# TO THE THEORY OF NORMAL FAMILIES OF SPACE MAPPINGS

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**Abstract**

Normal family properties of classes of $Q$-homeomorphisms are studied. In the terms of the majorant $Q(x)$ for the dilatation, a series of criteria for normality is obtained. These are based on estimates of the distortion of the spherical distance $h$ under $Q$-homeomorphisms. In particular, it is shown that the class $\mathfrak{F}_{Q,\delta}$ of all $Q$-homeomorphisms $f$ of a domain $D \subset \mathbb{R}^n$ into $\overline{\mathbb{R}^n}$, $n \geq 2$, with $h(\overline{\mathbb{R}^n} \setminus f(D)) \geq \delta > 0$ form a normal family if $Q(x)$ has finite mean oscillation in $D$. Normality of $\mathfrak{F}_{Q,\delta}$ is also proved if $Q(x)$ has singularities of the logarithmic type whose order are not greater than $n-1$ at every point $x \in D$.

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## 1 Introduction

Normal families play an important role in the local and boundary behavior of mappings. Normality is closely related with equicontinuity properties and, thus, with distortion estimates in the corresponding classes of mappings. We continue the study of $Q$–homeomorphisms that was started first in the plane [RSY$_1$]–[RSY$_4$] and then in the space [MRSY$_1$]–[MRSY$_4$] and [IR$_1$]–[IR$_2$].

Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \to [1, \infty]$ be a measurable function. A homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is said to be a **Q–homeomorphism** if

$$M(f\Gamma) \leq \int\limits_D Q(x) \cdot \rho^n(x)\, dm(x) \tag{1.1}$$

for every family $\Gamma$ of paths in $D$ and every admissible function $\rho$ for $\Gamma$. Thus, a $Q$–homeomorphism satisfies a modulus inequality well–known in the theory of quasiconformal mappings. However, we do not assume that $Q \in L^1_{loc}(D)$ and, thus, the right side of (1.1) can be infinite even if $M(\Gamma) < \infty$.

Recall that, given a family of paths $\Gamma$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called **admissible** for $\Gamma$, abbr. $\rho \in adm\, \Gamma$, if

$$\int\limits_\gamma \rho(x)\, |dx| \geq 1 \tag{1.2}$$

for each $\gamma \in \Gamma$. The **modulus** of $\Gamma$ is the quantity

$$M(\Gamma) = \inf_{\rho \in adm\,\Gamma} \int\limits_D \rho^n(x)\, dm(x). \tag{1.3}$$

## 2 Preliminaries

First recall some general facts on normal families of mappings in metric spaces. Let $(X, d)$ and $(X^{'}, d^{'})$ be metric spaces with distances $d$ and $d^{'}$, respectively. A family $\mathfrak{F}$ of continuous mappings $f : X \to X^{'}$ is said to be **normal** if every sequence of mappings $f_m \in \mathfrak{F}$ has subsequence $f_{m_k}$ converging uniformly on each compact set $C \subset X$ to a continuous mapping. Normality is closely related to the following. A family $\mathfrak{F}$ of mappings $f : X \to X^{'}$ is said to be **equicontinuous at a point** $x_0 \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d^{'}(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathfrak{F}$ and $x \in X$ with $d(x, x_0) < \delta$. The family $\mathfrak{F}$ is **equicontinuous** if $\mathfrak{F}$ is equicontinuous at every point $x_0 \in X$. It is well-known that if $(X, d)$ and $(X^{'}, d^{'})$ are metric spaces and $\mathfrak{F}$ is a normal family of mappings $f : X \to X^{'}$, then $\mathfrak{F}$ is equicontinuous.

A family $\mathfrak{F}$ of mappings $f : X \to X^{'}$ is said to be **uniformly equicontinuous on a set** $E \subset X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d^{'}(f(x), f(x^{'})) < \varepsilon$ for all $f \in \mathfrak{F}$ and for all $x$ and $x^{'} \in E$ with $d(x, x^{'}) < \delta$. The following results are also well-known.

**2.1. Lemma.** *Let $(X, d)$ and $(X^{'}, d^{'})$ be metric spaces and let $\mathfrak{F}$ be a family of equicontinuous mappings $f : X \to X^{'}$. Then $\mathfrak{F}$ is uniformly equicontinuous on every compact set $C \subset X$.*

**2.2. Corollary.** *Normal families of mappings between metric spaces are uniformly equicontinuous on compacts.*

The function

$$\omega_E(t) = \omega_E^{\mathfrak{F}}(t) = \sup\, d^{'}(f(x), f(z)) \tag{2.3}$$

where the supremum is taken over all $x, z \in E$ such that $d(x, z) \le t$ and $f \in \mathfrak{F}$ is called the **modulus of continuity of the family $\mathfrak{F}$ on the set** $E$.

Similarly, the function

$$\omega_{x_0}(t) = \omega_{x_0}^{\mathfrak{F}}(t) = \sup\, d^{'}(f(x_0), f(x)) \tag{2.4}$$

where the supremum is taken over all $x \in X$ and $f \in \mathfrak{F}$ such that $d(x, x_0) \le t$ is called the **modulus of continuity of $\mathfrak{F}$ at the point** $x_0 \in X$.

Note that by definition $\omega_E$ and $\omega_{x_0}$ are nonnegative, nondecreasing and continuous from the right. Note also that $\omega_{x_0}(t) \to 0$ as $t \to 0$ for every $x_0 \in X$ if the family $\mathfrak{F}$ is equicontinuous. Moreover, the following statement follows from Lemma 2.1.

**2.5. Corollary.** *If a family $\mathfrak{F}$ of mappings $f : X \to X^{'}$ is equicontinuous, then $\omega_C(t) \to 0$ as $t \to 0$ for every compact set $C \subset X$.*

The next statement is also obvious.

**2.6. Proposition.** *Let $(X, d)$ and $(X', d')$ be metric spaces and let $\overline{\mathfrak{F}}$ be a closure of a family $\mathfrak{F}$ of mappings $f : X \to X'$ with respect to the pointwise convergence in $X$. Then the moduli of continuity (2.3) and (2.4) of $\overline{\mathfrak{F}}$ and $\mathfrak{F}$ coincide.*

The following theorem shows that the locally uniform convergence and the pointwise convergence are equivalent for equicontinuous families of mappings in arbitrary metric spaces.

**2.7. Theorem.** *Let $(X, d)$ and $(X', d')$ be metric spaces and let $\mathfrak{F}$ be an equicontinuous family of mappings $f : X \to X'$. Then a sequence $f_m \in \mathfrak{F}$ converges uniformly on every compact set if and only if $f_m(x)$ converges at every point $x \in X$.*

**2.8. Lemma.** *Let $(X, d)$ be a metric space and a set $E \subset X$ be a dense subset of $X$ and let $(X', d')$ be a complete metric space. If an equicontinuous sequence of mappings $f_m : X \to X'$ is pointwise convergent on the set $E$, then $f_m$ converges uniformly on every compact set $C \subset X$.*

Since every compact metric space is complete we obtain the following consequence of Lemma 2.8.

**2.9. Corollary.** *If $(X, d)$ is a separable metric space and $(X', d')$ is compact metric space, then a family $\mathfrak{F}$ of mappings $f : X \to X'$ is normal if and only if $\mathfrak{F}$ is equicontinuous.*

In what follows, we use in $\overline{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ the **spherical (chordal) metric** $h(x, y) = |\pi(x) - \pi(y)|$ where $\pi$ is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$:

$$(2.10)\quad h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}, \quad h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}, \quad x \neq \infty \neq y.$$

Thus, by definition $h(x, y) \leq 1$ for all $x$ and $y \in \overline{\mathbb{R}^n}$. The **spherical (chordal) diameter** of a set $E \subset \overline{\mathbb{R}^n}$ is

$$(2.11)\qquad h(E) = \sup_{x,y \in E} h(x, y).$$

Note that

$$(2.12)\qquad h(x, y) \leq |x - y|$$

for all $x$,$y \in \mathbb{R}^n$ and

$$(2.13)\qquad h(x, y) \geq \frac{1}{2}|x - y|$$

for all $x$ and $y$ in the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ with the equality in (2.13) on $\partial\mathbb{B}^n$.

## 3 The main lemma on distortion for $Q$ - homeomorphisms.

A compact set $E \subset \mathbb{R}^n$ is called a **continuum** if $E$ is connected. The continuum is called degenerate if it consits of a single point. Later on, for sets $E$, $F$ and $G \subset \overline{\mathbb{R}^n}$, $\triangle(E,F,G)$ denotes the family of all nondegenerate (nonconstant) paths $\gamma : [a,b] \to \overline{\mathbb{R}^n}$ joining $E$ and $F$ in $G$, i.e., $\gamma \in \triangle(E,F,G)$ if and only if $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in G$ for all $t \in (a,b)$. The family $\triangle(E,F)$ denotes $\triangle(E,F,\mathbb{R}^n)$ or $\triangle(E,F,\overline{\mathbb{R}^n})$. A domain $D$ in $\overline{\mathbb{R}^n}$ is said to be a ring if $\overline{\mathbb{R}^n}\backslash D$ has two (connected) components $C_1$ and $C_2$. In the case we write $D = R(C_1, C_2)$. Here $cap\, R(C_1,C_2)$ denotes the $n$–modulus of $\triangle(C_1,C_2)$, see (1.3). The **Teichmuller ring** $R_{T,n}(s)$ is the ring $R([-e_1,0],[se_1,\infty])$, $s>0$.

By the well-known Gehring lemma

$$cap\ R(C_1,C_2) \geq \tau\left(\frac{1}{h(C_1)h(C_2)}\right) \tag{3.1}$$

where

$$\tau(s) = \tau_n(s) = cap\ R_{T,n}(s), \tag{3.2}$$

see e.g. 7.37 in [Vu] or [Ge$_1$].

It is also known, see e.g. (7.19) and (7.22) in [Vu] or [Ge$_1$] that

$$\tau_n(s) = \frac{\omega_{n-1}}{[\log\Phi(s)]^{n-1}} \tag{3.3}$$

where $\omega_{n-1}$ is an area of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$,

$$s+1 \leq \Phi(s) \leq \lambda_n^2\cdot(s+1),\ \ s>1. \tag{3.4}$$

Here $\lambda_n \geq 4$. More precisely, $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$ and $\lambda_n^{1/n} \to e$ as $n\to\infty$. Hence (3.1) implies that

$$M(\triangle(E,F)) \geq \frac{\omega_{n-1}}{\left[\log\frac{2\lambda_n^2}{h(E)h(F)}\right]^{n-1}} \tag{3.5}$$

for every continua $E$ and $F$ in $\overline{\mathbb{R}^n}$.

**3.6. Lemma.** *Let $f : D\to\overline{\mathbb{R}^n}$, $n\geq 2$, be a $Q$-homeomorphism such that $D^{'} = f(D)$ omits at least two points $v$ and $w\in\overline{\mathbb{R}^n}$ with $h(v,w)\geq\delta>0$. If*

$$\int\limits_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x)\cdot\psi_\varepsilon^n(|x-x_0|)\ dm(x) \leq c\cdot I^p(\varepsilon),\ \ \varepsilon\in(0,\varepsilon_0), \tag{3.7}$$

*for $x_0\in D$ and $0<\varepsilon_0\leq dist(x_0,\partial D)$ where $p\leq n$ and $\psi_\varepsilon(t)$ is a nonnegative on $(0,\infty)$ such that*

$$0 < I(\varepsilon) = \int\limits_{\varepsilon}^{\varepsilon_0}\psi_\varepsilon(t)dt < \infty,\ \ \varepsilon\in(0,\varepsilon_0), \tag{3.8}$$

*then*

$$h(f(x),f(x_0)) \leq \frac{\alpha_n}{\delta}\, exp\{-\beta_n I^{\gamma_{n,p}}(|x-x_0|)\} \tag{3.9}$$

*for all* $x \in B(x_0, \varepsilon_0)$ *where*

$$\alpha_n = 2\lambda_n^2, \quad \beta_n = \left(\frac{\omega_{n-1}}{c}\right)^{\frac{1}{n-1}}, \quad \gamma_{n,p} = 1 - \frac{p-1}{n-1}, \tag{3.10}$$

$\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$ *and* $\lambda_n^{1/n} \to e$ *as* $n \to \infty$.

Here we use standard balls in the Euclidean metric in $\mathbb{R}^n$

$$B(x_0, \varepsilon_0) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon_0\}. \tag{3.11}$$

*Proof.* Let $\Gamma_\varepsilon$ be the family of all paths joining $\overline{B(x_0,\varepsilon)}$ and $\overline{\mathbb{R}^n}\backslash B(x_0,\varepsilon_0)$ in the ring $A_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0\}$. By the Lusin theorem, see e.g. 2.3.5 in [Fu] and [Sa], p. 69, there is Borel function $\psi_\varepsilon^*(t) = \psi_\varepsilon(t)$ a.e. and the function

$$\rho_\varepsilon(x) = \begin{cases} \psi_\varepsilon^*(|x - x_0|)/I(\varepsilon), & x \in A_\varepsilon, \\ 0, & x \in \mathbb{R}^n \backslash A_\varepsilon \end{cases} \tag{3.12}$$

is admissible for $\Gamma_\varepsilon$. By the definition of $Q$-homeomorphisms and (3.7) we have

$$M(f\Gamma_\varepsilon) \le \int\limits_{\mathbb{R}^n} Q(x)\rho_\varepsilon^n(x)\, dm(x) \le c \cdot I^{p-n}(\varepsilon). \tag{3.13}$$

By the Jordan-Brouwer theorem $\overline{\mathbb{R}^n}\backslash fA_\varepsilon$ consists of two connected components. Let $E = f\left(\overline{B(x_0,\varepsilon)}\right)$ and $F = \overline{\mathbb{R}^n}\backslash f\left(B(x_0,\varepsilon_0)\right)$ and let $\Gamma_\varepsilon^*$ be the family of all paths joining $E$ and $F$ in $\overline{\mathbb{R}^n}$. Now

$$M\left(f\Gamma_\varepsilon\right) = M\left(\Gamma_\varepsilon^*\right). \tag{3.14}$$

Indeed, by definition $\gamma \in f\Gamma_\varepsilon$ joints $E$ and $F$ and hence $f\Gamma_\varepsilon \subset \Gamma_\varepsilon^*$ and $M\left(f\Gamma_\varepsilon\right) \le M\left(\Gamma_\varepsilon^*\right)$. On the other hand, $f\Gamma_\varepsilon < \Gamma_\varepsilon^*$ because $fA_\varepsilon$ separates $E$ and $F$ and, consequently, $M\left(f\Gamma_\varepsilon\right) \ge M\left(\Gamma_\varepsilon^*\right)$, see e.g. Theorem 1(c) in [Fu] and 6.4 in [Va].

Thus, (3.9) follows by (3.13), (3.14) and (3.5).

**3.15. Corollary.** *Under the conditions of Lemma 3.6 and p=1*

$$h(f(x), f(x_0)) \le \frac{\alpha_n}{\delta}\, exp\left\{-\beta_n I(|x - x_0|)\right\}. \tag{3.16}$$

## 4 Estimates of distortion under $Q$-homeomorphisms

**4.1. Theorem.** *Let* $f : D \to \overline{\mathbb{R}^n}$, $n \ge 2$, *be a* $Q$*-homeomorhism such that* $D' = f(D)$ *omits at least two points* $v$ *and* $w \in \overline{\mathbb{R}^n}$ *with* $h(v, w) \ge \delta > 0$. *Then, for every* $x_0 \in D$ *and* $x \in B(x_0, \varepsilon(x_0))$, $\varepsilon(x_0) \le dist(x_0, \partial D)$, *and each* $\beta \ge 1/(n-1)$,

$$h(f(x), f(x_0)) \le \frac{\alpha_n}{\delta}\, exp\left\{-\int\limits_{|x-x_0|}^{\varepsilon(x_0)} \frac{dr}{rq_{x_0}^{\beta}(r)}\right\} \tag{4.2}$$

*where* $\alpha_n$ *is given by (3.10) and* $q_{x_0}(r)$ *is the mean integral value of* $Q(x)$ *over the sphere* $|x - x_0| = r$.

*Proof.* Indeed, for the function

$$\psi(t)=\begin{cases}\frac{1}{tq_{x_0}^{\beta}(t)}, & t\in(0,\varepsilon_0),\\ 0, & \mathrm{t}\in[\varepsilon_0,\infty),\end{cases} \tag{4.3}$$

we have that

$$\int\limits_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x)\cdot\psi^n(|x-x_0|)\;dm(x)=\omega_{n-1}\int\limits_{\varepsilon}^{\varepsilon_0}\frac{dr}{rq_{x_0}^{\beta n-1}(r)}\le\omega_{n-1}\int\limits_{\varepsilon}^{\varepsilon_0}\frac{dr}{rq^{\beta}(r)} \tag{4.4}$$

and, thus, the conclusion follows immediately from Lemma 3.6 with $p=1$.

**4.5. Remark.** Of course, the mean value $q_{x_0}(r)$ of $Q(x)$ over some spheres $|x-x_0|=r$ can be infinite. However, $q_{x_0}(r)$ is measurable in the parameter $r$ because $Q(x)$ is measurable in $x$. Moreover,

$$\int\limits_{|x-x_0|}^{\varepsilon(x_0)}\frac{dr}{rq_{x_0}^{\beta}(r)}<\infty \tag{4.6}$$

for $x\ne x_0$ because $q_{x_0}(r)\ge 1$. The integral can be equal to 0 if $q_{x_0}(r)=\infty$ a.e. but then the inequality (4.2) is obvious because $\alpha_n\ge 32$ and $\delta$ as well as $h(f(x),f(x_0))$ is less or equal to 1.

**4.7. Corollary.** *Under the conditions of Theorem 4.1*

$$h(f(x),f(x_0))\le\frac{\alpha_n}{\delta}\,exp\left\{-\int\limits_{|x-x_0|}^{\delta(x_0)}\frac{dr}{rq(r)}\right\} \tag{4.8}$$

*where* $\delta(x_0)=dist(x_0,\partial D)$.

Choosing in Theorem 4.1 $\beta=1/(n-1)$ we conclude.

**4.9. Corollary.** *If*

$$q_{x_0}(r)\le\left[\log\frac{1}{r}\right]^{n-1} \tag{4.10}$$

*for* $r<\varepsilon(x_0)\le\delta(x_0)$, *then*

$$h(f(x),f(x_0))\le\frac{\alpha_n}{\delta}\left[\log\frac{\varepsilon(x_0)}{|x-x_0|}\right]^{-1} \tag{4.11}$$

*for all* $x\in B(x_0,\varepsilon(x_0))$.

**4.12. Corollary.** *If*

$$Q(x)\le\left[\log\frac{1}{|x-x_0|}\right]^{n-1},\quad x\in B(x_0,\varepsilon(x_0)), \tag{4.13}$$

*then (4.11) holds in the ball $B(x_0, \varepsilon(x_0))$.*

**4.14. Remark.** If instead of (4.10) and (4.13) we have the conditions

$$q_{x_0}(r) \le c \cdot \left[\log \frac{1}{r}\right]^{n-1} \tag{4.15}$$

and, correspondingly,

$$Q(x) \le c \cdot \left[\log \frac{1}{|x-x_0|}\right]^{n-1}, \tag{4.16}$$

then

$$h(f(x), f(x_0)) \le \frac{\alpha_n}{\delta}\left[\log \frac{\varepsilon_{x_0}}{|x-x_0|}\right]^{-1/c^{1/(n-1)}}. \tag{4.17}$$

**4.18. Theorem.** *Let $f : D \to \overline{\mathbb{R}^n}$, $n \ge 2$, be a $Q$-homeomorphism with $h\left(\overline{\mathbb{R}^n} \backslash f(D)\right) \ge \delta > 0$. Then*

$$h(f(x), f(x_0)) \le \frac{\alpha_n}{\delta} exp\left\{-\int\limits_{|x-x_0|}^{\varepsilon(x_0)} \frac{dr}{rK^{\beta}_{x_0}(r)}\right\} \tag{4.19}$$

*for every $\beta \ge 1/(n-1)$, $x_0 \in D$, $x \in B(x, \varepsilon(x_0))$, $\varepsilon(x_0) \le dist(x_0, \partial D)$, and*

$$K_{x_0}(r) = \sup_{t\in[r,\varepsilon(x_0)]} Q_{x_0}(t) \tag{4.20}$$

*where $Q_{x_0}(t)$ is the mean value of $Q$ over $B(x_0, t)$,*

$$Q_{x_0}(t) = \fint_{B(x_0,t)} Q(x)\, dm(x). \tag{4.21}$$

*Proof.* By the theorem on the differentiability of indefinite integral applied to $\int\limits_0^t \varphi(\rho)d\rho$, where $\varphi(\rho) = \int\limits_{|x|=\rho} Q(x)$ and $|B(x_0, t)| = \int\limits_0^t S(x_0, \rho)d\rho$ where $S(\rho)$ is the area of the sphere $|x - x_0| = \rho$ we have inequality

$$q_{x_0}(r) \le K_{x_0}(r) \tag{4.22}$$

for a.e. $r \in (0, \varepsilon_0)$ and hence (4.2) implies (4.19).

**4.23. Remark.** We could use above, instead of $q_{x_0}(r)$, $K_{x_0}(r)$ and $Q_{x_0}(t)$,

$$q_{x_0,\alpha}(r) = \left(\fint_{|x|=r} Q^{\alpha}(x)\right)^{1/\alpha}, \tag{4.24}$$

$$K_{x_0,\alpha}(r) = \sup_{t\in[r,\varepsilon(x_0)]} Q_{x_0,\alpha}(r), \tag{4.25}$$

$$Q_{x_0,\alpha}(t) = \left(\fint_{B(x_0,t)} Q^{\alpha}(x)\, dm(x)\right)^{1/\alpha}, \tag{4.26}$$

because the Jensen inequality yields $q_{x_0,\alpha}(r) \geq q_{x_0}(r)$ and $K_{x_0,\alpha}(r) \geq K_{x_0}(r)$.

Choosing in Lemma 3.6 $\psi(t) = \frac{1}{t}$ and $p = 1$ we also have the following conclusion.

**4.27. Corollary.** *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a $Q$-homeomorphism such that $f(0) = 0$ and*

$$\int\limits_{\varepsilon<|x|<1} Q(x) \ \frac{dm(x)}{|x|^n} \leq c \log \frac{1}{\varepsilon}, \ \varepsilon \in (0,1). \tag{4.28}$$

*Then*

$$|f(x)| \leq \alpha_n \cdot |x|^{\beta_n}. \tag{4.29}$$

**4.30. Lemma.** *Let $f : D \to \overline{\mathbb{R}^n}$, $n \geq 2$, be a $Q$-homeomorphism with $h\left(\overline{\mathbb{R}^n} \backslash f\left(D\right)\right) \geq \delta > 0$ and let in $U \subset D$*

$$M(t) \ = \ m(\{x \in U : \, Q(x) > t\,\}) \leq \varphi(t) \ , \quad t \ \geq \ 1 \tag{4.31}$$

*where the function $\varphi : [1,\infty] \to [0,\infty]$, $\varphi(\infty) = 0$, is nonincreasing and continuous from the right and*

$$\liminf_{t\to\infty} \ t \cdot \varphi(t) \ = \ 0 \ . \tag{4.32}$$

*Then*

$$s(f(y), f(x_0)) \leq \frac{32}{\delta} \cdot exp \left( - \ \frac{1}{n} \int\limits_{\Omega_n |y-x_0|^n}^{\Omega_n {\varepsilon_0}^n} \frac{d\rho}{\int\limits_0^{\rho} \varphi^{-1}(\tau) d\tau} \right) \tag{4.33}$$

*for all $y \in B(x_0, \varepsilon_0) \subset U$ where $\Omega_n$ is the volume of the unit ball in $\mathbb{R}^n$ and*

$$\varphi^{-1}(\tau) \ = \ \begin{cases} \sup\limits_{\varphi(t)\geq\tau} \ t, & \text{if} \quad \tau \leq \varphi(1) \ , \\ \qquad 1, & \text{if} \quad \tau > \varphi(1) \ . \end{cases} \tag{4.34}$$

Note that the inverse function $\varphi^{-1} : [0,\infty] \to [1,\infty]$ is also nonincreasing but continuous from the left.

*Proof.* Set $Q_0^*(x) = Q_0(x) - t(\varepsilon_0)$ on $B(x_0, \varepsilon_0)$ where

$$t(\varepsilon_0) \ = \ \varphi^{-1}(\Omega_n {\varepsilon_0}^n) \tag{4.35}$$

and

$$Q_0(x) = \begin{cases} Q(x), & \text{if} \quad Q(x) \geq t(\varepsilon_0), \\ t(\varepsilon_0), & \text{if} \quad Q(x) < t(\varepsilon_0). \end{cases} \tag{4.36}$$

Note that $Q_0^*(x) \geq 0$ and $M_0^*(t) = M(t + t(\varepsilon_0))$, $t \geq 0$, in $B(x_0, \varepsilon_0)$ and hence

$$\int\limits_{B(x_0,\varepsilon_0)} Q_0^*(x) \ dm(x) \ = \ \int\limits_0^{\infty} M_0^*(t) \ dt \ = \ \int\limits_{t(\varepsilon_0)}^{\infty} M(t) \ dt \ , \tag{4.37}$$

see e.g. [MZ], p. 3. The latter implies by (4.31) that

$$\int\limits_{B(x_0,\varepsilon_0)} Q(x)\ dm(x)\ \le\ t(\varepsilon_0)\cdot\Omega_n{\varepsilon_0}^2+\int\limits_{t(\varepsilon_0)}^{\infty}\varphi(t)\ dt \tag{4.38}$$

for every ball $B(x_0,\varepsilon_0)\subset U$ because $Q(x)\le Q_0(x)$ on $B(x_0,\varepsilon_0)$.

Further, applying integration by parts, we have from (4.38) that

$$\int\limits_{B(x_0,\varepsilon_0)} Q(x)\ dm(x)\ \le\ \liminf_{t\to\infty}\left(t\cdot\varphi(t)-\int\limits_{t(\varepsilon_0)}^{t}\eta\ d\varphi(\eta)\right)\ \le\ -\int\limits_{t(\varepsilon_0)}^{\infty}\eta\ d\varphi(\eta)\ , \tag{4.39}$$

see Theorem III(14.1) in [Sa].

Now, by the change of variables $\tau=\varphi(\eta)$ we obtain from (4.39) that

$$⨍_{B(x_0,\varepsilon_0)} Q(x)\ dm(x)\ \le\ ⨍_0^{\Omega_n{\varepsilon_0}^2}\varphi^{-1}(\tau)\ d\tau\ . \tag{4.40}$$

Note that the right side in (4.40) is nonincreasing with respect to the parameter $\varepsilon_0$ because $\varphi^{-1}$ is noninsreasing. Thus, the statement is a consequence of Theorem 4.18 under $\beta=1$.

**4.41. Remark.** If a function $Q(x):D\to[1,\infty]$, $D\subset\mathbb{C}$, is measurable and

$$\int\limits_U \Phi(Q(x))\ dm(x)\ \le\ C\ <\ \infty \tag{4.42}$$

where $U\subset D$ and $\Phi:[1,\infty]\to[0,\infty]$ is a nondecreasing function, then

$$M(t)\ =\ m(\{x\in U:\ Q(x)>t\,\})\le\ C/\Phi(t) \tag{4.43}$$

and, thus, the case of the integral constraint (4.42) is reduced to the case of the corresponding constraint by measure (4.43) on $Q$ with $\varphi(t)=C/\Phi(t)$.

## 5 Finite mean oscillation

Recall that a real valued function $\varphi\in\mathrm{L}^1_{loc}(D)$ is said to be of **bounded mean oscillation** in a domain $D\subset\mathbb{R}^n$, abbr. $\varphi\in$ BMO(D) or simply $\varphi\in$ **BMO**, if

$$\|\varphi\|_*=\sup_{B\subset D}⨍_B|\varphi(x)-\varphi_B|\,dm(x)\ <\ \infty \tag{5.1}$$

where the supremum is taken over all balls $B$ in $D$ and

$$\varphi_B=⨍_B\varphi(x)\,dm(x)=\frac{1}{|B|}\int\limits_B\varphi(x)\,dm(x) \tag{5.2}$$

is the mean value of the function $\varphi$ over $B$. It is well–known that $\mathrm{L}^\infty(\mathrm{D})\subset$ BMO(D) $\subset\mathrm{L}^p_{loc}(\mathrm{D})$ for all $1\le p<\infty$, see e.g. [JN] and [RR]. For connections of BMO functions and quasiconformal and quasiregular mappings, see e.g. [As], [AG], [Ge$_2$], [Jo], [MRV] and [RR]. Following [IR$_1$] we say that a function $\varphi : D \to \mathbb{R}$ has **finite mean oscillation** at a point $x_0 \in D$ if

$$\overline{\lim_{\varepsilon\to 0}} \ \ -\!\!\!\!\!\!\int_{B(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi_\varepsilon}|\, dm(x) < \infty \tag{5.3}$$

where

$$\overline{\varphi_\varepsilon} = -\!\!\!\!\!\!\int_{B(x_0,\varepsilon)} \varphi(x)\, dm(x) \tag{5.4}$$

is a mean value of the function $\varphi(x)$ over $B(x_0, \varepsilon)$. Note that under (5.3) it is possible that $\overline{\varphi_\varepsilon} \to \infty$ as $\varepsilon \to 0$. We also say that a function $\varphi : D \to \mathbb{R}$ is of finite mean oscillation in the domain D, abbr. $\varphi \in$ FMO(D) or simply $\varphi \in$ **FMO**, if $\varphi$ has a finite mean oscillation at every point $x \in D$.

Recall some facts on finite mean oscillation from [IR$_1$].

**5.5. Proposition.** *If, for some numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \ \ -\!\!\!\!\!\!\int_{B(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon|\, dm(x) < \infty\,, \tag{5.6}$$

*then $\varphi$ has finite mean osillation at $x_0$.*

**5.7. Corollary.** *If, for a point $x_0 \in D$,*

$$\overline{\lim_{\varepsilon\to 0}} \ \ -\!\!\!\!\!\!\int_{B(x_0,\varepsilon)} |\varphi(x)|\, dm(x) < \infty\,, \tag{5.8}$$

*then $\varphi$ has a finite mean oscillation at $x_0$.*

**5.9. Lemma.** *Let $\varphi : D \to \mathbb{R}, n \geq 2$, be a nonnegative function with a finite mean oscillation at $0 \in D$. Then*

$$\int\limits_{\varepsilon<|x|<\varepsilon_0} \frac{\varphi(x)\, dm(x)}{(|x| log \frac{1}{|x|})^n} = O\left(loglog \frac{1}{\varepsilon}\right) \tag{5.10}$$

*as $\varepsilon \to 0$ for some $\varepsilon_0 \leq dist\,(0, \partial D)$.*

*Thus, from Lemma 3.6 we obtain the following.*

**5.11. Theorem.** *Let $f : D \to \overline{\mathbb{R}^n}$, $n \geq 2$, be a $Q$-homeomorphism such that $D' = f(D)$ omits at least two points $v$ and $w \in \overline{\mathbb{R}^n}$ with $h(v, w) \geq \delta > 0$. If $Q(x)$ has finite mean oscillation at a point $x_0 \in D$, then*

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\delta} \left\{ \log \frac{\varepsilon_0}{|x - x_0|} \right\}^{-\beta_0} \tag{5.12}$$

*for $x \in B(x_0, \varepsilon_0)$ where $\alpha_n$ depends only on $n$, $\varepsilon_0 < dist(x_0, \partial D)$ and $\beta_0 > 0$ depend only on the function $Q$.*

## 6 On normal families of $Q$-homeomorphisms

Let $\mathfrak{F}_{Q,\delta}(D)$ be the class of all $Q$-homeomorphisms $f : D \to \overline{\mathbb{R}^n}$, $n \geq 2$, such that $h(\overline{\mathbb{R}^n} \setminus f(D)) \geq \delta > 0$. The above results now yield:

**6.1. Theorem.** *If $Q \in FMO$, then $\mathfrak{F}_{Q,\delta}(D)$ is a normal family.*

**6.2. Corollary.** *The class $\mathfrak{F}_{Q,\delta}(D)$ is normal if, for every $x_0 \in D$,*

$$\overline{\lim_{\varepsilon \to 0}} \fint_{B(x_0,\varepsilon)} Q(x) \quad dm(x) < \infty. \tag{6.3}$$

**6.4. Corollary.** *The class $\mathfrak{F}_{Q,\delta}(D)$ is normal if every $x_0 \in D$ is a Lebesgue point of $Q(x)$.*

**6.5. Theorem.** *The class $\mathfrak{F}_{Q,\delta}(D)$ is normal if the integral condition*

$$\int\limits_0^{\delta(x_0)} \frac{dr}{r q_{x_0}^{\beta}(r)} = \infty \tag{6.6}$$

*holds for some $\beta \geq 1/(n-1)$ and, in particular, for $\beta = 1$ at every point $x_0 \in D$.*

Here as above $\delta(x_0) = dist(x_0, \partial D)$ and $Q(x)$ denotes the mean integral value of $Q(x)$ over the sphere $|x - x_0| = r$. In view of Remark 4.14 we have also the following.

**6.7. Corollary.** *The class $\mathfrak{F}_{Q,\delta}(D)$ is normal if $Q(x)$ has singularities of the logarithmic type of the order which is not more then $n-1$ at every point $x \in D$.*

**6.8. Theorem.** *The class $\mathfrak{F}_{Q,\delta}(D)$ is normal if*

$$m(\{x \in U_{x_0} : \ Q(x) \ > \ t\}) \ \leq \varphi_{x_0}(t) \ , \quad t \in [1, \infty] \tag{6.9}$$

*in a neighborhood $U_{x_0}$ of every point $x_0 \in D$ for a function $\varphi_{x_0} : [1, \infty] \to [0, \infty]$ which is nonincreasing and continuous from the right such that*

$$\liminf_{t \cdot \to \infty} \ t \cdot \varphi_{x_0}(t) \ = \ 0 \ , \tag{6.10}$$

*and*

$$\int\limits_0^{\delta} \frac{d\rho}{\int\limits_0^{\rho} \varphi_{x_0}^{-1}(\tau) \ d\tau} \ = \ + \infty \ . \tag{6.11}$$

**6.12. Corollary.** *The class $\mathfrak{F}_{Q,\delta}(D)$ is normal if the conditions (6.9) and (6.10) hold for strictly decreasing functions $\varphi_{x_0}$ such that*

$$\int\limits_{\Delta}^{\infty} \frac{d\varphi_{x_0}(t)}{\int\limits_t^{\infty} \eta \ d\varphi_{x_0}(\eta)} \ = \ + \infty \tag{6.13}$$

*at every point* $x_0 \in D$.

**6.14. Remark.** In particular, all the results formulated in Sections 3-6 hold for homeomorphisms $f$ of the Sobolev class $W^{1,n}_{loc}$ with $f^{-1} \in W^{1,n}_{loc}$ under the condition that

$$K_I(x,f) \le Q(x) \;\; a.e. \tag{6.15}$$

where $K_I(x,f)$ is the so-called inner dilatation of the mapping $f$ at the point $x \in D$, see e.g. theorem 4.6 and 7.10 [MRSY$_4$]. Note that, for homeomorphisms $f \in W^{1,n}_{loc}$ with $K_I \in L^1_{loc}$, $f^{-1}$ belongs to $W^{1,n}_{loc}$, see Corollary 2.2 in [KO].

**Postscript.** As in the classical case of $Q$–quasiconformal mappings, cf. Theorem II.5.1 in [LV], a family $\mathfrak{F}$ of $Q$–homeomorphisms $f : D \to \overline{\mathbb{R}^n}$ in a domain $D \subset \mathbb{R}^n$, $n \ge 2$, is normal under every condition given above for $Q$ if there is a number $\Delta > 0$ such that one of the following conditions holds:

1) Every mapping $f \in \mathfrak{F}$ omits two values whose spherical distance is greater than $\Delta$.

2) Every mapping $f \in \mathfrak{F}$ omits one value $w_0$ and $h(w(x_i)), w_0) > \Delta$, $i = 1, 2$, at two fixed points $x_1$ and $x_2 \in D$.

3) At three fixed points $x_1, x_2$ and $x_3 \in D$, $h(w(x_i), w(x_j) > \Delta$, $i \ne j$, $i, j = 1, 2, 3$.

In particular, $\mathfrak{F}$ is normal if all mappings $f \in \mathfrak{F}$ omit two fixed values.

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