ON COMPACTNESS OF THE DIFFERENCE OF COMPOSITION OPERATORS

PEKKA J. NIEMINEN AND EERO SAKSMAN

ABSTRACT. Let ϕ and ψ be analytic self-maps of the unit disc, and denote by C_ϕ and C_ψ the induced composition operators. The compactness and weak compactness of the difference $T = C_\phi - C_\psi$ are studied on H^p spaces of the unit disc and L^p spaces of the unit circle. It is shown that the compactness of T on H^p is independent of $p \in [1, \infty)$. The compactness of T on L^1 and M (the space of complex measures) is characterized, and examples of ϕ and ψ are constructed such that T is compact on H^1 but non-compact on L^1. Other given results deal with L^∞, weakly compact counterparts of the previous results, and a conjecture of J. E. Shapiro.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disc of the complex plane and $\phi : \mathbb{D} \to \mathbb{D}$ an analytic map. It is well known that the composition $C_\phi f = f \circ \phi$ defines a linear operator C_ϕ which acts boundedly on various spaces of analytic or harmonic functions on $\mathbb{D}$, including the classical Hardy spaces H^p. During the past few decades much effort has been devoted to the study of these operators with the goal of explaining the operator-theoretic properties of C_ϕ, such as compactness and spectra, in terms of the function-theoretic properties of the symbol ϕ. We refer to the monographs by J. H. Shapiro [S2] and Cowen and MacCluer [CoM] for an overview of the field as of the early 1990s.

The mapping properties of the difference of two composition operators, i.e. an operator of the form

$$T = C_\phi - C_\psi$$

have also been studied. Primary motivation for this line of research has arisen from the urge to understand the topological structure of the set of composition operators in $L(H^2)$, the space of bounded linear operators on the Hilbert space H^2. Papers pursuing this theme include [M], [SS2], [Sh], [B] and [MT]. Properties of T acting on other function spaces have been studied in e.g. [MOZ] and [Go].

In the present paper we investigate the compactness of T on various classical spaces. In addition to the H^p spaces, we will consider L^p and M, the spaces of p-integrable functions and complex Borel measures on the unit circle $\mathbb{T} = \partial\mathbb{D}$. The definition of C_ϕ on these spaces was first given by Sarason [Sa]. The idea is simple: If $\mu \in M$, then the Poisson integral

$$u(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} \, d\mu(\zeta)$$

Date: December 19, 2003.

2000 *Mathematics Subject Classification.* Primary 47B33; Secondary 30D55, 47B07.

Key words and phrases. composition operator, Aleksandrov measure, compactness, difference.

The first author and (in part) the second author were supported by the Academy of Finland, project 49077.

is a harmonic function on $\mathbb{D}$. Since ϕ is analytic, the composition $v = u \circ \phi$ is also harmonic, and by expressing μ as a linear combination of positive measures one sees that v is the Poisson integral of a unique measure $\nu \in M$. One sets $C_\phi \mu = \nu$. Then $C_\phi : M \to M$ is bounded, and one may further show that the restriction of C_ϕ to L^p for $1 \le p \le \infty$ defines a bounded operator $L^p \to L^p$. Let us recall here that the functions in H^p correspond to those functions in L^p (or measures in M if $p = 1$) whose negative Fourier coefficients are all zero.

Some of our results make use of the notion of *Aleksandrov measures.* For any analytic map $\phi : \mathbb{D} \to \mathbb{D}$, these are the positive Borel measures μ_α supported on $\mathbb{T}$ and defined by the Poisson representation

$$\frac{1 - |\phi(z)|^2}{|\alpha - \phi(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2}\, d\mu_\alpha(\zeta) \tag{1.1}$$

for all $\alpha \in \mathbb{T}$. In other words, one has $C_\phi \delta_\alpha = \mu_\alpha$ if δ_α is the unit point mass at α. In [A] A. B. Aleksandrov used these measures to analyse the boundary values of inner functions.

Let us recall that in the case of a single composition operator, the compactness on H^p $(1 \le p < \infty)$ was first characterized by J. H. Shapiro [S1] in terms of the Nevanlinna counting function. Sarason's work [Sa] gave a different-looking compactness criterion for the case of L^1 and M, but soon after Shapiro and Sundberg [SS1] discovered that Shapiro's and Sarason's conditions are equivalent. Later Cima and Matheson [CM1] expressed the condition in terms of the Aleksandrov measures of ϕ: the operator C_ϕ is compact if and only if μ_α is absolutely continuous for each α (the correspondence of Nevanlinna counting functions and Aleksandrov measures was studied in greater detail in [NS]). Thus, interestingly enough, the same criterion characterizes the compactness of C_ϕ on each of the spaces mentioned above. One of the purposes of the present work is to investigate to what extent the same phenomenon exists for the difference of two composition operators, and whether natural analogues of the absolute continuity criterion still hold true.

We now give a brief description of the results obtained. In Section 2 we show that the compactness of T on H^p is independent of the exponent p in the range $1 \le p < \infty$. This generalizes the corresponding result for a single composition operator. We also provide a counterpart of a result of Sarason [Sa2] as we show that $T \in W(H^1)$ implies $T \in K(H^1)$. Here and throughout the paper we use $K(X)$ and $W(X)$ to denote the spaces of compact and weakly compact linear operators on a Banach space X.

In Section 3 we characterize in a relatively simple manner the compactness of T on L^1 and M. Let us denote by μ_α and ν_α the Aleksandrov measures of ϕ and ψ at α, respectively. Also let $\mu_\alpha = \mu_\alpha^a + \mu_\alpha^s$ be the Lebesgue decomposition of μ_α into absolutely continuous and singular parts with the analogous notation used for ν_α. We prove that

$$T \in K(L^1),\, K(M) \quad \text{iff} \quad \begin{cases} (1) & \mu_\alpha^s = \nu_\alpha^s \text{ for all } \alpha \in \mathbb{T}, \\ (2) & \{\mu_\alpha^a - \nu_\alpha^a : \alpha \in \mathbb{T}\} \text{ is uniformly integrable.} \end{cases}$$

We also show that this condition is equivalent both to $T \in W(L^1)$ and to $T \in W(M)$.

The above characterization leads to an interesting question: is $T \in K(L^1)$ equivalent to $T \in K(H^1)$ as it is in the case of a single composition operator? If the answer were affirmative, conditions (1) and (2) would yield a characterization for the compactness of T on H^1 and hence on all H^p for $1 \le p < \infty$. In Section 4 we answer the question negatively, which is a main result of this paper. The required counter-example is fairly complicated and relies, among other things, on rather delicate estimates involving the harmonic measure. However, we will find that the construction sheds some light on the different nature of T on H^1 and L^1.

The necessity of condition (1) above, which requires that the singular parts of the Aleksandrov measures agree at every point, may actually be deduced from the work of J. E. Shapiro [Sh]. In fact, Shapiro showed that (1) is necessary for $T \in K(H^2)$, and then he conjectured that it would also be sufficient. In Section 5 we provide a counter-example to this conjecture. Thus we also see that condition (2) above cannot be dispensed with.

Finally, in Section 6 we extend a result of MacCluer et al. [MOZ] by characterizing the compactness and weak compactness of T on L^∞.

A word about notation. The unit circle $\mathbb{T}$ is equipped with the one-dimensional Lebesgue measure, normalized to have total mass one and denoted by m. The L^p norms of functions on $\mathbb{T}$ will be computed in terms of m. The symbol λ is used to denote the planar Lebesgue measure, normalized so that the area of the unit disc $\mathbb{D}$ is one.

2. Compactness on H^p, $1 \leq p < \infty$

In the present section we consider compactness of the difference two composition operators on the scale of H^p spaces for $1 \leq p < \infty$. We show that the compactness of the difference is independent of the exponent p in the indicated range. For a single composition operator the analogous result was known already in the 1970s [ST]. In our case the classical proof does not work, and the argument below combines an algebraic trick with interpolation. We also show that the weak compactness on H^1 is equivalent to compactness. For a single composition operator this fact was proved by Sarason [Sa2].

Theorem 2.1. *Let $\phi, \psi : \mathbb{D} \to \mathbb{D}$ be analytic and put $T = C_\phi - C_\psi$. Then the following three conditions are equivalent:*

1. *$T \in K(H^p)$ for all $1 \leq p < \infty$,*
2. *$T \in K(H^p)$ for some $1 \leq p < \infty$,*
3. *$T \in W(H^1)$.*

Proof. Propositions 2.2 and 2.3 below isolate the two major steps of the proof. Assuming these results, the proof boils down to a standard interpolation argument. Namely, it is known that in the real interpolation method (see [BS]) the compactness of the operator on one of the end-point spaces implies its compactness on the interpolation space as well (the general result is due to Cwikel [Cw]). In addition, by a result of Fefferman et al. [FRS], for any given $1 \leq p < q$ we obtain the spaces H^s with $p < s < q$ as real interpolation spaces of the couple (H^p, H^q).

In the present situation, as T is bounded on each H^p with $1 \leq p \leq \infty$, it follows immediately that $T \in K(H^p)$ for some $1 < p < \infty$ implies that $T \in K(H^p)$ for all p in this range. In addition, $T \in K(H^1)$ implies $T \in K(H^p)$ for $1 < p < \infty$. Combining these facts with Propositions 2.2 and 2.3 we get the equivalence of the stated conditions. □

It should be remarked that it is possible to avoid the use of general (and rather involved) results of interpolation theory and give a more straightforward argument in the special case considered above.

Proposition 2.2. *If $T \in K(H^2)$, then $T \in K(H^1)$.*

Proof. We will employ the de la Vallee–Poussin operators $V_n : H^1 \to H^1$ defined by setting

$$V_n f(z) = \sum_{k=0}^{n} \hat{f}_k z^k + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n} \hat{f}_k z^k$$

for $f \in H^1$ with the Taylor expansion $f(z) = \sum_{k=0}^{\infty} \hat{f}_k z^k$. Viewed as acting on boundary values these are the convolutions $V_n f = (2K_{2n-1} - K_{n-1}) * f$, where K_n denotes the n:th Fejer kernel (see [K, I.2.13]). Thus $\|V_n\| \leq 3$. Each V_n is a finite-rank operator and hence compact on H^1.

We assume that $T \in K(H^2)$. Since $TV_{2n} \in K(H^1)$ for all n, it suffices to prove that $\|TR_{2n}\| \to 0$, where $R_n = I - V_n$. To this end we fix $f \in H^1$ with $\|f\|_1 = 1$ and note that we always have $R_{2n} f = z^{2n} g$ where $\|g\|_1 = \|R_{2n} f\|_1 \leq 4\|f\|_1 = 4$. By a routine application of the inner-outer factorization theorem of H^p functions we can further write $g = h_1^2 + h_2^2$ where $h_j \in H^2$ with $\|h_j\|_2^2 \leq \|g\|_1$, $j = 1, 2$. Thus, our claim will follow if we show that

$$\sup\{\|T(z^{2n} h^2)\|_1 : h \in H^2,\ \|h\|_2 \leq 1\} \to 0 \quad \text{as } n \to \infty.$$

Now let $h \in H^2$ with $\|h\|_2 \leq 1$. The main idea is to utilize the identity

$$T(z^{2n} h^2) = (C_\phi + C_\psi)(z^n h) \cdot T(z^n h).$$

Since $\|z^n h\|_2 = \|h\|_2$, an application of Hölder's inequality to this identity yields the estimate

$$\|T(z^{2n} h^2)\|_1 \leq M \|T(z^n h)\|_2,$$

where M is the sum of the operator norms of C_ϕ and C_ψ acting on H^2. Since $z^n \to 0$ in $\mathbb{D}$ and since the functions h in the unit ball of H^2 are uniformly bounded on compact subsets of $\mathbb{D}$, the compactness of T on H^2 implies by a standard argument that $\sup\{\|T(z^n h)\|_2 : \|h\|_2 \leq 1\} \to 0$ as $n \to \infty$. The desired conclusion obtains immediately. $\square$

Proposition 2.3. *If $T \in W(H^1)$, then $T \in K(H^1)$.*

The crux of the proof of this proposition is contained in the following lemma, just as in the case of a single composition operator. Here we will make use of the well-known fact that a sequence in L^1 that converges both weakly and almost everywhere converges also in L^1 norm (see [DS, IV.8.12] or the remarks at the beginning of Section 3).

Lemma 2.4. *If $T \in W(H^1)$ and $\phi \neq \psi$, then $|\phi(\zeta)| < 1$ and $|\psi(\zeta)| < 1$ for a.e. ζ.*

Proof. We will show that $|\phi(\zeta)| < 1$ for a.e. ζ. Assume to the contrary. Since $\phi(\zeta) \neq \psi(\zeta)$ for a.e. $\zeta \in \mathbb{T}$, it follows that there exists a set $F \subset \mathbb{T}$ of positive measure such that $|\phi(\zeta)| = 1$ and $|\phi(\zeta) - \psi(\zeta)| \geq \epsilon$ for all $\zeta \in F$ and some $\epsilon > 0$. Consequently, the Borel measure μ on $\mathbb{T}$ defined by $\mu(A) = m(F \cap \phi^{-1}(A))$ is positive and non-vanishing. Thus there exists a point $\zeta_0 \in \mathbb{T}$ such that if $I_n = \{e^{i\theta}\zeta_0 : |\theta| < \frac{1}{n}\}$, then

$$\lim_{n\to\infty} \frac{\mu(I_n)}{m(I_n)} = \lim_{n\to\infty} n\pi\mu(I_n) = c > 0. \tag{2.1}$$

In order to proceed we introduce "test functions" $Q_n \in H^1$ such that (i) $\|Q_n\|_1 = 1$, (ii) $|Q_n| \geq n$ on I_n, and (iii) $Q_n \to 0$ locally uniformly on $\overline{\mathbb{D}} \setminus \{\zeta_0\}$ as $n \to \infty$. These can be easily realized as outer functions of the form

$$\log Q_n(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log g_n(\zeta)\, dm(\zeta),$$

where g_n assumes constant values on I_n and on $\mathbb{T}\setminus I_n$. By (i) it is clear that the sequence (TQ_n) is bounded in H^1 norm, and (iii) implies that it converges to zero pointwise on $\mathbb{D}$. On the other hand, since $T\in W(H^1)$, every subsequence of (TQ_n) must have a weakly convergent subsequence. But by the preceding observation the only possible weak limit is zero, so the whole sequence (TQ_n) converges to zero weakly in H^1 and hence in L^1. In addition, property (iii) yields that $TQ_n\to 0$ almost everywhere on $\mathbb{T}$. Together these two facts imply that $TQ_n\to 0$ in L^1 norm.

To obtain a contradiction we consider the estimate

$$\|TQ_n\|_1 \geq \int_{F\cap\phi^{-1}(I_n)} |C_\phi Q_n|\,dm - \int_{F\cap\phi^{-1}(I_n)} |C_\psi Q_n|\,dm.$$

The first integral here equals $\int_{I_n}|Q_n|\,d\mu$, which is greater than $n\mu(I_n)$ by property (ii) of Q_n. The second integral tends to zero as $n\to\infty$ because for large n the boundary values of ψ are bounded away from ζ_0 in the set $F\cap\phi^{-1}(I_n)$ and thus property (iii) ensures that $C_\psi Q_n\to 0$ uniformly in that set. Hence, in view of (2.1), we have that $\liminf\|TQ_n\|_1 \geq \lim n\mu(I_n) = c/\pi > 0$, which is a contradiction. $\square$

Proof of Proposition 2.3. Let (f_n) be a bounded sequence in H^1. We need to show that the sequence (Tf_n) has a subsequence that converges in H^1. Since (f_n) is a normal family we may assume, by passing to a subsequence, that (f_n) converges locally uniformly to some function g on $\mathbb{D}$. It is easy to check that $g\in H^1$. Then $T(f_n-g)\to 0$ pointwise on $\mathbb{D}$ and almost everywhere on $\mathbb{T}$ due to the preceding lemma. On the other hand, since $T\in W(H^1)$, we may extract a subsequence (f_{n_k}) for which $T(f_{n_k}-g)\to 0$ weakly in H^1. Together these facts yield that $Tf_{n_k}\to Tg$ in H^1, and the proof is complete. $\square$

Remark 2.5. For $1<p<\infty$ one of course has that $T\in K(L^p)$ if and only if $T\in K(H^p)$ because the Riesz projection is bounded in this case and commutes with C_ϕ and C_ψ.

3. Compactness on L^1 and M

In his important work [Sa], Sarason considered the composition operator C_ϕ as an integral operator acting on the spaces L^1 and M of integrable functions and complex Borel measures on $\mathbb{T}$. He showed that the following four compactness conditions are all equivalent: $C_\phi\in K(M)$, $C_\phi\in W(M)$, $C_\phi\in K(L^1)$, and $C_\phi\in W(L^1)$. Moreover, he characterized all these by a condition which is easily seen to be equivalent to the absolute continuity of the Aleksandrov measures of ϕ (see [CM1]).

In this section we will give a generalization of Sarason's result to the setting of differences of composition operators. We recall from (1.1) that the Aleksandrov measure of ϕ at α can be defined as $\mu_\alpha = C_\phi\delta_\alpha$. Similarly we let $\nu_\alpha = C_\psi\delta_\alpha$ if ψ is another self-map of the unit disc $\mathbb{D}$. We also recall that a set $A\subset L^1$ is *uniformly integrable* if

$$\sup_{f\in A}\int_{\{|f|>L\}} |f|\,dm \to 0 \quad \text{as } L\to\infty.$$

According to the classical Dunford–Pettis theorem (see [W, III.C.12]) a set $A\subset L^1$ is relatively weakly compact if and only if it is uniformly integrable. We will also have an occasion to use Vitali's convergence theorem (see e.g. [R1, Exercise 6.10]), which asserts that if (f_n) is a uniformly integrable sequence in L^1 such that $f_n\to f$ almost everywhere, then $f_n\to f$ in L^1 norm.

Our characterization is the following.

Theorem 3.1. *Let $\mu_\alpha = \mu_\alpha^a + \mu_\alpha^s$ and $\nu_\alpha = \nu_\alpha^a + \nu_\alpha^s$ be the Lebesgue decompositions of the Aleksanrov measures of ϕ and ψ, respectively, so that*

$$\mu_\alpha^a(\zeta) = \frac{1 - |\phi(\zeta)|^2}{|\alpha - \phi(\zeta)|^2}, \qquad \nu_\alpha^a(\zeta) = \frac{1 - |\psi(\zeta)|^2}{|\alpha - \psi(\zeta)|^2},$$

and μ_α^s, ν_α^s are singular. The following conditions are equivalent for $T = C_\phi - C_\psi$:

1. $T \in K(M)$,
2. $T \in W(M)$,
3. $T \in K(L^1)$,
4. $T \in W(L^1)$,
5. $\mu_\alpha^s = \nu_\alpha^s$ *for all* $\alpha \in \mathbb{T}$ *and* $\{\mu_\alpha^a - \nu_\alpha^a : \alpha \in \mathbb{T}\}$ *is uniformly integrable.*

It should be emphasized that to guarantee the compactness of T on M and L^1, it is not sufficient to require only that $\mu_\alpha^s = \nu_\alpha^s$ for all α. This issue is discussed in greater detail in Section 5.

Note that (1) is the strongest and (4) is the weakest of the compactness conditions in Theorem 3.1. Therefore the proof of the theorem reduces to verifying implications (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1). The first of these depends on the fact that every composition operator (and hence T) is weak*-weak*-continuous as an operator on M. This fact is a consequence of the following easy observation.

Lemma 3.2. *Let (τ_n) be a bounded sequence in M and let (u_n) be the sequence of corresponding Poisson integrals. Then (τ_n) converges weak* to zero if and only if (u_n) converges pointwise to zero.*

For implication (5) $\Rightarrow$ (1) we require another lemma from functional analysis. This lemma is basically a consequence of the Krein–Milman theorem (see e.g. [R2, 3.23]), which ensures that the absolute convex hull of the set $\{\delta_\alpha : \alpha \in \mathbb{T}\}$ is weak*-dense in the unit ball of M. We omit the details of the argument.

Lemma 3.3. *Let $S : M \to M$ is a bounded linear operator which is weak*-weak*-continuous. If the set $\{S\delta_\alpha : \alpha \in \mathbb{T}\}$ is relatively compact in M, then $S \in K(M)$.*

Proof of Theorem 3.1. (4) *implies* (5): For every $\alpha \in \mathbb{T}$ and $0 < r < 1$, define $f_{\alpha,r} \in L^1$ by setting $f_{\alpha,r}(\zeta) = (1 - r^2)/|\alpha - r\zeta|^2$. Then $\|f_{\alpha,r}\|_1 = 1$ and, as $r \to 1-$, $f_{\alpha,r} \to \delta_\alpha$ in the weak* topology of M. Since T is weak*-weak*-continuous on M, it follows that $Tf_{\alpha,r} \to \mu_\alpha - \nu_\alpha$ weak*. Furthermore, since $T \in W(L^1)$, we can find some r_n increasing to 1 such that Tf_{α,r_n} converges weakly to an element of L^1 as $n \to \infty$. By the uniqueness of the limit, we conclude that $\mu_\alpha - \nu_\alpha \in L^1$, or equivalently, $\mu_\alpha^s = \nu_\alpha^s$. Moreover, our argument also shows that the differences $\mu_\alpha - \nu_\alpha = \mu_\alpha^a - \nu_\alpha^a$ belong to the weak closure of the relatively weakly compact set $\{Tf_{\alpha,r} : \alpha \in \mathbb{T},\ 0 < r < 1\}$. Therefore the set $\{g_\alpha - h_\alpha : \alpha \in \mathbb{T}\}$ is relatively weakly compact and, by the Dunford–Pettis theorem, uniformly integrable.

(5) *implies* (1): Observe first that the function $\alpha \mapsto \mu_\alpha^a(\zeta) - \nu_\alpha^a(\zeta)$ is continuous for almost all $\zeta \in \mathbb{T}$. Therefore, since the set $\{\mu_\alpha^a - \nu_\alpha^a : \alpha \in \mathbb{T}\}$ is assumed to be uniformly integrable, Vitali's convergence theorem can be applied to show that the map $\alpha \mapsto \mu_\alpha^a - \nu_\alpha^a$ is continuous with respect to the norm topology of L^1. Hence $\{\mu_\alpha^a - \nu_\alpha^a : \alpha \in \mathbb{T}\}$ is a compact subset of L^1. Because $T\delta_\alpha = \mu_\alpha - \nu_\alpha = \mu_\alpha^a - \nu_\alpha^a$, Lemma 3.3 implies that $T \in K(M)$. ☐

Remark 3.4. The weak*-weak* continuity of C_ϕ on M indicates that C_ϕ is an adjoint of some operator acting on C, the space of continuous functions on $\mathbb{T}$. Using the identity $C_\phi \delta_\alpha = \mu_\alpha$ and an approximation argument (see [CM2]) one finds that this operator is the *Aleksandrov operator* A_ϕ defined by the integral formula

$$A_\phi f(\alpha) = \int_{\mathbb{T}} f \, d\mu_\alpha, \qquad \alpha \in \mathbb{T}.$$

The operator A_ϕ was introduced by Aleksandrov [A], who showed that it defines a bounded linear operator on many function spaces, including C and L^p for $1 \le p \le \infty$. Also, one may show that $A_\phi : L^p \to L^p$ represents the adjoint (or preadjoint) of $C_\phi : L^q \to L^q$ when q is the conjugate exponent of p. Since an operator is compact (resp. weakly compact) if and only if its adjoint is, these observations provide an alternative approach to the proof of Theorem 3.1.

4. Comparison between the cases of L^1 and H^1

After Theorems 3.1 and 2.1 it becomes natural to ask whether a complete analogue of the case of one composition operator holds. That is, whether $C_\phi - C_\psi \in K(H^1)$ implies $C_\phi - C_\psi \in K(L^1)$. If it were so, the compactness of the difference on each of the spaces H^p, L^p $(1 \le p < \infty)$ and M would be equivalent and characterized by condition (5) of Theorem 3.1. Our next theorem, which can be seen as a main result of the present paper, answers this question negatively. The counter-example is fairly complicated, but it gives some intuition on the difference between the cases of L^1 and H^1 (cf. Remark 4.6 below).

Theorem 4.1. *There exist two analytic functions $\phi, \psi : \mathbb{D} \to \mathbb{D}$ such that $T = C_\phi - C_\psi$ satisfies $T \in K(H^1)$ but $T \notin K(L^1)$.*

Before we turn to the actual proof, we collect a number of auxiliary notions and lemmas. First we have a useful compactness condition, which reminds [SS2, Theorem 3.2]. Let us recall that a bounded linear operator T on a (separable) Hilbert space is *Hilbert–Schmidt* if its *Hilbert–Schmidt norm*

$$\|T\|_{\mathrm{HS}} = \left(\sum_{k=0}^{\infty} \|Te_k\|^2 \right)^{1/2}$$

is finite, where (e_k) is any orthonormal basis of the underlying Hilbert space. Every Hilbert–Schmidt operator is compact.

Lemma 4.2. *Let $\phi, \psi : \mathbb{D} \to \mathbb{D}$ be analytic functions such that $|\phi| < 1$ and $|\psi| < 1$ almost everywhere on $\mathbb{T}$, and let $E \subset \mathbb{T}$ be measurable. Then the Hilbert–Schmidt norm of the operator $T : H^2 \to L^2$ defined by*

$$Tf(\zeta) = \big(C_\phi f(\zeta) - C_\psi f(\zeta)\big)\chi_E(\zeta)$$

satisfies

$$\|T\|_{\mathrm{HS}}^2 \le \int_E \frac{|\phi - \psi|}{\min(1 - |\phi|, 1 - |\psi|)^2} \, dm.$$

Proof. We have

$$\|T\|_{\mathrm{HS}}^2 = \sum_{k=0}^{\infty} \|Tz^k\|_2^2 = \sum_{k=0}^{\infty} \int_E |\phi^k - \psi^k|^2 \, dm.$$

By writing $|a - b|^2 = |a|^2 + |b|^2 - 2 \operatorname{Re} a\bar{b}$ and summing the appropriate geometric series we obtain

$$\|T\|_{\mathrm{HS}}^2 = \int_E \left(\frac{1}{1 - |\phi|^2} + \frac{1}{1 - |\psi|^2} - 2 \operatorname{Re} \frac{1}{1 - \phi\overline{\psi}} \right) dm.$$

Fix $w, w' \in \mathbb{D}$ and consider the function

$$g(z) = \frac{1}{1-|z|^2} + \frac{1}{1-|w|^2} - 2\operatorname{Re}\frac{1}{1-z\overline{w}}$$

on the line segment connecting w and w'. On this segment we have the estimate $|\nabla g| \le C\min(1-|w|, 1-|w'|)^{-2}$. Moreover, $g(w) = 0$. The lemma follows immediately from these observations and the above expression for $\|T\|_{\mathrm{HS}}$. □

Next we recall the following well-known estimate for the H^2 norm of a function $f \in H^2$:

$$\|f\|_2^2 - |f(0)|^2 \sim \int_{\mathbb{D}} |f'(z)|^2 (1-|z|)\, d\lambda(z), \tag{4.1}$$

where λ denotes the normalized planar Lebesgue measure on $\mathbb{D}$. The symbol $\sim$ means that the left- and right-hand sides of (4.1) are comparable to each other with some positive constants. In fact, an exact identity rather than just an equivalent expression for the H^2 norm of f is obtained by replacing the weight $1-|z|$ with $2\log(1/|z|)$. This identity is known as the Littlewood–Paley identity.

Lemma 4.3. *Let (z_k) be a sequence of points in $\mathbb{D}$ and put $d_k = 1-|z_k|$. Suppose $d_{k+1} \le a d_k$ for all k and some constant $0 < a < 1$. Then*

$$\sum_{k=1}^{\infty} |f'(z_k)|^2 d_k^3 \le C\|f\|_2^2, \qquad f \in H^2,$$

where C depends only on a.

Proof. Let $c = \frac{1}{2}(1-a)$ and $D_k = B(z_k, c d_k)$. Since the function $|f'|^2$ is subharmonic, we have $c^2 d_k^2 |f'(z_k)|^2 \le \int_{D_k} |f'|^2\, d\lambda$ for each k. Thus

$$d_k^3 |f'(z_k)|^2 \le 2c^{-2} \int_{D_k} |f'(z)|^2 (1-|z|)\, d\lambda(z)$$

because $d_k \le 2(1-|z|)$ for $z \in D_k$. As the discs D_k are disjoint by the choice of c, the desired estimate is obtained by summing over k and applying (4.1). □

As a final preparatory step we give a technical lemma that estimates the harmonic measure in a domain obtained from $\mathbb{D}$ by removing a number of small discs. Here we let

$$\beta(z, w) = \left| \frac{z-w}{1-z\overline{w}} \right| \tag{4.2}$$

be the pseudo-hyperbolic distance between any two points $z, w \in \mathbb{D}$. The pseudo-hyperbolic disc with centre $z \in \mathbb{D}$ and radius r is denoted by $D(z, r)$, whereas $B(z, r)$ stands for the usual Euclidean disc.

Lemma 4.4. *Suppose $d_1, \dots, d_n$ are positive numbers with $d_1 < \frac{1}{4}$ and $d_j \le \frac{1}{10} d_{j-1}$ for $j = 2, \dots, n$. Define $B_j = \overline{B}(1-d_j, d_j e^{-20n})$ and $\Omega = \mathbb{D} \setminus \bigcup_{j=1}^n B_j$. Let a be a complex number with $|a| \le \frac{1}{3}$, and let γ_j be the harmonic measure of ∂B_j with respect to Ω at a. Then*

$$C_1 \frac{d_j}{n} \le \gamma_j \le C_2 \frac{d_j}{n}, \qquad 1 \le j \le n,$$

where C_1 and C_2 are absolute positive constants.

Proof. It is a consequence of the Harnack inequality that the harmonic measure for Ω at a is comparable (with absolute constants) to the corresponding harmonic measure at 0. So it is enough to consider the case $a = 0$.

For $b > 0$ define $v_b(z) = b^{-1}\log(1/|z|)$ and note that v_b is the radially decreasing harmonic function in $\mathbb{C}\setminus\{0\}$ that equals 1 on the circle $|z| = e^{-b}$ and vanishes on $\mathbb{T}$. Let us write $r_j = 1 - d_j$, fix k with $1 \le k \le n$, and consider the function

$$u(z) = v_{30n}\Big(\frac{z - r_j}{1 - r_j z}\Big) - \sum_{j=1}^{k-1}\frac{d_k}{5nd_j}v_{20n}\Big(\frac{z - r_j}{1 - r_j z}\Big) - \sum_{j=k+1}^{n} v_{20n}\Big(\frac{z - r_j}{1 - r_j z}\Big).$$

It is harmonic in a region containing $\overline{\Omega}$. We also claim that

$$u|\partial B_k \le 1 \qquad \text{and} \qquad u|\partial B_j \le 0 \quad \text{for } j \ne k. \tag{4.3}$$

To see this, we first note that by a simple estimate $D(r_j, e^{-30n}) \subset B_j \subset D(r_j, e^{-20n})$ for all j. Then the first claim as well as the case $j > k$ of the second one follow by inspection. For $j < k$ one just needs to observe that if $z \in \partial B_j$, then $|z| \le \frac{2}{3}r_j + \frac{1}{3}$ and hence

$$1 - \rho(r_k, z) \le 1 - \rho(r_k, \tfrac{2}{3}r_j + \tfrac{1}{3}) \le \frac{3d_k}{d_j}.$$

Consequently,

$$v_{30n}\Big(\frac{z - r_k}{1 - r_k z}\Big) = \frac{1}{30n}\log\frac{1}{\rho(r_k, z)} \le \frac{d_k}{5nd_j}.$$

Here we applied the right-hand side of the simple estimate $1 - x \le \log(1/x) \le 2(1 - x)$, valid for all $x \in (\frac{1}{2}, 1)$. According to (4.3) we now get

$$\begin{aligned}\gamma_k \ge u(0) &= \frac{1}{30n}\log\frac{1}{r_k} - \sum_{j=1}^{k-1}\frac{d_k}{5nd_j}\cdot\frac{1}{20n}\log\frac{1}{r_j} - \sum_{j=k+1}^{n}\frac{1}{20n}\log\frac{1}{r_j}\\ &\ge \frac{1}{20n}\Big(\frac{2}{3}d_k - \sum_{j=1}^{k-1}\frac{d_k}{5nd_j}\cdot 2d_j - \sum_{j=k+1}^{n} 2d_j\Big) \ge \frac{d_k}{20n}\Big(\frac{2}{3} - \frac{2}{5} - \frac{2}{9}\Big).\end{aligned}$$

Since the number in parentheses is positive, the required lower bound is obtained.

To get the upper bound we just observe that γ_j is less than the harmonic measure of the pseudo-hyperbolic circle $\partial D(r_j, e^{-20n})$ with respect to $\mathbb{D}\setminus\overline{D}(r_j, e^{-20n})$ at 0. This yields $\gamma_j \le (1/20n)\log(1/r_j) \le d_j/10n$. □

Remark 4.5. The above lemma may also be approached from a stochastic point of view. In this way one obtains a very intuitive explanation for the factor e^{-20n} in the radii of the discs. In fact, this choice ensures that the harmonic measure of ∂B_j is of order $\sim 1/n$ (with respect to the domain $\mathbb{D}\setminus B_j$). Hence, in the first approximation the Brownian motion started at zero hits the circle ∂B_j with probablility $\sim (1 - c/n)^{j-1}(c/n) \sim c'/n$, as is seen by considering the probability that it has not first hit any of the discs $B_1, \ldots, B_{j-1}$. Here one crudely assumes that the hits to different discs are independent of each other. This argument can be made rigorous to provide another proof of the lemma.

We are ready for the details of the proof of Theorem 4.1. We have divided the argument into three steps. First we define the map ϕ and investigate some of its properties. Then we construct the map ψ, and finally establish the compactness properties of the resulting operator $T = C_\phi - C_\psi$.

Step 1: the map ϕ. For each $k \geq 1$, let $A_k = B(\frac{1}{4}e^{i/k}, \frac{3}{4})$ and put $\Omega_0 = \bigcup_{k=k_0}^{\infty} A_k$. Then define the discs

$$D_{k,j} = \overline{B}\big((1-d_{k,j})e^{i/k}, d_{k,j}e^{-20\cdot 2^k}\big), \qquad 1 \leq j \leq 2^k,\ k \geq 1,$$

where $d_{k,j} = 10^{-k-j}$. One can easily check that these are pairwise disjoint and satisfy $D_{k,j} \subset A_k$ and $D_{k,j} \cap A_{k'} = \emptyset$ whenever $k \neq k'$. Now let $\Omega = \Omega_0 \setminus \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{2^k} D_{k,j}$. Clearly, Ω is a region contained in the unit disc whose boundary intersects the unit circle only at the points 1 and $e^{i/k}$, $k \geq k_0$. The map ϕ is now defined to be an analytic covering map from $\mathbb{D}$ onto Ω with $\phi(0) = 0$.

We will next obtain some information on the distribution of the boundary values of ϕ. Recall that since ϕ is a covering map, its radial boundary limits (which, by Fatou's theorem, exist at almost every boundary point) all lie in $\partial\Omega$. Moreover, their distribution is given by the harmonic measure for Ω at 0. Let us introduce the notation

$$E_0 = \phi^{-1}(\partial\Omega_0), \qquad E_{k,j} = \phi^{-1}(\partial D_{k,j}), \quad 1 \leq j \leq 2^k,\ k \geq 1.$$

In order to study the boundary value distribution of ϕ on $\partial\Omega_0$ we use the well-known fact that the boundary values of every analytic self-map of the unit disc induce a Carleson measure (see e.g. [CoM, Theorem 3.12]). This implies that there is a constant $c > 0$ such that

$$m\big(\{\zeta \in \mathbb{T} : \phi(\zeta) \in W\}\big) \leq c\gamma$$

for every "Carleson window"

$$W = W(e^{i\theta}, \gamma) = \{re^{it} : 1-\gamma \leq r < 1,\ |t-\theta| \leq \gamma\}.$$

A simple geometric reasoning shows that for $\delta > 0$ the union of $W(1, 4\delta^{1/4})$ and $W(e^{i/k}, 2\delta^{1/2})$, $1 \leq k \leq \delta^{-1/4}$, covers all points $z \in \partial\Omega_0$ whose distance to the unit circle is $\leq \delta$. Therefore

$$m\big(\{\zeta \in E_0 : 1 - |\phi(\zeta)| \leq \delta\}\big) \leq c \cdot 4\delta^{1/4} + \delta^{-1/4} \cdot c \cdot 2\delta^{1/2} = 6c\delta^{1/4}.$$

In particular, if we let

$$E_{0,j} = \{\zeta \in E_0 : 2^{-j} < 1 - |\phi(\zeta)| \leq 2^{1-j}\}, \qquad j \geq 1,$$

then

$$m(E_{0,j}) \leq c' 2^{-j/4}, \qquad j \geq 1, \tag{4.4}$$

with $c' = 6 \cdot 2^{1/4}c$. Moreover, $\bigcup_{j=1}^{\infty} E_{0,j}$ covers all of E_0 apart from a set of measure zero.

Then we estimate $m(E_{k,j})$, the harmonic measure of $\partial D_{k,j}$ with respect to Ω at 0. An upper bound is obtained as a direct application of Lemma 4.4 by considering the harmonic measure of $\partial D_{k,j}$ with respect to the region $\mathbb{D} \setminus \bigcup_{j=1}^{2^k} D_{k,j}$. This yields

$$m(E_{k,j}) \leq C_2 2^{-k} d_{k,j}. \tag{4.5}$$

To get a lower bound, we estimate the harmonic measure of $\partial D_{k,j}$ with respect to the region $A_k \setminus \bigcup_{j=1}^{2^k} D_{k,j}$. Using Lemma 4.4 plus a scaling argument we find

$$m(E_{k,j}) \geq C_1 2^{-k} d_{k,j}. \tag{4.6}$$

Step 2: the map ψ. Consider the positive function h defined almost everywhere on $\mathbb{T}$ by setting

$$h(\zeta) = \begin{cases} 2^{-2j} & \text{if } \zeta \in E_{0,j},\ j \geq 1, \\ \frac{1}{4}d_{k,j} & \text{if } \zeta \in E_{k,j},\ 1 \leq j \leq 2^k,\ k \geq 1. \end{cases}$$

As a consequence of the definitions one immediately obtains the inequality

$$|h| \le \tfrac{1}{2}(1-|\phi|) \quad \text{a.e. on } \mathbb{T}. \tag{4.7}$$

We also claim that

$$\int_{\mathbb{T}} \log h \, dm > -\infty, \tag{4.8}$$

and

$$\int_{E_0} \frac{h\,dm}{(1-|\phi|-h)^2} < \infty. \tag{4.9}$$

To verify the first claim, we use (4.4) to compute

$$\int_{E_0} \log h\,dm = \sum_{j=1}^{\infty} m(E_{0,j}) \log 2^{-2j} \ge 2(\log 2)c' \sum_{j=1}^{\infty} 2^{-j/4} j > -\infty.$$

Also, if $E_k = \bigcup_{j=1}^{2^k} E_{k,j}$, then (4.5) can be used to estimate

$$\int_{E_k} \log h\,dm = \sum_{j=1}^{2^k} m(E_{k,j}) \log \tfrac{1}{4} d_{k,j} \ge C_2 2^{-k} \sum_{j=1}^{2^k} d_{k,j} \log \tfrac{1}{4} d_{k,j} \ge C_2 d_{k,1} \log \tfrac{1}{4} d_{k,1}.$$

Substituting $d_{k,1} = 10^{-k-1}$ and summing over k yields (4.8). For the second claim we observe that on $E_{0,j}$ one has $1-|\phi|-h \ge 2^{-j}-2^{-2j}$ and hence

$$\int_{E_{0,j}} \frac{h\,dm}{(1-|\phi|-h)^2} \le \frac{c' 2^{-2j}\, 2^{-j/4}}{(2^{-j}-2^{-2j})^2} \le 4c' 2^{-j/4}.$$

Inequality (4.9) is obtained by summing over j.

For each $k \ge 1$ and $1 \le j \le 2^k$ we now define a function $h_{k,j}$ on $\mathbb{T}$ by setting

$$h_{k,j} = \Big(\frac{2^{-k-j}}{100} + \frac{\chi_{E_{k,j}}}{2}\Big) h.$$

We also let $H_{k,j}$ be an outer function satisfying $|H_{k,j}| = h_{k,j}$ almost everywhere on $\mathbb{T}$. Such a function exists due to (4.8). Then we set

$$H = \sum_{k,j} \rho_{k,j} H_{k,j},$$

where $\rho_{k,j}$ are unimodular constants to be specified in a moment. It is easy to check that the above series is convergent and defines an analytic function on $\mathbb{D}$. In addition, our definitions and (4.7) yield that

$$|H| < h \le \tfrac{1}{2}(1-|\phi|) \quad \text{a.e. on } \mathbb{T}.$$

Thus the formula

$$\psi = \phi + H$$

defines an analytic self-map of $\mathbb{D}$.

What still remains of the definition of ψ is the choice of the phase factors $\rho_{k,j}$. We claim that these can be chosen in such a way that

$$\int_{E_{k,j}} \left| \frac{1-|\psi|^2}{|e^{i/k}-\psi|^2} - \frac{1-|\phi|^2}{|e^{i/k}-\phi|^2} \right| dm \ge c \frac{m(E_{k,j})}{d_{k,j}} \tag{4.10}$$

with c a positive constant independent of k and j. For the verification of this fact we first observe from the definition of $H_{k,j}$ that

$$|H_{k,j}| \ge \frac{d_{k,j}}{10} \quad \text{on } E_{k,j} \tag{4.11}$$

and (independently of the choice of the phase factors)

$$|\psi - \phi - \rho_{k,j} H_{k,j}| \le \frac{d_{k,j}}{100} \quad \text{on } E_{k,j} \tag{4.12}$$

for all $k \ge 1$ and $1 \le j \le 2^k$. A direct computation shows for the norm of the gradient of the Poisson kernel that

$$\Big|\nabla \frac{1-|z|^2}{|\zeta - z|^2}\Big| = \frac{2}{|\zeta - z|^2}. \tag{4.13}$$

As a consequence we obtain

$$|\nabla u(0)| \le 2 \int_{\mathbb{T}} |u| \, dm$$

for any function u harmonic in a neighbourhood of the closed unit disc. Let us apply this estimate to the function

$$u(z) = \frac{1 - |\phi(\zeta) + zH_{k,j}(\zeta)|^2}{|e^{i/k} - \phi(\zeta) - zH_{k,j}(\zeta)|^2} - \frac{1 - |\phi(\zeta)|^2}{|e^{i/k} - \phi(\zeta)|^2}$$

with $\zeta \in E_{k,j}$ fixed. By (4.11) and (4.13) we get $|\nabla u(0)| \ge |H_{k,j}(\zeta)|/d_{k,j}^2 \ge 1/10d_{k,j}$, so an application of Fubini's theorem shows that

$$\int_{\mathbb{T}} \Big[\int_{E_{k,j}} \Big| \frac{1 - |\phi + \rho H_{k,j}|^2}{|e^{i/k} - \phi - \rho H_{k,j}|^2} - \frac{1 - |\phi|^2}{|e^{i/k} - \phi|^2} \Big| \, dm \Big] \, dm(\rho) \ge \frac{1}{20} \frac{m(E_{k,j})}{d_{k,j}}.$$

Therefore $\rho_{k,j} \in \mathbb{T}$ can be chosen such that

$$\int_{E_{k,j}} \Big| \frac{1 - |\phi + \rho_{k,j} H_{k,j}|^2}{|e^{i/k} - \phi - \rho_{k,j} H_{k,j}|^2} - \frac{1 - |\phi|^2}{|e^{i/k} - \phi|^2} \Big| \, dm \ge \frac{1}{20} \frac{m(E_{k,j})}{d_{k,j}}. \tag{4.14}$$

On the other hand, in view of inequality (4.12) we have the estimate

$$\begin{aligned} &\int_{E_{k,j}} \Big| \frac{1 - |\phi + \rho_{k,j} H_{k,j}|^2}{|e^{i/k} - \phi - \rho_{k,j} H_{k,j}|^2} - \frac{1 - |\psi|^2}{|e^{i/k} - \psi|^2} \Big| \, dm \\ &\qquad \le \frac{4}{d_{k,j}^2} \cdot \frac{d_{k,j}}{100} m(E_{k,j}) = \frac{1}{25} \frac{m(E_{k,j})}{d_{k,j}}. \end{aligned} \tag{4.15}$$

Here we used the fact that the gradient of the Poisson kernel on the line segment connecting the points involved is less than $4/d_{k,j}^2$. Combining (4.14) and (4.15) we now get (4.10) with $c = \frac{1}{20} - \frac{1}{25} = \frac{1}{100}$.

Step 3: compactness properties. Recall that we write $T = C_\phi - C_\psi$. First we check that $T \in K(H^2)$. We let $E_k = \bigcup_{j=1}^{2^k} E_{k,j}$ for $k \ge 1$ and define $T_k f = \chi_{E_k} T f$ for $k \ge 0$, so that T_k is an operator from H^2 to L^2. We obviously have

$$T = T_0 + T_1 + T_2 + \cdots$$

with convergence in the strong operator topology (i.e. with pointwise convergence). It is enough to show that each summand on the right-hand side is compact and that $\sum_k \|T_k\| < \infty$. The compactness of T_0 is a consequence of (4.9), the fact that $|H| \le h$ a.e. on $\mathbb{T}$ and Lemma 4.2. Fix $k \ge 1$. Since ϕ and ψ are bounded away from the unit circle on E_k, it is clear that T_k is compact. We next estimate the norm of T_k. Let $f \in H^2$. Since the values of ϕ and ψ on $E_{k,j}$ lie in the disc $B((1-d_{k,j})e^{i/k}, \frac{1}{2}d_{k,j})$, we see that there exists a point $w_{k,j}$ in the closure of that disc such that

$$|f \circ \phi - f \circ \psi| \le |f'(w_{k,j})| d_{k,j} \quad \text{on } E_{k,j}.$$

Applying inequality (4.5) and Lemma 4.3 we obtain

$$\|T_k f\|_2^2 \le C_2 2^{-k} \sum_{j=1}^{2^k} |f'(w_{k,j})|^2 d_{k,j}^3 \le CC_2 2^{-k} \|f\|_2^2.$$

Thus $\|T_k\| \le (CC_2)^{1/2} 2^{-k/2}$, and it follows that $\sum_k \|T_k\| < \infty$. Hence $T \in K(H^2)$.

Finally we verify that $T \notin K(L^1)$. Summing over j in (4.10) and applying estimate (4.6) we find

$$\int_{E_k} \left| \frac{1-|\psi|^2}{|e^{i/k}-\psi|^2} - \frac{1-|\phi|^2}{|e^{i/k}-\phi|^2} \right| dm \ge c \sum_{j=1}^{2^k} \frac{m(E_{k,j})}{d_{k,j}} \ge cC_1.$$

Since $m(E_k)$ tends to zero as $k \to \infty$, we conclude that condition (5) of Theorem 3.1 fails. Hence $T \notin K(L^1)$. The proof of Theorem 4.1 is now complete.

Remark 4.6. The above proof deals with H^2, but it might be more instructive to consider H^1 instead because it bears a close relation to L^1 and the compactness of T on H^1 is equivalent to compactness on H^2 by Theorem 2.1. Slightly heuristically speaking, one applies above the fact (essentially due to Paley) that in the dual of H^1 widely separated blocks with respect to the trigonometric basis generate L^2, whereas nothing like this is true for L^1.

5. Necessity of the uniform integrability condition in Theorem 3.1: a conjecture of J. E. Shapiro

In this section we show that the uniform integrability requirement in condition (5) of Theorem 3.1 is not superfluous. This matter is directly connected to a conjecture of J. E. Shapiro [Sh]. Shapiro's work contains, among other things, a number of interesting estimates for the norm and essential norm of the operator $T = C_\phi - C_\psi$ on H^2. In his Conjecture 5.4 it is conjectured that $T \in K(H^2)$ if the singular parts of the Aleksandrov measures of ϕ and ψ coincide at every point of $\mathbb{T}$. Our next result produces a counter-example to this conjecture and at the same time verifies the necessity of uniform integrability in condition (5) of Theorem 3.1.

Theorem 5.1. *There exist two analytic functions $\phi, \psi : \mathbb{D} \to \mathbb{D}$ such that the singular parts of the Aleksandrov measures of ϕ and ψ coincide at every point of $\mathbb{T}$ but $T = C_\phi - C_\psi$ is non-compact on all the spaces H^p $(1 \le p < \infty)$, L^1 and M.*

Note that it is sufficient to verify the non-compactness of T only on the space H^2 since Theorem 2.1 asserts that the compactness of T on H^p does not depend on p and since H^1 is a subspace of L^1 and M. We will actually provide two different examples to prove the theorem. The first one will be obtained as a simple application of a result by J. H. Shapiro and C. Sundberg [SS2]. Let $\kappa : \mathbb{R} \to [0,1)$ be a continuous, 2π-periodic function which is increasing and positive on $(0,\pi]$, decreasing and positive on $[-\pi, 0)$, and vanishes at the origin. Shapiro and Sundberg call such κ a *contact function.* It defines an approach region

$$\Omega(\kappa) = \{re^{i\theta} : 1 - r > \kappa(\theta)\},$$

whose boundary is a Jordan curve in $\overline{\mathbb{D}}$ that meets the unit circle only at the point 1. The following theorem is a slightly simplified version of [SS2, Theorem 4.1], as complemented by [SS2, Remark 5.1].

Theorem 5.2. *Suppose κ is a C^2 contact function and ϕ is a conformal map from $\mathbb{D}$ onto $\Omega(\kappa)$. If*

$$\int_0^\pi \log\kappa(\theta)\,d\theta = -\infty,$$

then C_ϕ is essentially isolated in the set of composition operators on H^2.

First proof of Theorem 5.1. Choose any contact function κ satisfying the conditions of the above theorem; for instance, let $\kappa(\theta) = e^{-1/|\theta|}$ when $0 < |\theta| \le \pi$, and $\kappa(0) = 0$. Also let ϕ be a conformal map from $\mathbb{D}$ onto $\Omega(\kappa)$ such that $\operatorname{Im}\phi(0) \neq 0$ and $\phi(1) = 1$. Here we consider ϕ as extended to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{\Omega(\kappa)}$. Since $\Omega(\kappa)$ touches the unit circle only at the point 1, we see that for all $\alpha \neq 1$ the function (1.1) is bounded and hence the singular part of the corresponding Aleksandrov measure vanishes: $\mu_\alpha^s = 0$. In addition, μ_1^s must be a multiple of δ_1 since in the case $\alpha = 1$ the function (1.1) is continuous on $\overline{\mathbb{D}}\setminus\{1\}$. Now define ψ by the formula $\psi(\overline{z}) = \overline{\phi(z)}$ and use ν_α to denote the Aleksandrov measure of ψ at α. By symmetry considerations it is clear that $\nu_\alpha^s = \mu_\alpha^s$ for all α. However, since $\phi(0) \neq \psi(0)$, Theorem 5.2 shows that $C_\phi - C_\psi$ is non-compact on H^2. □

Remark 5.3. Observe that in the above example the operators C_ϕ and C_ψ are both *essentially isolated* in the set of composition operators on H^2, that is, isolated in the topology induced by the essential norm. Moreover, both ϕ and ψ are univalent.

Since the proof of Theorem 5.2 is fairly long and technical, it seems desirable to establish Theorem 5.1 by a direct argument, which reveals in a more transparent manner how the continuous parts of the Aleksandrov measures influence the difference operator. We will spend the rest of the present section sketching such an example.

To prepare, we note that whenever ϕ is a univalent map on $\mathbb{D}$ we may perform a change of variables in (4.1) to get the estimate

$$\|C_\phi f\|_2^2 - |f(\phi(0))|^2 \sim \int_{\phi(\mathbb{D})} |f'(w)|^2 (1 - |\phi^{-1}(w)|)\,d\lambda(w) \tag{5.1}$$

for $f \in H^2$. A consequence of this is given by the next lemma.

Lemma 5.4. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be univalent with $\phi(0) = 0$, and assume that B is an open disc of radius $\frac{3}{4}$ contained in $\phi(\mathbb{D})$. Then, for all $f \in H^2$,*

$$\|C_\phi f\|_2^2 \ge c \int_B |f'(w)|^2 \operatorname{dist}(w, \partial B)\,d\lambda(w),$$

where $c > 0$ is a constant independent of ϕ, B and f.

Proof. Let ψ be a conformal map taking $\mathbb{D}$ onto B with $\psi(0) = 0$. Applying the Schwarz lemma to the map $\phi^{-1} \circ \psi$ one sees that $|\phi^{-1}(w)| \le |\psi^{-1}(w)|$ for $w \in B$. Moreover, since ψ is a Möbius transformation and $\operatorname{dist}(0, \partial B) \ge \frac{1}{4}$, it is not difficult to show that $1 - |\psi^{-1}(w)| \ge c' \operatorname{dist}(w, \partial B)$ where $c' > 0$ is an absolute constant. Thus $1 - |\phi^{-1}(w)| \ge c' \operatorname{dist}(w, \partial B)$ for $w \in B$, and the lemma follows from (5.1). □

Second proof of Theorem 5.1. For every integer $k \neq 0$ define

$$A_k = B\big((\tfrac{1}{4} - |k|^{-9})e^{i/k}, \tfrac{3}{4}\big),$$

so that A_k is an open disc contained in $\mathbb{D}$ with radius $\frac{3}{4}$. Its distance to $\mathbb{T}$ equals $|k|^{-9}$, the closest point on $\mathbb{T}$ being $e^{i/k}$. Let $\Omega = \bigcup_{k=2}^\infty A_k$. Then Ω is a simply connected Jordan region that touches the unit circle only at the point 1. The map ϕ is now defined to be the conformal map taking $\mathbb{D}$ onto Ω with $\phi(0) = 0$ and $\phi(1) = 1$;

again we consider ϕ as extended to a homeomorphism between $\overline{\mathbb{D}}$ and $\overline{\Omega}$. Finally define the map ψ through the formula $\psi(\overline{z}) = \overline{\phi(z)}$, so that ψ becomes a conformal map from $\mathbb{D}$ onto the region $\Omega' = \bigcup_{k=2}^{\infty} A_{-k}$, the reflection of Ω with respect to the real axis.

Let μ_α and ν_α be the Aleksandrov measures of ϕ and ψ at $\alpha \in \mathbb{T}$, respectively. Also, for every $a \in \mathbb{D}$, define $f_a \in H^2$ to be the normalized reproducing kernel function

$$f_a(z) = \frac{\sqrt{1-|a|^2}}{1-\overline{a}z}.$$

Then $\|f_a\|_2 = 1$ and $f_a \to 0$ weakly in H^2 as $|a| \to 1-$. With this notation, the crucial properties of ϕ and ψ can be summarized as follows:

(1) $\mu_\alpha^s = \nu_\alpha^s = 0$ for $\alpha \neq 1$, and $\mu_1^s = \nu_1^s = \gamma\delta_1$ with $\gamma \geq 0$;
(2) if $a_k = (1-k^{-9})e^{i/k}$, then

$$\liminf_{k\to\infty} \|C_\phi f_{a_k}\|_2 > 0 \qquad \text{and} \qquad \lim_{k\to\infty} \|C_\psi f_{a_k}\|_2 = 0.$$

Notice that property (2) ensures that the difference $C_\phi - C_\psi$ is non-compact on H^2 since it does not map the weakly null sequence (f_{a_k}) into a norm-null sequence.

Property (1) is verified by exactly the same reasoning as used in the paragraph following Theorem 5.2. To establish the first part of (2), we let $k \geq 2$ and apply Lemma 5.4 to get

$$\|C_\phi f_{a_k}\|_2^2 \geq c \int_{A_k} |f'_{a_k}(w)|^2 \operatorname{dist}(w, \partial A_k)\, d\lambda(w),$$

where $c > 0$ is a constant. Write $G_k = B((1-3k^{-9})e^{i/k}, k^{-9})$. Then $G_k \subset A_k$ and an easy estimate shows that for $w \in G_k$ one has $|1 - \overline{a_k}w| \leq 5k^{-9}$ and hence $|f'_{a_k}(w)|^2 \geq c'k^{27}$ with some constant $c' > 0$. Since $\operatorname{dist}(w, \partial A_k) \geq k^{-9}$ for $w \in G_k$, we obtain

$$\|C_\phi f_{a_k}\|_2^2 \geq cc'k^{27}\lambda(G_k) = cc',$$

and the first part of (2) follows.

For the proof of the second part of (2) we begin with the estimate

$$\|C_\psi f_{a_k}\|_2^2 \leq |f_{a_k}(0)|^2 + c\int_{\Omega'} |f'_{a_k}(w)|^2\, d\lambda(w),$$

which trivially follows from (5.1). Clearly $f_{a_k}(0) \to 0$ as $k \to \infty$. To estimate the integral observe that by the definition of the region Ω' we have

$$\operatorname{dist}(1/\overline{a_k}, \partial\Omega') \geq \operatorname{dist}(e^{i/k}, \partial B(\tfrac{1}{4}, \tfrac{3}{4})) \geq \frac{1}{16k^2}.$$

Hence, if $w \in \Omega'$, one has

$$|f'_{a_k}(w)|^2 = \frac{1-|a_k|^2}{|a_k|^2|1/\overline{a_k} - w|^4} \leq \frac{2k^{-9}}{(\frac{1}{2})^2(1/16k^2)^4} = 2^{19}/k,$$

and it follows that $\int_{\Omega'} |f'_{a_k}|^2\, dA \to 0$ as $k \to \infty$. This establishes the second part of (2) and finishes the second proof of Theorem 5.1. $\square$

6. Compactness on H^∞ and L^∞

In [MOZ] B. MacCluer et al. studied the topological structure and compact differences of composition operators on the space H^∞ of bounded analytic functions. Their results involve the pseudo-hyperbolic metric β, defined by (4.2). In particular they showed that the operator $T = C_\phi - C_\psi$ is compact on H^∞ if and only if

$$\beta(\phi(z), \psi(z)) \to 0 \quad \text{as } \max\big(|\phi(z)|, |\psi(z)|\big) \to 1. \tag{6.1}$$

In this section we revisit this result and generalize it slightly by considering the case of L^∞ and weakly compact differences. Observe that [MOZ] established the equivalence of conditions (3) and (5) of the following result.

Theorem 6.1. *Let $\phi, \psi : \mathbb{D} \to \mathbb{D}$ be analytic and put $T = C_\phi - C_\psi$. Then the following five conditions are equivalent:*

(1) $T \in K(L^\infty)$,
(2) $T \in W(L^\infty)$,
(3) $T \in K(H^\infty)$,
(4) $T \in W(H^\infty)$,
(5) *condition* (6.1) *holds.*

Note that it is enough to verify that (4) implies (5) and (5) implies (1). The latter implication is a straightforward adaptation of the argument given in [MOZ] and it is dealt with in Proposition 6.3 below. The former implication is more involved and will be established as Proposition 6.5.

We begin with an easy lemma. Here we use ρ to denote the hyperbolic metric on $\mathbb{D}$; it is related to the pseudo-hyperbolic metric by the formula

$$\rho(z,w) = \log \frac{1+\beta(z,w)}{1-\beta(z,w)}.$$

(See, for example, [G, §I.1].)

Lemma 6.2. *If u is the Poisson integral of a function $f \in L^\infty$, then $|u(z)-u(w)| \le \|f\|_\infty \rho(z,w)$ for $z, w \in \mathbb{D}$.*

Proof. An application of equality (4.13) yields that

$$|\nabla u(z)| \le \int_{\mathbb{T}} \frac{2\|f\|_\infty}{|\zeta - z|^2}\, dm(\zeta) = \frac{2\|f\|_\infty}{1-|z|^2}.$$

The lemma follows since $2|dz|/(1-|z|^2)$ is the element of arc length in the hyperbolic metric. ☐

Proposition 6.3. *If* (6.1) *holds, then $T \in K(L^\infty)$.*

Proof. Let (f_n) be a bounded sequence in L^∞ and let (u_n) be the sequence of corresponding Poisson integrals. We should show that a subsequence of (Tf_n) converges in L^∞. Invoking a normal family argument (or the weak* compactness of the closed unit ball of L^∞), we may further assume (cf. the proof of Proposition 2.3) that $u_n \to 0$ locally uniformly in $\mathbb{D}$.

Let $\epsilon > 0$. By condition (6.1) and the above lemma we can find $0 < r < 1$ such that for all n

$$|u_n(\phi(z)) - u_n(\psi(z))| \le \epsilon \quad \text{when } \max\bigl(|\phi(z)|, |\psi(z)|\bigr) > r.$$

On the other hand, since $u_n \to 0$ locally uniformly, we have for n large enough

$$|u_n(\phi(z)) - u_n(\psi(z))| \le \epsilon \quad \text{when } \max\bigl(|\phi(z)|, |\psi(z)|\bigr) \le r.$$

Combining these two inequalities yields that $\|Tf_n\|_\infty = \|u_n \circ \phi - u_n \circ \psi\|_\infty \le \epsilon$ for all sufficiently large n. Hence $Tf_n \to 0$ in L^∞ and the proof is complete. ☐

In order to prove that condition (6.1) is implied by the weak compactness of T on H^∞, we recall some notions from the Banach space theory. A Banach space X is said to have the *Dunford–Pettis property* if $x_n^*(x_n) \to 0$ whenever $x_n \to 0$ weakly in X and $x_n^* \to 0$ weakly in the dual X^*. Equivalently, this means that every weakly compact linear operator from X into some Banach space is completely continuous, i.e. maps weakly null sequences into norm-null sequences. A well-known example of a space with the Dunford–Pettis property is c_0, the space of null sequences of scalars under the supremum norm. For a survey of the Dunford–Pettis property we refer to [D].

The special auxiliary functions provided by the next lemma will be crucial to our argument. We leave the simple verification of the lemma to the reader.

Lemma 6.4. *Suppose (a_n) is a sequence of points in $\mathbb{D}$ such that $a_n \to 1$. Then there exist numbers $0 < \epsilon_n < 1$ and $0 < \delta_n < \delta_n' < \pi$ such that $\epsilon_n \to 0$, $\delta_n' \to 0$, and if*

$$h_n(e^{i\theta}) = \begin{cases} 1 & \text{for } \delta_n < |\theta| < \delta_n' \\ \epsilon_n & \text{otherwise,} \end{cases}$$

then the outer functions

$$Q_n(z) = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log h_n(e^{i\theta})\, d\theta \right\}$$

satisfy $\|Q_n\|_\infty = 1$ and $|Q_n(a_n)| \geq \frac{1}{2}$ for every n.

We have now reached our objective.

Proposition 6.5. *If $T \in W(H^\infty)$, then (6.1) holds.*

Proof. Suppose to the contrary that (6.1) fails. This means that we can find a number $\epsilon > 0$ and points $z_n \in \mathbb{D}$ such that if $a_n = \phi(z_n)$ and $b_n = \psi(z_n)$, then

$$\max(|a_n|, |b_n|) \to 1 \qquad \text{and} \qquad \beta(a_n, b_n) \geq \epsilon \text{ for all } n.$$

By passing to a subsequence and interchanging the roles of ϕ and ψ, if necessary, we may assume that $a_n \to \alpha$ for some $\alpha \in \mathbb{T}$. Without loss of generality, take $\alpha = 1$. Let (Q_n) be the sequence of outer functions corresponding to (a_n) as given by Lemma 6.4. By passing to a further subsequence we may assume that $\delta_{n+1}' \leq \delta_n$ and $\epsilon_n \leq 2^{-n-1}$ for all n.

Now define

$$f_n(z) = Q_n(z) \cdot \frac{z - b_n}{1 - z\overline{b_n}},$$

so that $f_n \in H^\infty$ with $\|f_n\|_\infty = 1$, $|f_n(a_n)| \geq \frac{1}{2}\epsilon$ and $f_n(b_n) = 0$. Because the sets $\{\zeta \in \mathbb{T} : |f_n(\zeta)| > \epsilon_n\}$ are pairwise disjoint and $\sum_n \epsilon_n \leq \frac{1}{2}$, it is easy to check that the mapping $(\xi_n) \mapsto \sum_n \xi_n f_n$ is an isomorphic embedding of c_0 into H^∞. Thus $f_n \to 0$ weakly and since T was assumed weakly compact, the Dunford–Pettis property of c_0 implies that $\|Tf_n\|_\infty \to 0$. However, it follows from the definition of f_n that

$$\|Tf_n\|_\infty \geq |Tf_n(z_n)| = |f_n(a_n) - f_n(b_n)| = |f_n(a_n)| \geq \tfrac{1}{2}\epsilon$$

for every n. This contradiction completes the proof of the proposition and, as noted before, the proof of Theorem 6.1. $\square$

References

[A] A. B. Aleksandrov, *The multiplicity of boundary values of inner functions* (Russian), Izv. Akad. Nauk Armyan. SSR Ser. Mat. **22** (1987), 490–503.

[BS] C. Bennet and R. Sharpley, *Interpolation of Operators*, Pure and Appl. Math. vol. 129, Academic Press, Boston, MA, 1988.

[B] P. Bourdon, *Components of linear-fractional composition operators*, J. Math. Anal. Appl. **279** (2003), 228–245.

[CM1] J. A. Cima and A. L. Matheson, *Essential norms of composition operators and Aleksandrov measures*, Pacific J. Math. **179** (1997), 59–63.

[CM2] J. A. Cima and A. L. Matheson, *Cauchy transforms and composition operators*, Illinois J. Math. **42** (1998), 58–69.

[CoM] C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions.* CRC Press, Boca Raton, 1995.

[Cw] M. Cwikel, *Real and complex interpolation and extrapolation of compact operators*, Duke Math. J. **65** (1992), 333–343.

[D] J. Diestel, *A survey of results related to the Dunford–Pettis property*, Contemp. Math. **2** (1980), 15–60.

[DS] N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1958.

[FRS] C. Fefferman, N. M. Rivière and Y. Sagher, *Interpolation between H^p spaces: the real method*, Trans. Amer. Math. Soc. **191** (1974), 75–81.

[G] J. B. Garnett, *Bounded Analytic Functions.* Academic Press, New York, 1981.

[Go] T. E. Goebeler, *Composition operators acting between Hardy spaces*, Integral Equations Operator Theory **41** (2001), 389–395.

[K] Y. Katznelson, *An Introduction to Harmonic Analysis*, Wiley, New York, 1968; reprinted by Dover, New York, 1976.

[M] B. D. MacCluer, *Components in the space of composition operators*, Integral Equations Operator Theory **12** (1989), 725–738.

[MOZ] B. MacCluer, S. Ohno and R. Zhao, *Topological structure of the space of composition operators on H^∞*, Integral Equations Operator Theory **40** (2001), 481–494.

[MT] J. Moorhouse and C. Toews, *Differences of composition operators*, Trends in Banach spaces and operator theory (Memphis, TN, 2001), Contemp. Math. **321** (2003), 207–213.

[NS] P. J. Nieminen and E. Saksman, *Boundary correspondence of Nevanlinna counting functions for self-maps of the unit disc*, Trans. Amer. Math. Soc. (to appear).

[R1] W. Rudin, *Real and Complex Analysis* (3rd ed.), McGraw-Hill, New York, 1987.

[R2] W. Rudin, *Functional Analysis* (2nd ed.), McGraw-Hill, New York, 1991.

[Sa] D. Sarason, *Composition operators as integral operators*, Analysis and Partial Differential Equations, Marcel Dekker, New York, 1990.

[Sa2] D. Sarason, *Weak compactness of holomorphic composition operators on H^1*, Functional analysis and operator theory (New Delhi, 1990), 75–79, Lecture Notes in Math. 1511, Springer, Berlin, 1992.

[Sh] J. E. Shapiro, *Aleksandrov measures used in essential norm inequalities for composition operators*, J. Operator Theory **40** (1998), 133–146.

[S1] J. H. Shapiro, *The essential norm of a composition operator*, Ann. Math. **125** (1987), 375–404.

[S2] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993.

[SS1] J. H. Shapiro and C. Sundberg, *Compact composition operators on L^1*, Proc. Amer. Math. Soc. **108** (1990), 443–449.

[SS2] J. H. Shapiro and C. Sundberg, *Isolation amongst the composition operators*, Pacific J. Math. **145** (1990), 117–152.

[ST] J. H. Shapiro and P. D. Taylor, *Compact, nuclear, and Hilbert–Schmidt composition operators on H^2*, Indiana Univ. Math. J. **23** (1973), 471–496.

[W] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Univ. Press, Cambridge, 1991.

Department of Mathematics and Statistics, University of Helsinki, P.O. Box 4 (Yliopistonkatu 5), FIN-00014 University of Helsinki, Finland

E-mail address: `pekka.j.nieminen@helsinki.fi`

Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (Mattilanniemi D), FIN-40014 University of Jyväskylä, Finland

E-mail address: `saksman@maths.jyu.fi`