POINTWISE BEHAVIOUR OF SEMICONTINUOUS SUPERSOLUTIONS TO A QUASILINEAR PARABOLIC EQUATION

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Abstract. A kind of supersolutions of the so-called p-parabolic equation are studied. These p-superparbolic functions are defined as lower semicontinuous functions obeying the comparison principle. Incidentally, they are precisely the viscosity supersolutions. One of our results guarantees the existence of a spatial Sobolev gradient. For $p = 2$ we have the supercaloric functions and the ordinary heat equation.

1. Introduction

The solutions of the partial differential equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \qquad 1 < p < \infty, \tag{1.1}$$

form a similar basis for a prototype of a nonlinear parabolic potential theory as the solutions of the heat equation do in the classical theory. Especially, the celebrated Perron method can be applied even in the nonlinear situation $p \neq 2$; see [KL] for this and related results concerning more general equations. For the regularity theory of such equations the reader is asked to consult the monograph by DiBenedetto [D]. See also Chapter 2 of the recent book [WZYL]. The equation is often called the *p-parabolic equation*, but is also known as the *evolutionary p-Laplace equation* and the *non-Newtonian filtration equation* in the literature.

In this connection the so-called *p-superparabolic functions* are essential. They are defined as lower semicontinuous functions obeying the comparison principle with respect to the solutions of (1.1). The p-superparabolic functions are of actual interest also because they are the *viscosity supersolutions* of (1.1), see [JLM]. Thus there is an alternative definition in the modern theory of viscosity solutions, but here we are content to mention that our results automatically hold for the viscosity supersolutions. One should pay attention to the fact that, in their definition, the p-superparabolic functions are not required to have any derivatives, and, consequently, it is not evident how to directly relate them to the differential inequality

$$\frac{\partial u}{\partial t} \geq \operatorname{div}(|\nabla u|^{p-2}\nabla u). \tag{1.2}$$

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The weak solutions of (1.2), defined with the aid of test functions under the integral sign, are called *supersolutions* and, since they are required to have Sobolev derivatives, they constitute a more tractable class of functions. The reader should carefully distinguish between p-superparabolic functions and supersolutions. For example consider the *Barenblatt solution* $\mathcal{B}_p\colon \mathbf{R}^{n+1} \to [0,\infty)$,

$$
(1.3)\quad \mathcal{B}_p(x,t) = \begin{cases} t^{-n/\lambda}\left(C - \dfrac{p-2}{p}\lambda^{1/(1-p)}\left(\dfrac{|x|}{t^{1/\lambda}}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)}, & t>0,\\ 0, & t\le 0, \end{cases}
$$

where $\lambda = n(p-2)+p$, $p>2$, and the constant C is usually chosen so that

$$
\int_{\mathbf{R}^n} \mathcal{B}_p(x,t)\,dx = 1
$$

for every $t>0$. This function is not a supersolution in an open set that contains the origin. It is the a priori summability of $\nabla\mathcal{B}_p$ that fails. Indeed,

$$
\int_{-1}^{1}\int_Q |\nabla\mathcal{B}_p(x,t)|^p\,dx\,dt = \infty,
$$

where $Q = [-1,1]^n \subset \mathbf{R}^n$. However, the Barenblatt solution is a p-superparabolic function in $\mathbf{R}^{n+1}$. In the case $p=2$ we have the heat kernel

$$
W(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, & t>0\\ 0, \quad t\le 0. \end{cases}
$$

In contrast with the heat kernel, which is strictly positive, the Barenblatt solution has a bounded support at a given instance $t>0$. Hence the disturbancies propagate with finite speed when $p>2$. -The Barenblatt solution describes the propagation of the heat after the explosion of a hydrogen bomb in the atmosphere. This function was discovered in [B].

Supersolutions and p-superparabolic functions are often identified in the literature, even though this is not strictly speaking correct, as the Barenblatt solution shows. However, we show that there are no other locally bounded p-superparabolic functions than supersolutions. Indeed, locally bounded p-superparabolic functions have Sobolev derivatives with respect to the spatial variable and we can substitute them into the weak form of (1.2). This is the content of the following theorem.

1.4. Theorem. *Let $p \ge 2$. Suppose that v is a locally bounded p-superparabolic function in an open set $\Omega \subset \mathbf{R}^{n+1}$. Then the Sobolev derivative*

$$
\nabla v = \left(\frac{\partial v}{\partial x_1},\dots,\frac{\partial v}{\partial x_n}\right)
$$

exists and the local summability

$$
\int_{t_1}^{t_2}\int_Q |\nabla v|^p\,dx\,dt < \infty
$$

holds for each $\overline{Q} \times [t_1, t_2] \subset \Omega$. *Moreover, we have*

$$\int_{t_1}^{t_2} \int_Q \Big(|\nabla v|^{p-2} \nabla v \cdot \nabla \varphi - v \frac{\partial \varphi}{\partial t} \Big)\, dx\, dt \geq 0$$

whenever $\varphi \in C_0^\infty(Q \times (t_1, t_2))$ *with* $\varphi \geq 0$.

The proof of Theorem 1.4 is presented in Chapter 4. For unbounded p-superparabolic functions some immediate results follow, because, if v is a p-superparabolic function, so are functions

$$v_L = v_L(x) = \min(v(x), L),$$

and thus Theorem 1.4 is valid for each v_L.

In the case $p = 2$ the proof of Theorem 1.4 can be extracted from the linear representation formulas in [W1-2]. Then all superparabolic functions can be represented in terms of the heat kernel. For $p > 2$ the principle of superposition is not available. Instead we use an obstacle problem in the calculus of variations to construct supersolutions which approximate a given p-superparabolic function. *A priori* estimates for the approximants are derived and these estimates are passed over to the limit.

In order to bypass some technical difficulties related to the time derivative u_t we use the reqularized equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\big((|\nabla u|^2 + \varepsilon^2)^{(p-2)/2} \nabla u \big), \tag{1.5}$$

which does not degenerate at the critical points where $\nabla u = 0$. Here ε is a real parameter. The solutions of (1.5) are smooth, provided $\varepsilon \neq 0$. In the case $\varepsilon = 0$ the equation (1.5) reduces to the true p-parabolic equation (1.1). See Chapter 2 of [WZYL].

A distinct feature is that the p-superparabolic functions are defined at every point of their domain. Thus the study of the pointwise behaviour is relevant. If the value is changed even at a single point, then the obtained function is not p-superparabolic anymore. At each point in the domain a p-superparabolic function v has the value given by

$$v(x,t) = \operatorname*{ess\,lim\,inf}_{\substack{(y,\tau)\to(x,t)\\ \tau<t}} v(y,\tau).$$

This is the content of Theorem 5.2. Here the essential limes inferior means that any set of $(n+1)$-dimensional measure zero can be neglected in the calculation of the limes inferior. It is worthwhile to observe the role of the time. What is to happen in the future will have no influence at the present time: the instances τ with $\tau \geq t$ are not necessary to include.

Our argument is based on a general principle and it applies to other equations as well. It can be extended to include equations like

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \bigg(\Big| \sum_{k,m=1}^{n} a_{km}(x) \frac{\partial u}{\partial x_k} \frac{\partial u}{\partial x_m} \Big|^{(p-2)/2} a_{ij}(x) \frac{\partial u}{\partial x_j} \bigg),$$

where the matrix (a_{ij}) with bounded measurable coefficients satisfies the standard condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq |\xi|^2$$

for all $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ in $\mathbf{R}^n$.

A comment on our approach to deal with the time derivative u_t is appropriate. Strictly speaking this quantity does not exist in the same sense as ∇u; it is not a derivative in Sobolev's sense. For supersolutions, even simple examples of the form $u(x,t) = g(t)$ where $g(t)$ is an increasing lower semicontinuous stepfunction, illuminate this feature. The regularized equation and some averaging procedures are here needed only to overcome this difficulty. Let us however mention that u_t can be interpreted as an object in the theory of J.-L. Lions et consortes, see [L]. The use of this powerful theory enables us to calculate rather freely with u_t so that a much shorter exposition is possible. Although we have not followed this path, the reader may find it instructive to skip the regularized equation and the averaging procedures at the first reading. We have also deliberately decided to exclude the case $p < 2$. On the other hand, we think that some features might be interesting even for the ordinary heat equation, to which everything reduces when $p = 2$.

2. Preliminaries

In what follows, Q will always stand for a parallelepiped

$$Q = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n), \quad a_i < b_i, \quad i = 1, 2, \ldots, n,$$

in $\mathbf{R}^n$ and the abbreviations

$$Q_T = Q \times (0, T), \qquad Q_{t_1,t_2} = Q \times (t_1, t_2),$$

where $T > 0$ and $t_1 < t_2$, are used for the space-time boxes in $\mathbf{R}^{n+1}$. The parabolic boundary of Q_T is

$$\Gamma_T = (\overline{Q} \times \{0\}) \cup (\partial Q \times [0, T]).$$

Observe that the interior of the top $\overline{Q} \times \{T\}$ is not included. Similarly, Γ_{t_1,t_2} is the parabolic boundary of Q_{t_1,t_2}. Let $1 \leq p < \infty$. In order to describe the appropriate function spaces, we recall that $W^{1,p}(Q)$ denotes the Sobolev space of functions $u \in L^p(Q)$, whose first distributional partial derivatives belong to $L^p(Q)$ with the norm

$$\|u\|_{W^{1,p}(Q)} = \|u\|_{L^p(Q)} + \|\nabla u\|_{L^p(Q)}.$$

The Sobolev space with zero boundary values, denoted by $W_0^{1,p}(Q)$, is the completion of $C_0^\infty(Q)$ in the norm $\|u\|_{W^{1,p}(Q)}$. We denote by $L^p(t_1, t_2; W^{1,p}(Q))$, $t_1 < t_2$, the space of functions such that for almost every t, $t_1 \leq t \leq t_2$, the function $x \to u(x,t)$ belongs to $W^{1,p}(Q)$ and

$$\int_{t_1}^{t_2} \int_Q \big(|u(x,t)|^p + |\nabla u(x,t)|^p\big)\, dx\, dt < \infty.$$

Notice that the time derivative u_t is deliberately avoided. The definition for $L^p(t_1, t_2; W_0^{1,p}(Q))$ is analogous.

The Sobolev inequality is valid in the following form, see [D, Chapter I, Proposition 3.1].

2.1. Lemma. *Suppose that* $u \in L^p(0,T;W_0^{1,p}(Q))$*. Then there is* $c = c(n,p) > 0$ *such that*

$$\int_0^T \int_Q |u|^{(1+2/n)p}\,dx\,dt \le c \int_0^T \int_Q |\nabla u|^p\,dx\,dt \Big(\operatorname*{ess\,sup}_{0<t<T} \int_Q |u|^2\,dx \Big)^{p/n}. \tag{2.2}$$

To be on the safe side we give the definition of the (super)solutions, interpreted in the weak sense.

2.3. Definition. Let Ω be an open set in $\mathbf{R}^{n+1}$ and suppose that u belongs to $L^p(t_1,t_2;W^{1,p}(Q))$ whenever $\overline{Q}_{t_1,t_2} \subset \Omega$. Then u is called a *solution* of (1.5), if

$$\int_{t_1}^{t_2} \int_Q \Big((|\nabla u|^2 + \varepsilon^2)^{(p-2)/2} \nabla u \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \Big)\,dx\,dt = 0 \tag{2.4}$$

whenever $\overline{Q}_{t_1,t_2} \subset \Omega$ and $\varphi \in C_0^\infty(Q_{t_1,t_2})$. If, in addition, u is continuous, then u is called *p-parabolic* in the case $\varepsilon = 0$. Further, we say that u is a *supersolution* of (1.5), if the integral (2.4) is non-negative for all $\varphi \in C_0^\infty(Q_{t_1,t_2})$ with $\varphi \ge 0$. If this integral is non-positive instead, we say that u is a *subsolution*.

Several remarks are related to the definition. By parabolic regularity theory the solutions in the case $\varepsilon \ne 0$ are smooth, after a possible redefinition on a set of measure zero. The existence of the continuous time derivative u_t is of vital importance to us. In the p-parabolic case $\varepsilon = 0$ the solutions and their spatial gradients are Hölder continuous but this does not concern u_t, see [D] and [WZYL].

2.5. Remark. If the test function φ is required to vanish only on the lateral boundary $\partial Q \times [t_1,t_2]$, then the boundary terms

$$\int_Q u(x,t_1)\varphi(x,t_1)\,dx = \lim_{\sigma \to 0} \frac{1}{\sigma} \int_{t_1}^{t_1+\sigma} \int_Q u(x,t)\varphi(x,t)\,dx\,dt$$

and

$$\int_Q u(x,t_2)\varphi(x,t_2)\,dx = \lim_{\sigma \to 0} \frac{1}{\sigma} \int_{t_2-\sigma}^{t_2} \int_Q u(x,t)\varphi(x,t)\,dx\,dt$$

have to be included. In the case of a supersolution to the p-parabolic equation the condition becomes

$$\begin{aligned} &\int_{t_1}^{t_2} \int_Q \Big(|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \Big)\,dx\,dt \\ &\qquad + \int_Q u(x,t_2)\varphi(x,t_2)\,dx - \int_Q u(x,t_1)\varphi(x,t_1)\,dx \ge 0 \end{aligned} \tag{2.6}$$

for almost all $t_1 < t_2$ with $\overline{Q}_{t_1,t_2} \subset \Omega$.

There is a principal, well-recognized difficulty with the definition, present in the case $\varepsilon = 0$. Namely, in proving estimates we usually need a test function φ that depends on the solution itself, for example $\varphi = u\zeta$ where ζ is a smooth cutoff function. Then one cannot avoid that the "forbidden quantity" u_t shows up in the calculation of φ_t. In most cases one can easily overcome this difficulty by using an equivalent definition in terms of Steklov averages, as in [D, pp. 18 and 25] and [WZYL, Chapter 2]. Alternatively, one can proceed using convolutions with smooth mollifiers as in [AS, pp. 199–121]. This remark also concerns our estimates but we have one noteworthy exception. Indeed, the proof of a convergence result (Lemma 4.15) is delicate when it comes to the "forbidden quantity" and there is a true complication, not easy to detect and not easy to dismiss.

We have found the convolution

$$u^*(x,t) = \frac{1}{\sigma}\int_0^t e^{(\tau-t)/\sigma} u(x,\tau)\, d\tau, \qquad \sigma > 0, \tag{2.7}$$

to be expedient, see [N, p. 36]. It is essential in the proof of Lemma 4.7. The notation hides the dependece on σ. For continuous or bounded and semicontinuous functions u the averaged function u^* is defined at each point. Observe that

$$u^* + \sigma \frac{\partial u^*}{\partial t} = u. \tag{2.8}$$

Some properties are listed in the following lemma.

2.9. Lemma. (i) *If $u \in L^p(Q_T)$, then*

$$\|u^*\|_{p,Q_T} \leq \|u\|_{p,Q_T}$$

and

$$\frac{\partial u^*}{\partial t} = \frac{u - u^*}{\sigma} \in L^p(Q_T).$$

Moreover, $u^* \to u$ in $L^p(Q_T)$ as $\sigma \to 0$.

(ii) *If, in addition, $\nabla u \in L^p(Q_T)$, then $\nabla(u^*) = (\nabla u)^*$ componentwise,*

$$\|\nabla u^*\|_{p,Q_T} \leq \|\nabla u\|_{p,Q_T},$$

and $\nabla u^* \to \nabla u$ in $L^p(Q_T)$ as $\sigma \to 0$.

(iii) *Furthermore, if $u_k \to u$ in $L^p(Q_T)$, then also*

$$u_k^* \to u^* \qquad \text{and} \qquad \frac{\partial u_k^*}{\partial t} \to \frac{\partial u^*}{\partial t}$$

in $L^p(Q_T)$.

(iv) *If $\nabla u_k \to \nabla u$ in $L^p(Q_T)$, then $\nabla u_k^* \to \nabla u^*$ in $L^p(Q_T)$.*

(v) *Analogous results hold for weak convergence in $L^p(Q_T)$.*

(vi) *Finally, if $\varphi \in C(\overline{Q}_T)$, then*

$$\varphi^*(x,t) + e^{-t/\sigma}\varphi(x,0) \to \varphi(x,t)$$

uniformly in Q_T as $\sigma \to 0$.

Proof. The proof is an excercise. □

The averaged equation for a supersolution u in Ω is the following. If $\overline{Q}_T \subset \Omega$, then

$$
\begin{aligned}
\int_0^T \int_Q \Big((\nabla u|^{p-2} \nabla u)^* \cdot \nabla \varphi - u^* \frac{\partial \varphi}{\partial t} \Big)\, dx\, dt& \\
+ \int_Q u^*(x,T) \varphi(x,T)\, dx \geq \int_Q u(x,0) \Big(\frac{1}{\sigma} \int_0^T \varphi(x,s) e^{-s/\sigma}\, ds \Big)\, dx&
\end{aligned}
\tag{2.10}
$$

for all test functions $\varphi \geq 0$ vanishing on the parabolic boundary Γ_T of Q_T. If, in addition, $u \geq 0$ in Q_T and $\varphi(x,T) = 0$, this implies that

$$
\int_0^T \int_Q \Big((\nabla u|^{p-2} \nabla u)^* \cdot \nabla \varphi - u^* \frac{\partial \varphi}{\partial t} \Big)\, dx\, dt \geq 0. \tag{2.11}
$$

Notice that we do not have $|\nabla u^*|^{p-2} \nabla u^*$ in (2.10), except in the favourable case $p = 2$. For the proofs of (2.10) and (2.11), we observe that (2.6) implies that

$$
\begin{aligned}
\int_s^T \int_Q \Big(&|\nabla u(x,t-s)|^{p-2} \nabla u(x,t-s) \cdot \nabla \varphi(x,t) \\
&- u(x,t-s) \frac{\partial \varphi}{\partial t}(x,t) \Big)\, dx\, dt + \int_Q u(x,T-s) \varphi(x,T)\, dx \\
&\geq \int_Q u(x,0) \varphi(x,s)\, dx
\end{aligned}
\tag{2.12}
$$

when $0 \leq s \leq T$. Notice that $(x, t-s) \in \overline{Q}_T$. Multiply by $\sigma^{-1} e^{-s/\sigma}$, integrate over $[0,T]$ with respect to s and, finally, change the order of integration between s and t. This yields (2.10).

The averaging procedure (2.7) has the advantage that values taken outside Q_T are not evoked. That is not the case with the more conventional convolution

$$
(u * \rho_\sigma)(x,t) = \int_{-\infty}^{\infty} u(x, t-\tau) \rho_\sigma(\tau)\, d\tau \tag{2.13}
$$

where Friedrichs' mollifier

$$
\rho_\sigma(\tau) = \begin{cases} \dfrac{C}{\sigma} e^{-\sigma^2/(\sigma^2-\tau^2)}, & |\tau| < \sigma, \\ 0, & |\tau| \geq \sigma, \end{cases} \tag{2.14}
$$

is involved. The result is that the inequality

$$
\iint_\Omega \Big((|\nabla u|^{p-2} \nabla u) * \rho_\sigma \cdot \nabla \varphi - (u * \rho_\sigma) \frac{\partial \varphi}{\partial t} \Big)\, dx\, dt \geq 0 \tag{2.15}
$$

holds for all $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, under the restriction that the parameter σ has to be strictly less than the distance from the support of φ to the boundary of Ω. This has the effect that the virtual domain is shrinked. On the other hand, an advantage of this approach is that it is not limited to the space-time boxes.

The next lemma is well-known, see [D].

2.16. Lemma. *Let $u \in L^p(0,T;W^{1,p}(Q))$ be a subsolution of the equation (1.5). Assume that $u \geq 0$ in Q_T. Then there is a constant $c = c(p)$ such that*

$$\begin{aligned}\int_0^T \int_Q |\nabla u|^p \zeta^p \, dx \, dt + \operatorname*{ess\,sup}_{0<t<T} \int_Q u^2 \zeta^p \, dx &\leq c \int_0^T \int_Q u^2 \left| \frac{\partial(\zeta^p)}{\partial t} \right| dx \, dt \\ &+ c \int_0^T \int_Q u^p |\nabla \zeta|^p \, dx \, dt + |\varepsilon|^{p-2} \int_0^T \int_Q \zeta^p \, dx \, dt\end{aligned} \tag{2.17}$$

holds for all $\zeta \in C^\infty(Q_T)$, $\zeta \geq 0$, vanishing on the parabolic boundary Γ_T.

Proof. The proof is included only for instructive purposes. First, consider the case $\varepsilon = 0$. Let $0 \leq \tau \leq T$. The equation reads

$$\int_0^\tau \int_Q \Big(|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \Big) \, dx \, dt + \int_Q u(x,\tau) \varphi(x,\tau) \, dx \leq 0$$

where we want to use the test function $\varphi = \zeta^p u$, which strictly speaking, is not admissible. However, here the use of the averaged function u^* or other convolution approximants presents no difficulties. After a few integrations by parts we arrive at

$$\begin{aligned}&- \int_0^\tau \int_Q u \frac{\partial \varphi}{\partial t} \, dx \, dt + \int_Q u(x,\tau) \varphi(x,\tau) \, dx \\ &\qquad = \frac{1}{2} \int_Q u(x,\tau)^2 \zeta(x,\tau)^p \, dx - \frac{1}{2} \int_0^\tau \int_Q u^2 \frac{\partial(\zeta^p)}{\partial t} \, dx \, dt.\end{aligned} \tag{2.18}$$

We also have

$$\begin{aligned}&\int_0^\tau \int_Q |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \, dt \\ &\qquad = \int_0^\tau \int_Q |\nabla u|^p \zeta^p \, dx \, dt + p \int_0^\tau \int_Q \zeta^{p-1} u |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta \, dx \, dt\end{aligned} \tag{2.19}$$

and, using Young's inequality

$$p \zeta^{p-1} |\nabla u|^{p-1} u |\nabla \zeta| \leq (p-1) \beta^q \zeta^p |\nabla u|^p + \beta^{-p} u^p |\nabla \zeta|^p$$

where $q = p/(p-1)$ and β is small enough ($\beta^q = 1 - 1/p$ will do), we can estimate the absolute value of the last integral so that the integral of $(p-1)\beta^q \zeta^p |\nabla u|^p$ is absorbed by the left-hand side.

Combining this with (2.18) and (2.19) we obtain

$$\begin{aligned}&\int_0^\tau \int_Q |\nabla u|^p \zeta^p \, dx \, dt + \frac{1}{2} \int_Q \zeta(x,\tau)^p u(x,\tau)^2 \, dx \\ &\qquad \leq c \int_0^T \int_Q u^2 \left| \frac{\partial(\zeta^p)}{\partial t} \right| dx \, dt + c \int_0^T \int_Q u^p |\nabla \zeta|^p \, dx \, dt\end{aligned} \tag{2.20}$$

The parameter τ, $0 \leq \tau \leq T$, is at our disposal. Inequality (2.20) is to be used twice. First, we select τ so that

$$\int_Q \zeta(x,\tau)^p u(x,\tau)^2 \, dx \geq \frac{1}{2} \operatorname*{ess\,sup}_{0<t<T} \int_Q \zeta^p(x,t) u^2(x,t) \, dx.$$

This shows that the essential supremum term in question is less than four times the right-hand side of (2.20). Second, we take $\tau = T$ in (2.20) to estimate the first integral in (2.17). Adding the two estimates, we arrive at the desired result in the case $\varepsilon = 0$.

Let us turn our attention to the case $\varepsilon \neq 0$. The case is even simpler, since no averaging of the test function $\varphi = \zeta^p u$ is needed. Otherwise the proof is rather similar. An additional feature is that with the aid of the inequalities

$$\frac{1}{2}\zeta^p|\nabla u|^p + \frac{1}{2}|\varepsilon|^{p-2}|\nabla u|^2 \leq (|\nabla u|^2+\varepsilon^2)^{(p-2)/2}|\nabla u|^2$$

and

$$(|\nabla u|^2+\varepsilon^2)^{(p-2)/2}|\nabla u\cdot\nabla\zeta| \leq 2^{(p-2)/2}|\nabla u|^{p-1}|\nabla\zeta| + 2^{(p-2)/2}|\varepsilon|^{p-2}|\nabla u||\nabla\zeta|$$

not only the integral of $\zeta^p|\nabla u|^p$ but also the integral of $|\varepsilon|^{p-2}\zeta^p|\nabla u|^2$ can be absorbed. $\square$

For solutions with zero boundary values on some part of the boundary the previous estimate can be improved. To be more specific, consider the domain $R = Q\setminus Q'$, $Q' \subset\subset Q$, between two parallelepipeds. Suppose that h is p-parabolic in the domain

$$R_T = (Q\setminus Q')\times(0,T).$$

Assume further that $h \in C(\overline{R}_T)$, $h \geq 0$, and that h vanishes on the parabolic boundary Γ_T of Q_T. Then

$$\int_0^T\int_R |\nabla h|^p\zeta^p\,dx\,dt \leq c\int_0^T\int_R h^p|\nabla\zeta|^p\,dx\,dt \tag{2.21}$$

for all smooth $\zeta = \zeta(x) \geq 0$ depending only on the spatial variable and vanishing on the lateral boundary $\partial Q'\times[0,T]$. (There is no requirement on $\partial Q\times[0,T]$.)

The important improvement over (2.17) is that we obtain an estimate of the integral of $|\nabla h|^p$ over a domain of the form $(Q\setminus Q'')\times(0,T)$, where $Q' \subset\subset Q'' \subset\subset Q$. The proof can be extracted from the proof of Lemma 2.16. This will be used in the proof of Lemma 4.15.

3. The obstacle problem

The obstacle problem in the calculus of variations is a basic tool in the study of the p-superparabolic functions, to be defined in Section 4 as lower semicontinuous functions obeying a comparison principle. As we will see later, a p-superparabolic function can be approximated from below with solutions of obstacle problems.

Let $\psi \in C^\infty(\mathbf{R}^{n+1})$ and consider the class $\mathcal{F}_\psi$ of all functions $w \in C(\overline{Q}_T)$ such that

$$w \in L^p(0,T;W^{1,p}(Q)),\quad w=\psi \text{ on } \Gamma_T,\quad \text{and } w\geq\psi \text{ in } Q_T.$$

The function ψ acts as an obstacle and also prescribes the boundary values.

The following existence theorem will be useful for us later. The case $\varepsilon = 0$ is treated in [AL].

3.1. Lemma. *There is a unique $w \in \mathcal{F}_\psi$ such that*

$$\begin{aligned}(3.2)\qquad &\int_0^T \int_Q \Big((|\nabla w|^2 + \varepsilon^2)^{(p-2)/2} \nabla w \cdot \nabla(\phi - w) + (\phi - w)\frac{\partial v}{\partial t}\Big)\,dx\,dt \\ &\qquad \geq \frac{1}{2}\int_Q |\phi(x,T) - w(x,T)|^2\,dx\end{aligned}$$

for all smooth functions ϕ in the class $\mathcal{F}_\psi$. In particular, w is a continuous supersolution of (1.5)*. Moreover, in the open set $\{w > \psi\}$ the function w is a solution of* (1.5)*. In the case $\varepsilon \neq 0$ we have $w \in C^\infty(Q_T)$.*

Let w_ε denote the solution of (3.2) with $\varepsilon \neq 0$ and let v denote the one with $\varepsilon = 0$. We keep the obstacle ψ fixed and let $\varepsilon \to 0$ in (3.2). The question is about the convergence of the solutions of the obstacle problems: Do the w_ε's converge to v in some sense?

3.3. Lemma. *The inequality*

$$(3.4)\qquad \int_0^T \int_Q \big((|\nabla w_\varepsilon|^2 + \varepsilon^2)^{(p-2)/2} \nabla w_\varepsilon - |\nabla v|^{p-2}\nabla v\big) \cdot \nabla(w_\varepsilon - v)\,dx\,dt \leq 0$$

holds. In particular,

$$(3.5)\qquad \lim_{\varepsilon \to 0} \int_0^T \int_Q |\nabla w_\varepsilon - \nabla v|^p\,dx\,dt = 0.$$

Proof. We write $w = w_\varepsilon$. We begin the proof with some remarks. The estimate

$$\int_0^T \int_Q |\nabla w|^p\,dx\,dt \leq c\int_0^T \int_Q |\nabla v|^p\,dx\,dt + c|\varepsilon|^{p-2}\int_0^T \int_Q |\nabla v|^2\,dx\,dt$$

with $c = c(p)$ follows from (3.4) with the aid of Young's inequality.

We also have

$$(3.6)\qquad \int_0^T \int_Q |\nabla w|^p\,dx\,dt \leq c\int_0^T \int_Q |\nabla v|^p\,dx\,dt + c|\varepsilon|^{p-2}T|Q|$$

with another $c = c(p)$, if $|\varepsilon| \leq 1$, for instance.

The elementary inequalities

$$|a - b|^p \leq 2^{p-2}(|a|^{p-2}a - |b|^{p-2}b)\cdot(a - b)$$

and

$$\big|(|a|^2 + \varepsilon^2)^{(p-2)/2}a - |a|^{p-2}a\big| \leq \begin{cases} \dfrac{p-2}{2}|\varepsilon|^{p-1}, & 2 \leq p \leq 3, \\ \dfrac{p-2}{2}\varepsilon^2(|a|^2 + \varepsilon^2)^{(p-3)/2}, & p \geq 3, \end{cases}$$

and (3.4) imply that

$$(3.7)\qquad \int_0^T \int_Q |\nabla w - \nabla v|^p\,dx\,dt \leq c(p-2)|\varepsilon|^{p-1}T|Q|,$$

when $2 \le p \le 3$ and

$$\int_0^T \int_Q |\nabla w - \nabla v|^p \, dx \, dt \le c\varepsilon^2 \Big(\int_0^T \int_Q |\nabla v|^p \, dx \, dt + T|Q| \Big), \tag{3.8}$$

when $p \ge 3$. Here we also used (3.6). In particular, this implies that (3.5) holds. Thus it is enough to prove (3.4).

Now we proceed to the actual proof. For simplicity we assume that the obstacle $\psi \ge 0$ and that ψ vanishes on the parabolic boundary Γ_T. (We will only need this case.) The difficulty is that v_t is forbidden. However, w_t is available. Our aim is to take full advantage of the fact that w and v are solutions of the corresponding equations in the open sets where the obstacle does not hinder.

We replace v with the supersolution

$$v_\alpha = v + \frac{\alpha}{T - t},$$

where $\alpha > 0$. Then $v_\alpha \to \infty$ as $t \to T$ and $v_\alpha \ge \psi + \alpha/T$. Since v_α is continuous, we conclude that $v_\alpha^* \to v_\alpha$ uniformly in $\overline{Q}_{T-\delta}$, $\delta > 0$, as $\sigma \to 0$; recall Lemma 2.9. It follows that

$$v_\alpha^* > \psi + \frac{\alpha}{2T}$$

in $\overline{Q}_T$, when σ is small enough (depending on α). The set $\{w > v_\alpha^*\}$ is open and it cannot touch the Euclidean boundary of Q_T. The set $\{w > v_\alpha^*\}$ is contained in $\{w > \psi + \alpha/(2T)\}$, that is, the obstacle does not hinder w. Thus w is a solution of (1.5) in $\{w > v_\alpha^*\}$.

We choose the test function $\varphi = w - v_\alpha^*$ and subtract the regularized equation

$$\iint_{\{w > v_\alpha^*\}} \Big((|\nabla v_\alpha|^{p-2} \nabla v_\alpha)^* \cdot \nabla \varphi - v_\alpha^* \frac{\partial \varphi}{\partial t} \Big) \, dx \, dt \ge 0$$

(see (2.11)) from the equation for w. We obtain

$$\begin{aligned}
\iint_{\{w > v_\alpha^*\}} &\big((|\nabla w|^2 + \varepsilon^2)^{(p-2)/2} \nabla w - (|\nabla v_\alpha|^{p-2} \nabla v_\alpha)^* \big) \cdot \nabla (w - v_\alpha^*) \, dx \, dt \\
&\le \iint_{\{w > v_\alpha^*\}} (w - v_\alpha^*) \frac{\partial}{\partial t} (w - v_\alpha^*) \, dx \, dt \\
&= \frac{1}{2} \iint_{\{w > v_\alpha^*\}} \frac{\partial}{\partial t} (w - v_\alpha^*)^2 \, dx \, dt = 0.
\end{aligned}$$

This estimate is free of derivatives with respect to time. Moreover, $\nabla v_\alpha = \nabla v$ and $\nabla v_\alpha^* = \nabla v^*$. First, we let $\sigma \to 0$ and then $\alpha \to 0$. We arrive at

$$\iint_{\{w \ge v\}} \big((|\nabla w|^2 + \varepsilon^2)^{(p-2)/2} \nabla w - |\nabla v|^{p-2} \nabla v \big) \cdot \nabla (w - v) \, dx \, dt \le 0. \tag{3.9}$$

Strictly speaking the limit set of integration contains the set $\{w > v\}$ and is itself contained in $\{w \ge v\}$, but, because the integrand vanishes a.e. in $\{w = v\}$, this does not matter. This is the desired estimate in $\{w \ge v\}$.

To obtain the same estimate for $\{w \le v\}$, we reverse the roles of the functions in the previous construction. However, the situation is not completely symmetric. It is important that no points outside $\{v > \psi\}$ are evoked in the averaging procedure.

Since v is a solution of (1.1) in $\{v > \psi\}$ we have

$$\iint_{\{v>\psi\}} \Big((|\nabla v|^{p-2}\nabla v) * \rho_\sigma \cdot \nabla\varphi - (v * \rho_\sigma)\frac{\partial \varphi}{\partial t}\Big)\, dx\, dt = 0 \tag{3.10}$$

when $\varphi \in C_0^\infty(\{v > \psi\})$ and σ is smaller than a number depending on the test function φ, see (2.15).

This time we consider the supersolution

$$w_\alpha = w + \frac{\alpha}{T-t},$$

where $\alpha > 0$. The set $\{v*\rho_\sigma > w_\alpha\}$ is contained in $\{v > \psi + \alpha/(2T)\}$ for all σ small enough (depending on α). The latter set has a positive distance to the coincidence set $\{v = \psi\}$. For σ small enough the function

$$\varphi = v * \rho_\sigma - w_\alpha$$

will, consequently, do as test function in (3.10), when we integrate only over the set $\{\varphi > 0\}$. It follows that

$$\begin{aligned}
&\iint_{\{v*\rho_\sigma>w_\alpha\}} \Big((|\nabla v|^{p-2}\nabla v) * \rho_\sigma \\
&\qquad\qquad - (|\nabla w_\alpha|^2 + \varepsilon^2)^{(p-2)/2}\nabla w_\alpha\Big) \cdot \nabla(v * \rho_\sigma - w_\alpha\big)\, dx\, dt \\
&\le \iint_{\{v*\rho_\sigma>w_\alpha\}} (v * \rho_\sigma - w_\alpha)\frac{\partial}{\partial t}(v * \rho_\sigma - w_\alpha)\, dx\, dt \\
&= \frac{1}{2}\iint_{\{v*\rho_\sigma>w_\alpha\}} \frac{\partial}{\partial t}(v * \rho_\sigma - w_\alpha)^2\, dx\, dt = 0.
\end{aligned}$$

The time derivative has disappeared, so that we can let $\sigma \to 0$ and then $\alpha \to 0$. Again we arrive at (3.9), though integrated over the set $\{v \ge w\}$ this time. Hence we have obtained the desired estimate. □

4. The p-superparabolic functions and their approximants

The supersolutions of the p-parabolic equation do not form a good closed class of functions. The Barenblatt solution defined in (1.3) is not a supersolution of (1.1) in $\mathbf{R}^{n+1}$. However, the Barenblatt solution is a p-superparabolic function according to the following definition.

4.1. Definition. A function $v\colon \Omega \to (-\infty, \infty]$ is called *p-superparabolic* if

(1) v is lower semicontinuous,
(2) v is finite in a dense subset of Ω,
(3) v satisfies the following comparison principle on each Q_{t_1,t_2} with $\overline{Q}_{t_1,t_2} \subset \Omega$: if h is p-parabolic in Q_{t_1,t_2} and continuous in $\overline{Q}_{t_1,t_2}$ and if $h \le v$ on Γ_{t_1,t_2}, then $h \le v$ in Q_{t_1,t_2}.

It follows immediately from the definition that if u and v are p-superparabolic functions so are their pointwise minimum $\min(u, v)$ and $u + \alpha$, $\alpha \in \mathbf{R}$. Observe, that $u + v$ and βu, $\beta \in \mathbf{R}$, are not superparabolic in general. However, with some effort we can see that if v is p-superparabolic in Q_T, then

$$v + \frac{\alpha}{T - t},$$

with $\alpha > 0$, is a p-superparabolic function in Q_T. This is well in accordance with the corresponding properties of supersolutions.

Notice that a p-superparabolic function is defined at every point in its domain. No differentiability is presupposed in the definition. The only tie to the differential equation is through the comparison principle. It was established in [KL] that (3) can be replaced by the following "elliptic" comparison principle.

4.2. Lemma. *Let Ξ be any domain with compact closure in Ω. If h is p-parabolic in Ξ and continuous in $\overline{\Xi}$ and if $h \le v$ on the Euclidean boundary $\partial\Xi$, then $h \le v$ in the whole Ξ.*

Observe that there is no reference to the parabolic boundary. We will only need the fact that the condition (3) in Definition 4.1 implies Lemma 4.2. This is a rather immediate consequence of the definition. The opposite implication is deeper.

Of course, there is a relation between supersolutions and p-superparabolic functions. Roughly speaking, the supersolutions are p-superparabolic, provided the issue about lower semicontinuity is properly handled. We refer to [KL, Lemma 4.2]. In particular, a continuous supersolution is p-superparabolic.

The Barenblatt solution clearly shows that the class of p-superparabolic functions contains more than supersolutions. Nevertheless, it turns out that a p-superparabolic function can be approximated pointwise with an increasing sequence of supersolutions, constructed through successive obstacle problems. Let us describe this procedure.

4.3. Lemma. *Suppose that v is a p-superparabolic function in Ω and let $\overline{Q}_{t_1,t_2} \subset \Omega$. Then there is a sequence of supersolutions*

$$v_k \in C(\overline{Q}_{t_1,t_2}) \cap L^p(t_1, t_2; W^{1,p}(Q)), \quad k = 1, 2, \dots,$$

of (1.1) *such that $v_1 \le v_2 \le \dots \le v$ and $v_k \to v$ pointwise in Q_{t_1,t_2} as $k \to \infty$. If, in addition, v is locally bounded in Ω, then the Sobolev derivative ∇v exists and $\nabla v \in L^p_{\mathrm{loc}}(\Omega)$.*

Proof. The lower semicontinuity implies that there is a sequence of functions $\psi_k \in C^\infty(\Omega)$, $k = 1, 2, \dots$, such that

$$\psi_1 \le \psi_2 \le \dots \qquad \text{and} \qquad \lim_{k\to\infty} \psi_k = v$$

at every point of Ω. Using the functions ψ_k as obstacles we construct supersolutions of (1.1) that approximate v from below. This has to be done locally, say in a given box Q_{t_1,t_2} with $\overline{Q}_{t_1,t_2} \subset \Omega$. To simplify the notation we consider Q_T, assuming that $\overline{Q}_T \subset \Omega$. Let $v_k \in C(\overline{Q}_T) \cap L^p(0, T; W^{1,p}(Q))$, $k = 1, 2, \dots$, denote the solution of the obstacle problem in Q_T with the obstacle ψ_k, see Lemma 3.1 with $\varepsilon = 0$. Then $v_k \ge \psi_k$ and $v_k = \psi_k$ on Γ_T.

We claim that

$$v_1 \le v_2 \le \dots \qquad \text{and} \qquad v_k \le v, \qquad k = 1, 2, \dots,$$

in Q_T. Consider the set $\{v_k > \psi_k\}$. In this set v_k is a p-parabolic function with boundary values ψ_k, except possibly when $t = T$. The function

$$v + \frac{\alpha}{T - t},$$

with $\alpha > 0$, is a p-superparabolic function in Q_T. This function has larger boundary values than v_k on the boundary of $\{v_k > \psi_k\}$. By the "elliptic" comparison principle (Lemma 4.2) it can be shown that

$$v + \frac{\alpha}{T - t} \ge v_k$$

in $\{v_k > \psi_k\}$. Letting $\alpha \to 0$ we obtain $v \ge v_k$ in $\{v_k > \psi_k\}$. But certainly $v \ge v_k$ in $\{v_k = \psi_k\}$. Thus $v \ge v_k$ and consequently

$$v = \lim_{k\to\infty} \psi_k \le \lim_{k\to\infty} v_k \le v$$

in Q_T. Notice how the comparison principle was used. The fact that $v_{k+1} \ge v_k$ can be proved in the same manner.

So far we have constructed an increasing sequence of continuous supersolutions of the p-parabolic equation converging pointwise to the given p-superparabolic function. Assume, in addition, that there is $L < \infty$ such that

$$0 \le v(x, t) \le L$$

for every $(x, t) \in Q_T$. Then also $0 \le v_k \le L$, $k = 1, 2, \dots$, and Lemma 2.16, applied to the subsolution $L - v_k$, provides us with the bound

$$\int_0^T \int_Q |\nabla v_k|^p \zeta^p \, dx\, dt \le cL^2 \int_0^T \int_Q \left| \frac{\partial(\zeta^p)}{\partial t} \right| dx\, dt + cL^p \int_0^T \int_Q |\nabla \zeta|^p \, dx\, dt$$

where ζ, $0 \le \zeta \le 1$, is a test function vanishing on the parabolic boundary Γ_T. By weak compactness we can, via weakly convergent subsequences of $\zeta \nabla v_k$, conclude that ∇v exists in Sobolev's sense and that $\nabla v \in L^p_{\mathrm{loc}}(Q_T)$. The weak lower semicontinuity implies

$$\begin{aligned}\int_0^T \int_Q |\nabla v|^p \zeta^p \, dx\, dt &\le \liminf_{k\to\infty} \int_0^T \int_Q |\nabla v_k|^p \zeta^p \, dx\, dt \\ &\le cL^2 \int_0^T \int_Q \left| \frac{\partial(\zeta^p)}{\partial t} \right| dx\, dt + cL^p \int_0^T \int_Q |\nabla \zeta|^p \, dx\, dt.\end{aligned}$$

As a matter of fact we have established that $\nabla v \in L^p_{\mathrm{loc}}(Q_T)$ provided v is locally bounded in Ω. $\square$

4.4. Remark. As a supersolution each v_k satisfies the equation

$$\int_0^T \int_Q \Big(|\nabla v_k|^{p-2} \nabla v_k \cdot \nabla \varphi - v_k \frac{\partial \varphi}{\partial t} \Big)\, dx\, dt \geq 0. \tag{4.5}$$

The passage to the limit under the integral sign as $k \to \infty$ requires much more information than the so far established weak convergence, except in the case $p = 2$. A comment is appropriate now. For $p > 2$ the inequality

$$2^{2-p} |\nabla (v^* - v_k^*)|^p \leq (|\nabla v^*|^{p-2} \nabla v^* - |\nabla v_k^*|^{p-2} \nabla v_k^*) \cdot (\nabla v^* - \nabla v_k^*)$$

is still available, but the averaging procedure leads to the quantity $(|\nabla v_k|^{p-2} \nabla v_k)^*$ instead of $|\nabla v_k^*|^{p-2} \nabla v_k^*$. All attempts to adjust the situation cause, as it were, subtle difficulties disturbing the double limit procedure as $k \to \infty$ and $\sigma \to 0$.

Let us resume the study of the approximants

$$v_1 \leq v_2 \leq \ldots \qquad \text{with} \qquad v = \lim_{k \to \infty} v_k.$$

obtained from the obstacle problem in Q_T with obstacles ψ_k as in the proof of Lemma 4.3. We assume that

$$0 \leq \psi_k \leq L, \qquad k = 1, 2, \ldots \tag{4.6}$$

Our aim is to prove that, under this assumption, $\nabla v_k \to \nabla v$ locally in $L^p(Q_T)$. To achieve this we utilize the corresponding property for the regularized equation (1.5), which is easier to handle. Let $\varepsilon \neq 0$ be fixed and let w_k denote the solution to the obstacle problem in Q_T with the obstacle ψ_k. In the notation of Lemma 3.1, we have $w_k \in \mathcal{F}_{\psi_k}$. Recall that $w_k \geq \psi_k$ in Q_T and that $w_k = \psi_k$ on Γ_T. Since $\varepsilon \neq 0$, we also have $w_k \in C(\overline{Q}_T) \cap C^\infty(Q_T)$. It follows that

$$w_1 \leq w_2 \leq \ldots \qquad \text{and} \qquad w = \lim_{k \to \infty} w_k$$

as in the case $\varepsilon = 0$; see the proof of Lemma 4.3. The dependence of ε is suppressed in the notation for w_k and w. Here w is, to begin with, merely a lower semicontinuous function in $L^p(0, T; W^{1,p}(Q))$. Recall that the obstacles ψ_k were induced by v. Therefore $w \geq v$. We also have $0 \leq w \leq L$. We claim that w is, in fact, a supersolution of (1.5). The proof is rather involved, although we deal with the regularized problem with a fixed $\varepsilon \neq 0$. The complication is visible in the averaging procedure.

4.7. Lemma. *Under the assumption* (4.6) *for the obstacles,*

$$\nabla w_k \to \nabla w \qquad \textit{in} \qquad L^p_{\mathrm{loc}}(Q_T).$$

Moreover, w is a supersolution to (1.5) *in Q_T.*

Proof. Let $\zeta \in C_0^\infty(Q_T)$, $0 \leq \zeta \leq 1$, where it is essential that the support of ζ does not touch the Euclidean boundary ∂Q_T. Write $\theta = \zeta^p$. The uniform bound for $u = L - w_k$ provided by Lemma 2.16 allows us to conclude that

$$\int_0^T \int_Q |\nabla w_k|^p \theta \, dx\, dt \leq c$$

where $c = c(n, p, L, \theta, \varepsilon)$. Observe that the bound is independent of k. By weak compactness it follows that ∇w exists and that $\nabla w \in L^p_{\text{loc}}(Q_T)$. We can extract a subsequence for which ∇w_{k_j} converges to ∇w weakly in $L^p_{\text{loc}}(Q_T)$. (Actually, this holds for the original sequence.)

Let $\sigma > 0$ and use the averaged function w^* introduced in (2.7). Recall that

$$\frac{\partial w^*}{\partial t} = \frac{w - w^*}{\sigma}. \tag{4.8}$$

Using the test function $(w^* - w_k)\theta$ in the equation for w_k, we obtain

$$\begin{aligned}\int_0^T \int_Q &\big((|\nabla w^*|^2 + \varepsilon^2)^{(p-2)/2} \nabla w^* - (|\nabla w_k^*|^2 + \varepsilon^2)^{(p-2)/2} \nabla w_k^*\big) \\ &\cdot \nabla\big((w^* - w_k)\theta\big)\, dx\, dt \\ &\le \int_0^T \int_Q (|\nabla w^*|^2 + \varepsilon^2)^{(p-2)/2} \nabla w^* \cdot \nabla\big((w^* - w_k)\theta\big)\, dx\, dt \\ &\quad - \int_0^T \int_Q w_k \frac{\partial}{\partial t}\big((w^* - w_k)\theta\big)\, dx\, dt.\end{aligned} \tag{4.9}$$

We aim at first letting $k \to \infty$, which causes a difficulty in the last term, namely, the appearance of the time derivative w_t, which has to be avoided.

Let us begin with the crucial last term

$$\begin{aligned}&- \int_0^T \int_Q w_k \frac{\partial}{\partial t}\big((w^* - w_k)\theta\big)\, dx\, dt \\ &= - \int_0^T \int_Q w_k (w^* - w_k) \frac{\partial \theta}{\partial t}\, dx\, dt - \int_0^T \int_Q \theta w_k \frac{\partial}{\partial t}(w^* - w_k)\, dx\, dt.\end{aligned}$$

Using

$$\begin{aligned}\int_0^T \int_Q \theta (w^* - w_k) \frac{\partial}{\partial t}(w^* - w_k)\, dx\, dt &= \frac{1}{2} \int_0^T \int_Q \theta \frac{\partial}{\partial t}(w^* - w_k)^2\, dx\, dt \\ &= -\frac{1}{2} \int_0^T \int_Q (w^* - w_k)^2 \frac{\partial \theta}{\partial t}\, dx\, dt\end{aligned}$$

we can write the last term in (4.9) as

$$\begin{aligned}- \int_0^T \int_Q w_k \frac{\partial}{\partial t}\big((w^* - w_k)\theta\big)\, dx\, dt &= - \int_0^T \int_Q \theta w^* \frac{\partial}{\partial t}(w^* - w_k)\, dx\, dt \\ &\quad - \frac{1}{2} \int_0^T \int_Q (w^* - w_k)(w^* + w_k) \frac{\partial \theta}{\partial t}\, dx\, dt.\end{aligned}$$

One more integration by parts exposes the factor $w^* - w_k$ and we have

$$\begin{aligned}- \int_0^T \int_Q w_k \frac{\partial}{\partial t}\big((w^* - w_k)\theta\big)\, dx\, dt &= \int_0^T \int_Q \theta (w^* - w_k) \frac{\partial w^*}{\partial t}\, dx\, dt \\ &\quad + \frac{1}{2} \int_0^T \int_Q (w^* - w_k)^2 \frac{\partial \theta}{\partial t}\, dx\, dt.\end{aligned} \tag{4.10}$$

Needless to say, the calculations leading to this formula can be arranged in various ways. In view of (4.8) we have the important estimate

$$\begin{aligned}
\lim_{k\to\infty} & \int_0^T \int_Q \theta(w^* - w_k)\frac{\partial w^*}{\partial t}\, dx\, dt \\
&= \int_0^T \int_Q \frac{\theta(w^* - w)(w - w^*)}{\sigma}\, dx\, dt \le 0
\end{aligned}$$

and from (4.10) we finally obtain

$$\begin{aligned}
\limsup_{k\to\infty} & \Big(-\int_0^T \int_Q w_k \frac{\partial}{\partial t}\big((w^* - w_k)\theta\big)\, dx\, dt \Big) \\
& \le \frac{1}{2}\int_0^T \int_Q (w^* - w)^2 \frac{\partial \theta}{\partial t}\, dx\, dt
\end{aligned} \tag{4.11}$$

for the last term in (4.9).

Rearranging (4.9) we have

$$\begin{aligned}
2^{2-p} & \int_0^T \int_Q \theta |\nabla w^* - \nabla w_k|^p\, dx\, dt \\
\le & \int_0^T \int_Q \theta\big((|\nabla w^*|^2 + \varepsilon^2)^{(p-2)/2} \nabla w^* - (|\nabla w_k^*|^2 + \varepsilon^2)^{(p-2)/2} \nabla w_k^* \big) \\
& \cdot \nabla (w^* - w_k)\, dx\, dt \\
\le & \int_0^T \int_Q (w^* - w_k)(|\nabla w_k^*|^2 + \varepsilon^2)^{(p-2)/2} \nabla w_k^* \cdot \nabla \theta\, dx\, dt \\
& + \int_0^T \int_Q \theta (|\nabla w^*|^2 + \varepsilon^2)^{(p-2)/2} \nabla w^* \cdot \nabla (w^* - w_k)\, dx\, dt \\
& + \int_0^T \int_Q -w_k \frac{\partial}{\partial t}\big((w^* - w_k)\theta\big)\, dx\, dt \\
= & A_k + B_k + C_k,
\end{aligned} \tag{4.12}$$

where the algebraic inequality

$$2^{2-p}|a-b|^p \le \big((|a|^2 + \varepsilon^2)^{(p-2)/2} a - (|b|^2 + \varepsilon^2)^{(p-2)/2} b \big) \cdot (a - b),$$

$a, b \in \mathbf{R}^n$, $p \ge 2$, was used. We estimate the last three integrals in (4.12). Recall that $\theta = \zeta^p$. Hölder's inequality and (2.17), with the subsolution $L - w_k$, give

$$\begin{aligned}
|A_k| \le & p 2^{(p-2)/2} \Big(\int_0^T \int_Q |w^* - w_k|^p |\nabla \zeta|^p\, dx\, dt \Big)^{1/p} \\
& \cdot \Big(\int_0^T \int_Q \big(2|\nabla w_k|^p + |\varepsilon|^{p-2} \big) \zeta^p\, dx\, dt \Big)^{1/q} \\
\le & c \Big(\int_0^T \int_Q |w^* - w_k|^p |\nabla \zeta|^p\, dx\, dt \Big)^{1/p}
\end{aligned}$$

where $q = p/(p-1)$ and $c = c(n, p, L, \zeta, \varepsilon)$ is independent of k and σ. Thus

$$\limsup_{k\to\infty} |A_k| \le c\Big(\int_0^T \int_Q |w^* - w|^p |\nabla\zeta|^p \, dx \, dt\Big)^{1/p}. \tag{4.13}$$

In the second integral B_k we may proceed to the limit in view of the weak convergence and we obtain

$$\lim_{k\to\infty} B_k = \int_0^T \int_Q \zeta^p (|\nabla w^*|^2 + \varepsilon^2)^{(p-2)/2} \nabla w^* \cdot \nabla(w^* - w) \, dx \, dt. \tag{4.14}$$

Recall that C_k was handled in (4.11).

Each of the bounds (4.11), (4.13), and (4.14) approaches zero as $\sigma \to 0$. Therefore we can select $\sigma > 0$ so small that

$$\limsup_{k\to\infty} \int_0^T \int_Q \theta |\nabla w^* - \nabla w_k|^p \, dx \, dt$$

is as small as we please. Then also

$$\begin{aligned} &\limsup_{k\to\infty} \|\theta(\nabla w - \nabla w_k)\|_{p,Q_T} \\ &\qquad \le \|\theta(\nabla w^* - \nabla w)\|_{p,Q_T} + \limsup_{k\to\infty} \|\theta(\nabla w^* - \nabla w_k)\|_{p,Q_T} \end{aligned}$$

can be made smaller than any preassigned quantity. This proves that $\nabla w_k \to \nabla w$ locally in $L^p(Q_T)$. Now we may pass to the limit under the integral in

$$\int_0^T \int_Q \Big((|\nabla w_k|^2 + \varepsilon^2)^{(p-2)/2} \nabla w_k \cdot \nabla\varphi - w_k \frac{\partial\varphi}{\partial t} \Big) \, dx \, dt \ge 0$$

as $k \to \infty$. The result is the required equation for ∇w. □

The main objective of this section is to give a proof of Theorem 1.4 stating that bounded p-superparabolic functions are, in fact, supersolutions. We have already established that $\nabla v \in L^p_{\mathrm{loc}}(\Omega)$ (see Lemma 4.3). The proof of Theorem 1.4 follows from the approximation result below.

4.15. Lemma. *Suppose that v is a locally bounded p-superparabolic function in Ω. For every Q_{t_1,t_2} with $\overline{Q}_{t_1,t_2} \subset \Omega$, there exists a sequence of supersolutions v_k of (1.1), $v_1 \le v_2 \le \cdots \le v$, in Q_{t_1,t_2} such that $v_k \to v$ at each point of Q_{t_1,t_2} and*

$$\nabla v_k \to \nabla v \qquad \text{in} \qquad L^p(Q_{t_1,t_2})$$

as $k \to \infty$.

Proof. We may assume that $\overline{Q}_T \subset \Omega$, and that $\overline{Q}_{t_1,t_2} \subset Q_T$ We can also assume that $1 \le v \le L$ in $\overline{Q}_T$ by adding a constant to v. Let $\theta \in C_0^\infty(Q_T)$, $0 \le \theta \le 1$ and $\theta = 1$ on Q_{t_1,t_2}. Again, consider the obstacles ψ_k and the two sequences of supersolutions

$$v_1 \le v_2 \le \ldots, \qquad v = \lim_{k\to\infty} v_k$$

and

$$w_1 \le w_2 \le \dots, \qquad w = \lim_{k\to\infty} w_k$$

in Q_T. These are the same functions as before. Recall that v_k are solutions to the obstacle problem related to the true p-parabolic equation (1.1) whereas w_k are the corresponding solutions related to the regularized equation (1.5), all with the same ε.

We want to show that

$$\lim_{k\to\infty} \int_0^T \int_Q \theta |\nabla v_k - \nabla v|^p \, dx \, dt = 0. \tag{4.16}$$

To this end, we use Minkowski's inequality and obtain

$$\begin{aligned} &\|\theta(\nabla v - \nabla v_k)\|_{p,Q_T} \\ &\quad \le \|\theta(\nabla v - \nabla w)\|_{p,Q_T} + \|\theta(\nabla w - \nabla w_k)\|_{p,Q_T} + \|\theta(w_k - \nabla v_k)\|_{p,Q_T}. \end{aligned} \tag{4.17}$$

There is a technical difference between the cases $2 \le p \le 3$ and $p > 3$.

First we study the case $2 \le p \le 3$. This is simple because (3.7) implies that

$$\begin{aligned} \int_0^T \int_Q \theta |\nabla w_k - \nabla v_k|^p \, dx \, dt &\le \int_0^T \int_Q |\nabla w_k - \nabla v_k|^p \, dx \, dt \\ &\le c|\varepsilon|^{p-1} T |Q| \end{aligned}$$

for the last term in (4.17). By the weak lower semicontinuity of the integral also the first term is bounded by $c|\varepsilon|^{p-1}T|Q|$. Finally, we use Lemma 4.7 to conclude that for the intermediate term

$$\lim_{k\to\infty} \|\theta(\nabla w - \nabla w_k)\|_{p,Q_T} = 0.$$

Therefore

$$\limsup_{k\to\infty} \|\theta(\nabla v - \nabla v_k)\|_{p,Q_T} \le c|\varepsilon|^{p-1} T|Q|$$

for every $\varepsilon \ne 0$. The left-hand side is independent of ε. The desired result (4.16) follows in the case $2 \le p \le 3$.

Let us then consider the case $p > 3$. Now we have to use (3.8) instead of (3.7). Thus there is an extra difficulty with the term $\|\nabla v_k\|_{p,Q_T}$ in the inequality

$$\int_0^T \int_Q \theta |\nabla w_k - \nabla v_k|^p \, dx \, dt \le c\varepsilon^2 \Big(\int_0^T \int_Q |\nabla v_k|^p \, dx \, dt + T|Q| \Big) \tag{4.18}$$

since a bound independent of the index k is called for. In order to avoid the conflict between the global nature of Lemma 3.3 and the local result in Lemma 2.16, we make an adjustment.

Notice that we need $v_k \to v$ only in Q_{t_1,t_2}, not in the whole of Q_T. We select the obstacles ψ_k so that $\psi_k = 0$ in $\overline{Q}_T \setminus \overline{Q}_{t_1,t_2}$, and $\psi_k \to v$ only in Q_{t_1,t_2}. Since $v \ge 1$, we may assume that $\psi_k > 0$ in Q_{t_1,t_2}. With these arrangements it is clear that the obstacle cannot hinder in the outer region $\overline{Q}_T \setminus \overline{Q}_{t_1,t_2}$. Therefore v_k is even p-parabolic there. Thus the estimate (2.21) is available and together with (2.17) this exhibits a bound of the form $\varepsilon^2 c(L, Q_T, Q_{t_1,t_2})$, when $|\varepsilon| \le 1$, for the quantities in (4.18). Now one only has to replace $c|\varepsilon|^{p-1}T|Q|$ in the previous case by $\varepsilon^2 c(L, Q_T, Q_{t_1,t_2})$. This concludes the proof. $\square$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\overline{Q}_{t_1,t_2} \subset \Omega$. By Lemma 4.16 we have

$$\begin{aligned}\int_{t_1}^{t_2}\int_Q \Big(|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi - v\frac{\partial\varphi}{\partial t}\Big)\,dx\,dt \\ = \lim_{k\to\infty}\int_{t_1}^{t_2}\int_Q \Big(|\nabla v_k|^{p-2}\nabla v_k\cdot\nabla\varphi - v_k\frac{\partial\varphi}{\partial t}\Big)\,dx\,dt \geq 0\end{aligned}$$

whenever $\varphi \in C_0^\infty(Q_{t_1,t_2})$ with $\varphi \geq 0$. □

5. Pointwise behaviour

For a lower semicontinuous function v in Ω we have

$$\begin{aligned}v(x,t) &\leq \liminf_{(y,\tau)\to(x,t)} v(y,\tau) \\ &\leq \operatorname*{ess\,lim\,inf}_{(y,\tau)\to(x,t)} v(y,\tau) \leq \operatorname*{ess\,lim\,inf}_{\substack{(y,\tau)\to(x,t)\\ \tau<t}} v(y,\tau),\end{aligned} \tag{5.1}$$

when $(x,t) \in \Omega$. We show that for a p-superparabolic function also the reverse inequalities hold.

5.2. Theorem. *Suppose that v is a p-superparabolic function Ω. Then*

$$v(x,t) = \operatorname*{ess\,lim\,inf}_{\substack{(y,\tau)\to(x,t)\\ \tau<t}} v(y,\tau) \tag{5.3}$$

holds at every point $(x,t) \in \Omega$.

The proof is based on the following lemma.

5.4. Lemma. *Suppose that v is a p-superparabolic function Ω. Assume that $\overline{Q}_T \subset \Omega$ and that*

(1) *$v \leq 0$ at every point in Q_T and*
(2) *$v = 0$ at almost every point in Q_T.*

Then $v = 0$ at every point in $Q \times (0,T]$.

Proof. Let $\psi_k \in C^\infty(\Omega)$ be such that

$$\psi_1 \leq \psi_2 \leq \dots \qquad \text{and} \qquad \lim_{k\to\infty}\psi_k = v$$

at every point in Ω. Let v_k be the solution of the obstacle problem with the obstacle ψ_k as in the proof of Lemma 4.3. Then

$$v_1 \leq v_2 \leq \dots \qquad \text{and} \qquad \psi_k \leq v_k \leq v$$

at every point in Q_T. To be on the safe side concerning the final result also at the instant $t = T$, we can solve the obstacle problem in a slightly larger domain, say $Q_{T+\delta}$ with a small $\delta > 0$.

Select an arbitrary $Q' \subset\subset Q$ and choose t', $0 < t' < T$. Let h_k denote the unique p-parabolic function in $Q'_{t',T+\delta}$ with the values $h_k = v_k$ on the parabolic boundary of $Q'_{t',T+\delta}$. At every point in $Q'_{t',T+\delta}$ we have

$$h_1 \le h_2 \le \dots \qquad \text{and} \qquad h_k \le v_k \le v.$$

By Harnack's convergence theorem (see [KL, Remark 3.2]), the limit function

$$h = \lim_{k\to\infty} h_k$$

is p-parabolic in $Q'_{t',T+\delta}$. Then the function

$$w = \begin{cases} h & \text{in} \quad Q'_{t',T+\delta}, \\ v & \text{otherwise}, \end{cases}$$

is, indeed, p-superparabolic in Ω. For the verification of the comparison principle the fact that $h \le v$ is essential, see [KL, p. 671].

We know that $w \le v \le 0$ everywhere in Q_T and, in particular, that $h \le v \le 0$ everywhere in $Q'_{t',T+\delta}$. We claim that $h = 0$ in $Q'_{t',T}$. Then we could conclude that $v = 0$ everywhere in $Q' \times (t', T)$. Moreover, $h(x,T) \le v(x,T)$ for every $x \in Q'$ (this holds up to the instant $t = T+\delta$) and since h is continuous, we could also conclude that $v(x,T) \ge 0$, when $x \in Q'$. By lower semicontinuity the alternative $v(x,T) > 0$ is out of the question. Therefore it is sufficient to prove the claim.

First we observe that if h itself were an admissible test function in the equation for h, we could easily conclude that $\nabla h = 0$ almost everywhere. The averaged equation for h reads

$$\begin{aligned} \int_{t'}^{T} \int_{Q'} \Big((|\nabla h|^{p-2} \nabla h)^* \cdot \nabla\varphi - h^* \frac{\partial\varphi}{\partial t} \Big) \, dx\, dt + \int_{Q'} h^*(x,T)\varphi(x,T)\, dx \\ \ge \int_{Q'} h(x,t') \Big(\frac{1}{\sigma} \int_{t'}^{T} \varphi(x,s) e^{-s/\sigma}\, ds \Big)\, dx \end{aligned}$$

where we have taken into account that the diffusion starts at the instance t' by regarding $\varphi(x,t)$ as zero when $0 \le t \le t'$. We choose the test function

$$\varphi_k = (h_k - v_k)^* = h_k^* - v_k^*$$

and regard $h_k - v_k$ as zero when $0 \le t \le t'$ in (2.7). Inserting the test function and letting $k \to \infty$ we obtain

$$\begin{aligned} \int_{t'}^{T} \int_{Q'} \Big((|\nabla h|^{p-2} \nabla h)^* \cdot \nabla\varphi - h^* \frac{\partial h^*}{\partial t} \Big) \, dx\, dt + \int_{Q'} h^*(x,T)^2\, dx \\ \ge \int_{Q'} h^*(x,t') \Big(\frac{1}{\sigma} \int_{t'}^{T} h^*(x,s) e^{-s/\sigma}\, ds \Big)\, dx \end{aligned} \tag{5.5}$$

in view of Lemma 2.9, since $h_k - v_k \to h$ in $L^p(Q'_{t',T})$ as $k \to \infty$. We have

$$-\int_{t'}^{T} \int_{Q'} h^* \frac{\partial h^*}{\partial t}\, dx\, dt = \frac{1}{2} \int_{Q'} h^*(x,t')^2\, dx - \frac{1}{2} \int_{Q'} h^*(x,T)^2\, dx.$$

Now we let $\sigma \to 0$. It follows that

$$\frac{1}{2}\int_{Q'} \left(h^*(x,T)^2 + h^*(x,t')^2\right) dx + \int_{t'}^{T}\int_{Q'} |\nabla h|^p \, dx\, dt = 0 \tag{5.6}$$

because the right-hand side of (5.5) converges to zero as $\sigma \to 0$. This can be easily seen in view of the fact $t' > 0$ (the excluded case $t' = 0$ produces an extra term which we now avoid).

From (5.6) it follows immediately that $\nabla h = 0$ almost everywhere in $Q'_{t',T}$. Therefore there is a constant c such that

$$\int_{t'}^{T} h(x,t)\, dt = c \tag{5.7}$$

when $x \in Q'$.

We have to show that the constant c is zero. To this end, notice that $\nabla w \in L^p_{\mathrm{loc}}(\Omega)$ if v is locally bounded in Ω. We may assume that, to begin with, we have the function $\min(v, 1)$, which is locally bounded. For the Sobolev derivative we can conclude that $\nabla w = 0$ almost everywhere in Q_T (because $\nabla v = 0$ almost everywhere by assumption and $\nabla h = 0$). Then also

$$\int_{t'}^{T} w(x,t)\, dx = c$$

for almost every $x \in Q$. In particular

$$\int_{t'}^{T} v(x,t)\, dx = c$$

for almost every $x \in Q \setminus Q'$. The assumption $v = 0$ almost everywhere implies that $c = 0$. Because $h \leq 0$ and because h is continuous, the integral (5.7) cannot vanish, unless $h = 0$ everywhere. This concludes the proof of the lemma. □

5.8. Lemma. *Suppose that v is p-superparabolic in Ω and that $Q_T \subset\subset \Omega$. If $v(x,t) > \lambda$ for almost every $(x,t) \in Q_T$, then $v(x,t) \geq \lambda$ for every $(x,t) \in Q \times (0,T]$.*

Proof. The auxiliary function

$$u(x,t) = \min(v(x,t), \lambda) - \lambda,$$

in place of v, satisfies the assumptions in Lemma 5.4. Hence $u = 0$ everywhere in $Q \times (0,T]$. This is equivalent to the assertion. □

Proof of Theorem 5.2. Denote

$$\lambda = \operatorname*{ess\,lim\,inf}_{\substack{(y,\tau)\to(x,t)\\ \tau<t}} v(y,\tau)$$

in (5.3). According to (5.1) it is sufficient to prove that $\lambda \leq v(x_0,t_0)$. Thus we can assume that $\lambda > -\infty$. First we consider the case $\lambda < \infty$. Given $\varepsilon > 0$, we can find a $\delta > 0$ and a parallelepiped Q with the center x_0 such that the closure of $Q \times (t_0 - \delta, t_0)$ is comprised in Ω and

$$v(x,t) > \lambda - \varepsilon$$

for almost every $(x,t) \in Q \times (t_0 - \delta, t_0)$. According to Lemma 5.4,

$$v(x,t) \geq \lambda - \varepsilon$$

for every $(x,t) \in Q \times (t_0 - \delta, t_0]$. In particular, we can take $(x,t) = (x_0, t_0)$. Hence $v(x_0,t_0) \geq \lambda - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have established that $\lambda \leq v(x_0,t_0)$, as desired.

The case $\lambda = \infty$ is easily reached via the functions $v_k = \min(v,k)$, $k = 1, 2, \ldots$ Indeed,

$$v(x_0,t_0) \geq v_k(x_0,t_0) \geq \min(\infty, k) = k,$$

$k = 1, 2, \ldots$, in view of the previous case. This concludes the proof of Theorem 5.2. □

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