Olli Martio, Vladimir Miklyukov, Matti Vuorinen

Removable singularities of $\mathcal{WT}$-differential forms and quasiregular mappings

File: `remsin5.tex`, March 8, 2004

Abstract

A theorem on removable singularities of $\mathcal{WT}$-differential forms is proved and applied to quasiregular mappings.

1 Main Theorem

Our notation is as in [2]. We recall some facts on differential forms and quasiregular mappings.

Let $\mathcal{M}$ be a Riemannian manifold of the class C^3, $\dim \mathcal{M} = n$, without boundary. Each differential form α can be written in terms of the local coordinates $x^1, \ldots, x^n$ as the linear combination

$$\alpha = \sum_{1 \le i_1 < \ldots < i_k \le n} \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} .$$

Let α be a differential form defined on an open set $D \subset \mathcal{M}$. If $\mathcal{F}(D)$ is a class of functions defined on D, then we say that the differential form α is in this class provided that $\alpha_{i_1 \ldots i_k} \in \mathcal{F}(D)$. For instance, the differential form α is in the class $L^p(D)$ if all its coefficients are in this class.

A differential form α of degree k on the manifold $\mathcal{M}$ with coefficients $\alpha_{i_1 \ldots i_k} \in L^p_{\mathrm{loc}}(\mathcal{M})$ is called *weakly closed* if for each differential form β, $\deg \beta = k+1$, with compact support $\operatorname{supp} \beta = \overline{\{m \in \mathcal{M} : \beta \neq 0\}}$ in $\mathcal{M}$ and with coefficients in the class $W^1_{q,\mathrm{loc}}(\mathcal{M})$, $1/p + 1/q = 1$, $1 \le p, q \le \infty$, we have

$$\int_{\mathcal{M}} \langle \alpha, \delta\beta \rangle * \mathbb{1}_{\mathcal{M}} = 0. \tag{1.1}$$

Here the operator $\star$ and the exterior differentiation d define the codifferential operator δ by the formula

$$\delta\alpha = (-1)^k \star^{-1} d \star \alpha$$

for a differential form α of degree k.

Clearly, $\delta\alpha$ is a differential form of degree $k-1$. For smooth differential forms α condition (1.1) agrees with the traditional condition of closedness $d\alpha = 0$.

Let

$$w \in L^p_{\mathrm{loc}}(\mathcal{M}),\ \deg w = k,\ 0 \le k \le n,\ p > 1, \tag{1.2}$$

and

$$\theta \in L^q_{\mathrm{loc}}(\mathcal{M}),\ \deg\theta = n-k,\ \frac{1}{p} + \frac{1}{q} = 1, \tag{1.3}$$

be weakly closed differential forms on $\mathcal{M}$.

We say that the pair of forms (1.2) and (1.3) satisfy a $\mathcal{WT}$-condition on $\mathcal{M}$ if there exist constants $\nu_1, \nu_2 > 0$ such that almost everywhere on $\mathcal{M}$

$$\nu_1\,|w|^p \le \ \langle w, *\theta\rangle \tag{1.4}$$

and

$$|\theta| \le \nu_2\,|w|^{p-1}. \tag{1.5}$$

Our main removability result for differential forms is the following.

1.6. Theorem. *Let $\mathcal{M}$ be a Riemannian C^3-manifold,* $\dim M = n \ge 2$*, and let $E \subset \mathcal{M}$ be a compact set of $p-$capacity zero, $1 \le p \le n$. Let Z and θ be forms on $\mathcal{M}\setminus E$ and of degrees $k-1$, $n-k$ respectively. Suppose that the pair $w = dZ \in L^p_{\mathrm{loc}}(\mathcal{M}\setminus E)$ and $\theta \in L^q_{\mathrm{loc}}(\mathcal{M}\setminus E)$, $1/p + 1/q = 1$, satisfies a $\mathcal{WT}$-condition on $\mathcal{M}\setminus E$.*

If

$$\operatorname*{ess\,sup}_{m\in\mathcal{M}\setminus E} |Z(m)| < \infty\,, \tag{1.7}$$

then there exist forms $\tilde{Z}$, $\tilde{\theta}$ such that the pair $\tilde{w} = d\tilde{Z} \in L^p_{\mathrm{loc}}(\mathcal{M})$, $\tilde{\theta} \in L^q_{\mathrm{loc}}(\mathcal{M})$ satisfies the $\mathcal{WT}$-condition on $\mathcal{M}$ and their restrictions to $\mathcal{M}\setminus E$ coincide with Z, θ respectively.

2 p-Capacity

Recall some concepts in the capacity theory. Let D be an open set on $\mathcal{M}$ and let $A, B \subset D$, be such that $\overline{A}$ and $\overline{B}$ are compact in D and $\overline{A} \cap \overline{B} = \emptyset$. Each triple $(A, B; D)$ is called a *condenser* on $\mathcal{M}$.

We fix $p \geq 1$. The p-capacity of the condenser $(A, B; D)$ is defined by

$$\operatorname{cap}_p(A, B; D) = \inf \int\limits_D |\nabla\varphi|^p * \mathbb{1}_{\mathcal{M}}, \tag{2.1}$$

where the infimum is taken over the set of all continuous functions φ of class $W^1_{p,\mathrm{loc}}(D)$ such that $\varphi|_A = 0$, $\varphi|_B = 1$. It is easy to see that for a pair $(A, B; D)$ and $(A_1, B_1; D)$ with $A_1 \subset A$, $B_1 \subset B$ we have

$$\operatorname{cap}_p(A_1, B_1; D) \leq \operatorname{cap}_p(A, B; D).$$

A standard appoximation argument shows that the quantity $\operatorname{cap}_p(A, B; D)$ does not change if one restricts the class of functions in the variational problem (2.1) to Lipschitz functions φ equal to 0 and 1 in the sets A and B, respectively, and $\nabla\varphi \neq 0$ a.e. on $\mathcal{M} \setminus (A \cup B)$.

We say that a compact set $E \subset \mathcal{M}$ is of p-capacity zero, if $\operatorname{cap}_p(E, U; \mathcal{M}) = 0$ for all open sets $U \subset \mathcal{M}$ such that $E \cap U = \emptyset$.

We will need

2.2. Lemma. *A set $E \subset \mathcal{M}$ is of 1-capacity zero if and only if*

$$\mathcal{H}^{n-1}(E) = 0\,.$$

Proof. Fix $\varepsilon > 0$ and an open set $U \subset \mathcal{M}$ such that $\operatorname{cap}_1(E, U; M) = 0$. Choose a Lipschitz function $\varphi : \mathcal{M} \to [0, 1]$ such that $\varphi|_E = 0$, $\varphi|_U = 1$, $\nabla\varphi \neq 0$ a.e. on $\mathcal{M} \setminus (E \cup U)$ and

$$\int\limits_{\mathcal{M}} |\nabla\varphi| * \mathbb{1}_{\mathcal{M}} \leq \varepsilon.$$

By the co-area formula we have

$$\int\limits_{\mathcal{M}} |\nabla\varphi| * \mathbb{1}_{\mathcal{M}} = \int\limits_0^1 dt \int\limits_{G_t} d\mathcal{H}^{n-1} = \int\limits_0^1 \mathcal{H}^{n-1}(G_t)\,,$$

where $G_t = \{m \in \mathcal{M} : \varphi(m) = t\}$ is a level set of φ [1, Section 3.2].

Thus we obtain

$$\inf_t \mathcal{H}^{n-1}(G_t) \le \varepsilon$$

and there exist G_t of arbitrarily small $(n-1)$-measure.

Since U is open it is possible only for the set E of $(n-1)$-measure zero.

If a compact set $E \subset \mathcal{M}$ is of p-capacity zero, then E is of q-capacity zero for all $q \in [1, p]$. By Lemma 2.2 we conclude that a set E of p-capacity zero, $p \ge 1$, satisfies $\mathcal{H}^{n-1}(E) = 0$. In particular, such a set has n-measure zero.

3 Applications to quasiregular mappings

Let $\mathcal{M}$ and $\mathcal{N}$ be Riemannian manifolds of dimension n. A continuous mapping $F : \mathcal{M} \to \mathcal{N}$ of the class $W^1_{n,\mathrm{loc}}(\mathcal{M})$ is called a *quasiregular* mapping [3, Section 14] if F satisfies

$$|F'(m)|^n \le K J_F(m)$$

almost everywhere on $\mathcal{M}$. Here $F'(m) : T_m(\mathcal{M}) \to T_{F(m)}(\mathcal{N})$ is the formal derivative of $F(m)$, further, $|F'(m)| = \max_{|h|=1} |F'(m)h|$. We denote by $J_F(m)$ the Jacobian of F at the point $m \in \mathcal{M}$, i.e. the determinant of $F'(m)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be Riemannian manifolds of dimensions $\dim \mathcal{A} = k$, $\dim \mathcal{B} = n-k$, $1 \le k \le n$. The Cartesian product $\mathcal{N} = \mathcal{A} \times \mathcal{B}$ has the natural structure of a Riemannian manifold. We denote by $\pi : \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ and $\eta : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ the natural projections of the manifold $\mathcal{N}$ onto submanifolds.

If $v_{\mathcal{A}}$ and $v_{\mathcal{B}}$ are volume forms on $\mathcal{A}$ and $\mathcal{B}$, respectively, then the differential form $v_{\mathcal{N}} = \pi^* v_{\mathcal{A}} \wedge \eta^* v_{\mathcal{B}}$ is a volume form on $\mathcal{N}$.

Theorem 1.6 can be used to study removable singularities of quasiregular mappings.

3.1. Theorem. *Let $\mathcal{A}$, $\mathcal{B}$ be Riemannian manifolds,* $\dim \mathcal{A} = k$, $\dim \mathcal{B} = n-k$, $1 \le k \le n$ *and $\mathcal{N} = \mathcal{A} \times \mathcal{B}$. Let $\mathcal{M}$ be an n-dimensional manifold without boundary and let $E \subset \mathcal{M}$ be a compact subset of the n/k-capacity zero.*

If $F : \mathcal{M} \setminus E \to \mathcal{N}$ is a quasiregular mapping, $f = \pi \circ F$, and the form Z, $dZ = f^* v_A$, satisfies (1.7), then there exists a quasiregular mapping $\tilde{F} : \mathcal{M} \to \mathcal{N}$ with $\tilde{F}|_{\mathcal{M}\setminus E} = F$.

Proof. It is enough to use the following statement, see [2].

3.2. Lemma. *If $F : \mathcal{M} \to \mathcal{N}$ is a quasiregular mapping and $f = \pi \circ F : \mathcal{M} \to \mathcal{A}$, $g = \eta \circ F : \mathcal{M} \to \mathcal{B}$ then the pair of forms $w = f^* v_{\mathcal{A}}$, $\theta = g^* v_{\mathcal{B}}$ satisfy a $\mathcal{WT}$-condition on $\mathcal{M}$ with the structure constants $\nu_1 = \nu_1(n, k, K)$, $\nu_2 = \nu_2(n, k, K)$ and $p = n/k$.*

By Theorem 1.6 we obtain the necessary conclusion.

We point out some special cases of Theorem 3.1.

3.3. Corollary. *Let $D \subset \mathbf{R}^n$ be a domain, $1 \le k \le n$, and let $E \subset D$ be a compact set of the n/k-capacity zero. Suppose that a quasiregular mapping*

$$F = (F_1, \ldots, F_k, F_{k+1}, \ldots, F_n) : D \setminus E \to \mathbf{R}^n$$

satisfies (1.7) with

$$Z(x) = \sum_{i=1}^{k} (-1)^{i-1} c_i \, F_i dF_1 \wedge dF_2 \wedge \ldots \wedge \widetilde{dF_i} \wedge \ldots \wedge dF_k \, ,$$

where the symbol $\widetilde{dF_i}$ means that this factor is omitted and $c_i = \mathrm{const}$, $\sum_{i=1}^{k} c_i = 1$.

Then there exists a quasiregular mapping $\tilde{F} : D \to \mathbf{R}^n$ for which $\tilde{F}\Big|_{D\setminus E} = F$.

Proof. We have

$$\begin{aligned} dZ &= \textstyle\sum_{i=1}^{k} (-1)^{i-1} c_i \, dF_i \wedge dF_1 \wedge dF_2 \wedge \ldots \wedge \widetilde{dF_i} \wedge \ldots \wedge dF_k = \\ &= dF_1 \wedge \ldots \wedge dF_k \, . \end{aligned}$$

If we put

$$\theta = dF_{k+1} \wedge \ldots \wedge dF_n \, ,$$

then by Lemma 3.2 the pair of forms $w = dZ$ and θ satisfies (1.4) and (1.5) on $D \setminus E$. Using Theorem 3.1 we can conclude that forms Z and θ have extensions to D. Moreover for an arbitrary subdomain D', $E \subset D' \subset\subset D$ it follows

$$\begin{aligned}
\int\limits_{D'\setminus E} J_F(x)\,dx_1\ldots dx_n &= \int\limits_{D'\setminus E} dF_1 \wedge \ldots \wedge dF_n = \\
&= \int\limits_{D'\setminus E} dZ \wedge \theta \le \\
&\le C \int\limits_{D'\setminus E} |dZ|\,|\theta|\,dx_1\ldots dx_n \le \\
&\le C\,\|dZ\|_{L^p(D'\setminus E)}\|\theta\|_{L^q(D'\setminus E)}\,,
\end{aligned}$$

where $C = \mathrm{const} < \infty$ [1, §1.7] and $p = n/k$, $q = n/(n-k)$.

From this it is easy to see that the vector-function F belongs to $W^1_{n,\mathrm{loc}}$ in D and E is removable for the quasiregular mapping F. Note that in the definition of a quasiregular mapping continuity is not needed, see Reshetnyak [6].

The case $k = 1$ reduces to the well-known case, see [5, Miklyukov, 1969].

3.4. Corollary. *Let $D \subset \mathbf{R}^n$ be a domain, and let $E \subset D$ be a compact set of n-capacity zero. Suppose that*

$$F = (F_1, F_2, \ldots, F_n) : D \setminus E \to \mathbf{R}^n$$

is a quasiregular mapping such that

$$\sup_{x \in D\setminus E} |F_1(x)| < \infty\,.$$

Then there exists a quasiregular mapping $\tilde{F} : D \to \mathbf{R}^n$ for which $\tilde{F}\big|_{D\setminus E} = F$.

For $k = n$ we have

3.5. Corollary. *Let $D \subset \mathbf{R}^n$ be a domain, and let $E \subset D$ be a compact set of Hausdorff $(n-1)$-measure zero. Suppose that*

$$F = (F_1, F_2, \ldots, F_n) : D \setminus E \to \mathbf{R}^n$$

is a quasiregular mapping such that

$$\operatorname{ess}\sup_{x \in D \setminus E} J_F(x) < \infty\,.$$

Then there exists a quasiregular mapping $f^* : D \to \mathbf{R}^n$ for which $f^*|_{D\setminus E} = f$.

Proof. Since the Jacobian determinant of F is bounded and E is of $(n-1)$measure zero, the quasiregularity of F implies that F and the form

$$\sum_{i=1}^{n} (-1)^i F_i\, dF_1\, dF_2 \wedge \ldots \widetilde{dF_i} \ldots \wedge dF_n$$

belong to $L^\infty_{\mathrm{loc}}(D)$. Hence the corollary follows from Corollary 3.3.

3.6. Remark. Observe that Corollary 3.5 has an easy alternative proof. Since $J_F(x)$ is bounded and E is of $(n-1)$-measure zero, the quasiregularity of F implies that the derivative of F belongs to $L^\infty_{\mathrm{loc}}(D)$ and F is a Lipschitz mapping in $D \setminus E$. This shows that F can be extended to a Lipschitz mapping on D. It is clear that the extended mapping is quasiregular in D.

Corollary 3.5 gives the following version of the well-known Painleve's Theorem (see, for example, [4, Mattila]).

3.7. Corollary. *Let $E \subset D \subset \mathbf{C}$ be a compact set of linear measure zero. Let $F : D \setminus E \to \mathbf{C}$ be a holomorphic function. The set E is removable for F if and only if*

$$\sup_{z \in K \setminus E} |F'(z)| < \infty\,,$$

for each compact set $K \subset D$.

4 Proof of Theorem 1.6

We will need the following partial integration formula for differential forms.

4.1. Lemma. *Let $\alpha \in W^1_{p,\mathrm{loc}}(\mathcal{M})$ and $\beta \in W^1_q(\mathcal{M})$ be differential forms, $\deg\alpha + \deg\beta = n-1$, $1/p + 1/q = 1$, $1 \le p,\ q \le \infty$, and let β have a compact support. Then*

$$\int\limits_{\mathcal{M}} d\alpha \wedge \beta = (-1)^{\deg\alpha+1} \int\limits_{\mathcal{M}} \alpha \wedge d\beta. \tag{4.2}$$

In particular, the form α is weakly closed if and only if $d\alpha = 0$ a.e. on $\mathcal{M}$.

See **proof** in [2],

Let $D \subset \mathcal{M}$ be a domain containing E and with a compact closure in $\mathcal{M}$. Let $\{U_k\}_{k=1}^{\infty}$ be a sequence of open sets $U_k \subset \mathcal{M}$ such that

$$E \subset U_k, \quad \overline{U}_k \subset D, \quad \cap_{k=1}^{\infty} U_k = E.$$

Fix a nonnegative Lipschitz function $\psi : \mathcal{M} \to \mathbf{R}$, $0 \le \psi \le 1$, with a compact support and $\psi \equiv 1$ on D. Fix a $k = 1, 2, \ldots$ and a Lipschitz function $\varphi : \mathcal{M} \to \mathbf{R}$, $0 \le \varphi \le 1$, with the properties

$$\varphi|_E = 0, \quad \varphi \subset U_k, \quad \varphi = 1 \quad \text{for all} \quad m \in \mathcal{M} \setminus U_k.$$

The form $\psi^p \varphi^p Z \wedge \theta$ has a compact support in $\mathcal{M} \setminus E$. Using (4.2) we have

$$\int\limits_{\mathcal{M}\setminus E} d\left(\psi^p \varphi^p Z \wedge \theta\right) = 0.$$

This yields

$$\int\limits_{\mathcal{M}\setminus E} \psi^p \varphi^p \, dZ \wedge \theta + (-1)^{\deg Z+1} \int\limits_{\mathcal{M}\setminus E} \psi^p \varphi^p Z \wedge d\theta = -\int\limits_{\mathcal{M}\setminus E} d\left(\psi^p \varphi^p\right) \wedge Z \wedge \theta.$$

Observe that

$$dZ \wedge \theta = \langle dZ, *\theta \rangle * \mathbb{1}_{\mathcal{M}}.$$

The form θ is closed and, consequently, from (1.4) and (1.5) we get

$$\begin{aligned}\nu_1 \int\limits_{\mathcal{M}\setminus E} \psi^p\varphi^p\,|dZ|^p * \mathbb{1} &\le \int\limits_{\mathcal{M}\setminus E} \psi^p\varphi^p\,\langle dZ, *\theta\rangle * \mathbb{1} = \\ &= -\int\limits_{\mathcal{M}\setminus E} d\,(\psi^p\varphi^p)\wedge Z\wedge\theta = \\ &= -\int\limits_{\mathcal{M}\setminus E} \langle d\,(\psi^p\varphi^p)\wedge Z, *\theta\rangle\, * \mathbb{1} \le \\ &\le \nu_2 \int\limits_{\mathcal{M}\setminus E} |d\,(\psi^p\varphi^p)\wedge Z||dZ|^{p-1}\, * \mathbb{1}.\end{aligned}$$

By (1.7) there exists a constant $0 < M < \infty$ such that

$$|Z(m)| < M \quad \text{for a.e. in} \quad \mathcal{M}\setminus E.$$

Thus, we obtain

$$\nu_1 \int\limits_{\mathcal{M}\setminus E} \psi^p\varphi^p\,|dZ|^p * \mathbb{1} \le \nu_2\, M \int\limits_{\mathcal{M}\setminus E} |d\,(\psi^p\varphi^p)\,|\,|dZ|^{p-1}\, * \mathbb{1}.$$

However,

$$|d\,(\psi^p\varphi^p)\,| \le p\,\varphi^p\psi^{p-1}|\nabla\psi| + p\,\varphi^{p-1}\psi^p|\nabla\varphi|$$

and

$$\begin{aligned}\nu_1 \int\limits_{\mathcal{M}\setminus E} \psi^p\varphi^p\,|dZ|^p * \mathbb{1} &\le p\nu_2\, M \int\limits_{\mathcal{M}\setminus E} \varphi^p\psi^{p-1}|\nabla\psi|\,|dZ|^{p-1}\, * \mathbb{1} + \\ &+ p\nu_2\, M \int\limits_{\mathcal{M}\setminus E} \psi^p\varphi^{p-1}|\nabla\varphi|\,|dZ|^{p-1}\, * \mathbb{1}.\end{aligned} \tag{4.3}$$

Next we use the Cauchy inequality

$$a\,b^{p-1} \le \frac{\varepsilon^p}{p}\,a^p + \frac{p-1}{p}\,\varepsilon^{p/(1-p)}\,b^p, \quad a,\,b,\,\varepsilon > 0,\,p > 1. \tag{4.4}$$

For $\varepsilon > 0$ this implies two estimates:

$$\begin{aligned}
\int\limits_{\mathcal{M}\setminus E} \varphi^p \psi^{p-1} |\nabla \psi| \, |dZ|^{p-1} \, * \mathbb{1} \;\; &\le \frac{p-1}{p} \, \varepsilon^{p/(1-p)} \int\limits_{\mathcal{M}\setminus E} \varphi^p \psi^p \, |dZ|^p \, * \mathbb{1} + \\
&+ \frac{\varepsilon^p}{p} \int\limits_{\mathcal{M}\setminus E} \varphi^p |\nabla \psi|^p \, * \mathbb{1} \,, \\
\int\limits_{\mathcal{M}\setminus E} \varphi^{p-1} \psi^p |\nabla \varphi| \, |dZ|^{p-1} \, * \mathbb{1} \;\; &\le \frac{p-1}{p} \, \varepsilon^{p/(1-p)} \int\limits_{\mathcal{M}\setminus E} \varphi^p \psi^p \, |dZ|^{p-1} \, * \mathbb{1} + \\
&+ \frac{\varepsilon^p}{p} \int\limits_{\mathcal{M}\setminus E} \psi^p |\nabla \varphi|^p \, * \mathbb{1} .
\end{aligned}$$

Now from (4.3) it follows

$$\begin{aligned}
\nu_1 \int\limits_{\mathcal{M}\setminus E} \psi^p \varphi^p \, |dZ|^p * \mathbb{1} \;\; &\le C_1 \int\limits_{\mathcal{M}\setminus E} \psi^p \varphi^p \, |dZ|^p * \mathbb{1} + \\
&+ C_2 \int\limits_{\mathcal{M}\setminus E} \varphi^p |\nabla \psi|^p \, * \mathbb{1} + \\
&+ C_2 \int\limits_{\mathcal{M}\setminus E} \psi^p |\nabla \varphi|^p \, * \mathbb{1},
\end{aligned}$$

where

$$C_1 = 2\,(p-1)\,\nu_2\, M\, \varepsilon^{p/(1-p)} \quad \text{and} \quad C_2 = \nu_2\, M\, \varepsilon^p .$$

Choose $\varepsilon = \varepsilon_0 > 0$ such that $C_1 = \frac{1}{2}\nu_1$. Then we obtain

$$\begin{aligned}
&\tfrac{1}{2}\nu_1 \int\limits_{\mathcal{M}\setminus E} \psi^p \varphi^p \, |dZ|^p * \mathbb{1} \le \\
&\le \nu_2 \, M \, \varepsilon_0^p \int\limits_{\mathcal{M}\setminus E} \varphi^p |\nabla \psi|^p \, * \mathbb{1} + \nu_2 \, M \, \varepsilon_0^p \int\limits_{\mathcal{M}\setminus E} \psi^p |\nabla \varphi|^p \, * \mathbb{1} = \\
&= \nu_2 \, M \, \varepsilon_0^p \int\limits_{U_k \setminus E} |\nabla \varphi|^p \, * \mathbb{1} + \nu_2 \, M \, \varepsilon_0^p \int\limits_{\mathcal{M}\setminus D} |\nabla \psi|^p \, * \mathbb{1}
\end{aligned}$$

and since $0 \le \psi$, $\varphi \le 1$,

$$\begin{aligned}
\tfrac{1}{2}\nu_1 \int\limits_{D\setminus U_k} |dZ|^p * \mathbb{1} \quad &\le \nu_2 \, M \, \varepsilon_0^p \int\limits_{U_k\setminus E} |\nabla \varphi|^p \, * \mathbb{1} + \\
&+ \nu_2 \, M \, \varepsilon_0^p \int\limits_{\mathcal{M}\setminus D} |\nabla \psi|^p \, * \mathbb{1}.
\end{aligned}$$

The properties of φ and ψ imply

$$\frac{1}{2}\nu_1 \int\limits_{D\setminus U_k} |dZ|^p * \mathbb{1} \le \nu_2 \, M \, \varepsilon_0^p \, \mathrm{cap}_p(E, U_k; \mathcal{M}) + \nu_2 \, M \, \varepsilon_0^p \, \mathrm{cap}_p(D, \mathcal{M}; \mathcal{M}).$$

However, $\mathrm{cap}_p(E, U_k; \mathcal{M}) = 0$ and thus we arrive at the estimate

$$\frac{1}{2}\nu_1 \int\limits_{D\setminus U_k} |dZ|^p * \mathbb{1} \le \nu_2 \, M \, \varepsilon_0^p \, \mathrm{cap}_p(D, \mathcal{M}; \mathcal{M})$$

and

$$\frac{1}{2}\nu_1 \int\limits_{D} |dZ|^p * \mathbb{1} \le \nu_2 \, M \, \varepsilon_0^p \, \mathrm{cap}_p(D, \mathcal{M}; \mathcal{M}) \tag{4.5}$$

because the measure of E is zero.

Since by Lemma 2.2 the set E is of $(n-1)$-measure zero, the coefficients of Z can be extended to $W^1_{p,\mathrm{loc}}$-functions in $\mathcal{M}$. This is due to the estimate

(4.5) and to the ACL-property of W_p^1-functions; note that the ACL-property can be easily transformed to the manifold $\mathcal{M}$ since $\mathcal{M}$ is in the class C^3.

Thus, Z can be extended up to some form $\tilde{Z}$. Moreover clearly, $d\tilde{Z} \in L^p_{\mathrm{loc}}(\mathcal{M})$.

By (1.5) the form θ belongs to $L^q(\mathcal{M}\setminus E)$. The extension of θ is analogous. Theorem is completely proved.

O. Martio and M. Vuorinen:
Department of Mathematics
P.O.Box 4 (Yliopistonkatu 5)
FIN-00014 University of Helsinki, FINLAND
Fax: 3589191 23213
Email: martio@cc.helsinki.fi and vuorinen@csc.fi

V.M. Miklyukov
Department of Mathematics
Volgograd State University
2Prodolnaya 30, Volgograd 400062
RUSSIA
Fax + tel: (7-8442)471608
Email: miklyuk@vlink.ru

References

[1] H. Federer: Geometric measure theory. – Die Grundlehren der math. Wiss. Vol. 153, Springer-Verlag, Berlin-Heidelberg-New York, 1969

[2] D. Franke, O. Martio, V. M. Miklyukov, M. Vuorinen, and R. Wisk: *Quasiregular mappings and $\mathcal{WT}$ -classes of differential forms on Riemannian manifolds.* – Pacific J. Math., v. 202, n. 1, 2002, 73-92.

[3] J. Heinonen, T. Kilpeläinen, and O. Martio: Nonlinear potential theory of degenerate elliptic equations. – Clarendon Press, 1993.

[4] P. Mattila: Search for geometric criteria for removable sets of bounded analytic functions, in prep.

[5] V.M. Miklyukov: Removable singularities of quasi-conformal mappings in space. (Russian) – Dokl. Akad. Nauk SSSR (1969), n.3, 525–527.

[6] Yu.G. Reshetnyak: Space Mappings with Bounded Distortion. — Translations of Mathematical Monographs 73, American Mathematical Society, Providence, RI, 1989.