, then $C_\varphi(x_0 z) = x_o\varphi \in VMOA(X)$, where $x_0 \in X$ is non-zero. Clearly this implies that $\varphi \in VMOA$. Conversely, suppose that $\varphi \in VMOA$. Then

$$\lim_{|a|\to 0} \|p\circ\varphi\circ\sigma_a - p(\varphi(a))\|_{H^1(X)} = 0$$

for every analytic X-valued polynomial p (by the proof of [1, Thm. 12]). By Fatou's theorem $p\circ\varphi \in BMOA_{\mathbb{T}}(X)$, so that $p\circ\varphi \in VMOA(X)$ for every analytic X-valued polynomial p. Since such polynomials are dense in $VMOA(X)$ it follows that C_φ maps $VMOA(X)$ into itself. □

4. Weak compactness of C_φ on $BMOA(X)$

Recall that a bounded linear map $T\colon X \to X$ is called compact (respectively weakly compact) if it maps the closed unit ball of X onto a relatively compact (respectively relatively weakly compact) set in X. It was noted in [24, p. 296] that C_φ can be compact on $H^p(X)$ only if X is finite dimensional and C_φ is compact on H^p (here $1 \le p \le \infty$). Moreover, if the composition operator is weakly compact on $H^p(X)$, then X must be reflexive. These facts actually hold for various spaces of vector-valued analytic functions [8, Prop. 1] including $BMOA(X)$.

Fact 4.1. *Suppose that $\mathcal{I}$ is an operator ideal such that the composition operator $C_\varphi\colon BMOA(X) \to BMOA(X)$ belongs to $\mathcal{I}$. Then the identity operator $id_X\colon X \to X$ and $C_\varphi\colon BMOA \to BMOA$ belong to $\mathcal{I}$.*

We refer to [26] for the definition of an operator ideal. Consequently, if C_φ is weakly compact on $BMOA(X)$, then X is reflexive and C_φ is weakly compact on $BMOA$. Our main theorem provides a sufficient condition for the weak compactness of C_φ on $BMOA(X)$.

Theorem 4.2. *Let X be a reflexive Banach space and suppose that $\varphi\colon \mathbb{D} \to \mathbb{D}$ is an analytic map such that $C_\varphi\colon BMOA \to BMOA$ is compact. Then $C_\varphi\colon BMOA(X) \to BMOA(X)$ is weakly compact.*

We split the proof of Theorem 4.2 into two parts. The main idea is to approximate C_φ in the operator norm by suitable weakly compact operators that are provided by Lemma 4.3 below. For the approximation we need Smith's characterization of the compact composition operators C_φ on $BMOA$. The key step is contained in Proposition 4.6.

Lemma 4.3. *There are linear operators $(V_n)_{n=0}^{\infty}$ on $BMOA(X)$ satisfying the following properties:*

(i) $\|V_n f\|_{BMOA(X)} \le 3\|f\|_{BMOA(X)}$ *for* $n \ge 0$.

(ii) *For every* $0 < r < 1$ *one has*
$$\sup_{f \in B_{BMOA(X)}} \sup_{|z| \le r} \|((I - V_n)f)(z)\|_X \to 0,$$
as $n \to \infty$, *where* I *is the identity operator on* $BMOA(X)$.

(iii) *If* X *is reflexive, then* V_n *is weakly compact on* $BMOA(X)$ *for* $n \ge 0$.

Proof. We use the de la Vallée-Poussin operators V_n defined by setting
$$V_n f(z) = \sum_{k=0}^{n} \widehat{f}_k z^k + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n} \widehat{f}_k z^k$$
for analytic functions $f\colon \mathbb{D} \to X$ with the Taylor expansion $f(z) = \sum_{k=0}^{\infty} \widehat{f}_k z^k$ (as in [24, Prop. 2]). Note that $V_n f = 2k_{2n-1}(f) - k_{n-1}(f)$, where
$$k_n(f)(z) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \widehat{f}_k z^k = \frac{1}{2\pi} \int_0^{2\pi} K_n(\theta) f(ze^{-i\theta}) d\theta$$
and K_n is the Fejér kernel (cf. [22, I.2.13]).

The fact that the operators V_n satisfy (ii) and (iii) is seen as in [24]. We will only check here that (i) holds for every V_n. In fact, by the triangle in equality and the fact $(V_n f)(0) = f(0)$, it is enough to show that $\|k_n(f)\|_{*,X} \le \|f\|_{*,X}$ for $n \ge 0$. Let $n \ge 0$. Then
$$\begin{aligned}
&\int_0^{2\pi} \|(k_n(f)(\sigma_a(re^{it})) - k_n(f)(a)\|_X \frac{dt}{2\pi} \\
&= \int_0^{2\pi} \| \int_0^{2\pi} K_n(\theta)[f(e^{-i\theta}\sigma_a(re^{it})) - f(e^{-i\theta}a)] \frac{d\theta}{2\pi} \|_X \frac{dt}{2\pi} \\
&\le \int_0^{2\pi} K_n(\theta) \int_0^{2\pi} \|f(e^{-i\theta}\sigma_a(re^{it})) - f(e^{-i\theta}a)\|_X \frac{dt}{2\pi} \frac{d\theta}{2\pi} \\
&\le \|f\|_{*,X},
\end{aligned}$$
since $\int_0^{2\pi} K_n(\theta) \frac{d\theta}{2\pi} = 1$ and
$$\int_0^{2\pi} \|f(e^{-i\theta}\sigma_a(re^{it})) - f(e^{-i\theta}a)\|_X \frac{dt}{2\pi} \le \sup_{\theta \in [0,2\pi)} \|f(e^{-i\theta}\cdot)\|_{*,X} = \|f\|_{*,X}$$
by the rotation invariance of the seminorm $\|\cdot\|_{*,X}$. We get the inequality $\|k_n(f)\|_{*,X} \le \|f\|_{*,X}$ by taking the supremum over $r \in (0,1)$ and $a \in \mathbb{D}$. $\square$

Remark 4.4. In the scalar case the uniform boundedness of the operators k_n on $BMOA$ was shown in [19, Thm. 4].

The compact composition operators C_φ on $BMOA$ were characterized by Smith [28, Theorem 1.1] as follows. The analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$ induces a compact composition operator on $BMOA$ if and only if φ satisfies both of the following conditions:

$$\lim_{r \to 1} \sup_{\{a\colon |\varphi(a)| > r\}} \sup_{0 < |w| < 1} |w|^2 N(\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a, w) = 0, \tag{1}$$

and

$$\lim_{t\to 1} \sup_{\{a:|\varphi(a)|\le R\}} m(\{\zeta \in \mathbb{T}\colon |(\varphi\circ\sigma_a)^*(\zeta)| > t\}) = 0 \tag{2}$$

for every $R < 1$, where m denotes the Lebesgue measure on $\mathbb{T}$. Condition (2) can actually be replaced by the condition

$$\lim_{t\to 1} \sup_{\{a:|\varphi(a)|\le R\}} \sup_{0<r<1} m(\{\zeta \in \mathbb{T}\colon |(\varphi\circ\sigma_a)(r\zeta)| > t\}) = 0 \tag{3}$$

for every $R < 1$; that is, C_φ is compact on $BMOA$ if and only if both (1) and (3) hold. Since (3) is useful later on, we include for the convenience of the reader a proof of the necessity of (3) (this is a simple modification of the argument on [28, p. 2720]). In fact, if (3) does not hold, then there exist $R < 1$, $\varepsilon > 0$, $t_n < 1$, $r_n \in (0, 1)$ and $a_n \in \mathbb{D}$ such that $t_n^n \to 1$, $|\varphi(a_n)| \le R$ and $m(E_n) \ge \varepsilon$, where $E_n = \{\zeta\colon |(\varphi\circ\sigma_{a_n})(r_n\zeta)| > t_n\}$. Let $f_n(z) = z^n$, so that $\|f_n\|_{BMOA} \le 1$ and (f_n) converges to 0 uniformly on compact subsets of $\mathbb{D}$. It suffices to check that $C_\varphi f_n$ does not converge to 0 in $BMOA$. Choose n_0 such that $t_n^n \ge \frac{2}{3}$ and $R^n \le \frac{1}{3}\varepsilon$ for $n \ge n_0$. Then

$$\begin{aligned}\|f_n\circ\varphi\|_{BMOA} &\ge \frac{1}{2\pi}\int_{\mathbb{T}} |(\varphi\circ\sigma_{a_n})^n(r_n\zeta) - \varphi^n(a_n)|dm(\zeta)\\ &\ge \frac{1}{2\pi}\int_{E_n} |(\varphi\circ\sigma_{a_n})(r_n\zeta)|^n dm(\zeta) - R^n\\ &\ge t_n^n m(E_n) - \varepsilon/3 \ge \varepsilon/3,\end{aligned}$$

for such n, which proves the necessity of (3).

We note that compact composition operators on $BMOA$ were also characterized in [9] in terms of Carleson measures. Compact composition operators on $VMOA$ were earlier characterized in [31].

The following lemma refines condition (1). It is a slight modification of [28, Lemma 2.1].

Lemma 4.5. *Let φ be an analytic self-map of the unit disk with $\varphi(0) = 0$. If*

$$\sup_{0<|w|<1} |w|^2 N(\varphi, w) \le \delta^4,$$

where $\delta < e^{-1/2}$, then

$$N(\varphi, z) \le 2\delta^2 \log(1/|z|)$$

for $\delta \le |z| < 1$.

Proof. For $\delta \le |z| \le e^{-1/2}$ the estimate $N(\varphi, z) \le \delta^2 \le 2\delta^2 \log(1/|z|)$ follows from the assumption. For $r \in (0, 1)$ the subharmonic function $N_r(\varphi, z)$ is bounded by the harmonic function $2e\delta^4 \log(1/|z|)$ on the annulus $\{w \in \mathbb{D}\colon e^{-1/2} < |w| < 1\}$, by the assumption and the fact that $N_r(\varphi, z) \le N(\varphi, z)$. Thus

$$N(\varphi, z) = \lim_{r\to 1} N_r(\varphi, z) \le 2e\delta^4 \log(1/|z|) \le 2\delta^2 \log(1/|z|)$$

for $e^{-1/2} < |z| < 1$. □

We are now ready to prove the key step of Theorem 4.2.

Proposition 4.6. *Let φ be an analytic self-map of the unit disk satisfying conditions (1) and (3). Then*

$$\|C_\varphi - C_\varphi V_n\|_{\mathcal{L}(BMOA(X))} \to 0$$

as $n \to \infty$, where the operators V_n are those of Lemma 4.3.

Proof. Let $\varepsilon > 0$ and let $f \in BMOA(X)$ be arbitrary. We need to show that there exists $n_0 \in \mathbb{N}$ so that

$$\|C_\varphi(I - V_n)f\|_{BMOA(X)} \leq \varepsilon \|f\|_{BMOA(X)}$$

for every $n \geq n_0$. We introduce the following abbreviations:

(i) $S_n = I - V_n$,
(ii) $\varphi_a = \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a$,
(iii) $g_{a,n} = (S_n f) \circ \sigma_{\varphi(a)} - (S_n f)(\varphi(a))$,

for $n \geq 0$ and $a \in \mathbb{D}$. Note that $\|g_{a,n}\|_{H^1(X)} \leq \|S_n f\|_{*,X} \leq 4\|f\|_{BMOA(X)}$ for $n \geq 0$, by Lemma 4.3 (i). By Lemma 4.3 (ii), one has $\|(C_\varphi S_n f)(0)\|_X = \|(S_n f)(\varphi(0))\|_X \leq \varepsilon \|f\|_{BMOA(X)}$ for n large enough. Hence, according to the identity $(\sigma_{\varphi(a)} \circ \sigma_{\varphi(a)})(z) = z$, it suffices to show that

$$\sup_{a \in \mathbb{D}} \|g_{a,n} \circ \varphi_a\|_{H^1(X)} = \|C_\varphi S_n f\|_{*,X} \leq \varepsilon \|f\|_{BMOA(X)}, \tag{4}$$

for $n \geq n_0$. Choose $\delta = \delta(\varepsilon) \in (0, \frac{1}{4})$ such that $\max\{8\delta^2,\ 48\delta \log(1/\delta)\} < \varepsilon$. By the assumption that φ satisfies conditions (1) and (3) there exist a number $R = R(\varepsilon) \in (0,1)$ such that

$$\sup_{0<|w|<1} |w|^2 N(\varphi_a, w) < \delta^4 \tag{5}$$

for every $a \in \mathbb{D}$ satisfying $|\varphi(a)| > R$, and a number $t_0 = t_0(\varepsilon) \in (0,1)$ such that

$$m(\{\zeta \in \mathbb{T} \colon |(\varphi \circ \sigma_a)(r\zeta)| > t_0\}) < \varepsilon^2 \tag{6}$$

for every $r \in (0,1)$ and $a \in \mathbb{D}$ satisfying $|\varphi(a)| \leq R$.

Consider first $a \in \mathbb{D}$ satisfying $|\varphi(a)| > R$. From Lemma 3.1 and the fact that $g_{a,n}(\varphi_a(0)) = 0$ we get

$$\begin{aligned}\|g_{a,n} \circ \varphi_a\|_{H^1(X)} &= \int_{\delta \leq |w| < 1} N(\varphi_a, w) d(\triangle \|g_{a,n}\|_X)(w) \\ &\quad + \int_{|w|<\delta} N(\varphi_a, w) d(\triangle \|g_{a,n}\|_X)(w) =: A + B.\end{aligned}$$

From (5) and Lemma 4.5 we get the estimate $N(\varphi_a, w) \leq 2\delta^2 \log(1/|w|)$ for $\delta \leq |w| < 1$. Using Lemma 3.1 once more, and recalling the choice of δ, we have

$$\begin{aligned}A &\leq 2\delta^2 \int_{\delta \leq |w| < 1} \log\left(\frac{1}{|w|}\right) d(\triangle \|g_{a,n}\|_X)(w) \\ &\leq 2\delta^2 \|g_{a,n}\|_{H^1(X)} \leq \varepsilon \|f\|_{BMOA(X)}.\end{aligned}$$

To estimate B, note that $2\log(2\delta/|w|) \geq 1$ and $\log(1/\delta) \geq 1$ for $|w| < \delta < \frac{1}{4}$. From these estimates and Littlewood's inequality [12, Theorem 2.29] we get

$$N(\varphi_a, w) \leq \log\left(\frac{1}{|w|}\right) \leq \log\left(\frac{2\delta}{|w|}\right) + \log\left(\frac{1}{\delta}\right) \leq 3\log\left(\frac{1}{\delta}\right) \log\left(\frac{2\delta}{|w|}\right),$$

for $0 < |w| < \delta$. Thus

$$\begin{aligned} B \leq\ & 3\log(1/\delta)\int_{|w|<\delta} \log\left(\frac{2\delta}{|w|}\right) d(\triangle\|g_{a,n}\|_X)(w) \\ \leq\ & 3\log(1/\delta)\int_{\mathbb{D}} \log^+\left(\frac{2\delta}{|w|}\right) d(\triangle\|g_{a,n}\|_X)(w). \end{aligned}$$

From Lemmas 3.1 and 3.2 we get that

$$\begin{aligned} B &\leq \frac{3\log(1/\delta)}{2\pi}\int_0^{2\pi} \|g_{a,n}(2\delta e^{i\theta}) - g_{a,n}(0)\|_X\, d\theta \\ &\leq 3\log(1/\delta)\frac{2\delta}{1-2\delta}\|g_{a,n}\|_{H^1(X)} \\ &\leq 12\delta\log(1/\delta)\|g_{a,n}\|_{H^1(X)}, \end{aligned}$$

so that $B \leq \varepsilon\|f\|_{BMOA(X)}$ in view of the choice of δ. Consequently,

$$\|g_{a,n}\circ\varphi_a\|_{H^1(X)} \leq A + B \leq 2\varepsilon\|f\|_{BMOA(X)}, \tag{7}$$

for $a \in \mathbb{D}$ satisfying $|\varphi(a)| > R$.

Consider next $a \in \mathbb{D}$ satisfying $|\varphi(a)| \leq R$. By Lemma 4.3 (ii) there is $n_0 = n_0(\varepsilon) \in \mathbb{N}$ so that for every $n \geq n_0$ and $|z| \leq t_0$ we have

$$\max\{\|(S_nf)(z)\|_X,\ \|(S_nf)(\varphi(a))\|_X\} \leq \varepsilon\|f\|_{BMOA(X)}.$$

Let $r \in (0,1)$ and put $E = \{\zeta \in \mathbb{T}\colon |(\varphi\circ\sigma_a)(r\zeta)| > t_0\}$, so that $m(E) < \varepsilon^2$ by (6). Then

$$\begin{aligned} &\frac{1}{2\pi}\int_{\mathbb{D}\setminus E}\|(g_{a,n}\circ\varphi_a)(r\zeta)\|_X\, dm(\zeta) \\ &\qquad = \frac{1}{2\pi}\int_{\mathbb{D}\setminus E}\|((S_nf)\circ\varphi\circ\sigma_a)(r\zeta) - (S_nf)(\varphi(a))\|_X\, dm(\zeta) \\ &\qquad \leq \sup_{|z|\leq t_0}\|(S_nf)(z)\|_X + \|(S_nf)(\varphi(a))\|_X \leq 2\varepsilon\|f\|_{BMOA(X)}, \end{aligned}$$

for $n \geq n_0$. On the other hand,

$$\begin{aligned} &\frac{1}{2\pi}\int_E\|(g_{a,n}\circ\varphi_a)(r\zeta)\|_X\, dm(\zeta) \\ &\qquad \leq m(E)^{1/2}\left(\frac{1}{2\pi}\int_{\mathbb{T}}\|(g_{a,n}\circ\varphi_a)(r\zeta)\|_X^2\, dm(\zeta)\right)^{1/2} \\ &\qquad \leq \varepsilon\|(S_nf)\circ\varphi\circ\sigma_a - (S_nf)(\varphi(a))\|_{H^2(X)} \end{aligned}$$

by Hölder's inequality and (6). By the analytic John-Nirenberg theorem [2, p. 15], which also holds in the vector-valued setting (with a similar proof as in the scalar case), there exists a constant C such that

$$\begin{aligned} &\frac{1}{2\pi}\int_E\|(g_{a,n}\circ\varphi_a)(r\zeta)\|_X\, dm(\zeta) \\ &\qquad \leq \varepsilon\sup_{b\in\mathbb{D}}\|(S_nf)\circ\varphi\circ\sigma_b - (S_nf)(\varphi(b))\|_{H^2(X)} \\ &\qquad \leq C\varepsilon\sup_{b\in\mathbb{D}}\|(S_nf)\circ\varphi\circ\sigma_b - (S_nf)(\varphi(b))\|_{H^1(X)} \\ &\qquad = C\varepsilon\|S_nf\circ\varphi\|_{*,X} \leq C\varepsilon\|S_nf\|_{*,X} \leq 4C\varepsilon\|f\|_{BMOA(X)}, \end{aligned}$$

where the last inequalities followed from Proposition 3.3 and Lemma 4.3 (i). By combining these estimates and taking the supremum over $r \in (0,1)$, we obtain

$$\|g_{a,n} \circ \varphi_a\|_{H^1(X)} \leq (2+4C)\varepsilon\|f\|_{BMOA(X)}$$

for $n \geq n_0$ and $a \in \mathbb{D}$ satisfying $|\varphi(a)| \leq R$. Together with (7) this proves (4). □

It is now easy to complete the proof of Theorem 4.2.

Proof of Theorem 4.2. Let X and φ be as assumed. Then the operators V_n are weakly compact on $BMOA(X)$ for $n \geq 0$, by Lemma 4.3 (iii). Since the weakly compact operators form a closed operator ideal, it suffices to verify that

$$\|C_\varphi - C_\varphi V_n\|_{\mathcal{L}(BMOA(X))} \to 0$$

as $n \to \infty$. Since by Smith's result φ satisfies conditions (1) and (3), this follows from Proposition 4.6. □

As a consequence we obtain an analogoue of Theorem 4.2 for $VMOA(X)$.

Corollary 4.7. *Let X be a reflexive Banach space and let φ be an analytic self-map of the unit disk such that $\varphi \in VMOA$. If C_φ is compact on $VMOA$, then C_φ is weakly compact on $VMOA(X)$.*

Proof. Let X and φ be as assumed. Then C_φ is compact on $BMOA$ by [28, Cor. 1.3], and C_φ is weakly compact on $BMOA(X)$ by Theorem 4.2. If (f_n) is a bounded sequence in $VMOA(X)$, then $(f_n \circ \varphi)$ has a weakly converging subsequence $(f_{n_k} \circ \varphi)$ in $BMOA(X)$. By Corollary 3.5 the subsequence belongs to $VMOA(X)$, and hence it converges weakly to a function $g \in VMOA(X)$. Thus C_φ is weakly compact on $VMOA(X)$. □

In the light of Fact 4.1 and Theorem 4.2 a complete characterization of the weakly compact composition operators on $BMOA(X)$ depends on whether all weakly compact composition operators on $BMOA$ are compact or not. Unfortunately the answer to this question is not known for arbitrary composition operators C_φ (see e.g. [11] for the discussion of this problem). However, by combining with some partial positive results from the literature we obtain the following consequence of Theorem 4.2.

Corollary 4.8. *Let φ be an analytic self-map of the unit disk such that φ satisfies one of the following conditions:*

(i) *φ is univalent, or*
(ii) *$\varphi \in VMOA$ and $\varphi(\mathbb{D})$ lies inside a polygon inscribed in the unit circle.*

Then C_φ is weakly compact on $BMOA(X)$ if and only if X is reflexive and C_φ is compact on $BMOA$.

Proof. Assume first that $\varphi\colon \mathbb{D} \to \mathbb{D}$ is univalent and C_φ is weakly compact on $BMOA(X)$. Then C_φ is weakly compact on $BMOA$ and X is reflexive by Fact 4.1. It is well-known that every bounded univalent map belongs to $VMOA$ (see for instance [13, Thm. 10]), so that φ induces a weakly compact composition operator on $VMOA$. By [11, Thm. 1] and [28, Thm. 4.1] the operator C_φ is actually compact on $VMOA$. Since C_φ on $BMOA$

is the second adjoint of C_φ on $VMOA$ (cf. [11, p. 939]), we get that C_φ is compact also on $BMOA$.

The proof is similar in the case where $\varphi \in VMOA$ maps $\mathbb{D}$ inside a polygon inscribed in the unit circle. Here we apply a result by Tjani (see the proof of [31, Thm. 3.15], or [25, Cor. 5.4]) stating that if such a map induces a weakly compact composition operator on $VMOA$, then C_φ is compact on $VMOA$.

In both cases the converse statement follows from Theorem 4.2. □

Remark 4.9. Weakly conditionally compact composition operators were characterized in [24] and [8] on various spaces of vector-valued analytic functions. Recall that a linear map $T\colon X \to X$ is weakly conditionally compact if for every bounded sequence $(x_k) \subset X$ the sequence (Tx_k) admits a weakly Cauchy subsequence. Rosenthal's l^1-theorem [23, 2.e.5] implies that T is weakly conditionally compact on X if and only if T is not an isomorphism on any isomorphic copy of l^1 in X. It is possible to modify the argument of Theorem 4.2 in the case where none of the subspaces of X are isomorphic to l^1. In fact, if C_φ is compact on $BMOA$, then C_φ is weakly conditionally compact on $BMOA(X)$ for such X. The details are left for the interested reader.

5. Weak vector-valued $BMOA$

In this section we discuss another interesting version of the vector-valued $BMOA$, the space $wBMOA(X)$ consisting of the weak X-valued $BMOA$ functions. The purpose of this section is to demonstrate that $wBMOA(X)$ differs from the space $BMOA(X)$ considered earlier in this paper. Weak vector-valued BMO was earlier considered e.g. in [4] and [21], and composition operators on various weak spaces were studied systematically in [8] by different methods.

Let $wBMOA(X)$ denote the space of analytic functions $f\colon \mathbb{D} \to X$ such that $x^* \circ f \in BMOA$ for every $x^* \in X^*$. The norm of $wBMOA(X)$ is given by

$$\|f\|_{wBMOA(X)} = \sup_{\|x^*\| \le 1} \|x^* \circ f\|_{BMOA}.$$

Similarly, for $1 \le p < \infty$, let $wH^p(X)$ denote the space of analytic functions $f\colon \mathbb{D} \to X$ such that $x^* \circ f \in H^p$ for every $x^* \in X^*$, equipped with the norm

$$\|f\|_{wH^p(X)} = \sup_{\|x^*\| \le 1} \|x^* \circ f\|_{H^p}.$$

Then $wBMOA(X)$ and $wH^p(X)$ are Banach spaces for every $1 \le p < \infty$ (cf. [8, Lemma 10]). Clearly

$$\|f\|_{wBMOA(X)} \le \|f\|_{BMOA(X)} \quad \text{and} \quad \|f\|_{wH^p(X)} \le \|f\|_{H^p(X)},$$

and the spaces coincide as sets whenever X is finite dimensional.

It is general result due to Bonet, Domański and Lindström [8, Proposition 11] that the counterpart of Theorem 4.2 for $wBMOA(X)$ holds: If X is a reflexive Banach space and φ induces a compact composition operator on $BMOA$, then C_φ is weakly compact on $wBMOA(X)$. This raises the question whether $BMOA(X)$ is a closed subspace of $wBMOA(X)$ for (some) infinite dimensional X. Actually it turns out that this is never the case. In the case where X is a Hilbert space an example of this type was given in [21, Lemma 2.3] (see also [4]). We include here a concrete example based on a known multiplier result (due to Girela) and Dvoretzky's l^2_n-theorem, that applies to any infinite dimensional Banach space. We refer to e.g. [15] for applications of Dvoretzky's theorem in parallel situations.

Example 5.1. *For any infinite dimensional complex Banach space X there exists a sequence $(f_n)_{n=1}^\infty$ of analytic functions $f_n\colon \mathbb{D} \to X$ such that*

$$\|f_n\|_{wBMOA(X)} \leq 1, \quad n \in \mathbb{N},$$

and

$$\|f_n\|_{H^1(X)} \to \infty, \quad \text{as } n \to \infty.$$

In particular, the norms $\|\cdot\|_{wBMOA(X)}$ and $\|\cdot\|_{BMOA(X)}$, as well as the norms $\|\cdot\|_{wH^p(X)}$ and $\|\cdot\|_{H^p(X)}$, are not equivalent for any $1 \leq p < \infty$.

Proof. We construct the desired example using a known characterization of multipliers from l^2 to $BMOA$. A sequence $(a_k)_{k=0}^\infty$ is said to be a multiplier from l^2 to $BMOA$ if $\sum_{k=0}^\infty a_k b_k z^k \in BMOA$ for every $(b_k)_{k=0}^\infty \in l^2$. In that case we say that $(a_k)_{k=0}^\infty$ belongs to $(l^2, BMOA)$. By [17, Theorem 9.7] a sequence $(a_k)_{k=0}^\infty$ belongs to $(l^2, BMOA)$ if and only if

$$\sum_{k=0}^n k^2 |a_k|^2 = O(n^2),$$

as $n \to \infty$. Thus the sequence $(a_k)_{k=0}^\infty$ given by setting $a_0 = 0$ and $a_k = 1/\sqrt{k}$ for $k = 1, 2, \ldots$ belongs to $(l^2, BMOA)$. In particular, by the closed graph theorem there is a constant C such that

$$\|\sum_{k=1}^\infty \frac{b_k}{\sqrt{k}} z^k\|_{BMOA} \leq C \left(\sum_{k=1}^\infty |b_k|^2\right)^{1/2}, \tag{8}$$

for $(b_k)_{k=1}^\infty \in l^2$.

Let X be an infinite dimensional complex Banach space and $n \in \mathbb{N}$. By Dvoretzky's theorem [14, Thm. 19.1] there exists an n-dimensional subspace E_n of X and a linear isomorphism $J_n\colon l^2_n \to E_n$ so that $\|J_n\| \leq 2$ and $\|J_n^{-1}\| = 1$. Let $x_k^{(n)} = J_n e_k^{(n)}$, where $e_k^{(n)}$ is the kth standard unit vector of l^2_n for $k = 1, \ldots, n$. Define the analytic function $f_n\colon \mathbb{D} \to X$ by

$$f_n(z) = \sum_{k=1}^n \frac{x_k^{(n)}}{\sqrt{k}} z^k.$$

Then

$$\|f_n(re^{i\theta})\|_X \geq \|\sum_{k=1}^n \frac{e_k^{(n)}}{\sqrt{k}} (re^{i\theta})^k\|_{l^2_n} = \left(\sum_{k=1}^n \frac{r^{2k}}{k}\right)^{1/2}$$

for $0 < r < 1$, so that

$$\|f_n\|^2_{H^1(X)} \geq \sup_{0<r<1} \left(\sum_{k=1}^n \frac{r^{2k}}{k}\right) = \sum_{k=1}^n \frac{1}{k} \geq \log n.$$

Suppose that $x^* \in X^*$ satisfies $\|x^*\|_{X^*} \leq 1$. Then $y_n^* = x^*|_{E_n} \in E_n^*$, and $J_n^* y_n^* \in (l_n^2)^*$ with $\|J_n^* y_n^*\|_{(l_n^2)^*} \leq \|J_n\| \|x^*\|_{X^*} \leq 2$, where J_n^* denotes the adjoint of J_n. We get from (8) that

$$\begin{aligned}
\|x^* \circ f_n\|_{BMOA} &= \| \sum_{k=1}^{n} \frac{y_n^*(x_k^{(n)})}{\sqrt{k}} z^k \|_{BMOA} \\
&\leq C \left(\sum_{k=1}^{n} |y_n^*(x_k^{(n)})|^2 \right)^{1/2} = C \left(\sum_{k=1}^{n} |y_n^*(J_n e_k^{(n)})|^2 \right)^{1/2} \\
&= C \left(\sum_{k=1}^{n} |(J_n^* y_n^*)(e_k^{(n)})|^2 \right)^{1/2} = C \|J_n^* y_n^*\|_{(l_n^2)^*} \leq 2C.
\end{aligned}$$

By taking the supremum over $x^* \in X$ satisfying $\|x^*\|_{X^*} \leq 1$, we get $\|f_n\|_{wBMOA(X)} \leq 2C$, where C is independent of n and X.

The fact that none of the norms are equivalent follows now from the continuous inclusions $wBMOA(X) \subset wH^p(X) \subset wH^1(X)$ and $BMOA(X) \subset H^p(X) \subset H^1(X)$ that hold for every $1 \leq p < \infty$. $\square$

Acknowledgements. I would like to thank my supervisor Hans-Olav Tylli for his comments, suggestions and support during the preparation of this work.

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