Calderon's inverse conductivity problem in the plane

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Abstract

We show that the Dirichlet to Neumann map for the equation $\nabla \cdot \sigma \nabla u = 0$ in a two dimensional domain uniquely determines the bounded measurable conductivity σ. This gives a positive answer to a question of A. P. Calderón from 1980. Earlier the result has been shown only for conductivities that are sufficiently smooth.

Contents

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1 Introduction and outline of the method

In 1980 A. P. Calderón [9] posed the following problem: Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with connected complement and $\sigma : \Omega \to (0, \infty)$ is measurable and bounded away from zero and infinity. Let $u \in H^1(\Omega)$ be the unique solution to

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \tag{1.1}$$

$$u\big|_{\partial\Omega} = \phi \in H^{1/2}(\partial\Omega). \tag{1.2}$$

The inverse conductivity problem of Calderón is then to recover σ from the boundary measurements, from the Dirichlet to Neumann map

$$\Lambda_\sigma : \phi \mapsto \sigma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

Here ν is the unit outer normal to the boundary and the derivative $\sigma \partial u / \partial \nu$ exists as an element of $H^{-1/2}(\partial\Omega)$, defined by

$$\langle \sigma \frac{\partial u}{\partial \nu}, \psi \rangle = \int_\Omega \sigma \nabla u \cdot \nabla \psi \, dm, \tag{1.3}$$

where $\psi \in H^1(\Omega)$ and dm denotes the Lebesgue measure.

The aim of this paper is to give a positive answer to Calderón's question in dimension two. More precisely, we prove

Theorem 1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain and $\sigma_i \in L^\infty(\Omega)$, $i = 1, 2$. Suppose that there is a constant $c > 0$ such that $c^{-1} \leq \sigma_i \leq c$. If*

$$\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$$

then $\sigma_1 = \sigma_2$.

Note, in particular, that no regularity is required for the boundary. Our approach to Theorem 1 yields, in principle, also a method to construct σ from the Dirichlet to Neumann operator Λ_σ. For this see Section 8.

The inverse problem to determine σ from Λ_σ is also known as *Electrical Impedance Tomography*. It has been proposed as a valuable diagnostic tool especially for detecting breast cancer [10]. A review for medical applications is given in [11]. For statistical methods in electrical impedance tomography see [17].

That Λ_σ uniquely determines σ was established in dimension three and higher for smooth conductivities by J. Sylvester and G. Uhlmann [26] in 1987.

In dimension two A. Nachman [19] produced in 1995 a uniqueness result for conductivities with two derivatives. For piecewise analytic conductivities the problem was solved by Kohn and Vogelius [15], [16].

The regularity assumptions have since been relaxed by several authors (cf. [20], [21]) but the original problem of Calderón has still remained unsolved. The largest class of potentials where the uniqueness has been shown so far is $W^{3/2,\infty}(\Omega)$ in dimensions three and higher [23] and $W^{1,p}(\Omega)$, $p > 2$, in dimension two [8].

The original approach in [26] and [19] was to reduce the conductivity equation (1.1) to the Schrödinger equation by substituting $v = \sigma^{1/2}u$. Indeed, after such a substitution v satisfies

$$\Delta v - qv = 0$$

where $q = \sigma^{-1/2}\Delta\sigma^{1/2}$. This explains why in this method one needs two derivatives. For the numerical implementation of [19] see [24].

Following the ideas of Beals and Coifman [6], Brown and Uhlman [8] found a first order elliptic system equivalent to (1.1). Indeed, by denoting

$$\begin{pmatrix} v \\ w \end{pmatrix} = \sigma^{1/2} \begin{pmatrix} \partial u \\ \bar{\partial} u \end{pmatrix}$$

one obtains the system

$$D \begin{pmatrix} v \\ w \end{pmatrix} = Q \begin{pmatrix} v \\ w \end{pmatrix},$$

where

$$D = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}$$

and $q = -\frac{1}{2}\partial \log \sigma$. This allowed Brown and Uhlman to work with conductivities with only one derivative. Note however, that the assumption $\sigma \in W^{1,p}(\Omega)$, $p > 2$, made in [8] implies that σ is Hölder continuous. From the viewpoint of applications this is still not satisfactory. Our starting point is to replace (1.1) with an elliptic equation that does not require any differentiability of σ.

We will base our argument to the fact that if $u \in H^1(\Omega)$ is a real solution of (1.1) then there exists a real function $v \in H^1(\Omega)$, called the *σ-harmonic conjugate* of u, such that $f = u + iv$ satisfies the $\mathbb{R}$-linear Beltrami equation

$$\overline{\partial} f = \mu \overline{\partial f}, \tag{1.4}$$

where $\mu = (1-\sigma)/(1+\sigma)$. In particular, note that μ is real valued. The assumptions for σ imply that $\|\mu\|_{L^\infty} \leq \kappa < 1$, and the symbol κ will retain this role throughout the paper.

The structure of the paper is the following:

Since the σ-harmonic conjugate is unique up to a constant we can define the μ-Hilbert transform $\mathcal{H}_\mu : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ by

$$\mathcal{H}_\mu : u\big|_{\partial\Omega} \mapsto v\big|_{\partial\Omega}.$$

We show in Section 2 that the Dirichlet to Neumann map Λ_σ uniquely determines $\mathcal{H}_\mu$ and vice versa. Theorem 1 now implies the surprising fact that $\mathcal{H}_\mu$ uniquely determines μ in equation (1.4) in the whole domain Ω.

Recall that a function $f \in H^1_{loc}(\Omega)$ satisfying (1.4) is called a *quasiregular mapping*; if it is also a homeomorphism then it is called *quasiconformal*. These have a well established theory, cf. [2], [5], [12], [18], that we will employ at several points in the paper. The H^1_{loc} -solutions f to (1.4) are automatically continuous and admit a factorization $f = \psi \circ H$, where ψ is $\mathbb{C}$-analytic and H is a quasiconformal homeomorphism. Solutions with less regularity may not share these properties [12]. The basic tools to deal with the Beltrami equation are two linear operators, the Cauchy transform $P = \overline{\partial}^{-1}$ and the Beurling transform $S = \overline{\partial}\partial^{-1}$. In Section 3 we recall the basic properties of these operators with some useful preliminary results.

It is not difficult to see, c.f. Section 2, that we can assume $\Omega = \mathbb{D}$, the unit disk of $\mathbb{C}$, and that outside Ω we can set $\sigma \equiv 1$, i.e., $\mu \equiv 0$.

In Section 4 we establish the existence of the geometric optics solution of (1.4) that have the form

$$f_\mu(z,k) = e^{ikz} M_\mu(z,k), \tag{1.5}$$

where

$$M_\mu(z,k) = 1 + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } |z| \to \infty. \tag{1.6}$$

As in the smooth case these solutions obey a $\overline{\partial}$-equation also in the k variable. However, their asymptotics as $|k| \to \infty$ are now more subtle and considerably more difficult to handle.

It turns out that it is instructive to consider the conductivities σ and σ^{-1}, or equivalently the Beltrami coefficients μ and $-\mu$, simultaneously. By defining

$$h_+ = \frac{1}{2}(f_\mu + f_{-\mu}), \quad h_- = \frac{i}{2}(\overline{f_\mu} - \overline{f_{-\mu}}) \tag{1.7}$$

we show in Section 5 that with respect to the variable k, h_+ and h_- satisfy the equations

$$\partial_{\overline{k}} h_+ = \tau_\mu \, \overline{h_-}, \quad \partial_{\overline{k}} h_- = \tau_\mu \, \overline{h_+} \tag{1.8}$$

where the scattering coefficient $\tau_\mu = \tau_\mu(k)$ is defined by

$$\overline{\tau_\mu(k)} = \frac{1}{4\pi i} \int \partial_{\bar z}\big(M_\mu - M_{-\mu}\big)\, d\bar z \wedge dz. \tag{1.9}$$

The remarkable fact in (1.8) is that the coefficient $\tau_\mu(k)$ does not depend on the space variable z; the idea to use such a phenomenon is due to Beals and Coifman [6]. In Section 6 we show that Λ_σ uniquely determines the scattering coefficient $\tau_\mu(k)$ as well as the geometric optics solutions f_μ and $f_{-\mu}$ outside $\mathbb{D}$.

The crucial problem in the proof of Theorem 1 is the behavior of the function $M_\mu(z,k) - 1 = e^{-ikz} f_\mu(z,k) - 1$ with respect to the k-variable. In the case of [19] and [8] the behaviour is roughly like $|k|^{-1}$. In the L^∞-case we cannot expect such a good behavior. Instead, we can show that $M_\mu(z,k)$ grows at most subexponentially in k. This is the key tool to our argument and it takes a considerable effort to prove this. Precisely, we show in Section 7 that

$$f_\mu(z,k) = \exp(ik\varphi(z,k))$$

where φ is a quasiconformal homeomorphism in the z-variable and satisfies the nonlinear Beltrami equation

$$\partial_{\bar z}\varphi = -\frac{\bar k}{k}\mu(z) e_{-k}(\varphi(z))\overline{\partial_z \varphi} \tag{1.10}$$

with the boundary condition

$$\varphi(z) = z + \mathcal{O}\left(\frac{1}{z}\right) \tag{1.11}$$

at infinity. Here the unimodular function e_k is given by

$$e_k(z) = e^{i(kz + \bar k \bar z)}. \tag{1.12}$$

The main result in Section 7 is that the unique solution of (1.10) and (1.11) obeys the property that

$$\varphi(z,k) - z \to 0 \quad \text{as } |k| \to \infty, \tag{1.13}$$

uniformly in z.

Section 8 is devoted to the proof of Theorem 1. Since

$$\mu = \bar\partial f_\mu / \overline{\partial f_\mu}$$

and ∂f for a non-constant quasiregular map f can vanish only in a set of Lebesgue measure zero, we are reduced to determine the function f_μ in the interior of $\mathbb{D}$. As said before, we already know these functions outside of $\mathbb{D}$. The key ingredient to solve this problem is the so-called *transport matrix* that transforms the solutions outside $\mathbb{D}$ to solutions inside. We show that this matrix is uniquely determined by Λ_σ. At this point one may work either with equation (1.1) or equation (1.4). We chose to go back to the conductivity equation since it slightly simplifies the formulas. More precisely, we set

$$u_1 = h_+ - ih_- \text{ and } u_2 = i(h_+ + ih_-). \tag{1.14}$$

Then u_1 and u_2 are complex solutions of the conductivity equations

$$\nabla \cdot \sigma \nabla u_1 = 0 \quad \text{and} \quad \nabla \cdot \frac{1}{\sigma} \nabla u_2 = 0, \tag{1.15}$$

respectively, and of the $\partial_{\overline{k}}$-equation

$$\frac{\partial}{\partial \overline{k}} u_j = -i\, \tau_\mu(k) \overline{u_j}, \quad j = 1, 2, \tag{1.16}$$

with the asymptotics $u_1 = e^{ikz}(1 + \mathcal{O}(1/z))$ and $u_2 = e^{ikz}(i + \mathcal{O}(1/z))$ in the z-variable. Uniqueness of (1.15) with these asymptotics gives that in the smooth case u_1 is exactly the exponentially growing solution of [19].

We then choose a point $z_0 \in \mathbb{C}$, $|z_0| > 1$. It is possible to write for each $z, k \in \mathbb{C}$

$$\begin{aligned} u_1(z,k) &= a_1 u_1(z_0,k) + a_2 u_2(z_0,k) \\ u_2(z,k) &= b_1 u_1(z_0,k) + b_2 u_2(z_0,k) \end{aligned} \tag{1.17}$$

where $a_j = a_j(z, z_0; k)$ and $b_j = b_j(z, z_0; k)$ are real valued. The transport matrix $T^\sigma_{z,z_0}(k)$ is now defined by

$$T^\sigma_{z,z_0}(k) = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}. \tag{1.18}$$

It is an invertible 2×2 real matrix depending on z, z_0 and k. The proof of Theorem 1 is thus reduced to

Theorem 2 *Assume that $\Lambda_\sigma = \Lambda_{\tilde{\sigma}}$ for two L^∞-conductivities σ and $\tilde{\sigma}$. Then for all $z, k \in \mathbb{C}$ and $|z_0| > 1$ the corresponding transport matrices $T^\sigma_{z,z_0}(k)$ and $T^{\tilde{\sigma}}_{z,z_0}(k)$ are equal.*

The idea behind the proof is to use the Beals-Coifman method in an efficient manner and to show that the functions

$$\alpha(k) = a_1(k) + ia_2(k) \text{ and } \beta(k) = b_1(k) + ib_2(k) \tag{1.19}$$

both satisfy with respect to the parameter k the Beltrami equation

$$\partial_{\overline{k}}\alpha = \nu_{z_0}(k)\overline{\partial_k \alpha}. \tag{1.20}$$

Here the coefficient

$$\nu_{z_0}(k) = i\frac{h_-(z_0,k)}{h_+(z_0,k)} \tag{1.21}$$

is determined by the data as proved in Section 6. Moreover it satisfies

$$|\nu_{z_0}(k)| \leq q < 1,$$

where the number q is independent of k (or z). These facts and the subexponential growth of the solutions serve as the key elements for the proof of Theorem 2.

2 Beltrami equation and Hilbert transform

In a general domain Ω we identify $H^{1/2}(\partial\Omega) = H^1(\Omega)/H^1_0(\Omega)$. When $\partial\Omega$ has enough regularity, trace theorems and extension theorems [27] readily yield the standard interpretation of $H^{1/2}(\partial\Omega)$. The Dirichlet condition (1.2) is consequently defined in the Sobolev sense, requiring that $u - \phi \in H^1_0(\Omega)$ for the element $\phi \in H^{1/2}(\partial\Omega)$. Furthermore, $H^{-1/2}(\partial\Omega) = H^{1/2}(\partial\Omega)^*$ and via (1.3) it is then clear that Λ_σ becomes a well-defined and bounded operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$.

In this setup Theorem 1 quickly reduces to the case where the domain Ω is the unit disk. In fact, let Ω be a simply connected domain with $\overline{\Omega} \subset \mathbb{D}$ and let σ and $\widetilde{\sigma}$ be two L^∞-conductivities on Ω with $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$. Continue both conductivities as the constant 1 outside Ω to obtain new L^∞-conductivities σ_0 and $\widetilde{\sigma}_0$. Given $\phi \in H^{1/2}(\partial\mathbb{D})$, let $u_0 \in H^1(\mathbb{D})$ be the solution to the Dirichlet problem $\nabla \cdot \sigma_0 \nabla u_0 = 0$ in $\mathbb{D}$, $u_{0|\partial\mathbb{D}} = \phi$. Assume also that $\widetilde{u} \in H^1(\Omega)$ is the solution to

$$\nabla \cdot \widetilde{\sigma}\nabla\widetilde{u} = 0 \ \text{ in } \Omega, \quad \widetilde{u} - u_0 \in H^1_0(\Omega).$$

Then $\widetilde{u}_0 = \widetilde{u}\chi_\Omega + u_0\chi_{\mathbb{D}\setminus\Omega} \in H^1(\mathbb{D})$ since zero extensions of $H^1_0(\Omega)$ functions remain in H^1. Moreover, an application of the definition (1.3) with the condition $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$ yields that $\widetilde{u}_0$ satisfies

$$\nabla \cdot \widetilde{\sigma}_0 \nabla \widetilde{u}_0 = 0$$

in the weak sense. Since in $\mathbb{D} \setminus \Omega$ we have $u_0 \equiv \widetilde{u}_0$ and $\sigma_0 \equiv \widetilde{\sigma}_0$, we obtain $\Lambda_{\widetilde{\sigma}_0}\phi = \Lambda_{\sigma_0}\phi$, and this holds for all $\phi \in H^{1/2}(\partial\mathbb{D})$. Thus if Theorem 1 holds for $\mathbb{D}$ we get $\widetilde{\sigma}_0 = \sigma_0$ and especially that $\widetilde{\sigma} = \sigma$.

From this on we assume that $\Omega = \mathbb{D}$, the unit disc in $\mathbb{C}$.

Let us then consider the complex analytic interpretation of (1.1). We will use the notations $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ and $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$; when clarity requires we may write $\overline{\partial} = \partial_{\overline{z}}$ or $\partial = \partial_z$. For derivatives with respect to the parameter k we always use the notations ∂_k and $\partial_{\overline{k}}$.

We start with a simple lemma

Lemma 2.1 *Assume $u \in H^1(\mathbb{D})$ is real valued and satisfies the conductivity equation* (1.1). *Then there exists a function $v \in H^1(\mathbb{D})$, unique up to a constant, such that $f = u + iv$ satisfies the $\mathbb{R}$-linear Beltrami equation*

$$\overline{\partial} f = \mu \overline{\partial f}, \tag{2.1}$$

where $\mu = (1-\sigma)/(1+\sigma)$.

Conversely, if $f \in H^1(\mathbb{D})$ satisfies (2.1) *with a $\mathbb{R}$-valued μ, then $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ satisfy*

$$\nabla \cdot \sigma \nabla u = 0 \quad \textit{and} \quad \nabla \cdot \frac{1}{\sigma} \nabla v = 0, \tag{2.2}$$

respectively, where $\sigma = (1-\mu)/(1+\mu)$.

Proof: Denote by w the vectorfield

$$w = (-\sigma \partial_2 u, \sigma \partial_1 u)$$

where $\partial_1 = \partial/\partial x$ and $\partial_2 = \partial/\partial y$ for $z = x + iy \in \mathbb{C}$. Then by (1.1) the integrability condition $\partial_2 w_1 = \partial_1 w_2$ holds for the distributional derivatives. Therefore there exists $v \in H^1(\mathbb{D})$, unique up to a constant, such that

$$\partial_1 v = -\sigma \partial_2 u \tag{2.3}$$

$$\partial_2 v = \sigma \partial_1 u. \tag{2.4}$$

It is a simple calculation to see that this is equivalent to (2.1). $\square$

We want to stress that every solution of (2.1) is also a solution of the standard $\mathbb{C}$-linear Beltrami equation

$$\overline{\partial} f = \tilde{\mu} \partial f \tag{2.5}$$

but with a different, $\mathbb{C}$-valued $\tilde{\mu}$ having though the same modulus as the old one. However, the uniqueness properties of (2.1) and (2.5) are quite different (cf. [28], [5]). Note also that the conditions for σ given in Theorem 1 imply the existence of a constant $0 \le \kappa < 1$ such that

$$|\mu(z)| \le \kappa$$

holds for almost every $z \in \mathbb{C}$.

Since the function v in Lemma 2.1 is defined only up to a constant we will normalize it by assuming

$$\int_{\partial\mathbb{D}} v \, ds = 0. \tag{2.6}$$

This way we obtain a unique map $\mathcal{H}_\mu : H^{1/2}(\partial\mathbb{D}) \to H^{1/2}(\partial\mathbb{D})$ by setting

$$\mathcal{H}_\mu : u\big|_{\partial\mathbb{D}} \mapsto v\big|_{\partial\mathbb{D}}. \tag{2.7}$$

The function v satisfying (2.3), (2.4) and (2.6) is called the *σ-harmonic conjugate* of u and $\mathcal{H}_\mu$ the *Hilbert transform* corresponding to equation (2.1).

Since v is the real part of the function $g = -if$ satisfying $\overline{\partial} g = -\mu \overline{\partial g}$, we have

$$\mathcal{H}_\mu \circ \mathcal{H}_{-\mu} u = \mathcal{H}_{-\mu} \circ \mathcal{H}_\mu u = -u + \fint_{\partial\mathbb{D}} u \, ds \tag{2.8}$$

where

$$\fint_{\partial\mathbb{D}} u \, ds = \frac{1}{2\pi} \int_{\partial\mathbb{D}} u \, ds.$$

So far we have only defined $\mathcal{H}_\mu(u)$ for real-valued u. By setting

$$\mathcal{H}_\mu(iu) = i\mathcal{H}_{-\mu}(u)$$

we have extended the definition of $\mathcal{H}_\mu(g)$ $\mathbb{R}$-linearly to all $\mathbb{C}$-valued $g \in H^{1/2}(\partial\mathbb{D})$. We also define $Q_\mu : H^{1/2}(\partial\mathbb{D}) \to H^{1/2}(\partial\mathbb{D})$ by

$$Q_\mu = \frac{1}{2}\left(I - i\mathcal{H}_\mu\right). \tag{2.9}$$

Then $g \mapsto Q_\mu(g) + \frac{1}{2}\fint_{\partial\mathbb{D}} g \, ds$ is a projection in $H^{1/2}(\partial\mathbb{D})$. In fact

$$Q_\mu^2(g) = Q_\mu(g) - \frac{1}{4}\fint_{\partial\mathbb{D}} g \, ds. \tag{2.10}$$

The proof of the following lemma is straight forward.

Lemma 2.2 *If $g \in H^{1/2}(\partial\mathbb{D})$, the following conditions are equivalent,*

a) $g = f\big|_{\partial\mathbb{D}}$, where $f \in H^1(\mathbb{D})$ and satisfies (2.1).

b) $Q_\mu(g)$ is a constant.

We close this section by

Proposition 2.3 *The Dirichlet to Neumann map Λ_σ uniquely determines $\mathcal{H}_\mu$, $\mathcal{H}_{-\mu}$ and $\Lambda_{\sigma^{-1}}$.*

Proof: Choose the counter clock-wise orientation for $\partial\mathbb{D}$ and denote by ∂_T the tangential (distributional) derivative on $\partial\mathbb{D}$ corresponding to this orientation. We will show that

$$\partial_T \mathcal{H}_\mu(u) = \Lambda_\sigma(u) \tag{2.11}$$

holds in the weak sense. This will be enough since by (2.8) $\mathcal{H}_\mu$ uniquely determines $\mathcal{H}_{-\mu}$. Note also that $-\mu = (1-\sigma^{-1})/(1+\sigma^{-1})$ and so $\Lambda_{\sigma^{-1}}(u) = \partial_T \mathcal{H}_{-\mu}(u)$.

By the definition of Λ_σ we have

$$\int_{\partial\mathbb{D}} \varphi \Lambda_\sigma u \, ds = \int_{\mathbb{D}} \nabla\varphi \cdot \sigma \nabla u \, dm, \quad \varphi \in C^\infty(\overline{\mathbb{D}}).$$

Thus, by (2.3), (2.4) and integration by parts, we get

$$\begin{aligned} \int_{\partial\mathbb{D}} \varphi \Lambda_\sigma u &= \int_{\mathbb{D}} \left(\partial_1\varphi\partial_2 v - \partial_2\varphi\partial_1 v\right) \, dm \\ &= -\int_{\partial\mathbb{D}} v \partial_T \varphi \, ds \end{aligned}$$

and (2.11) follows. $\square$

3 Beltrami operators

The Beltrami differential equation (1.4) and its solutions are effectively governed and controlled by two basic linear operators, the Cauchy transform and the Beurling transform. Any analysis of (1.4) requires basic facts of these operators. We briefly recall those in this section.

The Cauchy transform

$$Pg(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{g(\omega)}{\omega - z} \, dm(\omega) \tag{3.1}$$

acts as the inverse operator to $\overline{\partial}$; $P\overline{\partial}g = \overline{\partial}Pg = g$, for $g \in \mathbb{C}_0^\infty(\mathbb{C})$. We recall some mapping properties of P in appropriate Lebesgue, Sobolev and Lipschitz spaces. Below we denote

$$L^p(\Omega) = \left\{ g \in L^p(\mathbb{C}) \,\Big|\, g\big|_{\mathbb{C}\setminus\Omega} \equiv 0 \right\}.$$

Proposition 3.1 *Let $\Omega \subset \mathbb{C}$ be a bounded domain and let $1 < q < 2$ and $2 < p < \infty$. Then*

(i) $P : L^p(\mathbb{C}) \to \mathrm{Lip}_\alpha(\mathbb{C})$, *where* $\alpha = 1 - 2/p$;

(ii) $P : L^p(\Omega) \to W^{1,p}(\mathbb{C})$ *is bounded;*

(iii) $P : L^p(\Omega) \to L^p(\mathbb{C})$ *is compact;*

(iv) $P : L^p(\mathbb{C}) \cap L^q(\mathbb{C}) \to C_0(\mathbb{C})$ *is bounded, where C_0 is the closure of C_0^∞ in L^∞.*

For proof of Proposition 3.1 we refer to [28], but see also [19].

The Beurling transform is formally determined by $Sg = \partial P g$ and more precisely as a principal value integral

$$Sg(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{g(\omega)}{(\omega - z)^2}\, dm(\omega). \tag{3.2}$$

It is a Calderón-Zygmund operator with a holomorphic kernel. Since S is a Fourier multiplier operator with symbol

$$m(\xi) = -\frac{\xi}{\overline{\xi}}, \quad \xi = \xi_1 + i\xi_2 \tag{3.3}$$

we see, in particular, that S transforms the $\overline{\partial}$-derivatives to ∂-derivatives,

$$S(\overline{\partial}\varphi) = \partial\varphi, \text{ for } \varphi \in S'(\mathbb{C}). \tag{3.4}$$

Moreover, we have

$$S = -R_1^2 + 2iR_1R_2 - R_2^2,$$

where R_i's denote the Riesz-transforms. Also, it follows (cf. [25]) that

$$S : L^p(\mathbb{C}) \to L^p(\mathbb{C}), \quad 1 < p < \infty, \tag{3.5}$$

and $\lim_{p\to 2} \|S\|_{L^p \to L^p} = \|S\|_{L^2 \to L^2} = 1$.

It is because of (3.4) that the mapping properties of the Beurling transform control the solutions to the Beltrami equation (1.4). For instance, if supp(μ) is compact as it is in our case, finding a solution to (1.4) with asymptotics

$$f(z) = \lambda z + \mathcal{O}\left(\frac{1}{z}\right), \quad |z| \to \infty \tag{3.6}$$

is equivalent to solving

$$g = \mu\overline{Sg} + \overline{\lambda}\mu$$

and setting

$$f(z) = \lambda z + Pg(z),$$

where P is the Cauchy transform. Therefore, if we denote by $\overline{S}$ the $\mathbb{R}$-linear operator $\overline{S}(g) = \overline{S(g)}$, we need to understand the mapping properties of P and the invertibility of the operator $I - \mu\overline{S}$ in appropriate L^p-spaces in order to determine to which L^p-class the gradient of the solution to (1.4) belongs.

Recently, Astala, Iwaniec and Saksman established through the fundamental theory of quasiconformal mappings the precise L^p-invertibility range of these operators.

Theorem 3.2 *Let μ_1 and μ_2 be two $\mathbb{C}$-valued measurable functions such that*

$$|\mu_1(z)| + |\mu_2(z)| \leq \kappa \tag{3.7}$$

holds for almost every $z \in \mathbb{C}$ with a constant $0 \leq \kappa < 1$. Suppose that $1 + \kappa < p < 1 + 1/\kappa$. Then the Beltrami operator

$$B = I - \mu_1 S - \mu_2 \overline{S} \tag{3.8}$$

is bounded and invertible in $L^p(\mathbb{C})$, with norms of B and B^{-1} bounded by constants depending only on κ and p.

Moreover, the bound in p is sharp; for each $p \leq 1 + \kappa$ and for each $p \geq 1 + 1/\kappa$ there are μ_1 and μ_2 as above such that B is not invertible in $L^p(\mathbb{C})$.

For the proof see [4]. Since $\|S\|_{L^2 \to L^2} = 1$, all operators in (3.8) are invertible in $L^2(\mathbb{C})$ as long as $\kappa < 1$. Thus Theorem 3.2 determines the interval around the exponent $p = 2$ where the invertibility remains true. Note that it is a famous open problem [14] whether it holds

$$\|S\|_{L^p \to L^p} = \max\left\{p - 1, \frac{1}{p-1}\right\}.$$

If this turns out to be the case, then

$$\|\mu S\|_{L^p \to L^p} \leq \|\mu\|_{L^\infty} \|S\|_{L^p \to L^p} < 1$$

whenever $p < 1 + 1/\|\mu\|_{L^\infty}$. This would then give an alternative proof of Theorem 3.2.

Theorem 3.2 has also nonlinear counterparts [4] yielding solutions to nonlinear uniformly elliptic PDE's; here see also [13], [5]. On the other hand, in two dimensions the uniqueness of solutions to general nonlinear elliptic systems is typically reduced to the study of the pseudoanalytic functions of Bers (cf [7], [28]). In the sequel we will need the following version of this principle.

Proposition 3.3 *Let $F \in W^{1,p}_{loc}(\mathbb{C})$ and $\gamma \in L^p_{loc}(\mathbb{C})$ for some $p > 2$. Suppose that for some constant $0 \leq \kappa < 1$,*

$$\left|\overline{\partial} F(z)\right| \leq \kappa \left|\partial F(z)\right| + \gamma(z) \left|F(z)\right| \tag{3.9}$$

holds for almost every $z \in \mathbb{C}$. Then we have

a) If $F(z) \to 0$ as $|z| \to \infty$ and γ has a compact support then

$$F(z) \equiv 0.$$

b) If for large z, $F(z) = \lambda z + \varepsilon(z) z$ where the constant $\lambda \neq 0$ and $\varepsilon(z) \to 0$ as $|z| \to \infty$, then $F(z) = 0$ exactly in one point $z = z_0 \in \mathbb{C}$.

Proof: The result a) is essentially from [28]. For the convenience of the reader we will outline a proof for it after first proving b):

The continuity of $F(z) = \lambda z + \varepsilon(z) z$ and an application of the degree theory [29] or an appropriate homotopy argument show that F is surjective and consequently there exists at least one point $z_0 \in \mathbb{C}$ such that $F(z_0) = 0$.

To show that F can not have more zeros, let $z_1 \in \mathbb{C}$ and choose a large disk $B = B(0, R)$ containing both z and z_0. If R is so large that $\varepsilon(z) < \lambda/2$ for $|z| = R$, then $F\big|_{\{|z|=R\}}$ is homotopic to identity relative to $\mathbb{C} \setminus \{0\}$. Next we express (3.9) in the form

$$\overline{\partial} F = \nu(z) \partial F + A(z) F \tag{3.10}$$

where $|\nu(z)| \leq \kappa < 1$ and $|A(z)| \leq \gamma(z)$ for almost every $z \in \mathbb{C}$. Now $A\chi_B \in L^r(\mathbb{C})$ for all $1 \leq r \leq p' = \min\{p, 1 + 1/\kappa\}$ and we obtain from Theorem 3.2 that $(I - \nu S)^{-1}(A\chi_B) \in L^r$ for all $p'/(p'-1) < r < p'$.

Next we define, cf. [28], $\eta = P\big((I - \nu S)^{-1}(A\chi_B)\big)$. By Proposition 3.1, $\eta \in C_0(\mathbb{C})$ and clearly we have

$$\overline{\partial}\eta - \nu\partial\eta = A(z), \quad z \in B. \tag{3.11}$$

By a differentation we see that the function

$$g = e^{-\eta}F \tag{3.12}$$

satisfies

$$\overline{\partial}g - \nu\partial g = 0, \quad z \in B. \tag{3.13}$$

Since $\eta \in W^{1,r}(\mathbb{C})$ by Proposition 3.1, also $g \in W^{1,r}_{loc}(\mathbb{C})$ and thus g is quasiregular in B. As such, see e.g. [12] Theorem 1.1.1, $g = h \circ \psi$, where $\psi : B \to B$ is a quasiconformal homeomorphism and h holomorphic, both continuous up to a boundary.

Since η is continuous, (3.12) shows that $g\big|_{|z|=R}$ is homotopic to identity relative to $\mathbb{C} \setminus \{0\}$, and so is the holomorphic function $h\big|_{\{|z|=R\}}$. Therefore, h has by the principle of the argument ([22], Theorems V.7.1 and VIII.3.5) precisely one zero in $B = B(0,R)$. As already $h(\psi(z_0)) = e^{-\eta(z_0)}F(z_0) = 0$, there can be no further zeros for F either. This finishes the proof of b).

For the claim a) the condition $F(z) = \varepsilon(z)z$ is too weak to guarantee $F \equiv 0$ in general. But if γ has a compact support we may choose $\operatorname{supp}\gamma \subset B(0,R)$ and thus the function η solves (3.11) for all $z \in \mathbb{C}$. Consequently (3.13) holds in the whole plane and g in (3.11) is quasiregular in $\mathbb{C}$. But since F and η are bounded, also g is bounded and thus constant by Liouville's theorem. Now (3.12) gives

$$F = C_1 e^{\eta}, \quad \eta \in C_0(\mathbb{C}). \tag{3.14}$$

With the assumption $F(z) \to 0$ as $|z| \to \infty$ we then obtain $C_1 = 0$. □

We also have the following useful

Corollary 3.4 *Suppose $F \in W^{1,p}_{loc}(\mathbb{C}) \cap L^\infty(\mathbb{C})$, $p > 2$, $0 \leq \kappa < 1$ and that $\gamma \in L^p(\mathbb{C})$ has compact support. If*

$$\left|\overline{\partial}F(z)\right| \leq \kappa\,|\partial F(z)| + \gamma(z)\,|F(z)|\,, \quad z \in \mathbb{C},$$

then

$$F(z) = C_1 e^{\eta}$$

where C_1 is constant and $\eta \in C_0(\mathbb{C})$.

Proof: This is a reformulation of (3.14) from above. □

4 Complex geometric optics solutions

In this section we establish the existence of the solution to (1.4) of the form

$$f_\mu(z,k) = e^{ikz} M_\mu(z,k) \tag{4.1}$$

where

$$M_\mu(z,k) - 1 = \mathcal{O}\left(\frac{1}{z}\right) \text{ as } |z| \to \infty. \tag{4.2}$$

Moreover, it is demonstrated that

$$\mathrm{Re}\left(\frac{M_\mu(z,k)}{M_{-\mu}(z,k)}\right) > 0, \text{ for all } z,k \in \mathbb{C}. \tag{4.3}$$

The importance of (4.3) lies e.g. in the fact that

$$\nu_z(k) = -e_{-k}(z)\frac{\overline{M_\mu(z,k)} - \overline{M_{-\mu}(z,k)}}{M_\mu(z,k) + M_{-\mu}(z,k)} \tag{4.4}$$

appears as the coefficient in a Beltrami equation in the k-variable in Section 8. The result (4.3) clearly implies

$$|\nu_z(k)| < 1 \text{ for all } z,k \in \mathbb{C}. \tag{4.5}$$

We start with

Proposition 4.1 *Assume that $2 < p < 1 + 1/\kappa$, that $\alpha \in L^\infty(\mathbb{C})$ with $\mathrm{supp}(\alpha) \subset \mathbb{D}$ and that $|\nu(z)| \le \kappa\chi_{\mathbb{D}}(z)$ for almost every $z \in \mathbb{D}$. Define the operator $K : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ by*

$$Kg = P\left(I - \nu\overline{S}\right)^{-1}(\alpha\overline{g}).$$

Then $K : L^p(\mathbb{C}) \to W^{1,p}(\mathbb{C})$ and $I - K$ is invertible in $L^p(\mathbb{C})$.

Proof: First we note that by Theorem 3.2, $I - \nu\overline{S}$ is invertible in L^p and by Proposition 3.1 (iii) the operator $K : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ is well defined and compact. Note that $\mathrm{supp}\left((1-\nu\overline{S})^{-1}\alpha\overline{g}\right) \subset \mathbb{D}$. Thus, by Fredholm's alternative, we need to show that $I - K$ is injective. So suppose that $g \in L^p(\mathbb{C})$ satisfies

$$g = P\left(\left(I - \nu\overline{S}\right)^{-1}(\alpha\overline{g})\right). \tag{4.6}$$

By Proposition 3.1 (ii) $g \in W^{1,p}$ and thus by (4.6)

$$\overline{\partial} g = \left(I - \nu \overline{S}\right)^{-1} (\alpha \overline{g})$$

or equivalently

$$\overline{\partial} g - \nu \overline{\partial g} = \alpha \overline{g}. \tag{4.7}$$

Finally, from (4.7) it follows that g is analytic outside the unit disk. This together with $g \in L^p(\mathbb{C})$ implies

$$g(z) = \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \to \infty.$$

Thus the assumptions of Proposition 3.3 a) are fulfilled and we must have $g \equiv 0$. $\square$

It is not difficult to find examples showing that Proposition 4.1 fails for $p \leq 2$ and for $p \geq 1 + 1/\kappa$.

We are now ready to establish the existence of the complex geometrical optics solutions to (1.4).

Theorem 4.2 *For each $k \in \mathbb{C}$ and for each $2 < p < 1 + 1/\kappa$ the equation* (1.4) *admits a unique solution $f \in W^{1,p}_{loc}(\mathbb{C})$ of the form* (1.5) *such that the asymptotic formula* (1.6) *holds true.*

In particular, $f(z, 0) \equiv 1$.

Proof: If we write

$$f_\mu(z, k) = e^{ikz} M_\mu(z, k) = e^{ikz}(1 + \omega(z))$$

and plug this to (1.4) we obtain

$$\overline{\partial} \omega - e_{-k} \mu \overline{\partial \omega} = \alpha \overline{\omega} + \alpha \tag{4.8}$$

where e_{-k} is defined in (1.12) and

$$\alpha(z) = -i\overline{k} e_{-k}(z) \mu(z). \tag{4.9}$$

Hence

$$\overline{\partial} \omega = \left(I - e_{-k} \mu \overline{S}\right)^{-1} (\alpha \overline{\omega} + \alpha). \tag{4.10}$$

If now K is defined as in Proposition 4.1 with $\nu = e_{-k} \mu$ we get

$$\omega - K\omega = K(\chi_{\mathbb{D}}) \in L^p(\mathbb{C}). \tag{4.11}$$

Since by Proposition 4.1 the operator $I - K$ is invertible in $L^p(\mathbb{C})$, and by (4.8) ω is analytic in $\mathbb{C} \setminus \overline{\mathbb{D}}$ the claims follow by (4.10) and (4.11). $\square$

Next, let $f_\mu(z,k) = e^{ikz} M_\mu(z,k)$ and $f_{-\mu}(z,k) = e^{ikz} M_{-\mu}(z,k)$ be the solutions of Theorem 4.2 corresponding to conductivities σ and σ^{-1}, respectively.

Proposition 4.3 *For all $k, z \in \mathbb{C}$ we have*

$$\operatorname{Re}\left(\frac{M_\mu(z,k)}{M_{-\mu}(z,k)}\right) > 0. \tag{4.12}$$

Proof: Firstly, note that (1.4) implies for $M_{\pm\mu}$

$$\overline{\partial} M_{\pm\mu} \mp \mu e_{-k} \overline{\partial M_{\pm\mu}} = \mp i \bar{k} \mu e_{-k} \overline{M_{\pm\mu}}. \tag{4.13}$$

Thus we may apply Corollary 3.4 to get

$$M_{\pm\mu}(z) = \exp(\eta_\pm(z)) \neq 0 \tag{4.14}$$

and consequently $M_\mu / M_{-\mu}$ is well defined. Secondly, if (4.12) is not true the continuity of $M_{\pm\mu}$ and $\lim_{z\to\infty} M_{\pm\mu}(z,k) = 1$ imply the existence of $z_0 \in \mathbb{C}$ such that

$$M_\mu(z_0,k) = it M_{-\mu}(z_0,k)$$

for some $t \in \mathbb{R} \setminus \{0\}$. But then $g = M_\mu - it M_{-\mu}$ satisfies

$$\overline{\partial} g = \mu \overline{\partial(e_k g)},$$

$$g(z) = 1 - it + \mathcal{O}\left(\frac{1}{z}\right), \text{ as } z \to \infty.$$

According to Corollary 3.4 this implies

$$g(z) = (1 - it)\exp(\eta(z)) \neq 0,$$

contradicting the assumption $g(z_0) = 0$. $\square$

5 $\partial_{\overline{k}}$-equations

We will prove in this section the $\partial_{\overline{k}}$-equation (1.8) for the complex geometrical optics solutions. We begin by writing (1.4)-(1.6) in the form

$$\overline{\partial} M_\mu = \mu \overline{\partial(e_k M_\mu)}, \quad M_\mu - 1 \in W^{1,p}(\mathbb{C}). \tag{5.1}$$

By introducing a $\mathbb{R}$-linear operator L_μ,

$$L_\mu g = P\left(\mu \overline{\partial}(e_{-k}\overline{g})\right)$$

we see that (5.1) is equivalent to

$$(I - L_\mu)M_\mu = 1. \tag{5.2}$$

The following refinement of Proposition 4.1 will serve as the main tool in proving (1.8). Below we will study functions of the form $f =$ constant $+ f_0$, where $f_0 \in W^{1,p}(\mathbb{C})$. The corresponding Banach space is denoted by $W^{1,p}(\mathbb{C}) \oplus \mathbb{C}$.

Theorem 5.1 *Assume that $k \in \mathbb{C}$ and $\mu \in L^\infty_{comp}(\mathbb{C})$ with $\|\mu\|_\infty \leq \kappa < 1$. Then for $2 < p < 1 + 1/\kappa$ the operator*

$$I - L_\mu : W^{1,p}(\mathbb{C}) \oplus \mathbb{C} \to W^{1,p}(\mathbb{C}) \oplus \mathbb{C}$$

is bounded and invertible.

Proof: We write $L_\mu(g)$ as

$$L_\mu(g) = P\left(\mu e_{-k}\overline{\partial g} - i\overline{k}\mu e_{-k}\overline{g}\right) \tag{5.3}$$

Proposition 3.1 (ii) now yields that

$$L_\mu : W^{1,p}(\mathbb{C}) \oplus \mathbb{C} \to W^{1,p}(\mathbb{C}) \tag{5.4}$$

is bounded. Thus we need to show that $I - L_\mu$ is bijective on $W^{1,p}(\mathbb{C}) \oplus \mathbb{C}$. To this end assume

$$(I - L_\mu)(g + C_0) = h + C_1 \tag{5.5}$$

for $g, h \in W^{1,p}(\mathbb{C})$ and for constants C_0, C_1. This yields

$$C_0 - C_1 = g - h - L_\mu(g + C_0)$$

which by (5.4) gives $C_0 = C_1$. By differentiating, rearranging and by using the operator K_μ from Proposition 4.1 with $\alpha = -i\overline{k}\mu e_{-k}$ and $\nu = \mu e_{-k}$ we see that (5.5) is equivalent to

$$g - K_\mu(g) = K_\mu(C_0\chi_{\mathbb{D}}) + P\left[(1 - \mu e_{-k}\overline{S})^{-1}\overline{\partial} h\right]. \tag{5.6}$$

Since the right hand side belongs to $L^p(\mathbb{C})$ for each $h \in W^{1,p}(\mathbb{C})$ this equation has a unique solution $g \in W^{1,p}(\mathbb{C})$ by Proposition 4.1. □

As an immediate corollary we get the following important

Corollary 5.2 *The operator $I - L_\mu^2$ is invertible on $W^{1,p}(\mathbb{C}) \oplus \mathbb{C}$.*

Proof: Since $L_\mu = -L_{-\mu}$ we have $I - L_\mu^2 = (I - L_\mu)(I - L_{-\mu})$. $\square$

Next, we make use of the differentiability properties of the operator L_μ. For later purposes it will be better to work with L_μ^2 which can be written in the following convenient form

$$L_\mu^2 g = P\left(\mu\overline{\partial}(\partial + ik)^{-1}\mu(\partial + ik)g\right) \tag{5.7}$$

where the operator $(\partial + ik)^{-1}$ is defined by

$$(\partial + ik)^{-1} g = e_{-k}\partial^{-1}(e_k g). \tag{5.8}$$

Note that many mapping properties of this operator follow from Proposition 3.1. Moreover, we have

Lemma 5.3 *Let $p > 2$. Then the operator valued map $k \mapsto (\partial + ik)^{-1}$ is continuously differentiable in $\mathbb{C}$, in the uniform operator topology: $L^p(\mathbb{D}) \to W^{1,p}_{loc}(\mathbb{C})$.*

Proof: The lemma is a straightforward reformulation of [19], Lemma 2.2, where slightly different function spaces were used. Note that $W^{1,p}_{loc}(\mathbb{C})$ has the topology given by the seminorm $\|f\|_n = \|f\|_{W^{1,p}(B(0,n))}$, $n \in \mathbb{N}$. $\square$

Combining Lemma 5.3 with (5.7) shows that $k \to L_\mu^2$ is a C^1- family of operators $L_\mu^2 : W^{1,p}(\mathbb{C}) \oplus \mathbb{C} \to W^{1,p}(\mathbb{C}) \oplus \mathbb{C}$ in the uniform operator topology. If we iterate the equation (5.2) once we get

$$M_\mu = 1 + P(\mu\overline{\partial}e_{-k}) + L_\mu^2(M_\mu). \tag{5.9}$$

Therefore the above lemma shows that $k \mapsto M_\mu(z,k)$ is a continuously differentiable family of functions in $W^{1,p}(\mathbb{C}) \oplus \mathbb{C}$, $p > 2$. In particular, for each fixed $z \in \mathbb{C}$, $M_\mu(z,k)$ is continuously differentiable in k. An alternative way to see this is to note that $k \mapsto L_\mu$ is smooth in the operator norm topology of $\mathcal{L}\left(W^{1,p}(\mathbb{C}) \oplus \mathbb{C}\right)$ and then use Theorem 5.1. This gives by (5.2) that for fixed z the map $k \mapsto M_\mu(z,k)$ is, indeed, C^∞-smooth.

Furthermore, with respect to the first variable $M_\mu(z,k)$ is complex analytic in $\mathbb{C} \setminus \overline{\mathbb{D}}$ by (5.1), with development

$$M_\mu(z,k) = 1 + \sum_{n=1}^{\infty} b_n(k) z^{-n}, \text{ for } |z| > 1. \tag{5.10}$$

We define the *scattering amplitude* corresponding to M_μ to be

$$t_\mu(k) = \overline{b_1(k)}. \tag{5.11}$$

Equation (5.1) implies, as μ is real valued, that

$$t_\mu(k) = \frac{1}{\pi} \int\limits_{\mathbb{D}} \mu \partial(e_k M_\mu)\, dm. \tag{5.12}$$

Beals and Coifman [6] introduced the idea of studying the $\overline{k}$-dependence of operators associated to complex geometric optics solutions. We will use the Beals-Coifman principle in the following form:

Lemma 5.4 *Suppose $g \in W^{1,p}(\mathbb{C}) \oplus \mathbb{C}$ is fixed. Then*

$$\partial_{\overline{k}} \left(e_{-k} \partial^{-1} \mu \partial e_k g \right) = -i t_\mu(g;k) e_{-k} \tag{5.13}$$

where

$$t_\mu(g;k) = \frac{1}{\pi} \int\limits_{\mathbb{C}} \mu \partial(e_k g)\, dm. \tag{5.14}$$

Proof: For $f \in L^p_{comp}(\mathbb{C})$ we have

$$(e_{-k} \partial^{-1} e_k f)(z) = -\frac{1}{\pi} \int\limits_{\mathbb{C}} \frac{e_k(\xi - z)}{\overline{\xi} - \overline{z}} f(\xi)\, dm(\xi)$$

Using this representation [19], Lemma 2.2, shows that

$$\partial_{\overline{k}} (e_{-k} \partial^{-1} e_k f)(z) = \partial_{\overline{k}} \big((\partial + ik)^{-1} f \big)(z) = -i \widehat{f}(k) e_{-k}(z) \tag{5.15}$$

where

$$\widehat{f}(k) = \frac{1}{\pi} \int\limits_{\mathbb{C}} e_k(\xi) f(\xi)\, dm(\xi). \tag{5.16}$$

By rewriting the left hand side of (5.13) in the form $\partial_{\overline{k}} \big((\partial + ik)^{-1} \mu (\partial + ik) g \big)$ we see that the claim follows from (5.15). $\square$

To get rid of the second term on the right hand side of (5.9) we introduce

$$F_+ = \frac{1}{2} \left(M_\mu + M_{-\mu} \right), \tag{5.17}$$

$$F_- = \frac{i e_{-k}}{2} \left(\overline{M_\mu} - \overline{M_{-\mu}} \right). \tag{5.18}$$

In particular, (5.9) gives

$$F_+ = 1 + L_\mu^2 F_+. \tag{5.19}$$

From Lemma 5.4 and (5.7) one has $\partial_{\overline{k}} L_\mu^2(g) = -it_\mu(g;k)P(\mu\overline{\partial}e_{-k})$ for every $g \in W^{1,p}(\mathbb{C}) \oplus \mathbb{C}$. Hence a differentiation of (5.19) yields

$$(I - L_\mu^2)\,(\partial_{\overline{k}} F_+) = -i\tau_\mu(k)P(\mu\overline{\partial}e_{-k}) \tag{5.20}$$

where the scattering coefficient $\tau_\mu(k)$ is

$$\tau_\mu(k) \equiv t_\mu(F_+;k) = \frac{1}{2}\left(t_\mu(k) - t_{-\mu}(k)\right). \tag{5.21}$$

Note that this is consistent with (1.9).

One way to identify $\partial_{\overline{k}} F_+$ is by readily observing that the unique solution of (5.20) has also other realizations. Namely, if one subtracts the equation (5.9) applied to M_μ from the same equation applied to $M_{-\mu}$, one obtains after using $L_\mu^2 = L_{-\mu}^2$ that

$$(I - L_\mu^2)(e_{-k}\overline{F_-}) = -iP(\mu\overline{\partial}e_{-k}). \tag{5.22}$$

Thus by Corollary 5.2, (5.20) and (5.22) we have proven the first part of

Theorem 5.5 *For each fixed $z \in \mathbb{C}$, the functions $k \mapsto F_\pm(z,k)$ are continuously differentiable with*

a) $\partial_{\overline{k}} F_+(z,k) = \tau_\mu(k)e_{-k}(z)\overline{F_-(z,k)}$,

b) $\partial_{\overline{k}} F_-(z,k) = \tau_\mu(k)e_{-k}(z)\overline{F_+(z,k)}$.

Proof: The differentiability is clear since $M_{\pm\mu}(z,k)$ are continuously differentiable in k. Hence we are left proving b). We start by adding and subtracting (5.1) for M_μ and $M_{-\mu}$ to arrive at the equations

$$F_+ = 1 - i\overline{\partial}^{-1}\mu\overline{\partial}F_-, \tag{5.23}$$

$$F_- = ie_{-k}\partial^{-1}\mu\partial e_k F_+. \tag{5.24}$$

By differentiating the second equation with respect to $\overline{k}$ and by applying Lemma 5.4 we get

$$\partial_{\overline{k}} F_- = \tau_\mu(k)e_{-k} + ie_{-k}\partial^{-1}\mu\partial(e_k\partial_{\overline{k}} F_+). \tag{5.25}$$

Combining this with part a) we have

$$\partial_{\overline{k}} F_- = \tau_\mu(k)e_{-k}(1 + i\partial^{-1}\mu\partial\overline{F_-}). \tag{5.26}$$

This together with (5.23) yields b). $\square$

We close this section by returning to the functions

$$h_+ = \frac{1}{2}(f_\mu + f_{-\mu}) = e^{ikz} F_+ \tag{5.27}$$

and

$$h_- = \frac{i}{2}(\overline{f_\mu} - \overline{f_{-\mu}}) = e^{ikz} F_-. \tag{5.28}$$

These expressions with Theorem 5.5 give immediately the identities (1.8). Note that by Theorem 5.5, $k \mapsto h_\pm(z,k)$ is C^1 in $\mathbb{C}$, for each fixed z.

6 From Λ_σ to τ

We next prove that the Dirichlet to Neumann operator Λ_σ uniquely determines $f_\mu(z)$ and $f_{-\mu}(z)$ at the points z that lie outside $\mathbb{D}$ and moreover that Λ_σ determines $\tau_\mu(k)$ for all $k \in \mathbb{C}$.

Proposition 6.1 *If σ and $\widetilde{\sigma}$ are two conductivities satisfying the assumptions of Theorem 1, then if μ and $\widetilde{\mu}$ are the corresponding Beltrami coefficients, we have*

$$f_\mu(z) = f_{\widetilde{\mu}}(z) \text{ and } f_{-\mu}(z) = f_{-\widetilde{\mu}}(z) \tag{6.1}$$

for all $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$.

Proof: We assume $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$ which by Proposition 2.3 implies that $\mathcal{H}_\mu = \mathcal{H}_{\widetilde{\mu}}$. Since Λ_σ by the same proposition determines $\Lambda_{\sigma^{-1}}$ it is enough to prove the first claim of (6.1).

From (2.9) we see firstly that the projections $Q_\mu = Q_{\widetilde{\mu}}$ and thus by Lemma 2.2

$$Q_\mu(f - \widetilde{f}) = \text{ constant}$$

where we have written $f = f_\mu\big|_{\partial\mathbb{D}}$ and $\widetilde{f} = f_{\widetilde{\mu}}\big|_{\partial\mathbb{D}}$. By using Lemma 2.2 again we see that there exists a function $G \in H^1(\mathbb{D})$ such that G satisfies (2.1) in $\mathbb{D}$ and

$$G\big|_{\partial\mathbb{D}} = f - \widetilde{f}.$$

Define then G outside $\mathbb{D}$ by

$$G(z) = f_\mu(z) - f_{\widetilde{\mu}}(z), \quad |z| \geq 1,$$

to get a global solution to

$$\overline{\partial} G(z) - \mu(z)\overline{\partial G}(z) = 0, \quad z \in \mathbb{C}. \tag{6.2}$$

Since G is a H^1_{loc}-solution to (6.2), the general smoothness properties of quasiregular mappings [3] give $G \in W^{1,p}_{loc}(\mathbb{C})$ for all $2 \leq p < 2 + 1/\kappa$. This regularity can also be seen readily from Theorem 3.2, since the compactly supported function $h = f_\mu - f_{\widetilde{\mu}} - G$ satisfies

$$\overline{\partial} h = -(1 - \mu\overline{S})^{-1}\left(\chi_{\mathbb{D}}\overline{\partial} f_{\widetilde{\mu}} - \mu\overline{\partial f_{\widetilde{\mu}}}\right).$$

Finally, from the above we obtain that the function G_0, defined by

$$G_0(z) = e^{-ikz}G(z),$$

belongs to $W^{1,p}(\mathbb{C})$ and satisfies $G_0(z) = \mathcal{O}(1/z)$ with

$$\overline{\partial} G_0 - e_k\mu\overline{\partial G_0} = i\overline{k}\mu\overline{G_0}.$$

By Proposition 3.3 a) the function G_0 must hence vanish identically, which proves (6.1). □

Corollary 6.2 *The operator Λ_σ uniquely determines t_μ, $t_{-\mu}$ and τ_μ.*

Proof: The claim follows immediately from Proposition 6.1, (1.5) and from the definitions (5.11) of the scattering coefficients. □

From the results of Section 5 it follows that the coefficient τ_μ is continuously differentiable in k and vanishes at the origin. However, the global properties of τ_μ need a different approach. To this end we use a simple application of the Schwarz's lemma.

Proposition 6.3 *The complex geometric optics solutions*

$$f_{\pm\mu}(z,k) = e^{ikz}M_{\pm\mu}(z,k)$$

satisfy for $|z| > 1$ and for all $k \in \mathbb{C}$

$$\left|\frac{M_\mu(z,k) - M_{-\mu}(z,k)}{M_\mu(z,k) + M_{-\mu}(z,k)}\right| \leq \frac{1}{|z|}. \tag{6.3}$$

Moreover, for the scattering coefficient $\tau_\mu(k)$ we have

$$|\tau_\mu(k)| \leq 1 \quad \text{for all } k \in \mathbb{C}. \tag{6.4}$$

Proof: Fix the parameter $k \in \mathbb{C}$ and denote

$$m(z) = \frac{M_\mu(z,k) - M_{-\mu}(z,k)}{M_\mu(z,k) + M_{-\mu}(z,k)}.$$

Then by Proposition 4.3, $|m(z)| < 1$ for all $z \in \mathbb{C}$. Moreover, m is $\mathbb{C}$-analytic in $z \in \mathbb{C}\backslash\overline{\mathbb{D}}$, $m(\infty) = 0$, and thus by Schwarz's lemma we have $|m(z)| \le 1/|z|$ for all $z \in \mathbb{C}\backslash\overline{\mathbb{D}}$. Since (5.11) and (5.21) give $\lim_{z\to\infty} zm(z) = \overline{\tau_\mu}$, both claims of the proposition follow. $\square$

7 Subexponential growth

We know from Section 4 and (4.1), (4.14) that the complex geometric optics solution f_μ from (1.4) can be written in the exponential form. Here we begin by a more detailed analysis of this fact. For later purposes we also need to generalize the situation a bit by considering complex Beltrami coefficients μ_λ of the form $\mu_\lambda = \lambda\mu$, where the constant $\lambda \in \partial\mathbb{D}$ and μ is as before. Precisely as in Section 4 one can show the existence and uniqueness of $f_{\lambda\mu} \in W^{1,p}_{loc}(\mathbb{C})$ satisfying

$$\overline{\partial} f_{\lambda\mu} = \lambda\mu\overline{\partial f_{\lambda\mu}} \quad \text{and} \tag{7.1}$$

$$f_{\lambda\mu}(z,k) = e^{ikz}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \qquad \text{as } |z| \to \infty. \tag{7.2}$$

Lemma 7.1 *The function $f_{\lambda\mu}$ admits a representation*

$$f_{\lambda\mu}(z,k) = e^{ik\varphi_\lambda(z,k)}, \tag{7.3}$$

where for each fixed $k \in \mathbb{C} \setminus \{0\}$ and $\lambda \in \partial\mathbb{D}$, the function $\varphi_\lambda(\cdot,k) : \mathbb{C} \to \mathbb{C}$ is a quasiconformal homeomorphism that satisfies

$$\varphi_\lambda(z,k) = z + \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \to \infty \tag{7.4}$$

and

$$\overline{\partial}\varphi_\lambda(z,k) = -\frac{\overline{k}}{k}\mu_\lambda(z)\,(e_{-k}\circ\varphi_\lambda)(z,k)\,\overline{\partial\varphi_\lambda(z,k)}, \quad z \in \mathbb{C}. \tag{7.5}$$

Proof: Since the argument k is fixed we drop it from the notations and write simply $f_{\lambda\mu}(z,k) = f_{\lambda\mu}(z)$, $\varphi_\lambda(z,k) = \varphi_\lambda(z)$, etc. Denote

$$\mu_1(z) = \mu_\lambda(z)\frac{\overline{\partial f_\mu(z)}}{\partial f_\mu(z)}.$$

Then (1.4) gives

$$\overline{\partial} f_{\lambda\mu} = \mu_1 \partial f_{\lambda\mu}. \tag{7.6}$$

On the other hand, by the general theory of quasiconformal maps ([2], [12], [18]) there exists a unique quasiconformal homeomorphism $\varphi_\lambda \in H^1_{loc}(\mathbb{C})$ satisfying

$$\overline{\partial}\varphi_\lambda = \mu_1 \partial \varphi_\lambda \tag{7.7}$$

and having the asymptotics

$$\varphi_\lambda(z) = z + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \to \infty. \tag{7.8}$$

Moreover, any H^1_{loc}-solution to (7.6) is obtained from φ_λ by post-composing with an analytic function ([12], Theorem 11.1.2). In particular,

$$f_{\lambda\mu}(z) = h \circ \varphi_\lambda(z)$$

where $h : \mathbb{C} \to \mathbb{C}$ is an entire analytic function. But

$$\frac{h \circ \varphi_\lambda(z)}{\exp(ik\varphi_\lambda(z))} = \frac{f_{\lambda\mu}(z)}{\exp(ik\varphi_\lambda(z))}$$

has by (1.5), (1.6) and (7.8) the limit 1 as the variable $z \to \infty$. Thus

$$h(z) \equiv e^{ikz}.$$

Finally, (7.5) follows immediately from (1.4) and (7.3). $\square$

Note that the results in Section 4 show that (7.4), (7.5) has a unique solution. The existence of such a solution can also be directly verified by using Schauder's fixed point theorem [13], [5]. The result of Lemma 7.1 demonstrates that after a change of coordinates $z \mapsto \varphi(z)$ the complex geometric optics solution f_μ is simply an exponential function.

The main goal of this section is to show

Theorem 7.2 *If φ_λ satisfies* (7.4) *and* (7.5) *then*

$$\varphi_\lambda(z,k) \to z$$

uniformly in $z \in \mathbb{C}$ and $\lambda \in \partial\mathbb{D}$, as $k \to \infty$.

We have splitted the proof of Theorem 7.2 to several lemmata.

Lemma 7.3 *Suppose $\varepsilon > 0$ is given. Suppose also that for $\mu_\lambda(z) = \lambda\mu(z)$ we have*

$$f_n = \mu_\lambda S_n \mu_\lambda S_{n-1} \mu_\lambda \cdots \mu_\lambda S_1 \mu_\lambda \tag{7.9}$$

where $S_j : L^2(\mathbb{C}) \to L^2(\mathbb{C})$ are Fourier multiplier operators, each with a unimodular symbol. Then there is a number $R_n = R_n(\kappa, \varepsilon)$ depending only on μ, n and ε such that

$$|\widehat{f}_n(\xi)| < \varepsilon \text{ for } |\xi| > R_n. \tag{7.10}$$

Proof: Clearly it is enough to prove the claim for $\lambda = 1$.

Recall that for the Fourier transform $\widehat{\phi}$ we use the definition (5.16). By assumption

$$\widehat{S_j g}(\xi) = m_j(\xi)\widehat{g}(\xi)$$

where $|m_j(\xi)| = 1$ for $\xi \in \mathbb{C}$. We have by (7.9)

$$\|f_n\|_{L^2} \le \|\mu\|_{L^\infty}^n \|\mu\|_{L^2} \le \sqrt{\pi}\kappa^{n+1} \tag{7.11}$$

since $\operatorname{supp}(\mu) \subset \mathbb{D}$. Choose first ρ_n so that

$$\int\limits_{|\xi|>\rho_n} |\widehat{\mu}(\xi)|^2 \, dm(\xi) < \varepsilon^2. \tag{7.12}$$

After this choose $\rho_{n-1}, \rho_{n-2}, \ldots, \rho_1$ inductively so that for $l = n-1, \ldots, 1$

$$\pi \int\limits_{|\xi|>\rho_l} |\widehat{\mu}(\xi)|^2 \, dm(\xi) \le \varepsilon^2 \left(\prod_{j=l+1}^{n} \pi\rho_j \right)^{-2}. \tag{7.13}$$

Finally, choose ρ_0 so that

$$|\widehat{\mu}(\xi)| < \varepsilon\pi^{-n} \left(\prod_{j=1}^{n} \rho_j \right)^{-1}, \text{ when } |\xi| > \rho_0. \tag{7.14}$$

All these choices are possible since $\mu \in L^1 \cap L^2$.

Now, we set $R_n = \sum_{j=0}^n \rho_j$ and claim that (7.10) holds for this choice of R_n. Hence assume that $|\xi| > \sum_{j=0}^n \rho_j$. We have

$$|\widehat{f}_n(\xi)| \le \int\limits_{|\xi-\eta|\le\rho_n} |\widehat{\mu}(\xi-\eta)|\,|\widehat{f}_{n-1}(\eta)|\,dm(\eta) + \int\limits_{|\xi-\eta|\ge\rho_n} |\widehat{\mu}(\xi-\eta)|\,|\widehat{f}_{n-1}(\eta)|\,dm(\eta). \tag{7.15}$$

But if $|\xi-\eta| \le \rho_n$ then $|\eta| > \sum_{j=0}^{n-1} \rho_j$. Thus, if we denote

$$\Delta_n = \sup\left\{ |\widehat{f}_n(\xi)| \,:\, |\xi| > \sum_{j=0}^n \rho_j \right\}$$

it follows from (7.15) and (7.11) that

$$\begin{aligned}\Delta_n &\le \Delta_{n-1}(\pi\rho_n^2)^{1/2}\|\mu\|_{L^2} \\ &\quad + \left(\int\limits_{|\eta|\ge\rho_n} |\widehat{\mu}(\eta)|^2\,dm(\eta) \right)^{1/2} \|\widehat{f}_{n-1}\|_{L^2} \\ &\le \pi\rho_n\kappa\,\Delta_{n-1} + \kappa^n \left(\pi \int\limits_{|\eta|\ge\rho_n} |\widehat{\mu}(\eta)|^2\,dm(\eta) \right)^{1/2}\end{aligned}$$

for $n \ge 2$. Moreover, the same argument shows that

$$\begin{aligned}\Delta_1 &\le \pi\rho_1\kappa \sup\{|\widehat{\mu}(\xi)| \,:\, |\xi| > \rho_0\} \\ &\quad + \kappa \left(\pi \int\limits_{|\eta|>\rho_1} |\widehat{\mu}(\eta)|^2\,dm(\eta) \right)^{1/2}.\end{aligned}$$

In conclusion, after an iteration we have

$$\begin{aligned}\Delta_n &\le (\kappa\pi)^n \left(\prod_{j=1}^n \rho_j \right) \sup\{|\widehat{\mu}(\xi)| \,:\, |\xi| > \rho_0\} \\ &\quad + \kappa^n \sum_{l=1}^n \left(\prod_{j=l+1}^n \pi\rho_j \right) \left(\pi \int\limits_{|\eta|>\rho_l} |\widehat{\mu}(\eta)|^2\,dm(\eta) \right)^{1/2}.\end{aligned}$$

With the choices (7.12)–(7.14) this leads to

$$\Delta_n \le (n+1)\kappa^n \varepsilon \le \frac{\varepsilon}{1-\kappa},$$

which proves the claim. $\square$

Our next goal is to use Lemma 7.3 to obtain a similar asymptotic result as in Theorem 7.2 for a solution to a linear equation somewhat similar to (7.5).

Proposition 7.4 *Suppose $\psi \in H^1_{loc}(\mathbb{C})$ satisfies*

$$\begin{aligned} \overline{\partial}\psi &= -\lambda \frac{\overline{k}}{k}\mu(z) e_{-k}(z)\partial\psi, \quad \textit{and} \\ \psi(z) &= z + \mathcal{O}\left(\frac{1}{z}\right) \quad \textit{as } z \to \infty. \end{aligned} \tag{7.16}$$

Then $\psi(z,k) \to z$, uniformly in $z \in \mathbb{C}$ and $\lambda \in \partial\mathbb{D}$, as $k \to \infty$.

For Proposition 7.4 we need some preparations. First, as $\|S\|_{L^p \to L^p} \to 1$ when $p \to 2$, we can choose a $\delta_\kappa > 0$ so that $\kappa \|S\|_{L^p \to L^p} < 1$ whenever $2 - \delta_\kappa \le p \le 2 + \delta_\kappa$. With this notation we then have

Lemma 7.5 *Let ψ be the solution of (7.16) and let $\varepsilon > 0$. Then one can decompose $\overline{\partial}\psi$ in the following way: $\overline{\partial}\psi = g + h$ where*

(i) $\|h(\cdot,k)\|_{L^p} < \varepsilon$ *for* $2-\delta_\kappa \le p \le 2+\delta_\kappa$, *uniformly in k,*

(ii) $\|g(\cdot,k)\|_{L^p} \le C_0 = C_0(\kappa)$, *uniformly in k and*

(iii) $\widehat{g}(\xi,k) \to 0$ *as* $k \to \infty$,

where in (iii) convergence is uniform on compact subsets of the ξ-plane and also uniform in $\lambda \in \partial\mathbb{D}$. The Fourier transform is with respect to the first variable only.

Proof: We may solve (7.16) by Born-series which converge in L^p,

$$\overline{\partial}\psi = \sum_{n=0}^{\infty} \left(-\frac{\overline{k}}{k}\lambda\mu e_{-k}(z) S\right)^n \left(-\frac{\overline{k}}{k}\lambda\mu e_{-k}\right).$$

Let

$$h = \sum_{n=n_0}^{\infty} \left(-\frac{\overline{k}}{k}\lambda\mu e_{-k}(z) S\right)^n \left(-\frac{\overline{k}}{k}\lambda\mu e_{-k}\right).$$

Then

$$\|h\|_{L^p} \le \pi^{1/p} \frac{\kappa^{n_0+1}\|S\|_{L^p\to L^p}^{n_0}}{1-\kappa\|S\|_{L^p\to L^p}}.$$

We obtain (i) by choosing n_0 large enough.

The remaining part clearly satisfies (ii) with a constant C_0 that is independent of k and λ. To prove (iii) we first note that

$$S(e_{-k}\phi) = e_{-k}S_k\phi$$

where $(S_k\phi)\widehat{\ }(\xi) = m(\xi-k)\widehat{\phi}(\xi)$ and $m(\xi) = \xi/\overline{\xi}$. Consequently,

$$(\mu e_{-k}S)^n \mu e_{-k} = e_{-(n+1)k}\mu S_{nk}\mu S_{(n-1)k}\cdots \mu S_k\mu$$

and so

$$g = \sum_{j=1}^{n_0} \left(-\frac{\overline{k}}{k}\lambda\right)^j e_{-jk}\mu S_{(j-1)k}\mu\cdots\mu S_k\mu.$$

Therefore

$$g = \sum_{j=1}^{n_0} e_{-jk}G_j$$

where by Lemma 7.3, $|\widehat{G}_j(\xi)| < \widetilde{\varepsilon}$ whenever $|\xi| > R = \max_{j\le n_0} R_j$. As $(e_{-jk}G_j)\widehat{\ }(\xi) = \widehat{G}_j(\xi+jk)$, for any fixed compact set K_0 we can take k so large that $jk+K_0 \subset \mathbb{C}\setminus B(0,R)$ for each $1\le j\le n_0$. Then

$$\sup_{\xi\in K_0} |\widehat{g}(\xi,k)| \le n_0\widetilde{\varepsilon}.$$

This proves (iii). □

Proof of Proposition 7.4: We show first that when $k\to\infty$, $\overline{\partial}\psi \to 0$ weakly in L^p, $2-\delta_\kappa \le p \le 2+\delta_\kappa$. For this suppose $f_0 \in L^q$, $q = p/(p-1)$, is fixed and choose $\varepsilon > 0$. Then there exists $f\in C_0^\infty(\mathbb{C})$ such that $\|f_0 - f\|_{L^q} < \varepsilon$ and so by Lemma 7.5

$$|\langle f_0, \overline{\partial}\psi\rangle| \le \varepsilon C_1 + \Big|\int \widehat{f}(\xi)\widehat{g}(\xi,k)\,dm(\xi)\Big|.$$

Choose first R so large that

$$\int\limits_{\mathbb{C}\setminus B(0,R)} |\widehat{f}(\xi)|^2\,dm(\xi) \le \varepsilon^2$$

and then $|k|$ so large that $|\widehat{g}(\xi,k)| \le \varepsilon/(\sqrt{\pi}R)$ for all $\xi \in B(0,R)$. Now

$$\left|\int \widehat{f}(\xi)\widehat{g}(\xi,k)\,dm(\xi)\right| \le \int\limits_{B(0,R)} \widehat{f}(\xi)\widehat{g}(\xi,k)\,dm(\xi) + \int\limits_{\mathbb{C}\setminus B(0,R)} \widehat{f}(\xi)\widehat{g}(\xi,k)\,dm(\xi)$$

$$\le \varepsilon(\|f\|_{L^2} + \|g\|_{L^2}) \le C_2(f)\varepsilon.$$

The bound is the same for all λ, hence $\sup_{\lambda\in\partial\mathbb{D}} |\langle f_0, \overline{\partial}\psi\rangle| \to 0$ as $|k| \to \infty$.

To prove the uniform convergence of ψ itself we write

$$\psi(z,k) = z - \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{1}{\xi - z}\overline{\partial}\psi(\xi,k)\,dm(\xi). \tag{7.17}$$

Here note that $\operatorname{supp}(\overline{\partial}\psi) \subset \mathbb{D}$ and $\chi_{\mathbb{D}}(\xi)/(\xi - z) \in L^q$ for all $q < 2$. Thus by the weak convergence we get

$$\psi(z,k) \to z \text{ as } k \to \infty, \tag{7.18}$$

for each fixed $z \in \mathbb{C}$, but uniformly in $\lambda \in \partial\mathbb{D}$. On the other hand as $\sup_k \|\overline{\partial}\psi\|_{L^p} \le C_0(\kappa) < \infty$, for all z sufficiently large $|\psi(z,k) - z| < \varepsilon$, uniformly in $k \in \mathbb{C}$ and $\lambda \in \partial\mathbb{D}$. Moreover, (7.17) shows also that the family $\{\psi(\cdot,k) \,:\, k \in \mathbb{C},\ \lambda \in \partial\mathbb{D}\}$ is equicontinuous. Combining all these observations shows that the convergence in (7.18) is uniform in $z \in \mathbb{C}$ and $\lambda \in \partial\mathbb{D}$. $\square$

Finally we proceed to the nonlinear case: So assume that φ_λ satisfies (7.4) and (7.5). Since φ is a (quasiconformal) homeomorphism we may consider its inverse $\psi_\lambda : \mathbb{C} \to \mathbb{C}$,

$$\psi_\lambda \circ \varphi_\lambda(z) = z, \tag{7.19}$$

which also is quasiconformal. By differentiating (7.19) with respect to z and $\overline{z}$ one obtains that ψ satisfies

$$\overline{\partial}\psi_\lambda = -\frac{\overline{k}}{k}\lambda(\mu \circ \psi_\lambda)e_{-k}\partial\psi_\lambda \quad \text{and} \tag{7.20}$$

$$\psi_\lambda(z,k) = z + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \to \infty. \tag{7.21}$$

Proof of Theorem 7.2: It is enough to show that

$$\psi_\lambda(z,k) \to z \tag{7.22}$$

uniformly in z and λ as $k \to \infty$. To prove this we need to recall some further facts from quasiconformal mappings. Let us use the notation
(7.23)
$$\Sigma_\kappa = \{g \in H^1_{loc}(\mathbb{C}) : \overline{\partial} g = \nu \partial g,\ |\nu| \le \kappa \chi_{4\mathbb{D}} \text{ and } g = z + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \to \infty\}.$$

Note that with the above normalization at ∞ all elements $g \in \Sigma_\kappa$ are homeomorphisms [2]. Also, the use of $\chi_{4\mathbb{D}}$ will become clear soon.

Lemma 7.6 *a) The family Σ_κ is compact in the topology of uniform convergence on $\mathbb{C}$.*

b) Suppose that $f, g \in \Sigma_\kappa$, $1+\kappa < p < 1+1/\kappa$ and that $\varepsilon > 0$ is so small that $(1+\varepsilon)p < 1+1/\kappa$. Then

$$\int_{\mathbb{C}} |\overline{\partial} f - \overline{\partial} g|^p \, dm \le C(p,\varepsilon) \left(\int_{\mathbb{C}} |\nu_f - \nu_g|^{p\frac{1+\varepsilon}{\varepsilon}} dm \right)^{\frac{\varepsilon}{1+\varepsilon}}$$

where

$$\nu_f = \frac{\overline{\partial} f}{\partial f} \text{ and } \nu_g = \frac{\overline{\partial} g}{\partial g}.$$

Proof: The claim a) follows from [18], Theorem II.51. Furthermore, since $\partial g = 1 + S(\overline{\partial} g)$ for each $g \in \Sigma_\kappa$, we have

$$\overline{\partial} f - \overline{\partial} g = (I - \nu_f S)^{-1} \left(\nu_f - \nu_g + (\nu_f - \nu_g) S(\overline{\partial} g) \right), \quad f, g \in \Sigma_\kappa.$$

Applying Theorem 3.2 and Hölder's inequality gives then the claim b). $\square$

The support of $\mu \circ \psi_\lambda$ need not anymore be contained in $\mathbb{D}$. However, by Koebe's 1/4-theorem, see e.g. [1] Corollary 5.3, $\varphi_\lambda(\mathbb{D}) \subset 4\mathbb{D}$ and thus $\mathrm{supp}(\mu \circ \psi_\lambda) \subset 4\mathbb{D}$. Therefore by Lemma 7.6 a) we have sequences $k_n \to \infty$ and sequences $\lambda_n \to \lambda \in \partial\mathbb{D}$ such that $\psi_{\lambda_n}(\cdot, k_n) \to \psi_\infty$ uniformly, with $\psi_\infty \in \Sigma_\kappa$. To prove Theorem 7.2 it is enough to show that for any such sequence $\psi_\infty(z) \equiv z$.

Hence assume that we have such a limit function ψ_∞. We then consider the H^1_{loc}-solution $\Phi(z) = \Phi_\lambda(z,k)$ of

$$\overline{\partial} \Phi = -\frac{\overline{k}}{k} \lambda (\mu \circ \psi_\infty) e_{-k} \partial \Phi,$$
$$\Phi(z) = z + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \to \infty.$$

This is now a linear Beltrami equation which [2] has a unique solution $\Phi \in \Sigma_\kappa$ for each $k \in \mathbb{C}$. According to Proposition 7.4

$$\Phi_\lambda(z,k) \to z \text{ as } k \to \infty. \tag{7.24}$$

Secondly, when $2 < p < 1 + 1/\kappa$, by Lemma 7.6 b)

(7.25)
$$\begin{aligned}
|\psi_{\lambda_n}(z,k_n) - \Phi_\lambda(z,k_n)| &= \frac{1}{\pi}\left| \int\limits_{4\mathbb{D}} \frac{1}{\xi - z} \overline{\partial}\big(\psi_{\lambda_n}(\xi,k_n) - \Phi_\lambda(\xi,k_n)\big)\, dm(\xi) \right| \\
&\le C_1 \left\| \overline{\partial}\big(\psi_{\lambda_n}(\xi,k_n) - \Phi_\lambda(\xi,k_n)\big) \right\|_{L^p} \\
&\le C_2|\lambda_n - \lambda| + C_2 \left(\int\limits_{4\mathbb{D}} \left|\mu\big(\psi_{\lambda_n}(\xi,k_n)\big) - \mu\big(\psi_\infty(\xi)\big)\right|^{\frac{p(1+\varepsilon)}{\varepsilon}} dm(\xi) \right)^{\frac{\varepsilon}{p(1+\varepsilon)}}.
\end{aligned}$$

Lastly, we use the fact [3] that for all $2 < p < 1 + 1/\kappa$ and for all $g = \psi^{-1}$, $\psi \in \Sigma_\kappa$, we have for the Jacobian J_g that

$$\int\limits_{\mathbb{D}} J_g(z)^{p/2}\, dm(z) \le \int\limits_{\mathbb{D}} |\partial g|^p\, dm \le C(\kappa) < \infty. \tag{7.26}$$

where $C(\kappa)$ depends only on κ. Again, the bound can be deduced also from Theorem 3.2 since $\overline{\partial} g = (I - \nu_g S)^{-1}\nu_g$ and $\partial g = 1 + S(\overline{\partial} g)$. We use this estimate in the cases $\psi(z) = \psi_{\lambda_n}(z,k_n)$ and $\psi = \psi_\infty$. Namely, we have for each $\eta \in C_0^\infty(\mathbb{D})$ that

$$\begin{aligned}
\int\limits_{4\mathbb{D}} |\mu(\psi) - \eta(\psi)|^{\frac{p(1+\varepsilon)}{\varepsilon}} &= \int\limits_{\mathbb{D}} |\mu - \eta|^{\frac{p(1+\varepsilon)}{\varepsilon}} J_g \\
&\le \Big(\int\limits_{\mathbb{D}} |\mu - \eta|^{\frac{p^2(1+\varepsilon)}{\varepsilon(p-2)}} \Big)^{(p-2)/p} \Big(\int\limits_{\mathbb{D}} J_g^{p/2} \Big)^{2/p}.
\end{aligned} \tag{7.27}$$

Since μ can be approximated in the mean by smooth η, the last term in (7.27) can be made arbitrarily small. Since by uniform convergence $\eta(\psi_{\lambda_n}(z,k_n)) \to \eta(\psi_\infty(z))$ we see that the last bound in (7.25) converges to zero as $\lambda_n \to \lambda$ and $k_n \to k$. In view of (7.24) and (7.25) we have established that

$$\psi_{\lambda_n}(z,k_n) \to z$$

and that $\psi_\infty(z) \equiv z$. The theorem is proved. $\square$

8 Transport matrix

The gradient of a quasiregular map can vanish only on a set of Lebesque measure zero ([2], p. 34). By the equation (1.4) the map ∂f_μ can vanish only on points z where the whole gradient of f_μ vanishes. This means that we can recover μ and hence σ from f_μ by the formulae

$$\mu = (\overline{\partial f_\mu})^{-1}\overline{\partial} f_\mu \quad \text{and} \quad \sigma = \frac{1-\mu}{1+\mu}. \tag{8.1}$$

Next, let $u_1 = h_+ - ih_-$ and $u_2 = i(h_+ + ih_-)$ be defined as in (1.14). From Lemma 2.1 and (1.7) we see that u_1, u_2 satisfy the conductivity equations (1.15). With respect to the parameter k the u_j's are C^1-mappings, c.f. Theorem 5.5. A straight forward derivation using (1.8) shows also that at each point $z \in \mathbb{C}$, both $u_1(z,k)$ and $u_2(z,k)$ satisfy the $\partial_{\overline{k}}$-equation

$$\frac{\partial}{\partial \overline{k}} u(z,k) = -i\,\tau_\mu(k)\overline{u(z,k)}. \tag{8.2}$$

It is clear that the pair $\{u_1(z,k), u_2(z,k)\}$ determines the pair $\{f_\mu(z,k), f_{-\mu}(z,k)\}$ and vice versa. Thus by Proposition 6.1 the Dirichlet to Neumann map Λ_σ uniquely determines the functions u_1 and u_2 outside $\mathbb{D}$. To prove Theorem 1 it therefore suffices to transport these solutions from outside to inside $\mathbb{D}$ by using the data Λ_σ. For this purpose we will employ (8.2) and the fact from Corollary 6.2 that Λ_σ uniquely determines the scattering coefficient τ_μ.

We have only shown that τ_μ is bounded and that $u_j(z,k)e^{-ikz}$ behaves subexponentially as $k \to \infty$, and these facts alone are much too weak to guarantee the uniqueness of solutions to the pseudoanalytic equation (8.2). To remedy this we need to understand the transport matrix from (1.18).

We start by arguing that u_1 and u_2 are $\mathbb{R}$-linearly independent, i.e. that

$$u_2(z,k) \neq 0 \text{ and } u_1(z,k)/u_2(z,k) \notin \mathbb{R} \tag{8.3}$$

holds for all $z, k \in \mathbb{C}$. By Proposition 4.3 we have

$$\left|\frac{h_-(z,k)}{h_+(z,k)}\right| < 1 \text{ and } h_+(z,k) \neq 0 \text{ for all } z,k \in \mathbb{C}.$$

Since

$$\frac{u_1}{u_2} = -i\frac{h_+ - ih_-}{h_+ + ih_-}$$

this proves (8.3) and enables us to define the transport matrix $T^\sigma_{z,z_0}(k)$ as in (1.18). It also shows that $T^\sigma_{z,z_0}(k)$ is invertible.

It was discovered by Bers [7] that the coefficients connecting different solutions to the same pseudoanalytic equation as in (1.17) give rise to a quasiregular mapping. In [7] these mappings are called *pseudoanalytic functions of the second kind.* In our case this means that differentiating $u_1(z,k) = a_1(z,z_0;k)u_1(z_0,k) + a_2(z,z_0;k)u_2(z_0,k)$ with respect to $\overline{k}$ and using (8.2) with (1.14) gives for $\alpha = a_1 + ia_2$

$$\partial_{\overline{k}}\alpha(z,z_0;k) = \nu_{z_0}(k)\overline{\partial_k\alpha(z,z_0;k)}, \quad \nu_{z_0}(k) = i\frac{h_-(z_0,k)}{h_+(z_0,k)}. \tag{8.4}$$

Note that $\alpha(z,z_0;k)$ inherits the continuous differentiability with respect to k from the u_j's. For an explicit expression for α in terms of u_1 and $h_\pm$ see (8.8) below.

Moreover, the second row of $T^\sigma_{z,z_0}(k)$ gives similarly a solution $\beta = b_1 + ib_2$ for the same equation (8.4). In fact, if we write the μ dependence explicitely for u_i: $u_i = u_i(\mu)$, $i = 1,2$, then

$$u_2(\mu) = iu_1(-\mu) \text{ and } u_1(\mu) = -iu_2(-\mu). \tag{8.5}$$

Thus $i\beta_\mu = \alpha_{-\mu}$.

We have now shown that the rows of the transport matrix produce quasiregular mappings, with respect to k, satisfying (8.4). Our next task is to determine their asymptotic behaviour at ∞.

Proposition 8.1 *Suppose $z_0 \in \mathbb{C}$, $|z_0| > 1$. Then:*

a) For each fixed $k \neq 0$ and z_0 we have with respect to z

$$\alpha(z,z_0;k) = \exp(ikz + v(z))$$

where $v \in L^\infty$.

b) For each fixed z and z_0 we have with respect to k that

$$\alpha(z,z_0;k) = \exp(ik(z-z_0) + k\varepsilon(k)) \tag{8.6}$$

where $\varepsilon(k) \to 0$ as $k \to \infty$.

Up to the factor i, β has the same asymptotics.

Proof: The last claim is clear from (8.5).

For a) we write

$$\begin{aligned} u_1 &= \frac{1}{2}\left(f_\mu + f_{-\mu} + \overline{f_\mu} - \overline{f_{-\mu}}\right) \\ &= f_\mu\left(1 + \frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}}\right)^{-1}\left(1 + \frac{\overline{f_\mu} - \overline{f_{-\mu}}}{f_\mu + f_{-\mu}}\right). \end{aligned} \tag{8.7}$$

All factors in the product are nonvanishing. Taking the logarithm and applying (1.5), (1.6) leads to

$$u_1(z,k) = \exp\left(ikz + \mathcal{O}_k\left(\frac{1}{z}\right)\right).$$

On the other hand, dropping temporarily the fixed k from the notations,

$$u_1(z) = \alpha h_+(z_0) - i\overline{\alpha} h_-(z_0)$$

or

$$\frac{u_1(z)}{h_+(z_0)} = \alpha - i\overline{\alpha}\frac{h_-(z_0)}{h_+(z_0)}.$$

This gives

$$\alpha = \left(1 - \left|\frac{h_-(z_0)}{h_+(z_0)}\right|^2\right)^{-1}\left(1 + i\overline{\frac{h_-(z_0)}{h_+(z_0)}}\frac{\overline{u_1(z)}}{u_1(z)}\right)\frac{u_1(z)}{h_+(z_0)}. \tag{8.8}$$

According to Proposition 6.3, $|h_-(z_0)/h_+(z_0)| \leq 1/|z_0| < 1$ while $h_+(z_0)$ is constant for fixed k and z_0 . This proves a).

To prove b) note that

$$h_+ = \left(1 + i\frac{\overline{h_-}}{h_+}\right)^{-1} f_\mu \text{ and } u_1 = h_+\left(1 - i\frac{h_-}{h_+}\right).$$

Again, the factors are continuous and point wise non-vanishing. Therefore the identity (8.8) and Theorem 7.2 reduce the proof of b) to

Lemma 8.2 *For each fixed $z \in \mathbb{C}$,*

$$\left|\frac{h_-(z,k)}{h_+(z,k)}\right| \leq 1 - e^{-|k|\varepsilon(k)}.$$

Proof: By the definition of h_+ and h_- it suffices to show

$$\inf_t \left|\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} + e^{it}\right| \geq e^{-|k|\varepsilon(k)}. \tag{8.9}$$

For this, define

$$\Phi_t = e^{-it/2}(f_\mu \cos t/2 + i f_{-\mu}\sin t/2).$$

Then for each fixed k,

$$\Phi_t(z,k) = e^{ikz}\left(1 + \mathcal{O}_k\left(\frac{1}{z}\right)\right) \text{ as } z \to \infty$$

and

$$\overline{\partial}\Phi_t = \mu e^{-it}\overline{\partial\Phi_t}.$$

Thus $\Phi_t = f_{\lambda\mu}$, $\lambda = e^{-it}$, the exponentially growing solution of (7.1) and (7.2) from Section 7. But

$$\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} + e^{it} = \frac{2e^{it}\,\Phi_t}{f_\mu + f_{-\mu}} = \frac{f_{\lambda\mu}}{f_\mu}\,\frac{2e^{it}}{1 + M_{-\mu}/M_\mu}. \tag{8.10}$$

By Theorem 7.2

$$e^{-|k|\varepsilon_1(k)} \le |M_{\pm\mu}(z,k)| \le e^{|k|\varepsilon_1(k)} \tag{8.11}$$

and

$$e^{-|k|\varepsilon_2(k)} \le \inf_{\lambda\in\partial\mathbb{D}} \left|\frac{f_{\lambda\mu}(z,k)}{f_\mu(z,k)}\right| \le \sup_{\lambda\in\partial\mathbb{D}} \left|\frac{f_{\lambda\mu}(z,k)}{f_\mu(z,k)}\right| \le e^{|k|\varepsilon_2(k)} \tag{8.12}$$

where $\varepsilon_j(k) \to 0$ as $k \to \infty$. Since $\mathrm{Re}(M_{-\mu}/M_\mu) > 0$ the inequality (8.9) follows from (8.10), (8.11) and (8.12). This finishes the proof of Lemma 8.2 and thus also the proof of Proposition 8.1. $\square$

Now the last remaining obstacle is Theorem 2; that is, we need to prove that the data determines the transport matrices $T^\sigma_{z,z_0}(k)$. We know that in the k-variable the rows α and β of T^σ_{z,z_0} are quasiregular mappings satisfying (8.4) and having the asymptotics given by Proposition 8.1 b).

It is not clear if the asymptotics (8.6) are strong enough to determine the individual solution. However, if we consider the entire family $\{T^\sigma_{z,z_0} : z \in \mathbb{C}\}$, then the uniqueness does hold:

Proof of Theorem 2: Let $|z_0| > 1$ and $k \ne 0$. Let $\alpha_\mu(z, z_0, k)$ and $\alpha_{\widetilde{\mu}}(z, z_0, k)$ be defined by (1.17) and (1.19). Since neither of α_μ, $\alpha_{\widetilde{\mu}}$ vanishes at any point we can define the corresponding logarithms δ_μ and $\delta_{\widetilde{\mu}}$ by

$$\begin{aligned}\delta_\mu(z, z_0; k) &= \log \alpha_\mu(z, z_0, k) = ik(z - z_0) + k\varepsilon_1(k)\\ \delta_{\widetilde{\mu}}(z, z_0; k) &= \log \alpha_{\widetilde{\mu}}(z, z_0, k) = ik(z - z_0) + k\varepsilon_2(k)\end{aligned} \tag{8.13}$$

where for $|k| \to \infty$, $\varepsilon_j(k) \to 0$ by Proposition 8.1 b). Moreover, by Theorem 4.2,

$$\delta_\mu(z, z_0; 0) \equiv \delta_\mu(z, z_0; 0) \equiv 0$$

for all $z \in \mathbb{C}$, $|z_0| > 1$.

In addition, $z \mapsto \delta_\mu(z, z_0; k)$ is continuous, $\delta_\mu(z_0, z_0; k) = 0$ and we have

$$\delta_\mu(z, z_0; k) = ikz\left(1 + \frac{v_k(z)}{ikz}\right), \quad k \ne 0, \tag{8.14}$$

where by Proposition 8.1 a), $v_k \in L^\infty(\mathbb{C})$ for each fixed $k \in \mathbb{C}$. The use of degree theory or homotopy argument [29] gives that $z \mapsto \delta_\mu(z, z_0; k)$ is also surjective $\mathbb{C} \to \mathbb{C}$.

To prove the theorem it suffices to show that if $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$, then we have

$$\delta_{\widetilde{\mu}}(z, z_0; k) \neq \delta_\mu(w, z_o; k), \qquad \text{for } z \neq w \text{ and } k \neq 0. \tag{8.15}$$

Namely then (8.15) and the surjectivity of $z \mapsto \delta_\mu(z, z_0; k)$ show that necessarily we have $\delta_{\widetilde{\mu}}(z, z_0, k) = \delta_\mu(z, z_0, k)$ for all $k, z \in \mathbb{C}$ and $|z_0| > 1$. Hence $\alpha_\mu \equiv \alpha_{\widetilde{\mu}}$. By (8.5) we have $\beta_\mu = \beta_{\widetilde{\mu}}$ as well and hence that $T^\sigma_{z,z_0}(k) \equiv T^{\widetilde{\sigma}}_{z,z_0}(k)$.

To show (8.15) fix $z \neq w$ and note that by (8.4) δ_μ and $\delta_{\widetilde{\mu}}$ satisfy

$$\partial_{\overline{k}} \delta = \nu_{z_0}(k) e^{\overline{\delta} - \delta} \overline{\partial_k \delta}, \quad k \in \mathbb{C}, \tag{8.16}$$

where by Proposition 6.1 and the assumption $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$, the coefficient ν_{z_0} is the same for both δ_μ and $\delta_{\widetilde{\mu}}$. The difference

$$g(k) = \delta_\mu(w, z_o; k) - \delta_{\widetilde{\mu}}(z, z_0; k)$$

satisfies the equation

$$\partial_{\overline{k}} g - \nu_{z_0} e^{(\overline{\delta_\mu} - \delta_\mu)} \overline{\partial_k g} = \nu_{z_0} \overline{\partial_k \delta_{\widetilde{\mu}}} \left(e^{(\overline{\delta_\mu} - \delta_\mu)} - e^{(\overline{\delta_{\widetilde{\mu}}} - \delta_{\widetilde{\mu}})} \right).$$

In other words, there exist functions η and γ such that

$$\partial_{\overline{k}} g - \eta \partial_k g = \gamma g \tag{8.17}$$

with $|\eta| \leq |\nu_{z_0}| \leq 1/|z_0| < 1$ and $|\gamma| \leq 2|\nu_{z_0}||\partial_k \delta_{\widetilde{\mu}}| \leq 2|\partial_k \delta_{\widetilde{\mu}}|$. From (8.13) we have $g(k) = i(w - z)k + k\varepsilon(k)$. Since $\alpha_\mu, \alpha_{\widetilde{\mu}}$ are C^1 with respect to k, we see that γ is locally bounded with respect to k and we may apply Proposition 3.3 b) (with respect to k) to obtain that g vanishes only at $k = 0$. This shows (8.15) . $\square$

The proof of Theorem 1 is now immediate. If $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$, then by Proposition 6.1 we have $u_j^\sigma(z) = u_j^{\widetilde{\sigma}}(z)$ for $|z| > 1$ and $j = 1, 2$. Theorem 2 with (1.17) gives then $u_j^\sigma \equiv u_j^{\widetilde{\sigma}}$ and (8.1) that $\sigma \equiv \widetilde{\sigma}$.

Lastly, we describe how our uniqueness proof leads to a constructive procedure for recovering σ. First, by (2.9) and (2.11) the projection operator Q_μ can be calculated from Λ_σ. Next, by Lemma 2.2 and equation (5.1) the solution $M_\mu(\cdot, k)$ belongs to the space $\mathrm{Range}(P_\mu^k) \cap \mathrm{Range}(P_+)$ where the projections P_μ^k and P_+ are defined by

$$\begin{aligned} P_+(g)(z) &= \frac{1}{2}(I - i\mathcal{H}_0)(g)(z) \\ P_\mu^k(g)(z) &= e^{-ikz}(I - Q_\mu)(e^{ik\cdot} g)(z). \end{aligned} \tag{8.18}$$

By Theorem 4.2 the space $\mathrm{Range}(P_\mu^k) \cap \mathrm{Range}(P_+)$ is one dimensional. Since it is defined by the data, we may use the asymptotics (1.6) to construct M_μ on $\partial\mathbb{D}$ from the equations

$$(P_+ - P_\mu^k)g = 0, \quad P_+ g = cM_\mu. \tag{8.19}$$

By (2.8) this procedure also gives $M_{-\mu}$ on $\partial\mathbb{D}$.

The following step is to use the Fourier coefficients of $M_{\pm\mu}$ on $\partial\mathbb{D}$ to construct $M_{\pm\mu}$ in the exterior of $\mathbb{D}$. This gives by (1.7) and (1.21) the Beltrami coefficient $\nu_{z_0}(k)$ in (1.20). Finally by Theorem 2 we can uniquely solve equation (1.20) with the asymptotics (8.6) to obtain the transport matrix $T^\sigma_{z,z_0}(k)$ and hence $f_\mu(z,k)$ for $z \in \mathbb{D}$. Formula (8.1) yields then the conductivity σ.

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