DEGENERATE BELTRAMI EQUATION AND RADIAL Q–HOMEOMORPHISMS

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Abstract

Radial Q–homeomorphisms are introduced and studied and then applied in existence problems for degenerate Beltrami equations. Some general principles on the existence and uniqueness of homeomorphic ACL solutions for the Beltrami equations are formulated extending earlier results. We give a series of new criteria for existence in the terms of majorants of finite mean oscillation to various kinds of dilatations some of which are new.

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1 Introduction

Let D be a domain in the complex plane $\mathbb{C}$, i.e., open and connected subset of $\mathbb{C}$, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| \leq 1$ a.e. The equation

$$f_{\overline{z}} = \mu(z) \cdot f_z \tag{1.1}$$

where $f_{\overline{z}} = \overline{\partial} f = (f_x + if_y)/2$ and $f_z = \partial f = (f_x - if_y)/2$, $z = x + iy$, is the **Beltrami equation**, the function μ is its **complex coefficient** and

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{1.2}$$

is its (maximal) **dilatation**.

A mapping $f : D \to \mathbb{C}$ is said to be **absolutely continuous on lines**, abbr. $f \in$**ACL**, if, for every closed rectangle R in D with sides parallel to the coordinate axes, $f|R$ is absolutely continuous on almost all line segments in R which are parallel to the sides of R. In particular, f is ACL if it belongs to the Sobolev class $W^{1,1}_{loc}$, see e.g. [Ma], p. 8. Note that every f in ACL has partial derivatives f_x and f_y a.e. and thus by the well-known Gehring-Lehto theorem every ACL homemorphism $f : D \to \mathbb{C}$ is totally differentiable a.e., see [GL] or [LV], p. 128. For a sense-preserving ACL homeomorphism $f : D \to \mathbb{C}$, the Jacobian $J_f(z) = J(z, f) = |f_z|^2 - |f_{\overline{z}}|^2$ is nonnegative a.e., see [LV], p. 10, and the **dilatation** of f is $K_\mu(z)$, $K_\mu(z) \geq 1$ a.e. where the **complex dilatation** of f is defined by $\mu(z) = \mu_f(z) = f_{\overline{z}}/f_z$ if $f_z \neq 0$ and by $\mu(z) = 0$ otherwise and satisfies

$|\mu(z)| \leq 1$ a.e. Given a measurable function $Q : D \to [1, \infty)$, a sense-preserving ACL homeomorphism $f : D \to \overline{\mathbb{C}}$ is **Q(z)-quasiconformal**, abbr. **Q(z)-qc**, if

$$K_\mu(z) \leq Q(z) \quad \text{a.e.} , \tag{1.3}$$

and it is **quasiconformal**, abbr. **qc**, if $\|K_\mu\|_\infty < 1$. The following statement is well–known, see [Ah], p. 98, cf. [LV], p. 185 and 194.

THEOREM A. *For every measurable function $\mu : \mathbb{C} \to \mathbb{C}$ with*

$$\|\mu\|_\infty < 1 , \tag{1.4}$$

there is a homeomorphic ACL solution of the Beltrami equation (1.1).

In the present paper, we study the degenerate Beltrami equation (1.1) when

$$|\mu(z)| < 1 \quad \text{a.e.} \tag{1.5}$$

but, generally speaking, (1.4) is already not valid. For the goal, we use the approximate approach. The approach is based on convergence theorems and estimates of distortion following from the extremal length method.

The following convergence theorem is basic, see Theorem 3.1 and Corollary 5.12 in [RSY_1].

THEOREM B. *Let $f_n : D \to \overline{\mathbb{C}}$, $n = 1, 2, \ldots$ be a sequence of $Q(z)$-qc mappings with $Q \in L^1_{loc}(D)$. If $f_n \to f$ locally uniformly, then either f is constant or f is a $Q(z)$-qc mapping. If, in addition, $\mu_n(z) \to \nu(z)$ a.e., then $\mu(z) = \nu(z)$ a.e.*

In view of Theorems A and B the problem of the existence of an ACL homeomorphic solution for the Beltrami equation is reduced to the problem of normality of certain families of $Q(z)$-qc mappings. By the well-known Arzela-Ascoli theorem, the latter is related to suitable estimations of distortion under $Q(z)$-qc mappings. Below we use the following variant of the Arzela-Ascoli theorem, see e.g. [Du], p. 267, and [DS], p. 382.

THEOREM C. *Let (X, d) be a separable metric space and let (X', d') be a compact metric space. Then a family $\mathfrak{F}$ of mappings $f : X \to X'$ is normal if and only if $\mathfrak{F}$ is equicontinuous.*

Recall that a family $\mathfrak{F}$ of continuous mappings $f : X \to X'$ is said to be **normal** if every sequence of mappings $f_n \in \mathfrak{F}$ has a subsequence f_{n_k} converging uniformly on each compact set $C \subset X$ to a continuous mapping. A family $\mathfrak{F}$ of mappings $f : X \to X'$ is said to be *equicontinuous at a point* $x_0 \in X$ if, for every $\varepsilon > 0$, there is $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathfrak{F}$ and $x \in X$ with $d(x, x_0) < \delta$. It is said that $\mathfrak{F}$ is **equicontinuous** if $\mathfrak{F}$ is equicontinuous at every point $x_0 \in X$. Later on, (X, d) is a domain D in $\mathbb{C}$ with the usual (Euclidean) metric and (X', d') is the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with the spherical (chordal) metric

$$(1.6)\quad s(z,\infty) = \frac{1}{(1+|z|^2)^{\frac{1}{2}}},\ \ s(z,\zeta) = \frac{|z-\zeta|}{(1+|z|^2)^{\frac{1}{2}}(1+|\zeta|^2)^{\frac{1}{2}}},\ \ z \neq \infty \neq \zeta.$$

Recall also that a real valued function $\varphi \in \mathrm{L}^1_{loc}(D)$ is said to be of **bounded mean oscillation** in D, abbr. $\varphi \in$ BMO(D) or simply $\varphi \in$ **BMO**, if

$$(1.7)\qquad \|\varphi\|_* = \sup_{B \subset D} \ \fint_B |\varphi(z) - \varphi_B|\ dxdy\ \ <\ \ \infty$$

where the supremum is taken over all disks B in D and

$$(1.8)\qquad \varphi_B = \fint_B \varphi(z)\ dxdy = \frac{1}{|B|}\int_B \varphi(z)\, dxdy$$

is the **mean value** of the function φ over B. It is well–known that L^∞(D) $\subset$ BMO(D) $\subset \mathrm{L}^p_{loc}$(D) for all $1 \leq p < \infty$, see e.g. [RR]. A function φ in BMO is said to be have **vanishing mean oscillation**, abbr. $\varphi \in$ **VMO**, if the supremum in (1.7) taken over all disks B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$.

The BMO space was introduced by John and Nierenberg, see [JN], and soon became an important concept in harmonic analysis, complex analysis and partial differential equations. BMO functions are related in many ways to quasiconformal and quasiregular mappings, see e.g. [As], [AG], [Ge$_1$], [Jo], [MRV] and [Rei], as well as to the modern classes of mappings with finite distortion, see e.g. [AIKM], [IM], [KKM$_1$], [KKM$_2$], [IKO$_1$], [IKO$_2$] and [KO]. VMO has been introduced by Sarason, see [Sar]. Many papers are devoted to the existence and uniqueness of solutions of various kind of differential equations and, in particular, of elliptic type with coefficients in VMO, see e.g. [CFL], [IS], [Pa] and [Ra]. Note that, if the coefficients are in the Sobolev class $W^{1,2}_{loc}$, then they are in VMO, see e.g. [BN] and [CP].

Conditions for the existence and uniqueness of ACL homeomorphic solutions of the Beltrami equation can be given in terms of the dilatation $K_\mu(z)$. In particular, it was proved that, if $K_\mu(z)$ has a majorant in the class BMO_{loc}, then the Beltrami equation (1.1) has a homeomorphic ACL solution, see e.g. [RSY$_2$]. Many conditions for the existence of the Beltrami equation are formulated in terms of integral and measure constraints on dilatations, see e.g. [BJ], [Da], [GMSV], [IM], [Kr], [Le], [MM], [MS], [Pe] and [Tu]. These conditions assume either the exponential integrability or local integrability of the dilatations with a high degree. In the present paper, we prove the existence theorem when K_μ has a majorant $Q(z)$ of the class FMO (finite mean oscillation) which is a subclass of L^1_{loc} and contains the class BMO_{loc}.

2 Finite mean oscillation

We say that a function $\varphi : D \to \mathbb{R}$ has **finite mean oscillation** at a point $z_0 \in D$ if

$$d_\varphi(z_0) = \overline{\lim_{\varepsilon\to 0}} \ \fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \ dxdy \ < \ \infty \tag{2.1}$$

where

$$\overline{\varphi}_\varepsilon(z_0) = \fint_{D(z_0,\varepsilon)} \varphi(z) \ dxdy \tag{2.2}$$

is the mean value of the function $\varphi(z)$ over

$$D(z_0, \varepsilon) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\} \ , \tag{2.3}$$

see [IR$_1$]–[IR$_3$]. Note that the condition (2.1) includes the assumption that φ is integrable in a neighborhood of the point z_0. We call $d_\varphi(z_0)$ the **dispersion** of the function φ at the point z_0. We say that an $L^1_{loc}(D)$ function $\varphi : D \to \mathbb{R}$ is of **finite mean oscillation** in the domain D, abbr. $\varphi \in$ FMO(D) or simply $\varphi \in$ **FMO**, if φ has a finite dispersion at every point $z \in D$.

2.4. Proposition. *If for some numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{D(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \ dxdy \ < \infty \, , \tag{2.5}$$

then φ is of finite mean oscillation at z_0.

Proof. Indeed, by the triangle inequality

$$\fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \ dxdy \ \le$$

$$\le \ \fint_{D(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \ dxdy \ + \ |\varphi_\varepsilon - \overline{\varphi}_\varepsilon(z_0)| \ \le$$

$$\le \ 2 \cdot \fint_{D(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \ dxdy \, .$$

2.6. Corollary. *If*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{D(z_0,\varepsilon)} |\varphi(z)| \ dxdy \ < \ \infty \ , \tag{2.7}$$

then φ has finite mean oscillation at z_0.

2.8. Remark. By John and Nirenberg, the function $\varphi(z) = log\frac{1}{|z|}$ belongs to BMO in the unit disk Δ, see e.g. [RR], p. 5, but $\overline{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$. Thus, the condition (2.7) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0.

Note that, if a function $\varphi : D \to \mathbb{R}$ is integrable over $D(z_0, \varepsilon_0)$, then

$$\fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \ dxdy \ \le \ 2 \cdot \overline{\varphi}_\varepsilon(z_0) \tag{2.9}$$

and the right side in (2.9) is continuous in the parameter $\varepsilon \in (0,\varepsilon_0]$ by the absolute continuity of the indefinite integral. Thus, for every $\delta_0 \in (0,\varepsilon_0)$,

$$\sup_{\varepsilon\in[\delta_0,\varepsilon_0]} \quad ⨍_{D(z_0,\varepsilon)} |\varphi(z)-\overline{\varphi}_\varepsilon(z_0)|\; dxdy \ < \infty \ . \tag{2.10}$$

If (2.1) holds, then the **maximal dispersion** d_0 of the function φ in $D(z_0,\varepsilon_0)$,

$$d_0 = \sup_{\varepsilon\in(0,\varepsilon_0]} \quad ⨍_{D(z_0,\varepsilon)} |\varphi(z)-\overline{\varphi}_\varepsilon(z_0)|\; dxdy \ , \tag{2.11}$$

is finite, too.

A point $z_0 \in D$ is called a **Lebesgue point** of a function $\varphi : D \to \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon\to 0} \quad ⨍_{B(z_0,\varepsilon)} |\varphi(z)-\varphi(z_0)|\; dxdy \ = \ 0 \ . \tag{2.12}$$

It is well known that, given $\varphi \in L^1(D)$, almost all points D are its Lebesgue points.

2.13. Corollary. *Every function $\varphi : D \to \mathbb{R}$ which is locally integrable has finite mean oscillation at almost all points in D.*

Below we use the following notations

$$D(r) = D(0,r) = \{z\in\mathbb{C} : |z| < r\} \tag{2.14}$$

and

$$A(\varepsilon,\varepsilon_0) = \{z\in\mathbb{C} : \varepsilon < |z| < \varepsilon_0\}\,. \tag{2.15}$$

Versions of the next lemma have been first proved for BMO in [RSY$_1$]–[RSY$_2$] and then for FMO in [IR$_1$]–[IR$_3$].

2.16. Lemma. *Let $D \subset \mathbb{C}$ be a domain such that $D(1/2) \subset D$, and let $\varphi : D \to \mathbb{R}$ be a nonnegative function. If f is integrable in $D(1/2)$ and FMO at 0, then*

$$\int\limits_{A(\varepsilon,1/2)} \frac{\varphi(z)\, dxdy}{\left(|z|\log_2\frac{1}{|z|}\right)^2} \le\ C\cdot \log_2 log_2 \frac{1}{\varepsilon} \tag{2.17}$$

for $\varepsilon\in(0,1/4)$, where

$$C = 4\pi\ [\varphi_0 + 6d_0]\ , \tag{2.18}$$

φ_0 is the mean value of φ over the disk $D(1/2)$ and

$$d_0 = \sup_{r\in(0,1/2]} \quad ⨍_{D(r)} |\varphi(z)-\overline{\varphi}_r(0)|\; dxdy \ < \ \infty \ . \tag{2.19}$$

With respect to finiteness of the maximal dispersion d_0 in $D(2^{-1})$, see Remark 2.8.

Proof. Let $0<\varepsilon<2^{-2}$, $\varepsilon_k=2^{-k}$, $A_k=\{z\in D:\varepsilon_{k+1}\le|z|<\varepsilon_k\}$, $D_k=D(\varepsilon_k)$ and let φ_k be the mean value of $\varphi(z)$ over D_k, $k=1,2\ldots$. Take a natural number N such that $\varepsilon\in[\varepsilon_{N+1},\varepsilon_N)$ and denote $\alpha(t)=(t\log 1/t)^{-2}$. Then $A(\varepsilon,2^{-1})\subset A(\varepsilon)=\bigcup\limits_{k=1}^{N}A_k$ and

$$\eta(\varepsilon)=\int\limits_{A(\varepsilon)}\varphi(z)\,\alpha(|z|)\;dxdy\;\le\;|S_1|+S_2$$

where

$$S_1(\varepsilon)\;=\;\sum_{k=1}^{N}\int\limits_{A_k}(\varphi(z)-\varphi_k)\,\alpha(|z|)\;dxdy\;,$$

$$S_2(\varepsilon)\;=\;\sum_{k=1}^{N}\varphi_k\int\limits_{A_k}\alpha(|z|)\;dxdy\;.$$

Since $A_k\subset D_k$, $|z|^{-2}\le 4\pi/|D_k|$ for $z\in A_k$ and $\log\frac{1}{|z|}>k$ in A_k, then

$$|S_1|\le 4\pi d_0\sum_{k=1}^{N}\frac{1}{k^2}<8\pi d_0$$

because

$$\sum_{k=2}^{\infty}\frac{1}{k^2}<\int\limits_1^{\infty}\frac{dt}{t^2}=1\,.$$

Now,

$$\int\limits_{A_k}\alpha(|z|)\;dxdy\;\le\;\frac{1}{k^2}\int\limits_{A_k}\frac{dxdy}{|z|^2}=\frac{2\pi}{k^2}\;.$$

Moreover,

$$|\varphi_k-\varphi_{k-1}|=\frac{1}{|D_k|}\left|\;\int\limits_{D_k}(\varphi(z)-\varphi_{k-1})\;dxdy\;\right|\le$$

$$\le\frac{4}{|D_{k-1}|}\int\limits_{D_{k-1}}|\varphi(z)-\varphi_{k-1}|\;dxdy\le 4d_0$$

and by the triangle inequality, for $k\ge 2$,

$$\varphi_k=|\varphi_k|\le\varphi_1+\sum_{l=2}^{k}|\varphi_l-\varphi_{l-1}|\le\varphi_1+4kd_0=\varphi_0+4kd_0\,.$$

Hence

$$S_2=|S_2|\;\le\;2\pi\sum_{k=1}^{N}\frac{\varphi_k}{k^2}\;\le\;4\pi\varphi_0+8\pi d_0\sum_{k=1}^{N}\frac{1}{k}\;.$$

But

$$\sum_{k=2}^{N}\frac{1}{k}<\int\limits_1^{N}\frac{dt}{t}=\log N<\log_2 N$$

and, for $\varepsilon < \varepsilon_N$,

$$N = \log_2 \frac{1}{\varepsilon_N} < \log_2 \frac{1}{\varepsilon} \ .$$

Consequently,

$$\sum_{k=1}^{N} \frac{1}{k} < 1 + \log_2 \log_2 \frac{1}{\varepsilon} \ ,$$

and, thus, for $\varepsilon \in (0, 2^{-2})$,

$$\eta(\varepsilon) \ \leq \ 4\pi \left(2d_0 \ + \ \frac{4d_0 + \varphi_0}{\log_2 \log_2 \frac{1}{\varepsilon}} \right) \cdot \log_2 \log_2 \frac{1}{\varepsilon} \ \leq \ C \cdot \log_2 \log_2 \frac{1}{\varepsilon} \ .$$

Example. The well-known function

$$\varphi(z) = \begin{cases} e^{\frac{1}{|z|^2-1}}, & \text{if } |z| < 1, \\ 0, & \text{if } |z| \geq 1 \end{cases} \tag{2.20}$$

belongs to C_0^∞ and hence $\varphi \in \text{BMO}$. Fix $\delta > 0$ and consider the function

$$\varphi^*_\delta(z) = \begin{cases} \varphi_k(z), & \text{if } z \in D_k, \\ 0, & \text{if } z \in \mathbb{C} \setminus \cup D_k \end{cases} \tag{2.21}$$

where $D_k = D_k(z_k, r_k) = \{z \in \mathbb{C} : |z - z_k| < r_k\}$, $z_k = 2^{-k}$, $r_k = 2^{-(1+\delta)k^2}$, and

$$\varphi_k(z) = 2^{2k^2} \varphi\left(\frac{z - z_k}{r_k} \right), \quad z \in D_k, \quad k = 1, 2, \dots \ . \tag{2.22}$$

Note that $\|\varphi_k\|_* = 2^{2k^2} \|\varphi\|_*$ and hence φ^* does not belong to BMO.

On the other hand,

$$\int\limits_{D_k} \varphi_k(z) \ dxdy \ = \ 2^{-2\delta k^2} \int\limits_{\mathbb{C}} \varphi(z) \ dxdy \tag{2.23}$$

and, consequently,

$$\overline{\lim_{\varepsilon \to 0}} \ ⨍_{D(\varepsilon)} \varphi^*_\delta(z) \ dxdy \ < \ \infty \ . \tag{2.24}$$

Thus, by Corollary 2.6 $\varphi \in \text{FMO}$.

Note that

$$\int\limits_{D_k} \varphi_k^{1+\delta}(z) \ dxdy = \int\limits_{\mathbb{C}} \varphi^{1+\delta}(z) \ dxdy \tag{2.25}$$

and hence φ^*_δ is not locally integrable at the origin with the degree $1 + \delta$.

By the construction, every function φ^*_δ has a support in the unit ball $\mathbb{B}$. Hence the function

$$\varphi^*(z) \ = \ \sum_{m=1}^{\infty} \varphi^*_{m^{-1}} \ (z + 2m) \tag{2.26}$$

is in FMO and φ^* is not in L^p_{loc} for any $p > 1$ although $\varphi^* \in L^1_{loc}$.

3 On radial Q–homeomorphisms

Given a domain D in $\mathbb{C}$ and a measurable function $Q : D \to [1, \infty]$, a homeomorphism $f : D \to \overline{\mathbb{C}}$ is said to be a **Q-homeomorphism** if

$$M(f\Gamma) \leq \iint\limits_{D} Q(z) \cdot \rho^2(z)\ dxdy \tag{3.1}$$

holds for every path family Γ in $\overline{\mathbb{C}}$ and each $\rho \in adm\,\Gamma$. This term was introduced in [MRSY$_1$]. The modulus inequality is appeared and used earlier in [RSY$_1$] and [RSY$_2$] in the study of Q-qc mappings and in [LV]. Recall that, for given a family of paths Γ in $\overline{\mathbb{C}}$, a Borel function $\rho : \overline{\mathbb{C}} \to [0, \infty]$ is called **admissible** for Γ, abbr. $\rho \in adm\,\Gamma$, if

$$\int\limits_{\gamma} \rho(z)\,|dz| \ \geq\ 1 \tag{3.2}$$

for each $\gamma \in \Gamma$. The **modulus** of Γ is the quantity

$$M(\Gamma) = \inf_{\rho \in adm\,\Gamma} \int\limits_{\mathbb{C}} \rho^2(z)\ dxdy\ . \tag{3.3}$$

A homeomorphism $f : D \to \mathbb{C}$ is called **super Q–homeomorphisms** if (3.1) holds not only for all families Γ of continuous paths $\gamma : (0, 1) \to D$ but also **dashed lines** $\gamma : \Delta \to D$ where the mapping γ is continuous on an arbitrary open subset Δ of the real axis $\mathbb{R}$ consisting of a sequence of intervals $\Delta_i \subset \mathbb{R}$, $i = 1, 2, \ldots$, see [IR$_1$]. Below we show that super Q–homeomorphisms is an important subclass of Q–homeomorphisms including many mappings with finite distortion. The definitions of the modulus, and properties of rectifiability are similar and hence are omitted. For more general considerations of systems of measures, see [Fu].

We say that a property P holds for **almost every (a.e.)** dashed lines γ in a family Γ if the subfamily of all dashed lines in Γ for which P fails has modulus zero. For instance, almost all dashed lines in $\overline{\mathbb{C}}$ are rectifiable. We say that a family Γ of dashed lines is minorized by Γ^*, abbr. $\Gamma \geq \Gamma^*$, if for every line $\gamma \in \Gamma, \gamma : \Delta \to \mathbb{C}$, there is a line $\gamma^* \in \Gamma^*, \gamma^* : \Delta^* \to \mathbb{C}$, which is a restriction of γ, i.e., $\Delta^* \subset \Delta$ and $\gamma^* = \gamma|_{\Delta^*}$. Later on, the following property is useful, see Theorem 1(c) in [Fu], c. 178.

3.4. Proposition. *Let Γ and Γ^* be families of dashed lines in $\overline{\mathbb{C}}$. If $\Gamma \geq \Gamma^*$, then $M(\Gamma) \leq M(\Gamma^*)$.*

We recall here some general properties and definitions of homeomorphisms in the Sobolev class $W^{1,2}_{loc}$. If $f : D \to \mathbb{C}$ is differentiable with $J(z, f) \neq 0$, then x is called a **regular point** of the mapping f.

3.5. Proposition. *Let $f : D \to \mathbb{C}$ be a homeomorphism of the class $W^{1,2}_{loc}$. Then f is differentiable a.e. and satisfies Lusin's property (N). If, in addition,*

f^{-1} also belongs to the class $W^{1,2}_{loc}$, then f^{-1} is differentiable a.e. and f^{-1} has the property (N) and

$$J(z,f)\ \neq\ 0 \quad \text{a.e.} \tag{3.6}$$

The statement follows from the well–known results for $W^{1,2}_{loc}$ homeomorphisms, see e.g. [LV], p. 121, 128–130 and 150, and the equivalence of (N^{-1})-property to the property (3.6) for differentiable a.e. mappings f, see Theorem 1 in [Po]. Recall that a mapping $f : X \to Y$ between measurable spaces (X, Σ, μ) and (X', Σ', μ') is said to have the **Lusin (N)–property** if $\mu'(f(S)) = 0$ whenever $\mu(S) = 0$. Similarly, f has the **$(\mathbf{N}^{-1})$–property** if $\mu(S) = 0$ whenever $\mu'(f(S)) = 0$.

3.7. Theorem. *Let $f : D \to \mathbb{C}$ be a sense–preserving homeomorphism of the class $W^{1,2}_{loc}$ and $f^{-1} \in W^{1,2}_{loc}$. Then f is a super Q-homeomorphism with*

$$Q(z)\ =\ K_\mu(z)\ =\ \frac{1+|\mu(z)|}{1-|\mu(z)|} \tag{3.8}$$

where $\mu(z) = \mu_f(z)$.

Proof. $f^{-1} \in ACL^2$, see e.g. [Ma], p. 8. Hence by the Fuglede theorem the modulus of all locally rectifiable paths in $f(D)$ with at least one closed subpath where f^{-1} is not absolutely continuous has modulus zero, see [Fu] and [Va$_1$]. This family of paths minorizes the corresponding family of dashed lines and, thus, by Proposition 3.4 the latter family also has modulus zero.

Let Γ be an arbitrary family of dashed lines in D. Let us denote by Γ_* the family of all dashed lines $\gamma_* \in f\Gamma$ for which f^{-1} is absolutely continuous on every closed subpath of γ_*. Then $M(f\Gamma) = M(\Gamma_*)$. For $\rho \in adm\,\Gamma$, set

$$\rho_*(w) = \{\rho/(|f_z| - |f_{\overline{z}}|)\} \circ f^{-1}(w) \tag{3.9}$$

if f is regular at the point $z = f^{-1}(w)$, i.e., it is differentiable with $J_f(z) \neq 0$, and $\rho_*(w) = \infty$ otherwise at $w \in f(D)$ and $\rho_*(w) = 0$ outside of $f(D)$. Note that $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 \neq 0$ a.e. because $f^{-1} \in W^{1,2}_{loc}$ has the property (N), see e.g. Theorem III.6.1 in [LV] and Theorem 1 in [Po]. Then

$$\int\limits_{\gamma_*} \rho_*\ ds_* \geq \int\limits_{f^{-1}\circ\gamma^*} \rho\ ds \geq 1 \tag{3.10}$$

for all $\gamma_* \in \Gamma_*$, i.e., $\rho_* \in adm\,\Gamma_*$.

Since by Proposition 3.5 f is differentiable a.e. with $J(z,f) \neq 0$ and have the property (N), and hence

$$M(f\Gamma)\ =\ M(\Gamma_*)\ \leq\ \int\limits_{f(D)} \rho_*^2(w)\ dudv\ =\ \int\limits_{D} \rho^2(z)\ K_\mu(z)\ dxdy\ ,$$

i.e., f is a super Q–homeomorphism with $Q(z) = K_\mu(z)$.

It is known that if f is a homeomorphism in the class $W^{1,2}_{loc}$ with $K_\mu \in L^1_{loc}$, then $f^{-1} \in W^{1,2}_{loc}$, see e.g. [HK]. Thus, we have the next conclusion.

3.11. Corollary. *Let $f : D \to \mathbb{C}$ be a sense-preserving homeomorphism of the class $W^{1,2}_{loc}$ with $K_\mu \in L^1_{loc}$. Then f is a super Q-homeomorphism with $Q(z) = K_\mu(z)$.*

3.12. Remark. Note that the inequality

$$M(f\Gamma) \le \iint\limits_D K_\mu(z) \cdot \rho^2(z)\ dxdy \tag{3.13}$$

obtained in Theorem 3.7 cannot be improved by replacing K_μ by a smaller function unless we restrict to either special families Γ or special $\rho \in \operatorname{adm}\Gamma$. This is done next allowing us to use modulus techniques for wider classes of mappings and thus deriving new criteria for existence of homeomorphic solutions to the Beltrami equation.

Given a domain D and two sets E and F in $\overline{\mathbb{C}}$, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \to \overline{\mathbb{C}}$ which join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. We set $\Gamma(E, F) = \Gamma(E, F, \overline{\mathbb{C}})$ if $D = \overline{\mathbb{C}}$. Let $R = R(C_1, C_2)$ be a **ring domain,** or shortly a **ring** in $\overline{\mathbb{C}}$, i.e., a doubly connected domain in $\overline{\mathbb{C}}$ with the connected components C_1 and C_2 of $\overline{\mathbb{C}} \setminus R$. The **capacity** of R can be defined by

$$cap\ R(C_1, C_2) \ = \ M(\Gamma(C_1, C_2, R))\ , \tag{3.14}$$

see e.g. [Ge$_2$] and [Zi].

Let us localize and extend the Q–homemomorphism notation. Given a domain $D \subset \overline{\mathbb{C}}$, $z_0 \in D$, $\varepsilon_0 > 0$ with $D(z_0, \varepsilon_0) \subset D$, and $Q : D(z_0, \varepsilon_0) \to [1, \infty]$, we say that a homemomorphism $f : D \to \overline{\mathbb{C}}$ is a **radial Q–homeomorphism** at the point $z_0 \in D$ if

$$cap\ R(fC_1, fC_2, fD)\ \le \iint\limits_R Q(z) \cdot \eta^2(|z - z_0|)\ dxdy \tag{3.15}$$

for all circles $C_1 = \{z \in \mathbb{C} : |z - z_0| = \varepsilon_1\}$ and $C_2 = \{z \in \mathbb{C} : |z - z_0| = \varepsilon_2\}$ in D with $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$ and for every measurable function $\eta : (\varepsilon_1, \varepsilon_2) \to [0, \infty]$ such that

$$\int\limits_{\varepsilon_1}^{\varepsilon_2} \eta(r)\ dr \ = \ 1\ . \tag{3.16}$$

Here R is the ring

$$R = R(\varepsilon_1, \varepsilon_2, z_0) = \{z \in \mathbb{C} : \varepsilon_1 < |z - z_0| < \varepsilon_2\}\ . \tag{3.17}$$

3.18. Remark. Note that every Q–homeomorphism $f : D \to \overline{\mathbb{C}}$ is a radial Q–homeomorphism at each point $z_0 \in D$.

Below we give other conditions for radial Q–homeomorphisms.

Given $\omega \in \mathbb{C}$, $|\omega| = 1$, and a regular point z for a mapping $f : D \to \mathbb{C}$, the **derivative in the direction** ω of the mapping f at the point z is

$$\partial_\omega f(z) \;=\; \lim_{t\to+0} \frac{f(z+t\cdot\omega)-f(z)}{t}\;. \tag{3.19}$$

For given z_0 and $z \in D$, $z \neq z_0$, $\partial_r^{z_0} f(z)$ denotes the derivative of f at the point z in the **radial direction**

$$\omega_0 \;=\; \omega_0(z,z_0) \;=\; \frac{z-z_0}{|z-z_0|} \tag{3.20}$$

with respect to the center z_0. The **radial dilatation** of the mapping f at the point z with respect to the center z_0 is

$$K^r(z,z_0,f) \;=\; \frac{|J(z,f)|}{|\partial_r^{z_0} f(z)|^2} \tag{3.21}$$

The **big radial dilatation** of the mapping f at the point z with respect to the center z_0 is

$$K^R(z,z_0,f) = \frac{|J(z,f)|}{|\partial_R^{z_0} f(z)|^2} \tag{3.22}$$

where

$$|\partial_R^{z_0} f(z)| \;=\; \min_{\omega\in\mathbb{C},|\omega|=1} \frac{|\partial_\omega f(z)|}{|\mathrm{Re}\,\omega\overline{\omega_0}|}\;. \tag{3.23}$$

Note that the geometrical sense of the quantity $\mathrm{Re}\,\omega\overline{\omega_0}$ is a scalar product of ω and ω_0 interpreted as vectors in $\mathbb{R}^2$. In the other words, $\mathrm{Re}\,\omega\overline{\omega_0}$ is the projection of the vector ω onto the radial direction ω_0. It is clear that

$$|\partial_R^{z_0} f(z)| \;=\; \frac{|\partial_{\omega_*} f(z)|}{|\mathrm{Re}\,\omega_*\overline{\omega_0}|} \tag{3.24}$$

for some direction ω_* in $\mathbb{C}$ and hence

$$|\partial_r^{z_0} f(z)| \;\geq\; |\partial_R^{z_0} f(z)| \;\geq\; \min_{\omega\in\mathbb{C},|\omega|=1} |\partial_\omega f(z)|\;. \tag{3.25}$$

Consequently,

$$K^r(z,z_0,f) \;\leq\; K^R(z,z_0,f) \;\leq\; K_\mu(z) \tag{3.26}$$

for every z_0 and $z \in D$ and the equality in (3.26) holds if and only if the minimum in the right side of (3.25) is realized for the radial direction $\omega = \omega_0$.

The **tangent dilatation** of the mapping f at the point z with respect to the center z_0 is

$$K^T(z,z_0,f) \;=\; \frac{|\partial_\tau^{z_0} f(z)|^2}{|J(z,f)|} \tag{3.27}$$

where $\tau \in \mathbb{C}$, $|\tau| = 1$, is a perpendicular vector to ω_0 from (3.20), i.e., the **tangent direction** at z to the circle centered at z_0, cf. [GMSV]. Note that by the geometrical reasons $\partial_r^{z_0} f(z) \neq 0$, $|\partial_R^{z_0} f(z)| \neq 0$ and $\partial_\tau^{z_0} f(z) \neq 0$ at the regular points of the mapping f, see e.g. 1.2.1 in [Re].

We show that the tangent dilatation coincides with the big radial dilatation.

3.28. Lemma. *Let $z \in D$ be a regular point of a mapping $f : D \to \mathbb{C}$ with the complex dilatation $\mu(z) = f_{\overline{z}}/f_z$ such that $|\mu(z)| < 1$. Then*

$$K^r(z, z_0, f) \;=\; K^r_\mu(z, z_0) := \frac{1 - |\mu(z)|^2}{\left|1 + \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2} \tag{3.29}$$

and

$$K^T(z, z_0, f) \;=\; K^R_\mu(z, z_0) := \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2} \;=\; K^R(z, z_0, f)\ . \tag{3.30}$$

Proof. The equality (3.29) follows directly from the calculation:

$$\partial_r f \;=\; \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r} \;+\; \frac{\partial f}{\partial \overline{z}}\cdot\frac{\partial \overline{z}}{\partial r} \;=\; \frac{z - z_0}{|z - z_0|}\cdot f_z \;+\; \frac{\overline{z - z_0}}{|z - z_0|}\cdot f_{\overline{z}} \tag{3.31}$$

where $|z - z_0| = r$ because $J(z, f) = |f_z|^2 - |f_{\overline{z}}|^2$. Similarly, the first equality in (3.30) follows from the relations:

$$\partial_\tau f = \frac{1}{r}\left(\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \vartheta} + \frac{\partial f}{\partial \overline{z}}\cdot\frac{\partial \overline{z}}{\partial \vartheta}\right) = i\cdot\left(\frac{z - z_0}{|z - z_0|}\cdot f_z - \frac{\overline{z - z_0}}{|z - z_0|}\cdot f_{\overline{z}}\right) \tag{3.32}$$

where $\vartheta = \arg{(z - z_0)}$.

It remains to prove the right side equality in (3.30). The derivative of f at the regular point z in the arbitrary direction $\omega = e^{i\alpha}$ is the quantity $\partial_\omega f(z) = f_z + f_{\overline{z}}\cdot e^{-2i\alpha}$, see e.g. [LV], p. 17 and 182. Consequently,

$$X := \frac{|\partial^{z_0}_R f(z)|^2}{|f_z|^2} \;=\; \min_{\alpha\in[0,2\pi]} \frac{|\mu(z) + e^{2i\alpha}|^2}{\cos^2(\alpha - \vartheta)} \;=\; \min_{\beta\in[0,2\pi]} \frac{|\nu - e^{2i\beta}|^2}{\sin^2\beta} \;=$$

$$= \min_{\beta\in[0,2\pi]} \frac{1 + |\nu|^2 - 2|\nu|\cos(\kappa - 2\beta)}{\sin^2\beta} \;=$$

$$= \min_{t\in[-1,1]} \frac{1 + |\nu|^2 - 2|\nu|\cdot[(1 - 2t^2)\cos\kappa - 2t(1 - t^2)^{1/2}\sin\kappa]}{t^2}$$

where $t = \sin\beta$, $\beta = \alpha + \frac{\pi}{2} - \vartheta$, $\nu = \mu(z)e^{-2i\vartheta}$ and $\kappa = \arg\nu = \arg\mu - 2\vartheta$. Hence $X = \min\limits_{\tau\in[1,\infty]} \varphi_\pm(\tau)$ where $\tau = 1/\sin^2\beta$,

$$\varphi_\pm(\tau) = b + a\tau \pm c(\tau - 1)^{1/2}, \quad a = 1 + |\nu|^2 - 2|\nu|\cos\kappa, \quad b = 4|\nu|\cos\kappa \quad c = 4|\nu|\sin\kappa.$$

Since $\varphi'_\pm(\tau) = a \pm (\tau - 1)^{-1/2}c/2$ the minimum is realized for $\tau = 1 + c^2/4a^2$ under $(\tau - 1)^{1/2} = \mp c/2a$, correspondingly, where the signes are agreed. Thus,

$$X \;=\; b + \left(a + \frac{1}{4}\frac{c^2}{a}\right) - \frac{1}{2}\frac{c^2}{a} \;=\; \frac{(1 - |\nu|^2)^2}{1 + |\nu|^2 - 2|\nu|\cos\kappa}$$

that implies the right side equality in (3.30).

Some prototypes of the following theorem can be found in [RW], [Le] and [GMSV].

3.33. Theorem. *Let $f : D \to \mathbb{C}$ be a sense–preserving homeomorphism of the class $W^{1,2}_{loc}$ such that $f^{-1} \in W^{1,2}_{loc}$. Then f is a radial Q-homeomorphism at every point $z_0 \in D$ with $Q(z) = K^R_\mu(z, z_0)$, $\mu(z) = \mu_f$.*

Proof. Let $S_1 = \{z \in \mathbb{C} : |z - z_0| = \varepsilon_1\}$ and $S_2 = \{z \in \mathbb{C} : |z - z_0| = \varepsilon_2\}$ be an arbitrary pair of circles such that $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0 \leq \operatorname{diam} D$. Set $\Gamma = \Gamma(E, F, D)$ where $E = D \cap S_1$ and $F = D \cap S_2$ and denote by Γ_* the family of all rectifiable paths $\gamma_* \in f\Gamma$ for which f^{-1} is absolutely continuous on every closed subpath of γ_*. Then $M(f\Gamma) = M(\Gamma_*)$ by the Fuglede theorem, see [Fu] and [Va$_1$], because $f^{-1} \in ACL^2$, see e.g. [Ma], p. 8.

Note that along $\gamma = f^{-1} \circ \gamma_*$ at a regular point $z = \gamma(s)$ of f under differentiability of $\gamma(s)$ in s we have that , for $r = |z - z_0|$,

$$\left|\frac{dr}{ds_*}\right| = \frac{\frac{dr}{ds}}{\frac{ds_*}{ds}} = \lim_{\Delta s \to 0} \frac{|\Delta r|}{|\gamma_*'(s) \cdot \Delta s|} = \lim_{\Delta z \to 0} \frac{|\Delta r|}{|\partial_{\omega'} f(z)| \cdot |\Delta z|} = \frac{|\operatorname{Re} \omega' \overline{\omega_0}|}{|\partial_{\omega'} f(z)|}$$

where ω' is a tangent direction to the curve γ at the point $z = \gamma(s)$. Hence

$$\left|\frac{dr}{ds_*}\right| \leq \frac{1}{|\partial^{z_0}_R f(z)|} \tag{3.34}$$

where $|\partial^{z_0}_R f(z)|$ is defined by (3.23).

Let $\eta : (\varepsilon_1, \varepsilon_2) \to [0, \infty]$ be an arbitrary measurable function with the property (3.16). Then by the Lusin theorem there is a Borel function $\eta_* : (\varepsilon_1, \varepsilon_2) \to [0, \infty]$ such that $\eta_*(t) = \eta(t)$ a.e., see e.g. 2.3.5 in [Fe] and [Sa], p. 69. Let

$$\rho(z) = \eta_*(|z - z_0|)$$

if $z \in D \cap R$ where R is the ring defined by (3.17) and $\rho(z) = 0$ otherwise. Now, set

$$\rho_*(w) = \{\rho/|\partial^{z_0}_R f|\} \circ f^{-1}(w)$$

if f is differentiable and $|\partial^{z_0}_R f(z)| \neq 0$ at the point $z = f^{-1}(w)$ and $\rho_*(w) = \infty$ otherwise at $w \in f(D)$ and $\rho_*(w) = 0$ outside $f(D)$.

By Proposition 3.5 f and f^{-1} are regular a.e. and have the property (N). Then by (3.16) and (3.34) ρ_* is admissible for a.e. $\gamma_* \in \Gamma_*$,

$$\int_{\gamma_*} \rho_* ds_* = \int_{\gamma} \frac{\rho(z(s))}{|\partial^{z_0}_R f(z(s))|} \frac{ds_*}{ds} ds = \int_{\gamma} \frac{\rho(z(s))}{|\partial^{z_0}_R f(z(s))|} \frac{|\frac{dr}{ds}|}{|\frac{dr}{ds_*}|} ds \geq$$

$$\geq \int_{\gamma} \rho(z(s)) \left|\frac{dr}{ds}\right| ds \geq \int_{\varepsilon_1}^{\varepsilon_2} \eta(r)\, dr = 1 ,$$

see also Lemma 2.7 in [MRSY$_3$]. Hence, using the change of variables, see e.g. Theorem 6.4 in [Mu], we have that

$$M(f\Gamma) \leq \int_{f(D)} \rho_*(w)^2 dudv = \int_{D} \rho(z)^2\, K^R_\mu(z, z_0)\, dxdy =$$

$$= \int\limits_{D\cap R} K_\mu^R(z, z_0) \cdot \eta^2(|z - z_0|)\ dxdy\ ,$$

i.e., f is really a radial Q–homeomorphism with $Q(z) = K_\mu^R(z, z_0)$.

The following statement is a consequence of Corollary 3.11, see Remark 3.38.

3.35. Corollary. *Let $f : D \to \mathbb{C}$ be a sense-preserving homeomorphism of the class $W^{1,2}_{loc}$ with $K_\mu \in L^1_{loc}$. Then f is a radial Q-homeomorphism at the every point $z_0 \in D$ with $Q(z) = K_\mu(z)$.*

If f is a plane homeomorphism in $W^{1,2}_{loc}$ with a locally integrable $K_\mu(z)$, then $f^{-1} \in W^{1,2}_{loc}$, see e.g. [HK]. Hence we obtain the following consequence of Theorem 3.33.

3.36. Corollary. *Let $f : D \to \mathbb{C}$ be a sense-preserving homeomorphism of the class $W^{1,2}_{loc}$ and suppose that $K_\mu(z)$ is integrable in $D(z_0, \varepsilon_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon_0\} \subset D$ for some $z_0 \in D$ and $\varepsilon_0 > 0$. Then f is a radial Q-homeomorphism at the point $z_0 \in D$ with $Q(z) = K_\mu^R(z, z_0)$.*

3.37. Theorem. *Let $f_n : D \to \overline{\mathbb{C}}$, $n = 1, 2, \ldots$, be a sequence of radial Q–homemorphisms at a point $z_0 \in D$. If f_n converge locally uniformly to a homeomorphism $f : D \to \overline{\mathbb{C}}$, then f is also a radial Q–homeomorphism at the point z_0.*

Indeed, it follows from the uniform convergence of the rings $f_n R(C_1, C_2)$ to the ring $fR(C_1, C_2)$, see [Ge$_3$] and [Ge$_4$].

3.38. Remark. Note that the equality of the tangent dilatation and the big radial dilatation in the plane case does not mean that the minimum in the definition of the big radial dilatation (3.23) is realized for the tangent direction τ. Just $\operatorname{Re} \tau\overline{\omega_0} = 0$ and we have then ∞ in (3.23) at every regular point z. In fact, the corresponding direction α was calculated under the proof of Lemma 3.28 from where it is also clear that $\sin \beta \neq 0$.

Note also we have started to study the space case in collaboration with Professor Olli Martio and have found that the big radial dilatation plays the similar role there but not the corresponding tangent dilatation. In the space case, the coincidence of the big radial dilatation and the tangent dilatation does not hold in the contrary with Lemma 3.28 for the plane case.

4 Estimates of distortion in a domain

In accordance with the approximate scheme given in the introduction we need distortion estimates in the spherical metric for the corresponding classes of mappings. This problem is considered in the section.

By the well-known Gehring lemma

$$cap\ R(C_1, C_2)\ \geq\ cap\ R_T\left(\frac{1}{\delta(C_1)\delta(C_2)}\right) \tag{4.1}$$

for every disjoint continua C_1 and C_2 in $\overline{\mathbb{C}}$ where $\delta(E)$ denotes the spherical diameter of a set $E \subset \overline{\mathbb{C}}$ and $R_T(t)$ is the **Teichmüller ring**

$$R_T(t)\ =\ \overline{\mathbb{C}} \setminus ([-1, 0] \cup [t, \infty])\ , \qquad t\ >\ 1\ , \tag{4.2}$$

see e.g. 7.37 in [Vu] or [Ge$_5$]. It is also known that

$$cap\ R_T(t) = \frac{2\pi}{\log \Phi(t)} \tag{4.3}$$

where the function Φ admits the good estimates:

$$t + 1 \leq \Phi(t) \leq 16 \cdot (t + 1) < 32 \cdot t, \quad t > 1\ , \tag{4.4}$$

see e.g. (7.19) and (7.22) in [Vu]. Hence the Gehring inequality (4.1) implies that

$$M(\Gamma(E, F, R))\ >\ \frac{2\pi}{\log \frac{32}{\delta(E)\delta(F)}} \tag{4.5}$$

for every continua E and F in $\overline{\mathbb{C}}$.

Given $\Delta > 0$, a domain $D \subset \mathbb{C}$, a point $z_0 \in D$, and a measurable function $Q : D(z_0, \varepsilon_0) \to [1, \infty)$, $D(z_0, \varepsilon_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon_0, \}$, $\varepsilon_0 \leq \delta_0 = \operatorname{dist}(z_0, \partial D)$, let $\mathfrak{R}_Q^\Delta$ denote the class of all radial Q–homeomorphisms $f : D \to \overline{\mathbb{C}}$ at z_0 such that

$$\delta(\overline{\mathbb{C}} \setminus f(D)) \geq \Delta\ . \tag{4.6}$$

Let $\mathfrak{A}_Q^\Delta$ denote the class of all quasiconformal mappings with (4.6) and $\mu = \mu_f$ such that

$$K_\mu^R(z, z_0)\ =\ \frac{\left|1 - \frac{\overline{z - z_0}}{z - z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2}\ \leq\ Q(z) \quad \text{a.e.} \tag{4.7}$$

Similarly, let $\mathfrak{F}_Q^\Delta$ denote the class of all quasiconformal mappings f with (4.6) and

$$K_\mu(z)\ =\ \frac{1 + |\mu(z)|}{1 - |\mu(z)|}\ \leq\ Q(z) \quad \text{a.e.} \tag{4.8}$$

4.9. Remark. By Corollaries 3.35 and 3.36, the relations (3.26) and (3.30)

$$\mathfrak{F}_Q^\Delta\ \subset\ \mathfrak{A}_Q^\Delta\ \subset\ \mathfrak{R}_Q^\Delta\ . \tag{4.10}$$

Fix $z_0 \in D$, $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 \leq \operatorname{dist}(z_0, \partial D)$, let

$$A(\varepsilon) = \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}\ . \tag{4.11}$$

4.12. Lemma. *Let $f \in \mathfrak{R}^{\Delta}_{Q}$ and let $\psi_\varepsilon(t) : [0,\infty] \to [0,\infty]$ be a measurable function such that*

$$0 \ < \ I(\varepsilon) \ = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_\varepsilon(t) \ dt \ < \ \infty \ , \quad \varepsilon \in (0,\varepsilon_0) \ . \tag{4.13}$$

Then, for all $\zeta \in D(z_0,\varepsilon_0)$,

$$s(f(\zeta), f(z_0)) \ \leq \ \frac{32}{\Delta} \cdot \exp\left(-\frac{2\pi}{\omega(|\zeta - z_0|)}\right) \tag{4.14}$$

where

$$\omega(\varepsilon) \ = \ \frac{1}{I^2(\varepsilon)} \int\limits_{A(\varepsilon)} Q(z) \cdot \psi_\varepsilon^2(|z - z_0|) \ dxdy \ . \tag{4.15}$$

Proof. Let Γ_ε be a family of all paths joining the circles $C_\varepsilon = \{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$ and $C_0 = \{z \in \mathbb{C} : |z - z_0| = \varepsilon_0\}$ in $A(\varepsilon)$. Let also ψ_ε^* be a Borel function such that $\psi_\varepsilon^*(t) = \psi_\varepsilon(t)$ for a.e. $t \in [0,\infty]$. Such a function ψ_ε^* exists by the Lusin theorem, see e.g. 2.3.5. in [Fe] and [Sa], p. 69. Then the function

$$\rho_\varepsilon(z) = \begin{cases} \psi_\varepsilon^*(|z - z_0|)/I(\varepsilon), & \text{if } z \in A(\varepsilon), \\ 0, & \text{if } z \in \mathbb{C}\backslash A(\varepsilon), \end{cases}$$

is admissible for Γ_ε and hence

$$M(f\Gamma_\varepsilon) \leq \int\limits_{A(\varepsilon)} Q(z) \cdot {\rho_\varepsilon}^2(z) \ dxdy \ .$$

By the Jordan theorem each of the images fC_t of the circles C_t, $t \in (0,\varepsilon_0)$, splits the plane $\mathbb{C}$ into two components and, thus, $\mathbb{C}\backslash fA(\varepsilon)$ consists exactly of two components. Denote by Γ_ε^* the family of all paths in $\mathbb{C}$ joining the images of the spheres fC_ε and fC_0. Then

$$M(\Gamma_\varepsilon^*) = M(f\Gamma_\varepsilon)$$

because on the one hand $f\Gamma_\varepsilon \subset \Gamma_\varepsilon^*$ and hence $M(f\Gamma_\varepsilon) \leq M(\Gamma_\varepsilon^*)$, and on the other hand $f\Gamma_\varepsilon < \Gamma_\varepsilon^*$ (i.e., every path in Γ_ε^* contains a subpath in $f\Gamma_\varepsilon$ because $fA(\varepsilon)$ separates the two components of $\mathbb{C} \setminus fA(\varepsilon)$). Consequently, $M(f\Gamma_\varepsilon) \geq M(\Gamma_\varepsilon^*)$, see e.g. either Theorem 1(c) in [Fu] or 6.4 in [Va$_1$]. Finally, choosing $\varepsilon = |\zeta - z_0|$ by (4.5) we obtain (4.14).

4.16. Corollary. *If $f \in \mathfrak{R}^{\Delta}_{Q}$ with Q such that*

$$\int\limits_{A(\varepsilon)} Q(z) \cdot \psi_\varepsilon^2(|z - z_0|) \ \ dxdy \ \leq \ C \cdot I(\varepsilon) \tag{4.17}$$

where

$$0 \ < \ I(\varepsilon) \ = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_\varepsilon(t) \ dt < \infty \tag{4.18}$$

for measurable functions $\psi_\varepsilon(t) : [0,\infty] \to [0,\infty]$, $\varepsilon \in (0,\varepsilon_0)$, *then*

$$s(f(\zeta), f(z_0)) \ \le \ \frac{32}{\Delta} \cdot \exp\left(-\frac{2\pi}{C} \cdot I(|\zeta - z_0|)\right) \ . \tag{4.19}$$

4.20. Theorem. *Let* $f \in \mathfrak{R}^{\Delta}_{Q}$ *for* $\Delta > 0$ *and suppose that* Q *has finite mean oscillation at* z_0 *and that* f *is integrable over* $D(z_0, \varepsilon_0) \subset D$*. Then*

$$s(f(\zeta), f(z_0)) \ \le \ \frac{32}{\Delta} \cdot \left(\log \frac{2\varepsilon_0}{|\zeta - z_0|}\right)^{-\beta_0} \tag{4.21}$$

for every point $\zeta \in B(z_0, \varepsilon_0/2)$ *where*

$$2\beta_0 \ = \ [q_0 + 6d_0]^{-1} \ , \tag{4.22}$$

q_0 *is the mean value and* d_0 *is the maximal dispersion of* $Q(z)$ *in* $B(z_0, \varepsilon_0)$.

Proof. The mean value and the dispersion of a function over disks are invariant under linear transformations $w = (z - z_0)/2\varepsilon_0$. Hence, (4.21) follows by Corollary 4.16 and Lemma 2.16.

4.23. Theorem. *If* $f \in \mathfrak{R}^{\Delta}_{Q}$*, then*

$$s(f(\zeta), f(z_0)) \ \le \ \frac{32}{\Delta} \cdot \exp\left(-\int\limits_{|\zeta - z_0|}^{\varepsilon_0} \frac{dr}{rq(r)}\right) \tag{4.24}$$

for all $\zeta \in D(z_0, \varepsilon_0)$ *where* $q(r)$ *is the mean integral value of* $Q(z)$ *over the circle* $|z - z_0| = r$.

Proof. Indeed, for the function

$$\psi(t) = \begin{cases} 1/[tq(t)], & t \in (0, \varepsilon_0), \\ 0, & t \in [\varepsilon_0, \infty], \end{cases} \tag{4.25}$$

we have that

$$\int\limits_{A(\varepsilon)} Q(z) \cdot \psi^2(|z - z_0|) \ dxdy \ = \ 2\pi \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t) \ dt \tag{4.26}$$

and, thus, (4.24) follows from (4.19).

4.27. Remark. Note that the mean value $q(r)$ of $Q(z)$ over some circles $|z - z_0| = r$ can be infinite. However, $q(r)$ is measurable in the parameter r because $Q(z)$ is measurable in z. Moreover,

$$\int\limits_{|\zeta - z_0|}^{\varepsilon_0} \frac{dr}{rq(r)} < \infty \tag{4.28}$$

for $z \ne z_0$ because $q(r) \ge 1$. The integral can be equal to 0 if $q(r) = \infty$ a.e. but then the inequality (4.24) is obvious because Δ as well as $s(f(\zeta), f(z_0))$ is less or equal to 1.

4.29. Corollary. *If $f \in \mathfrak{R}_Q^{\Delta}$, then*

$$s(f(\zeta), f(z_0)) \ \leq \ \frac{32}{\Delta} \cdot exp\left(- \int\limits_{|\zeta - z_0|}^{\varepsilon_0} \frac{dr}{rK(r)} \right) \tag{4.30}$$

for all $\zeta \in D(z_0, \varepsilon_0)$ where

$$K(r) = \sup_{t \in [r, \varepsilon_0]} Q_t \tag{4.31}$$

and

$$Q_t \ = \ \fint_{B(z_0,t)} Q(z) \ dxdy \ . \tag{4.32}$$

Proof. Indeed, by the theorem on the differentiability of indefinite integral, see e.g. [Sa], p. 118, applied to $\int\limits_0^t \varphi(\rho) \ d\rho$ where $\varphi(\rho) = \int\limits_{|z-z_0|=\rho} Q(z)$ and $|B(z_0, t)| = \int\limits_0^t l(z_0, \rho) \ d\rho$ where $l(z_0, \rho)$ is the length of the circle $|z - z_0| = \rho$ we have the inequality

$$q(r) \ \leq \ K(r) \tag{4.33}$$

for a.e. $r \in (0, \varepsilon_0)$ and hence (4.24) implies (4.30).

Below we use the **mean logarithmic value** of Q over the annulus $A(\varepsilon)$:

$$M_{log}^Q(\varepsilon) = \fint\limits_{\varepsilon}^{\varepsilon_0} q(t) \ d\log t = \frac{1}{\log \varepsilon_0/\varepsilon} \int\limits_{\varepsilon}^{\varepsilon_0} q(t) \ \frac{dt}{t} \ . \tag{4.34}$$

As above, $q(t)$ denotes the mean value of Q over the circle $|z - z_0| = t$.

In particular, choosing $\psi_\varepsilon(t) = 1/t$ in Lemma 4.12, we have the next consequence.

4.35. Corollary. *If $f \in \mathfrak{R}_Q^{\Delta}$, then*

$$s(f(\zeta), f(z_0)) \ \leq \ \frac{32}{\Delta} \cdot \left(\frac{|\zeta - z_0|}{\varepsilon_0} \right)^{1/M_{log}^Q(|\zeta - z_0|)} \tag{4.36}$$

for all $\zeta \in D(z_0, \varepsilon_0)$.

As we see, Lemma 4.12 can serve as a rich source of various distortion estimates for radial Q–homeomorphisms.

5 Integral and measure conditions

The estimate of the distortion (4.30) is the base for the consideration of the classes of mappings with various integral and measure constraints for dilatations $K_\mu(z)$ and $K_\mu^R(z, z_0)$.

Let (X, Σ) and (X', Σ') be measurable spaces and let $g : X \to X'$ be a measurable transformation, i.e., $S = g^{-1}S' \in \Sigma$ for all $S' \in \Sigma'$. If m is a measure on the σ–algebra Σ, then the measure $m' = mg^{-1}$ on the σ–algebra Σ' is given by the equality

$$m'(S') \ = \ mg^{-1}(S') \ = \ m(g^{-1}(S')) \ , \quad \forall \ S' \in \Sigma' \ . \tag{5.1}$$

Below we use the following statement on the **change of variables**, see Theorem 39C in [Ha]. Here $\overline{\mathbb{R}}$ denotes the extended real axes $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

5.2. Proposition. *Let $g : X \to X'$ be a measurable transformation from the measurable space (X, Σ) with a measure m into a measurable space (X', Σ'). Then a function $f : X' \to \overline{\mathbb{R}}$ is integrable with respect to the measure $m' = mg^{-1}$ if and only if the function $f \circ g$ is integrable with respect to the measure m and, in the case of integrability, the equality*

$$\int\limits_{g^{-1}(S')} f(g(x)) \ dm \ = \ \int\limits_{S'} f(x') \ dm' \tag{5.3}$$

holds for every set $S' \in \Sigma'$.

Let (X, Σ) be a measurable space and let m be a measure on the σ–algebra Σ. Below we also use the **measure function** $M(t)$ for $g : X \to \overline{\mathbb{R}}$

$$M(t) \ = \ M_g(t) \ = \ m(\{x \in X : \ g(x) > t\}) \tag{5.4}$$

for which the following statement holds, see e.g. [MZ], p. 3.

5.5. Proposition. *Let (X, Σ) be a mesurable space and let $g : X \to [0, \infty]$ be a measurable function with respect to a measure m on Σ. Then*

$$\int\limits_{X} g \ dm \ = \ \int\limits_{0}^{+\infty} M(t) \ dt \ . \tag{5.6}$$

Now, if a function $Q(z) : D \to [1, \infty]$, $D \subset \mathbb{C}$, is measurable and $\Phi : [1, \infty] \to [0, \infty]$ is nondecreasing and

$$\int\limits_{U} \Phi(Q(z)) \ dxdy \ \le \ C \ < \ \infty \tag{5.7}$$

where $U \subset D$ is a neighborhood of a point $z_0 \in D$, then

$$M(t) \ \le \ \frac{C}{\Phi(t)} \tag{5.8}$$

where $M(t)$ is the measure function for Q with respect to the Lebesgue measure m in $\mathbb{C}$,

$$M(t) \ = \ M_{Q,U}(t) \ = \ m(\{z \in U : \ Q(z) > t\}) \ , \quad t \ \ge \ 1 \ . \tag{5.9}$$

Thus, later on we may restrict ourselves by the consideration of the constraints by measure only. Let $Q : D \to [1,\infty]$ be a measurable function, $z_0 \in D \subset \mathbb{C}$, U be a bounded neighborhood of z_0 and let

$$M(t) \ \le \varphi(t) \tag{5.10}$$

where we will always assume below that

$$\liminf_{t\to\infty} \ \varphi(t) \ = \ 0 \ . \tag{5.11}$$

The latter condition means that $Q(z) < \infty$ a.e.

Here we may also assume without loss of generality that the function $\varphi : [1,\infty] \to [0,\infty]$, $\varphi(\infty) = 0$, is nonincreasing and continuous from the right. Indeed, the same class of Q is defined by the **regular envelope** φ_r of φ :

$$\varphi_r(t) \ = \ \lim_{b\to t+0} \ \inf_{1\le a\le b} \ \varphi(a) \ . \tag{5.12}$$

This is the biggest function which is nonincreasing and continuous from the right such that $\varphi_r(t) \le \varphi(t)$ for all $t \in [1,\infty]$.

Note that the inverse function $\varphi^{-1} : [0,\infty] \to [1,\infty]$ can be correctly defined for any nonincreasing and continuous from the right $\varphi : [1,\infty] \to [0,\infty]$ with (5.11). More precisely, we set $\varphi^{-1}(\tau) = 1$ if $\tau > \varphi(1)$ and

$$\varphi^{-1}(\tau) \ = \ \sup_{\varphi(t)\ge\tau} \ t \tag{5.13}$$

if $\tau \le \varphi(1)$. It is clear that φ^{-1} is nonincreasing and continuous from the left.

5.14. Lemma. *Let $D \subset \mathbb{C}$ be a domain and let a function $Q : D \to [1,\infty]$ satisfy (5.10) in $U \subset D$. If the function φ in (5.10) is nonincreasing and continuous from the right and*

$$\liminf_{t\to\infty} \ t\cdot\varphi(t) \ = \ 0 \ , \tag{5.15}$$

then

$$\fint_{D(z_0,r)} Q(z) \ dxdy \ \le \ \fint_0^{\pi r^2} \varphi^{-1}(\tau) \ d\tau \tag{5.16}$$

in every disk $D(z_0, r) \subset U$.

Proof. By Proposition 5.5 we have the following estimate from the condition (5.10):

$$\int_{D(z_0,r)} Q(z) \ dxdy \ \le \ t(r)\cdot\pi r^2 + \int_{t(r)}^{\infty} \varphi(t) \ dt \tag{5.17}$$

for every disk $D(z_0, r) \subset U$ where

$$t(r) \ = \ \varphi^{-1}(\pi r^2) \ . \tag{5.18}$$

Indeed, set $Q_r^*(z) = Q_r(z) - t(r)$ on $D(z_0, r)$ where

$$Q_r(z) = \begin{cases} Q(z), & \text{if} \quad Q(z) \geq t(r), \\ t(r), & \text{if} \quad Q(z) < t(r). \end{cases} \tag{5.19}$$

Note that by definition $Q_r^*(z) \geq 0$ and, moreover, $M_r^*(t) = M(t + t(r))$, $t \geq 0$, in $D(z_0, r)$ and hence by (5.6)

$$\int\limits_{D(z_0,r)} Q_r^*(z)\ dxdy \ = \ \int\limits_0^\infty M_r^*(t)\ dt \ = \ \int\limits_{t(r)}^\infty M(t)\ dt \tag{5.20}$$

that just implies (5.17) because $Q(z) \leq Q_r(z)$ on $D(z_0, r)$.

Further, applying integration by parts, we have from (5.17) that

$$\int\limits_{D(z_0,r)} Q(z)\ dxdy \ \leq \ t(r)(\varphi(t(r)) - \varphi(t(r) - 0)) + \tag{5.21}$$

$$\liminf_{t \to \infty} \left(t \cdot \varphi(t) - \int\limits_{t(r)}^{t} \eta\ d\varphi(\eta) \right) \ \leq \ - \int\limits_{t(r)}^{\infty} \eta\ d\varphi(\eta)\ ,$$

see Theorem III(14.1) in [Sa]. Finally, (5.16) follows from (5.21) by the change of variables $\tau = \varphi(\eta)$ because φ is nonincreasing and $\varphi^{-1}(\varphi(\eta)) \geq \eta$.

Note that the right side in (5.16) is nonincreasing in the parameter r because φ^{-1} is noninsreasing. Hence we come from Lemma 5.14 to the following consequence of Corollary 4.29.

5.22. Theorem. *Let $f \in \mathfrak{R}_Q^\Delta$ and Q satisfy (5.10). If the function φ in (5.10) is nonincreasing and continuous from the right and (5.15) holds, then*

$$s(f(\zeta), f(z_0)) \ \leq \ \frac{32}{\Delta} \cdot exp \left(-\frac{1}{2} \int\limits_{\pi|\zeta - z_0|^2}^{\pi\varepsilon_0^2} \frac{d\rho}{\int_0^\rho \varphi^{-1}(\tau) d\tau} \right) \tag{5.23}$$

for all $\zeta \in D(z_0, \varepsilon_0)$ where ε_0 is from the definition of the class $\mathfrak{R}_Q^\Delta$.

6 Existence theorems

The next lemma is a very general principle on the existence of solutions for the Beltrami equations.

6.1. Lemma. *Let D be a domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. with $K_\mu \in L^1_{loc}$ and such that, for every point $z_0 \in D$,*

$$\int\limits_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^R(z, z_0) \cdot \psi^2_{z_0,\varepsilon}(|z - z_0|)\ dxdy = o(I^2_{z_0}(\varepsilon)) \tag{6.2}$$

as $\varepsilon \to 0$ for some $\varepsilon_0 < \delta(z_0) = dist\,(z_0, D)$ and measurable functions $\psi_{z_0,\varepsilon}$: $(0,\infty) \to (0,\infty)$ such that

$$0 < I_{z_0}(\varepsilon) = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt < \infty, \quad \varepsilon \in (0,\varepsilon_0)\ . \tag{6.3}$$

Then the Beltrami equation (1.1) has a homeomorphic ACL solution $f_\mu : D \to \mathbb{C}$ such that $f_\mu^{-1} \in W^{1,2}_{loc}$. Moreover, f is a radial Q-homeomorphism at every point $z_0 \in D$ with $Q(z) = K^R_\mu(z, z_0)$.

The rest existence theorems in the section are direct consequences of the lemma under the corresponding choice of the functions ψ. Another way to prove the existence theorems is to apply the distortion theorems from the previous sections and to repeat the general scheme given in Introduction and realised in the proof of Lemma 6.1 below.

Proof. For $n \in \mathbb{N}$, define $\mu_n : D \to \mathbb{C}$ by letting $\mu_n(z) = \mu(z)$ if $|\mu(z)| \le 1-1/n$ and 0 otherwise. Let $f_n : D \to \mathbb{C}$ be a homeomorphic ACL solution of (1.1), with μ_n instead of μ, which fixes z_1 and z_2, see Theorem A. By Theorem C and Lemma 4.12 it follows that the sequence f_n has a subsequence, denoted again by f_n, which converges locally uniformly to some non-constant mapping f in D. Then by Theorem B f is $Q(z)$-qc with $Q = K_\mu$ and satisfies (1.1) a.e.

Now, note that the local convergence $f_n \to f$ is equivalent to its continuous convergence, i.e., $f_n(z_n) \to f(z_0)$ if $z_n \to z_0$, see [Du, p. 268]. Since f is injective, it follows that $g_n = f_n^{-1} \to f^{-1} = g$ continuously, and hence locally uniformly. By direct computation we obtain that for large n

$$\iint\limits_B |\partial g_n|^2\ dudv\ = \int\limits_{g_n(B)}\!\!\int \frac{dxdy}{1-|\mu_n(z)|^2}\ \le \iint\limits_{B^*} Q(z)\ dxdy\ <\ \infty$$

where B and B^* are relatively compact domains in D and in $f(D)$, respectively, such that $g(\bar{B}) \subset B^*$. Note that the change of variables is allowed since g_n and f_n are in $\mathrm{W}^{1,2}_{loc}$. The last relation implies that the sequence g_n is bounded in $\mathrm{W}^{1,2}(B)$, and hence $f^{-1} \in \mathrm{W}^{1,2}_{loc}(f(D))$, see e.g. [Re], p. 319.

6.4. Corollary. *f_μ^{-1} is locally absolutely continuous and preserves nulls sets, and f_μ is regular, i.e., differentiable with $J_{f_\mu}(z) > 0$ a.e.*

Indeed, the assertion about f_μ^{-1} follows from the fact that $f_\mu^{-1} \in \mathrm{W}^{1,2}_{loc}$, see [LV], p. 131 and 150. For the regularity of f_μ, let E denote the set of points of D where f_μ is differentiable and $J_{f_\mu}(z) = 0$, and suppose that $|E| > 0$. Then $|f_\mu(E)| > 0$, since $E = f_\mu^{-1}(f_\mu(E))$ and f_μ^{-1} preserves null sets. Clearly f_μ^{-1} is not differentiable at any point of $f_\mu(E)$, contradicting the fact that f_μ^{-1} is differentiable a.e.

6.5. Corollary. *In particular, the conclusions hold if $K_\mu \in L^1_{loc}$ and at every point $z_0 \in D$*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu(z)\cdot\psi^2(|z-z_0|)\ dxdy\ \le\ C\cdot I(\varepsilon) \tag{6.6}$$

and $I(\varepsilon) \to \infty$ as $\varepsilon \to 0$ for a measurable function $\psi : (0, \infty) \to (0, \infty)$ with

$$0 < I(\varepsilon) = \int_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty, \qquad \varepsilon \in (0, \varepsilon_0)\ . \tag{6.7}$$

The following existence theorem is due to Lehto for the case where $K_\mu(z)$ is locally bounded outside of a closed set of the measure zero with some additional conditions, see Theorem 2 and Lemma 3 in [Le].

6.8. Theorem. *Let D be a domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{loc}$ and*

$$\int_0^{\delta(z_0)} \frac{dr}{r q^R_{z_0}(r)} \ = \ \infty \tag{6.9}$$

for every $z_0 \in D$ where $\delta(z_0) < dist\,(z_0, \partial D)$ and $q^R_{z_0}(r)$ is the mean integral value of $K^R_\mu(z, z_0)$ over the circle $|z - z_0| = r$. Then the Beltrami equation (1.1) has a homeomorphic ACL solution $f_\mu : D \to \mathbb{C}$ such that $f^{-1}_\mu \in W^{1,2}_{loc}$. Moreover, f_μ is a radial Q-homeomorphism at every point $z_0 \in D$ with $Q(z) = K^R_\mu(z, z_0)$.

The theorem follows from Lemma 6.1 with the function

$$\psi(t) = \begin{cases} 1/[t q^R_{z_0}(t)], & t \in (0, \delta(z_0)). \\ 0, & t \in [\delta(z_0), \infty], \end{cases} \tag{6.10}$$

see the equality (4.26).

6.11. Corollary. *In particular, if $K_\mu \in L^1_{loc}$ and*

$$q^R_{z_0}(r) \ = \ O\left(\log \frac{1}{r}\right) \tag{6.12}$$

as $r \to 0$ at every point $z_0 \in D$, then all the conclusions of Theorem 6.8 hold.

The next consequence of Theorem 6.8 is due to the work [MS] for the case $K_\mu \in L^p_{loc}$, $p > 1$, see Remark 6.39.

6.13. Corollary. *The conclusions are valid if (6.9) holds for $K_\mu(z)$ instead of $K^R_\mu(z, z_0)$ for every point $z_0 \in D$.*

6.14. Theorem. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{loc}$ and $K^R_\mu(z, z_0)$ is of finite mean oscillation at every point $z_0 \in D$. Then the Beltrami equation (1.1) has a homeomorphic ACL solution $f_\mu : D \to \mathbb{C}$ such that $f^{-1}_\mu \in W^{1,2}_{loc}$ and f_μ is a radial Q-homeomorphism at every point $z_0 \in D$ with $Q(z) = K^R_\mu(z, z_0)$.*

Theorem 6.14 follows from Lemma 6.1 with $\psi = 1/t \log 1/t$ correspondingly to Lemma 2.16. The following theorem is a consequence of Theorem 6.14 and Corollary 2.6.

6.15. Theorem. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{loc}$ and let*

$$\overline{\lim_{\varepsilon\to 0}} \;\;⨍_{D(z_0,\varepsilon)} \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2}\, dxdy \;<\; \infty \tag{6.16}$$

for every $z_0 \in D$. Then the Beltrami equation (1.1) has a homeomorphic ACL solution $f_\mu : D \to \mathbb{C}$ such that $f_\mu^{-1} \in W^{1,2}_{loc}$ and f_μ is a radial Q-homeomorphism at every point $z_0 \in D$ with $Q(z) = K^R_\mu(z, z_0)$.

The next theorem is one of the most important particular cases of Theorem 6.14 because $K^R_\mu(z, z_0) \le K_\mu(z)$, see the case BMO in [RSY$_1$]–[RSY$_2$].

6.17. Theorem. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that*

$$K_\mu(z) \;=\; \frac{1+|\mu(z)|}{1-|\mu(z)|} \;\le\; Q(z) \;\in\; FMO\,. \tag{6.18}$$

Then the Beltrami equation (1.1) has a homeomorphic ACL solution $f_\mu : D \to \mathbb{C}$ such that $f_\mu^{-1} \in W^{1,2}_{loc}$ and f_μ is a radial Q-homeomorphism at every point $z_0 \in D$.

The next consequence of Theorem 6.14 appeals to Proposition 2.4.

6.19. Corollary. *If $K_\mu(z)$ has a majorant $Q(z)$ for which every $z \in D$ is a Lebesgue point, then the Beltrami equation has a homeomorphic ACL solution f_μ with $f_\mu^{-1} \in W^{1,2}_{loc}$.*

6.20. Corollary. *If, for every $z \in D$, it holds the condition*

$$\overline{\lim_{\varepsilon\to 0}} \;\;⨍_{D(z,\varepsilon)} \frac{1+|\mu(z)|}{1-|\mu(z)|}\, dxdy \;<\; \infty\,, \tag{6.21}$$

then the Beltrami equation has a homeomorphic ACL solution f_μ which is regular, i.e., $J_{f_\mu}(z) > 0$ a.e.

Applying Lemma 6.1 with $\psi(t) = 1/t$, we also have the next statement in the terms of the mean logarithmic value of K^R_μ, cf. Corollary 4.35.

6.22. Theorem. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{loc}$ and*

$$M^{K^A_\mu}_{log}(|\zeta - z_0|) = o\left(\log \frac{1}{|\zeta - z_0|}\right) \tag{6.23}$$

at every point $z_0 \in D$ for some $\varepsilon_0 < dist(z_0, D)$. Then the Beltrami equation (1.1) has a homeomorphic ACL solution $f_\mu : D \to \mathbb{C}$ such that $f_\mu^{-1} \in W^{1,2}_{loc}$ and f_μ is a radial Q-homeomorphism at every point $z_0 \in D$ with $Q(z) = K^R_\mu(z, z_0)$.

6.24. Corollary. *The same conclusions hold if the mean logarithmic value of K_μ over the annulus $A(\varepsilon)$ does not converge to ∞ as $\varepsilon \to 0$, i.e.,*

$$\liminf_{\varepsilon\to 0} \; M_{log}^{K_\mu}(\varepsilon) \; < \; \infty \tag{6.25}$$

at every point $z_0 \in D$ for some $\varepsilon_0 < dist(z_0, D)$.

By Theorem 5.22 we have also the following statement.

6.26. Theorem. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{loc}$ and*

$$m(\{z \in U_{z_0} : \; K_\mu^R(z, z_0) \; > \; t\}) \; \leq \varphi_{z_0}(t) \; , \quad t \in [1, \infty] \tag{6.27}$$

in a neighborhood U_{z_0} of every point $z_0 \in D$ for a function $\varphi_{z_0} : [1, \infty] \to [0, \infty]$ which is nonincreasing and continuous from the right such that

$$\liminf_{t\cdot\to\infty} \; t \cdot \varphi_{z_0}(t) \; = \; 0 \; , \tag{6.28}$$

and

$$\int\limits_0^\delta \frac{d\rho}{\int\limits_0^\rho \varphi_{z_0}^{-1}(\tau) \; d\tau} \; = \; +\,\infty \; . \tag{6.29}$$

Then the Beltrami equation (1.1) has a homeomorphic ACL solution $f_\mu : D \to \mathbb{C}$ such that $f_\mu^{-1} \in W^{1,2}_{loc}$ and f_μ is a radial Q-homeomorphism at every point $z_0 \in D$ with $Q(z) = K_\mu^R(z, z_0)$.

Sometimes it is more convenient to use directly φ instead of φ^{-1}.

6.30. Corollary. *The conclusions are valid true if the conditions (6.27) and (6.28) hold for strictly decreasing continuous functions φ_{z_0} such that*

$$\int\limits_\Delta^\infty \frac{d\varphi_{z_0}(t)}{\int\limits_t^\infty \eta \; d\varphi_{z_0}(\eta)} \; = \; +\,\infty \tag{6.31}$$

at every point $z_0 \in D$.

Indeed, if the function φ_{z_0} is strictly decreasing, then $\varphi^{-1}(\varphi(\eta)) = \eta$ and the conditions (6.29) and (6.31) are equivalent by the change of variables $\tau = \varphi(\eta)$.

6.32. Theorem. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{loc}$ and*

$$\int\limits_{U_{z_0}} \Phi_{z_0}(K_\mu^R(z, z_0)) \; dxdy \; < \; \infty \tag{6.33}$$

for some neighborhood U_{z_0} at every point $z_0 \in D$ where the functions $\Phi_{z_0} : [1, \infty] \to [0, \infty]$ are nondecreasing and continuous from the right and

$$\limsup_{t \to \infty} \frac{\Phi_{z_0}(t)}{t} = \infty , \tag{6.34}$$

and (6.29) holds for $\varphi_{z_0}(t) = 1/\Phi_{z_0}(t)$. Then the Beltrami equation (1.1) has a homeomorphic ACL solution $f_\mu : D \to \mathbb{C}$ such that $f_\mu^{-1} \in W^{1,2}_{loc}$ and f_μ is a radial Q-homeomorphism at every point $z_0 \in D$ with $Q(z) = K^R_\mu(z, z_0)$.

Theorem 6.32 follows from Theorem 6.26 by (5.8).

6.35. Corollary. *In particular, the conclusions hold if $\mu : D \to \mathbb{C}$ is a measurable function with $|\mu(z)| < 1$ a.e. such that*

$$\int_D \Phi(K_\mu(z))\, dxdy < \infty \tag{6.36}$$

where the function $\Phi : [1, \infty] \to [0, \infty]$ is continuous and strictly increasing and

$$\limsup_{t \to \infty} \frac{\Phi(t)}{t} = \infty , \tag{6.37}$$

and, for $\varphi(t) = 1/\Phi(t)$,

$$\int_\Delta^\infty \frac{d\varphi(t)}{\int_t^\infty \eta \, d\varphi(\eta)} = +\infty . \tag{6.38}$$

Many well-known existence theorems for the degenerate Beltrami equation with constraints on K_μ of the integral and measure types such as David's and Tukia's and others can be derived directly from Theorem 6.26, see [Da] and [Tu], cf. also [BJ] and [IM]. A series of other existence theorems for the degenerate Beltrami equations in the terms of the angular (= tangent) dilatation have been obtained earlier in the work [GMSV].

6.39. Remark. Note that every homeomorphic ACL solution f of the Beltrami equation with $K_\mu \in L^1_{loc}$ belongs to the class $W^{1,1}_{loc}$ in all the above theorems. If, in addition, $K_\mu \in L^q_{loc}$, $q \in [1, \infty]$, then $f_\mu \in W^{1,s}_{loc}$ where $s = 2q/(1+q) \in [1, 2]$. Indeed,

$$|\partial f| + |\overline{\partial} f| = K_\mu^{1/2}(z) \cdot J_{f_\mu}^{1/2}(z),$$

and by Hölder's inequality, on every compact set $C \subset D$,

$$||\overline{\partial} f||_s \le ||\partial f||_s \le ||K_\mu^{1/2}||_p \cdot ||J_{f_\mu}^{1/2}||_2 = ||K_\mu||_q^{1/2} \cdot ||J_{f_\mu}||_1^{1/2} \le ||K_\mu||_q^{1/2} \cdot S(f(C))^{1/2},$$

see e.g. [LV], p. 131, where $\frac{1}{p} + \frac{1}{2} = \frac{1}{s}$ and $p = 2q$. Hence $f \in W^{1,s}_{loc}$, see e.g. [Ma], p. 8.

7 Uniqueness theorems

Iwaniec and Sverak [IS] proved that every $\mathrm{W}^{1,2}_{loc}$ solution g of the Beltrami equation with $K_\mu \in L^1_{loc}$ has a representation $g = h \circ f$ where h is holomorphic and f is a homeomorphism. We have a simple proof for the representation of $\mathrm{W}^{1,2}_{loc}$ solutions g of (1.1) through the regular homeomorphic solution f_μ when K_μ has a FMO majorant and under many other conditions.

7.1. Lemma. *Let μ, Q and f_μ be as in Lemma 6.1 and let g be a $W^{1,2}_{loc}$ solution of the Beltrami equation (1.1). Then g has the representation*

$$g = h \circ f_\mu \tag{7.2}$$

where h is holomorphic in $f_\mu(D)$.

In particular, (7.2) implies that g is regular, i.e., $J_g(z) > 0$ a.e.

Proof. Let $\varphi = f_\mu^{-1}$ and $h = g \circ \varphi$. Since $g \in \mathrm{W}^{1,2}_{loc}$ and $\varphi \in \mathrm{W}^{1,2}_{loc}$ it follows that $h \in \mathrm{W}^{1,1}_{loc}(f(D))$, see [LV], p. 151. Thus, by Weyl's lemma, see e.g. [Ah], p. 33, it suffices to show that $\partial h = 0$ a.e. in $f_\mu(D)$. Let E denote the set of points z in D where either f_μ or g do not satisfy (1.1) or $J_{f_\mu} = 0$. A direct computation, cf. [Ah], p. 9, shows that $\overline{\partial} h = 0$ in $f_\mu(D) \setminus f_\mu(E)$.

For $\overline{\partial} h = 0$ a.e. in $f_\mu(E)$, consider

$$\iint\limits_E \frac{dxdy}{1-|\mu(z)|^2} \ = \iint\limits_E K_{f_\mu}(z)\ \frac{dxdy}{(1+|\mu(z)|)^2} \ \leq \iint\limits_E Q(z)\ dxdy \ = \ 0 \tag{7.3}$$

which holds since $|E| = 0$ and $Q \in \mathrm{L}^1_{loc}$.

On the other hand, $\varphi \in \mathrm{W}^{1,2}_{loc}$ admits the change of variables, see e.g. [LV], p. 121, 128-130 and 150:

$$\int\limits_{f_\mu(E)}\int |\partial\varphi|^2\ dudv \ = \int\limits_{f_\mu(E)}\int J_\varphi(w)\ \frac{dudv}{1-|\mu(\varphi(w))|^2} \ = \iint\limits_E \frac{dxdy}{1-|\mu(z)|^2}$$

which by (7.3) implies that $|\partial\varphi| = 0$ a.e. on $f_\mu(E)$.

Now, a.e. $|\overline{\partial}\varphi| \leq |\partial\varphi|$ and

$$\overline{\partial} h \ = \ \overline{\partial}\varphi \cdot \partial g \circ \varphi + \overline{\partial\varphi} \cdot \overline{\partial} g \circ \varphi\ ,$$

hence $|\partial\varphi| = 0$ a.e. on $f_\mu(E)$ implying $|\overline{\partial} h| = 0$ a.e. on $f_\mu(E)$, and thus $\overline{\partial} h = 0$ a.e. in $f_\mu(D)$ and, consequently, h is holomorphic in $f_\mu(D)$ and (7.2) holds.

7.4. Theorem. *Let μ, Q and f_μ be as in Theorem 6.8 or 6.14 and let g be a $W^{1,2}_{loc}$ solution of the Beltrami equation (1.1). Then g has the representation (7.2) where h is holomorphic in $f_\mu(D)$.*

7.5. Corollary. *If (6.21) holds for each point $z \in D$, then the representation (7.2) holds for every $W^{1,2}_{loc}$ solution of the Beltrami equation.*

Iwaniec and Martin have constructed ACL solutions for the Beltrami equation outside of $\mathrm{W}^{1,2}_{loc}$ which are not open and discrete and, thus, are not generated by a homeomorphic solution in the sense of (7.2), see e.g. [IM]. By the well-known Stoilow theorem, it is easy to obtain the following conclusion, see Remark 6.39.

7.6. Proposition. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that*

$$K_\mu(z) \;=\; \frac{1+|\mu(z)|}{1-|\mu(z)|} \;\in\; L^1_{loc}\ . \tag{7.7}$$

Then any (continuous) discrete and open ACL solution g of the Beltrami equation has the representation $g = h\circ f$ where f is a homeomorphic $W^{1,1}_{loc}$ solution of the same equation and h is a meromorphic function on $f(D)$.

7.8. Remark. Thus, under the locally integrable K_μ, either the Beltrami equation has a homeomorphic $W^{1,1}_{loc}$ solution or has no continuous, discrete and open ACL solution at all. Note also that, for every $p \geq 1$, we have constructed examples of measurable functions $\mu : \mathbb{C} \to \mathbb{C}$, $|\mu(z)| < 1$ a.e., with $K_\mu \in L^p_{loc}$ for which the Beltrami equation has no homeomorphic ACL solution although $K_\mu \in C(\mathbb{C}\setminus\{0\})$, and hence $K_\mu \in \mathrm{FMO}(\mathbb{C}\setminus\{0\})$, see Proposition 6.3 in [RSY$_1$].

It ramains to study the problem on the uniqueness of homeomorphic ACL solutions.

Let (X, d) be a metric space and let $H = \{h_x(r)\}_{x\in X}$ be a family of positive functions h_x given on $(0, \rho_x)$, $\rho_x > 0$, such that $h_x(r) \to 0$ as $r \to 0$. Let

$$\Lambda^\rho_H(X) \;=\; \inf\ \Sigma\ h_{x_k}(r_k) \tag{7.9}$$

where the infimum is taken over all finite collections of $x_k \in X$ and $r_k \in (0, \rho)$ such that the balls

$$B(x_k, r_k) = \{x \in X : d(x, x_k) < r_k\} \tag{7.10}$$

cover X. The limit

$$\Lambda_H(X) = \lim_{\rho\to 0}\ \Lambda^\rho_H(X) \tag{7.11}$$

exists and we call it the **H-length** of X. In particular, $\Lambda_H(X)$ is called simply the **length** of X if $H = \{h_x(r)\}$ where $h_x(r)$ is the identity of $(0,\infty)$ onto $(0,\infty)$ for every $x \in X$.

Below we denote by S_μ the **singular set** of $\mu : D \to \mathbb{C}$, i.e., the set:

$$S_\mu = \{z \in D : \lim_{\varepsilon\to 0}\ \|K_\mu\|_{L^\infty(D(z,\varepsilon))} \;=\; \infty\}\ . \tag{7.12}$$

Immediately by the definition it is clear that the set S_μ is closed.

7.13. Lemma. *Let the hypotheses of in Lemma 6.1 hold and let the singular set S_μ be of the H-length zero for $H = \{h_{z_0}(r)\}_{z_0\in S_\mu}$ with*

$$h_{z_0}(r) \;= \exp\left(-\frac{2\pi}{\omega_{z_0}(r)}\right)\ ,\qquad z_0 \in S_\mu\ ,\qquad r \in (0, \delta(z_0))\ , \tag{7.14}$$

for $\delta(z_0) < dist\ (z_0, \partial D)$ where

$$\omega_{z_0}(\varepsilon) \;=\; \frac{1}{I^2_{z_0}(\varepsilon)} \int\limits_{A(\varepsilon)} K^R_\mu(z,z_0)\cdot\psi^2_\varepsilon(|z-z_0|)\ \ dxdy\ . \tag{7.15}$$

Then every homemorphic ACL solution f of the Beltrami equation has the representation $f = h\circ f_\mu$ where the mapping h is conformal.

Proof. First of all, S_μ is of the length zero and, consequently, does not locally disconnected $\mathbb{C}$ and, in particular, $G = D\setminus S_\mu$ is a domain, see e.g. [Va$_2$]. In the domain G the homeomorphisms f and f_μ are locally quasiconformal and hence $f\circ f_\mu^{-1}$ are conformal in the domain $f_\mu(G)$. Finally it remains to note that by Lemma 4.12 $f_\mu(S_\mu)$ is of the length zero and hence the set $f_\mu(S_\mu)$ is removable by the Painleve theorem, see e.g. [Be].

7.16. Theorem. *Let the hypotheses of Theorem 6.14 hold and let the singular set S_μ be of the H-length zero for $H = \{h_z(r)\}_{z\in S_\mu}$ with*

$$h_z(r) \;=\; \left(\log\frac{\delta(z)}{r}\right)^{-\beta(z)}\ ,\quad z\in S_\mu,\quad r\in(0,\delta(z))\ , \tag{7.17}$$

where $\delta(z) < dist\ (z,\partial D)$ and $2\beta(z) = (q(z)+6d(z))^{-1}$ where $q(z)$ is the mean value and $d(z)$ the maximal dispersion of $Q(\zeta)$ in the disk $D(z,\delta(z)/2)$.

Then every homemorphic ACL solution f of the Beltrami equation has the representation $f = h\circ f_\mu$ where the mapping h is conformal.

7.18. Remark. In view of Remark 2.8, if for every point $z\in D$,

$$Q_0(z) \;=\; \overline{\lim_{\varepsilon\to 0}}\ \fint_{D(z,\varepsilon)} |Q(z')|\ dx'dy'\ <\ \infty\ , \tag{7.19}$$

then one can take $\beta(z) = \gamma\cdot Q_0^{-1}(z)$ for every $\gamma < 1/26$.

On the base of Lemma 7.13, one can also render concrete the uniqueness conditions corresponding to all the existence theorems from the previous section.

8 One example

By (3.26) and (3.30), see also (3.29), we have the inequality

$$K^r(z,z_0,f)\ \le\ K^R(z,z_0,f)\ . \tag{8.1}$$

In this connection, the natural question is arised whether Theorem 3.33 can be improved in the sense that $K^R_\mu(z,z_0)$ can be changed by $K^r_\mu(z,z_0)$. The following example makes possible to give the negative answer to the question.

Example. Consider the quasiconformal automorphism $f:\mathbb{D}\to\mathbb{D}$ of the unit disk $\mathbb{D} = \{z\in\mathbb{C} : |z| < 1\}$ given in the polar coordinates:

$$f(re^{i\vartheta}) \;=\; \exp\left(i\vartheta - \int\limits_r^1 \frac{1+ik}{1-ik}\frac{dt}{t}\right)\ ,\qquad z = re^{i\vartheta}\ ,\qquad k\in(0,1), \tag{8.2}$$

see e.g. Proposition 6.4 in [RSY$_1$]. Then direct calculations with the application of the formula

$$\mu(re^{i\vartheta}) \;=\; e^{2i\vartheta}\,\frac{rf_r+if_\vartheta}{rf_r-if_\vartheta} \tag{8.3}$$

give the complex dilatation

$$\mu(re^{i\vartheta}) \;=\; ike^{2i\vartheta}\ . \tag{8.4}$$

From (8.2) we have that

$$|f(re^{i\vartheta})| \;=\; \exp\left(-\Re\int\limits_r^1 \frac{1+ik}{1-ik}\frac{dt}{t}\right) \;=\; r^{\frac{1-k^2}{1+k^2}}\ . \tag{8.5}$$

On the other hand, for $z_0=0$,

$$K_0 := \; K^R_\mu(z) := \; K^R_\mu(z,z_0) \;=\; \frac{|1+ik|^2}{1-k^2} \;=\; \frac{1+k^2}{1-k^2}\ , \tag{8.6}$$

i.e.,

$$|f(re^{i\vartheta})| \;=\; r^{\frac{1}{K_0}}\ . \tag{8.7}$$

The example show that, generally speaking, quasiconformal mappings are not radial Q–homeomorphisms with

$$Q(z) \;=\; K^r(z,z_0,f) \;=\; K^r_\mu(z) := \; K^r_\mu(z,z_0) \;=\; \frac{1-|\mu(z)|^2}{\left|1+\frac{\overline{z}}{z}\mu(z)\right|^2} \tag{8.8}$$

at $z_0=0$.

Indeed, let us assume the opposite. Then by Theorem 4.23 we would have the estimate

$$\frac{|f(z)|}{(1+|z|^2)^{1/2}} \;\le\; 32\cdot\exp\left(-\int\limits_{|z|}^1 \frac{dr}{rq(r)}\right)\ , \tag{8.9}$$

i.e.,

$$|f(z)| \;\le\; 64\cdot\exp\left(-\int\limits_{|z|}^1 \frac{dr}{rq(r)}\right) \tag{8.10}$$

where $q(r)$ is the mean value of $Q(z)$ over the circle $|z|=r$.

However, in the case of (8.4) and (8.8) we obtain that

$$Q(z) \;=\; K^r_\mu(z) \;=\; \frac{1-k^2}{1+k^2} \;=\; \frac{1}{K_0} \tag{8.11}$$

and, thus, (8.10) would imply the estimate

$$|f(re^{i\vartheta})| \;\le\; 64\cdot r^{K_0} \tag{8.12}$$

that contradicts (8.7) because $K_0>1$.

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