ON LINEAR DISTORTION OF HAUSDORFF DIMENSION

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Abstract

Let f be a mapping from a metric space X to a metric space Y, and let α be a positive real number. Write $\dim(E)$ and $\mathcal{H}^s(E)$ for the Hausdorff dimension and s-dimensional Hausdorff measure of a set E. We give sufficient conditions that the equality $\dim(f(E)) = \alpha \dim(E)$ holds for each $E \subseteq X$. The problem is studied also for the Cantor ternary function G. It is shown that there is a subset M of the Cantor ternary set such that $\mathcal{H}^s(M) = 1$ with $s = \frac{\lg 2}{\lg 3}$ and $\dim(G(E)) = \frac{\lg 3}{\lg 2} \dim(E)$ for every $E \subseteq M$.

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1 Statements of main results

Let f be a mapping from a metric space (X, ρ) to a metric space (Y, d). It is a simple fact that if the double inequality

$$c_1(\rho(x,y))^\alpha \le d(f(x), f(y)) \le c_2(\rho(x,y))^\alpha \tag{1.0}$$

holds for all x, y from X where $\alpha \in (0, \infty)$ and c_1 and c_2 are some positive constants, then every set $A \subseteq X$ satisfies

$$\dim(f(A)) = \frac{1}{\alpha} \dim(A).$$

We are interested in necessary and sufficient local conditions under which this equality holds for every $A \subseteq X$. The following theorem provides conditions for this.

Write $\mathrm{ac}(A)$ for the set of all accumulation points of a set A. Let f be a mapping from a metric space (X, ρ) to a metric space (Y, d). If x and a are the points of X and $x \ne a$ we put

$$K_f(x,a) := \begin{cases} \dfrac{\lg(d(f(x), f(a)))}{\lg(\rho(x,a))} & \text{if } \ f(x) \ne f(a) \\ \\ +\infty & \text{if } \ f(x) = f(a) \end{cases}$$

1.1. Theorem. *Let $f : (X, \rho) \to (Y, d)$ be a homeomorphism. Suppose that the limit*

$$\lim_{x \to a} K_f(x, a) = \alpha(a) \in (0, \infty) \tag{1.2}$$

exists for every $a \in \mathrm{ac}(X)$. Then the following statements are equivalent.

1.3. *There exists a set $X_0 \subseteq \mathrm{ac}X$ such that*

$$\alpha(a) = \alpha_0 \tag{1.4}$$

for all $\mathrm{ac}(X) \backslash X_0$ *and either* $\dim(Z) = 0$ *or* $\dim(Z) = \infty$ *for every* $Z \subseteq X_0$.

1.5. *For every $A \subseteq X$ the equality*

$$\dim(f(A)) = \frac{1}{\alpha_0} \dim(A) \tag{1.6}$$

holds. ■

1.7. Corollary. *Let $f : X \to Y$ be a homeomorphism and let* $\dim(X) < \infty$. *Suppose that the limit (1.2) exists for every $a \in \mathrm{ac}(X)$. Then*

$$\dim(f(A)) = \dim(A), \quad \forall\ A \subseteq X \tag{1.8}$$

if and only if

$$\alpha(a) = 1 \tag{1.9}$$

for every $a \in \mathrm{ac}(X) \backslash X_0$ where $X_0 \subset X$ is zero-dimensional. ■

1.10. Remark. Note that, as it follows from the proof of Theorem 1.1, the equivalence 1.3 $\iff$ 1.5 is also valid if (1.2) holds only in $X \backslash X_*$ where

$$\dim(X_*) = \dim(f(X_*)) = 0. \tag{1.11}$$

See further concequences of Theorem 1.1 in the end of Section 2. Note also that, if $f : X \to Y$ is a continuous bijection and (1.2) holds, then it does not follow that f is a homeomorphism. On the other hand, if $f : (X, \rho) \to (Y, d)$ is a homeomorphism and for each $a \in \mathrm{ac}(X)$ we have $\alpha(a) = 1$, then there need not exist positive constants α and c such that the inequality $d(f(x), f(y)) \leq c(\rho(x, y))^\alpha$ holds for all x and y in some ball $B(a, r) \subseteq X$; see Example 3.12.

In the third part, we investigate the following problem: Let $C \subset [0, 1]$ be the standard Cantor ternary set and let G be the Cantor function. Characterize the set of points $x \in C$ such that

$$\lim_{\substack{y \to x \\ y \in C}} \frac{\lg |G(x) - G(y)|}{\lg |x - y|} = \frac{\lg 2}{\lg 3}.$$

In Theorem 1.13 these points are characterized in terms of the spacing of $0's$ and $2's$ in ternary expansions.

Let x be a point of the Cantor ternary set C. Then x has a triadic representation

$$x = \sum_{m=1}^{\infty} \frac{2\alpha_m}{3^m}$$

where $\alpha_m \in \{0, 1\}$. Define a sequence $\{\mathcal{R}_x(n)\}_{n=1}^{\infty}$ by the rule

$$\mathcal{R}_x(n) := \begin{cases} \inf\{m - n : \alpha_m \neq \alpha_n, \quad m > n\} & \text{if } \exists\, m > n : \alpha_m \neq \alpha_n \\ 0 & \text{if } \forall\, m > n : \alpha_m = \alpha_n, \end{cases} \tag{1.12}$$

i.e.

$$\mathcal{R}_x(n) = 1 \iff (\alpha_n \neq \alpha_{n+1}),$$

$$\mathcal{R}_x(n) = 2 \iff (\alpha_n = \alpha_{n+1})\ \&\ (\alpha_{n+1} \neq \alpha_{n+2})$$

and so on.

1.13. Theorem. *Let x be a point of C. Then*

$$\lim_{\substack{y \to x \\ y \in C}} \frac{\lg |G(x) - G(y)|}{\lg |x - y|} = \frac{\lg 2}{\lg 3} \tag{1.14}$$

if and only if

$$\lim_{n \to \infty} \frac{\mathcal{R}_x(n)}{n} = 0. \blacksquare \tag{1.15}$$

This theorem and Theorem 1.1 imply the following result.

1.16. Theorem. *There exists a set $M \subseteq C$ such that $\mathcal{H}^s(M) = 1$ with $s = \frac{\lg 2}{\lg 3}$ and* $\dim(G(A)) = \frac{\lg 3}{\lg 2} \dim(A)$ *for every $A \subseteq M$.* $\blacksquare$

1.17. Remarks. It is well-known that the Cantor function G satisfies the inequality

$$|G(x) - G(y)| \leq 2|x - y|^{\frac{\lg 2}{\lg 3}} \tag{1.18}$$

for all x and y in $[0, 1]$. The proof can be found in [HT], see also [Gi]. The Hausdorff dimension of the Cantor ternary set equals $\frac{\lg 2}{\lg 3}$ and, moreover, $\mathcal{H}^s(C) = 1$ for $s = \frac{\lg 2}{\lg 3}$.

2 Linear distortion of Hausdorff dimension under mappings of metric spaces

We recall the definitions of the Hausdorff dimension and s-dimensional Hausdorff measure. Let (X, ρ) be a metric space and let

$$\operatorname{diam} A := \sup\{\rho(x, y) : x, y \in A\}$$

be the **diameter** of $A \subseteq X$ if $A \neq \emptyset$, diam $\emptyset = 0$. If $A \subseteq \bigcup_{i \in I} E_i$ with $0 < \mathrm{diam} E_i \leq \delta$ for each index $i \in I$, then $\{E_i\}_{i \in I}$ is called *a* δ-cover of A. If all δ-covers of A are uncountable, then

$$\mathcal{H}^s_\delta(A) := \infty$$

for each $s \geq 0$ and, in the opposite case,

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{i \in \mathbb{N}} (\mathrm{diam}\ E_i)^s : \{E_i\}_{i \in \mathbb{N}} \text{ is a countable } \delta\text{-cover of } A \right\} \tag{2.1}$$

for $s \geq 0$. The s**-dimensional Hausdorff measure** of A is defined by

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A). \tag{2.2}$$

Note that the limit exists because $\mathcal{H}^s_\delta(A)$ is nonincreasing function of δ and that the Hausdorff measure $\mathcal{H}^s$ is a regular Borel measure, see e.g. [Ro]. The **Hausdorff dimension** of A is the number $\dim(A)$ such that

$$\mathcal{H}^s(A) = \begin{cases} +\infty & \text{if} \quad s < \dim(A), \\ 0 & \text{if} \quad s > \dim(A). \end{cases}$$

2.3. Definition. Let (X, ρ) be a metric space. A family $\mathcal{B}$ of closed balls $B(a, \delta)$ in (X, ρ) is said to fulfil the **condition** (**V**) if $B(a, \delta) \in \mathcal{B}$ whenever $a \in X$ and $\delta \in (0, \Delta(a)]$ for some $\Delta(a) > 0$.

The proof of the next Lemma is well–known.

2.4. Lemma. *Let A be a subset of the metric space (X, ρ) such that $\mathcal{H}^s(A) = 0$ for some $s > 0$ and let a family $\mathcal{B}$ of closed balls in (X, ρ) fulfil the (V) condition. Then, for all positive numbers δ and η, there is a countable δ-cover $\{B_i\}_{i \in \mathbb{N}} \subseteq \mathcal{B}$ of the set A with*

$$\sum_{i \in \mathbb{N}} (\mathrm{diam}\ B_i)^s \leq \eta. \tag{2.5}$$

2.9. Proposition. *Suppose (X, ρ) and (Y, d) be metric spaces. Let $\beta \in (0, \infty)$ and let $f : X \to Y$ be a mapping such that*

$$\liminf_{x \to a} K_f(x, a) \geq \beta \tag{2.10}$$

for each $a \in \mathrm{ac}(X)$. Then we have

$$\dim(A) \geq \beta \dim(f(A)) \tag{2.11}$$

for every $A \subseteq X$.

Proof. If $\dim(A) = \infty$, then the inequality (2.11) is trivial. Suppose that $0 \leq \dim(A) < s < \infty$. Then by the definition of the Hausdorff dimension we have $\mathcal{H}^s(A) = 0$.

For each $\varepsilon \in (0, \beta)$, define a family $\mathcal{B}_\varepsilon$ of the closed balls $B(a, \delta)$ in (X, ρ) by the rule

$$(B(a,\delta) \in \mathcal{B}_\varepsilon) \iff (\forall\, x \in B(a,\delta) : d(f(x), f(a)) \leq (\rho(x,a))^{\beta-\varepsilon}). \tag{2.12}$$

It follows immediately from (2.10) that $\mathcal{B}_\varepsilon$ fulfils the condition (V). Hence by Lemma 2.4, for every $\eta > 0$, there is δ-cover $\{B_i(a_i, \delta_i)\}_{i\in\mathbb{N}} \subseteq \mathcal{B}_\varepsilon$ of A such that

$$\sum_{i\in\mathbb{N}} (\text{diam } B(a_i, \delta_i))^s \leq \eta. \tag{2.13}$$

It follows from (2.12) that

$$\text{diam } (f(B(a_i, \delta_i)) \leq 2(\text{diam } (B(a_i, \delta_i)))^{\beta-\varepsilon}.$$

The last inequality and (2.13) imply that $\{f(B(a_i, \delta_i))\}_{i\in\mathbb{N}}$ is $2\delta^{\beta-\varepsilon}$-cover of $f(A)$ and

$$\sum_{i\in\mathbb{N}} (\text{diam } f(B(a_i, \delta_i)))^{\frac{s}{\beta-\varepsilon}} \leq 2^{\frac{s}{\beta-\varepsilon}}\eta.$$

Consequently

$$\mathcal{H}^{\frac{s}{\beta-\varepsilon}}(f(A)) = 0,$$

that is

$$\dim(f(A)) \leq \frac{s}{\beta - \varepsilon}$$

for all $\varepsilon \in (0, \beta)$ and every $s > \dim(A)$. Letting $\varepsilon \to 0$ and $s \to \dim(A)$ we have (2.11). ■

2.14. Corollary. *Suppose (X, ρ) an (Y, d) be a metric spaces. Let $0 < \beta \leq \alpha < \infty$ and let $f : X \to Y$ be a homeomorphism such that*

$$\beta \leq \liminf_{x\to a} K_f(x, a) \leq \limsup_{x\to a} K_f(x, a) \leq \alpha \tag{2.15}$$

for every $a \in \text{ac}(X)$. Then the inequalities

$$\frac{1}{\alpha}\dim(A) \leq \dim(f(A)) \leq \frac{1}{\beta}\dim(A) \tag{2.16}$$

hold for every $A \subseteq X$.

Proof. By Proposition 2.9 it suffices to prove the first inequality in (2.16).
Since f is a homeomorphism, we have

$$\text{ac}(Y) = f(\text{ac}(X)).$$

Let f^{-1} be the inverse map of f and let $a \in \mathrm{ac}\,(X)$. Applying inequality (2.15) we obtain

$$\frac{1}{\alpha} \leq \{\limsup_{x \to a} K_f(x,a)\}^{-1} = \liminf_{y \to b} K_{f^{-1}}(y,b)$$

where $b = f(a) \in \mathrm{ac}(Y)$. Now the desired inequality follows from Proposition 2.9. ∎

2.17. Proof of Theorem 1.1.

1.3 $\Rightarrow$ **1.5** Suppose that there is $X_0 \subseteq \mathrm{ac}(X)$ such that $\alpha(a) = \alpha_0$ for every $a \in \mathrm{ac}(X) \backslash X_0$, and for every $Z \subseteq X_0$ we have either $\dim(Z) = 0$ or $\dim(Z) = \infty$. Let A be a subset of X. Then

$$\dim(A) = \max\{\dim(A \backslash X_0),\ \dim(A \cap X_0)\}$$

and

$$\dim(f(A)) = \max\{\dim(f(A \backslash X_0)),\ \dim(f(A \cap X_0))\}.$$

Thus, by Corollary 2.14 it remains to prove that either

$$\dim(A \cap X_0) = \dim(f(A \cap X_0)) = 0$$

or

$$\dim(A \cap X_0) = \dim(f(A \cap X_0)) = +\infty.$$

To prove this we represent $A \cap X_0$ in the form

$$A \cap X_0 = \bigcup_{n=1}^{\infty} A_n$$

where

$$A_n := \{a \in A \cap X_0 : \frac{1}{n} \leq \alpha(a) \leq n\}$$

for $n = 1, 2, \ldots$. From this representation we get the equalities

$$f(A \cap X_0) = \bigcup_{n=1}^{\infty} f(A_n),$$

$$\dim(A \cap X_0) = \sup_{1 \leq n < \infty} (\dim A_n),$$

$$\dim(f(A \cap X_0)) = \sup_{1 \leq n < \infty} \dim(f(A_n)).$$

Hence $\dim(A \cap X_0) = 0$ iff

$$\forall\, n \in \mathbb{N} : \dim(A_n) = 0.$$

It follows from the alternative

$$\dim(A_n) = 0 \quad \text{or} \quad \dim(A_n) = \infty$$

that we have $\dim(A \cap X_0) = \infty$ iff there exists $n_0 \in \mathbb{N}$ such that $\dim(A_{n_0}) = \infty$.

Consequently, the conclusion follows by Corollary 2.14.

1.5 $\Rightarrow$ **1.3** Suppose that (1.6) holds for every $A \subseteq X$. Put

$$X_0^+ := \{a \in \mathrm{ac}X : \alpha(a) > \alpha_0\}, \quad X_0^- := \{a \in \mathrm{ac}X : \alpha(a) < \alpha_0\}, \quad \text{and} \quad X_0 := X_0^+ \cup X_0^-.$$

By definition $\alpha(a) = \alpha_0$ for each $a \in (\mathrm{ac}X)\backslash X_0$. It remains to prove that

$$\dim(Z) = 0 \quad \text{or} \quad \dim(Z) = +\infty$$

for each $Z \subseteq X_0$.

Suppose that for some $Z \subseteq X_0$

$$0 < \dim(Z) < \infty.$$

Then we have

$$0 < \dim(Z \cap X_0^+) < \infty \tag{2.18}$$

or

$$0 < \dim(Z \cap X_0^-) < \infty. \tag{2.19}$$

Consider the case (2.18) first. Set for each positive integer n

$$X_n^+ := \{a \in \mathrm{ac}(X) : \alpha(a) \in (\alpha_0 + \frac{1}{n}, \alpha_0 + n)\}. \tag{2.20}$$

Then

$$X_0^+ = \bigcup_{n=1}^{\infty} X_n^+$$

and (2.18) implies that for some $n_0 \in \mathbb{N}$

$$0 < \dim(Z \cap X_{n_0}^+) < \infty.$$

Hence, by Corollary 2.14 and (2.20), we have

$$\dim(f(Z \cap X_{n_0}^+)) \leq \frac{1}{\alpha_0 + \frac{1}{n_0}} \dim(Z \cap X_{n_0}^+) < \frac{1}{\alpha 0} \dim(Z \cap X_{n_0}^+).$$

This contradicts (1.6) with $A = Z \cap X_{n_0}^+$. The case (2.19) can be proved analogously. ■

2.21. Proof of Corollary 1.7.

In order to prove this corollary, it suffices to take $\alpha_0 = 1$. It should be observed here that the inequality $\dim(X) < \infty$ implies the equality $\dim(Z) = 0$ for each $Z \subseteq X_0$. ■

2.22. Corollary. *Suppose that (X, ρ) and (Y, d) are metric spaces, and*

$$X = X^\circ \cup X^1, \quad X^\circ \cap X^1 = \emptyset.$$

Let $\alpha \in (0, \infty)$ and let $\varphi : X \to Y$ be a mapping such that:

$$\dim(X^\circ) = \dim(\varphi(X^\circ)) = 0. \tag{2.23}$$

2.24. *For every* $a \in \mathrm{ac}(X^1) \cap X^1$,

$$\lim_{\substack{x \to a \\ x \in X^1}} K_\varphi(x, a) = \alpha.$$

2.25. *The restriction* $\varphi|_{X^1} : X^1 \to \varphi(X^1)$ *is a homeomorphism.*
Then

$$\dim(\varphi(A)) = \frac{1}{\alpha} \dim(A) \tag{2.26}$$

for each $A \subseteq X$.

Proof. It follows from (2.23) that $\dim(A) = \dim(A \cap X^1)$ and $\dim(\varphi(A)) = \dim(\varphi(A \cap X^1))$ for every $A \subseteq X$. Consequently, it suffices to prove (2.26) for $A \subseteq X^1$.
For this purpose we can use Theorem 1.1 with $X = X^1$, $Y = \varphi(X^1)$, $X_0 = \emptyset$ and f equals $\varphi|_{X^1} : X^1 \to \varphi(X^1)$. ■

2.27. Corollary. *Suppose that all the conditions of Corollary 2.22 with an exception of 2.25. If* X^1 *can be represented in the form* $X^1 = \bigcup_{j \in \mathbb{N}} X^1_j$ *such that* $\varphi|_{X^1_j} : X^1_j \to \varphi(X^1_j)$ *is a homeomorphism for every* $j \in \mathbb{N}$, *then equality (2.26) holds for every* $A \subseteq X$.

Proof. Reasoning as in the proof of Corollary 2.22 we can easily show that

$$\dim(\varphi(A)) = \sup_{j \in \mathbb{N}}(\dim(\varphi(A \cap X^1_j))) = \frac{1}{\alpha} \sup_{j \in \mathbb{N}}(\dim(A \cap X^1_j)) = \frac{1}{\alpha} \dim(A). \ ■$$

2.28. Corollary. *Suppose all the conditions of Corollary 2.22 hold except 2.25. If* X^1 *is separable and* $\varphi|_{X^1} : X^1 \to \varphi(X^1)$ *is a local homeomorphism, then equality (2.26) holds for each* $A \subseteq X$.

Proof. Every separable metric space has a countable base, see e.g. [Ku, §21, II, Theorem 2]. Hence we can use Corollary 2.27. ■

Let $cp(X)$ denote the set of all **condensation points** of a metric space X, i.e. points whose neighborhoods are not countable sets.

2.29. Corollary. *Suppose that* X *and* Y *are metric spaces,* $f : X \to Y$ *is a local homeomorphism and* X *is separable. Let* $\alpha \in (0, \infty)$ *and let*

$$\lim_{x \to a} K_f(x, a) = \alpha$$

for every $a \in cp(X)$. *Then (2.26) holds for every* $A \subseteq X$.

Proof. The set $X \backslash cp\,(X)$ is a countable set in every separable metric space X, [Ku, §23, III]. ■

3 The Cantor ternary function

We recall the definitions of the Cantor ternary set C and ternary Cantor function G. Let $x \in [0,1]$, then x belongs C if and only if x has a base 3 expansion using only the digits 0 and 2, i.e.

$$x = \sum_{m=1}^{\infty} \frac{2\alpha_m}{3^m}, \quad \alpha_m \in \{0,1\}. \tag{3.1}$$

The Cantor function G may be defined on C by the following rule. If $x \in C$ has the ternary expansion (3.1) then

$$G(x) := \sum_{m=1}^{\infty} \frac{\alpha_m}{2^m}.$$

In this section we will denote by C^1 the set of all endpoints of complementary intervals of C. We also write $C^\circ := C \backslash C^1$.

The proof of Theorem 1.13 needs the following lemma.

3.2. Lemma. *Let $x \in C$ be a point with the representation (3.1) and let $\mathcal{R}_x(n)$ be the sequence from (1.12). Then*

$$\mathcal{R}_x(n) = \mathcal{R}_{1-x}(n). \tag{3.3}$$

If x is not a right endpoint of a complementary interval of C, then

$$\sum_{m=n+1}^{\infty} \alpha_m 2^{-(m-n)} \geq 2^{-(1+\mathcal{R}_x(n+1))}, \quad \forall\, n \in \mathbb{N}. \tag{3.4}$$

Proof. Since $1-x = \sum\limits_{m=1}^{\infty} \frac{2(1-\alpha_m)}{3^m}$ we have (3.3). It remains to prove (3.4). If $\alpha_{n+1} = 1$, then (3.4) is obvious. In the opposite case, $2^{-(1+\mathcal{R}_x(n+1))}$ is the first positive element of the series $\sum\limits_{m=n+1}^{\infty} \alpha_m 2^{-(m-n)}$. ■

3.5. Proof of Theorem 1.13.

Consider first the case where x is not an endpoint of some complementary interval of C. Suppose that x has representation (3.1), y tends to x, and

$$y = \sum_{m=1}^{\infty} \frac{2\beta_m}{3^m}$$

where $\beta_m = \beta_m(y) \in \{0,1\}$. Let $n_0 = n_0(x,y)$ be the smallest index m with $|\beta_m - \alpha_m| \neq 0$. Then using the definition of the Cantor function we have

$$\frac{\lg 3}{\lg 2} \lim_{\substack{y \to x \\ y \in C}} \frac{\lg |G(x) - G(y)|}{\lg |x-y|} = \lim_{\substack{y \to x \\ y \in C}} \left(1 - \frac{\lg \left| \operatorname{sign}(\alpha_{n_0} - \beta_{n_0}) + \sum\limits_{m=n_0+1}^{\infty} (\alpha_m - \beta_m) 2^{-(m-n_0)} \right|}{n_0 \lg 2} \right).$$

Writing

$$z(x,y) := \left|\operatorname{sign}(\alpha_{n_0}-\beta_{n_0}) + \sum_{m=n_0+1}^{\infty}(\alpha_m-\beta_m)2^{-(m-n_0)}\right|$$

we see that the limit relation (1.14) is equivalent to

$$\lim_{\substack{y\to x\\ y\in C}} \frac{\lg z(x,y)}{n_0(x,y)} = 0. \tag{3.6}$$

Next we obtain bounds for $z(x,y)$. If

$$\operatorname{sign}(\alpha_{n_0}-\beta_{n_0}) = 1,$$

then

$$z(x,y) = 1 + \sum_{m=n_0+1}^{\infty}(\alpha_m-\beta_m)2^{-(m-n_0)}$$

and hence

$$\sum_{m=n_0+1}^{\infty}\alpha_m 2^{-(m-n_0)} \le z(x,y) \le 2.$$

Consequently, by (3.4) we obtain

$$2^{-(1+\mathcal{R}_x(n_0+1))} \le z(x,y) \le 2. \tag{3.7}$$

If

$$\operatorname{sign}(\alpha_{n_0}-\beta_{n_0}) = -1,$$

then

$$z(x,y) = 1 - \sum_{m=n_0+1}^{\infty}(\alpha_m-\beta_m)2^{-(m-n_0)},$$

and hence

$$1 - \sum_{m=n_0+1}^{\infty}\alpha_m 2^{-(m-n_0)} \le z(x,y) \le 2. \tag{3.8}$$

Since

$$1 - \sum_{m=n_0+1}^{\infty}\alpha_m 2^{-(m-n_0)} = \sum_{m=n_0+1}^{\infty}(1-\alpha_n)2^{-(m-n_0)},$$

relations (3.8), (3.4) and (3.3) imply that

$$z(x,y) \ge 2^{-(1+\mathcal{R}_{1-x}(n_0+1))} = 2^{-(1+\mathcal{R}_x(n_0+1))}.$$

Consequently, as in the first case, we have (3.7). Now, (3.6) follows from (3.7) and (1.15). Thus, the implication (1.15) $\Rightarrow$ (1.14) follows.

Suppose now that (1.15) does not hold. Then there is a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty}\frac{\mathcal{R}_x(1+n_j)}{1+n_j}=\limsup_{n\to\infty}\frac{\mathcal{R}_x(n)}{n}=a\in(0,\infty] \tag{3.9}$$

and

$$\alpha_{n_j}\neq\alpha_{n_j+1},\quad \forall\ n_j \tag{3.10}$$

where α_{n_j} and α_{n_j+1} are digits in the representation (3.1).

Let $\{\alpha_m\}_{m=1}^{\infty}$ be a sequence of digits in (3.1). Putting

$$\beta_m^{(j)}:=\begin{cases}1-\alpha_m & \text{if } n_j\le m\le n_j+\mathcal{R}_x(1+n_j)\\ \alpha_m & \text{otherwise}\end{cases} \tag{3.11}$$

and

$$y_j:=\sum_{m=1}^{\infty}\frac{2\beta_m^{(j)}}{3^m},$$

we claim that

$$\lim_{j\to\infty}\frac{\lg|G(x)-G(y_j)|}{\lg|x-y_j|}=\frac{\lg 2}{\lg 3}\,(1+a).$$

Indeed, (3.10) and (3.11) imply the equalities

$$\frac{1}{2}|x-y_j|=\left(\frac{1}{3}\right)^{n_j}-\left(\frac{1}{3}\right)^{n_j+1}-\ldots-\left(\frac{1}{3}\right)^{n_j+\mathcal{R}_x(n_j+1)},$$

$$|G(x)-G(y_j)|=\left(\frac{1}{2}\right)^{n_j}-\left(\frac{1}{2}\right)^{n_j+1}-\ldots-\left(\frac{1}{2}\right)^{n_j+\mathcal{R}_x(n_j+1)}.$$

Hence, by (3.9), we have

$$\lim_{j\to\infty}\frac{\lg|G(x)-G(y_j)|}{\lg|x-y_j|}=\frac{\lg 2}{\lg 3}\lim_{j\to\infty}\frac{n_j+\mathcal{R}_x(1+n_j)}{n_j}=(1+a)\frac{\lg 2}{\lg 3}.$$

Consider now the case where $x\in C^1$. In this case there is $n_0\in\mathbb{N}$ such that $\mathcal{R}_x(n)=0$ for every $n\ge n_0$. It remains only to show that

$$\lim_{\substack{y\to x\\ y\in C}}\frac{\lg|G(x)-G(y)|}{\lg|x-y|}=\frac{\lg 2}{\lg 3}.$$

Suppose that x is a right endpoint of a complementary interval, (i.e. $\alpha_n=0$ for all large enough n), y tends to x, and

$$y=\sum_{m=1}^{\infty}\frac{2\beta_m(y)}{3^m},\quad \beta_m(y)\in\{0,1\}.$$

Then there are positive integers m_1 and $m_2 = m_2(y)$ such that $x = \sum\limits_{m=1}^{m_1} \frac{2\alpha_m}{3^m}$, $\beta_m(y) = \alpha_m$ if $1 \le m \le m_1$, $m_2(y) > m_1$, $\beta_{m_2}(y) = 1$, and $\beta_m(y) = 0$ if $m_1 < m < m_2(y)$.

Hence

$$\lim_{\substack{y\to x\\ y\in C}} \frac{\lg|G(x)-G(y)|}{\lg|x-y|} = \lim_{\substack{y\to x\\ y\in C}} \frac{\lg\left(\sum\limits_{m=m_2(y)}^{\infty} \beta_m 2^{-m}\right)}{\lg\left(2\sum\limits_{m=m_2(y)}^{\infty} \beta_m 3^{-m}\right)} = \lim_{\substack{y\to x\\ y\in C}} \frac{-m_2(y)\lg 2}{-m_2(y)\lg 3} = \frac{\lg 2}{\lg 3}.$$

The case where x is a left endpoint of a complementary interval is similar. ■

3.12. Example. Here we give an example of a homeomorphism $f : (X, \rho) \to (Y, d)$ such that:

(i) For every $a \in \operatorname{ac}(X)$

$$\lim_{x\to a} K_f(x, a) = 1.$$

(ii) For arbitrary positive α, c and for every ball $B(a,\delta) \subseteq X$ there exist $x, y \in B(a,\delta)$ for which

$$d(f(x), f(y)) \ge c(\rho(x,y))^\alpha.$$

The example is constructed with aid of the Cantor ternary set C and the Cantor function G.

Set

$$M := \left\{ x \in C : \lim_{\substack{y\to x\\ y\in C}} \frac{\lg|G(x)-G(y)|}{\lg|x-y|} = \frac{\lg 2}{\lg 3} \right\}$$

and

$$I^\circ := G(M \cap C^\circ). \tag{3.13}$$

Let $F_1 : I^\circ \to M \cap C^\circ$ be a function such that $F(G(y)) = y$ for every $y \in M \cap C^\circ$. It is easy to see that F_1 is a homeomorphism and by the definition of M

$$\lim_{y\to a} K_{F_1}(a, y) = \frac{\lg 3}{\lg 2} \tag{3.14}$$

for every $a \in I^\circ$.

Let $\mathcal{E}$ be the space of infinite strings from the two-letters alphabet $\{0, 1\}$. We may define a metric d on $\mathcal{E}$ by setting

$$d(\alpha, \beta) = \max_{1\le n<\infty} \frac{1}{2^n}|\alpha_n - \beta_n|$$

if $\alpha = \{\alpha_n\}_{n=1}^\infty$, $\beta = \{\beta_n\}_{n=1}^\infty$ are elements of $\mathcal{E}$.

Evidently, if $d(\alpha,\beta) \ne 0$, then there is a positive integer n such that $d(\alpha, \beta) = 2^{-n}$.

The space $(\mathcal{E}, d)$ is an ultrametric space and the map

$$C \ni \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n} \xrightarrow{\Phi} \{\varepsilon_n\} \in \mathcal{E}$$

is a homeomorphism. (The proof can be found in [Ed, Chapter 2].)

We claim that the inequality

$$\frac{1}{2}|x-y|^{\frac{\lg 2}{\lg 3}} \leq d(\Phi(x), \Phi(y)) \leq |x-y|^{\frac{\lg 2}{\lg 3}} \tag{3.15}$$

holds for all $x, y \in C$. Really, if x and y have the representations

$$x = \sum_{i=1}^{\infty} 2\alpha_i 3^{-i}, \quad y = \sum_{i=1}^{\infty} 2\beta_i 3^{-i}, \quad \alpha_i, \ \beta_i \in \{0, 1\},$$

and if $d(\Phi(x), \Phi(y)) = 2^{-n}$, then we get

$$\frac{1}{2}|x-y| = \left|\sum_{i=n}^{\infty}(\alpha_i - \beta_i)3^{-i}\right| \geq 3^{-n}\left(1 - \sum_{i=1}^{\infty} 3^{-i}\right) = \frac{1}{2}(d(\Phi(x), \Phi(y)))^{\frac{\lg 3}{\lg 2}}.$$

A similar argument yields

$$|x-y| \leq 3(d(\Phi(x), \Phi(y)))^{\frac{\lg 3}{\lg 2}}.$$

These inequalities imply (3.15).

Let $F_2 : M \cap C^\circ \to \Phi(M \cap C^\circ)$ be the restriction of the map Φ on the set $M \cap C^\circ$. Obviously, F_2 is a homeomorphism and it follows from (3.15) that

$$\lim_{z \to a} K_{F_2}(a, z) = \frac{\lg 2}{\lg 3} \tag{3.16}$$

for each $a \in (M \cap C^\circ)$. Now set, for every $x \in I^\circ$,

$$f(x) := F_2(F_1(x)).$$

The function f is a homeomorphism from I° to $\Phi(M \cap C^\circ)$ and the limits (3.14), (3.16) imply that

$$\lim_{x \to a} K_f(x, a) = 1$$

for each $a \in I^\circ$, i.e., we get (i).

Suppose α, c are arbitrary positive constants and O ia an open interval in $[0, 1]$. To prove (ii) we can choose $x \in C$ such that (3.9) holds and

$$\alpha > \frac{2}{1+a}, \quad a \in (0, \infty), \quad G(x) \in O.$$

Since M is a dense subset of a perfect set C, there are sequences $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ in M such that

$$\lim_{j \to \infty} x_j = \lim_{j \to \infty} y_j = x, \quad x_j \neq y_j \quad \forall \ j \in \mathbb{N},$$

and

$$\lim_{j\to\infty} \frac{\lg|F_1(x_j)-F_1(y_j)|}{\lg|x_j-y_j|} = \frac{\lg 3}{\lg 2}\frac{1}{(1+a)},$$

see the proof of Theorem 1.13. The last relation and (3.15) imply that

$$\lim_{j\to\infty} \frac{\lg d(f(x_j),f(y_j))}{\lg|x_j-y_j|} = \frac{1}{(1+a)}.$$

Hence there is $N_0 \in \mathbb{N}$ such that

$$d(f(x_j),f(y_j)) \geq |x_j-y_j|^{\frac{2}{1+a}}$$

for $j \geq N_0$. Since $\frac{2}{1+a} < \alpha$ and $\lim\limits_{j\to 0}|x_j-y_j| = 0$, we have (ii). ■

To prove Theorem 1.16 we use the following two propositions.

3.17. Proposition. *Let I° be the subset of the unit interval $[0,1]$ from (3.13). Then each number, simply normal to base 2, belongs to I°.* ■

We recall the definition. Suppose x belongs to $[0,1]$ and has the following base 2 representation

$$x = \sum_{n=1}^{\infty}\frac{a_n}{2^n}, \quad a_n \in \{0,1\}.$$

This number x is called a **simply normal** to base 2 if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} a_n = \frac{1}{2}. \tag{3.18}$$

3.19. Proof of Proposition 3.17. Let $x_0 \in C$ be a point with the ternary expansion

$$x_0 = \sum_{n=1}^{\infty}\frac{2\alpha_n}{3^n}, \quad \alpha_n \in \{0,1\}.$$

Suppose $G(x_0)$ is a simply normal to base 2 number. Since $G(x_0) = \sum\limits_{n=1}^{\infty}\frac{\alpha_n}{2^n}$, formula (3.18) implies that $x_0 \in C^\circ$. (If $x_0 \in C^1$, i.e. x is an endpoint of a complementary interval of C, then $\lim\limits_{N\to\infty}\frac{1}{N}\sum\limits_{n=1}^{N}\alpha_n = 0$ or $\lim\limits_{N\to\infty}\frac{1}{N}\sum\limits_{n=1}^{N}\alpha_n = 1$.) Hence, it suffices to show that

$$\lim_{n\to\infty}\frac{\mathcal{R}_{x_0}(n)}{n} = 0. \tag{3.20}$$

Put $\mathbb{N}_0 := \{n : \alpha_n = 0\}$ and $\mathbb{N}_1 := \{n : \alpha_n = 1\}$. Since $x_0 \in C^\circ$ we have $\operatorname{card}(\mathbb{N}_0) = \operatorname{card}(\mathbb{N}_1) = \infty$. It follows from (3.18) that

$$\frac{1}{2} = \lim_{\substack{m\to\infty\\ m\in\mathbb{N}_1}}\frac{1}{m+\mathcal{R}_{x_0}(m)}\sum_{n=1}^{m+\mathcal{R}_{x_0}(m)}\alpha_n =$$

$$= \lim_{\substack{m\to\infty \\ m\in\mathbb{N}_1}} \frac{m}{m+\mathcal{R}_{x_0}(m)} \frac{1}{m} \left(\mathcal{R}_{x_0}(m) - 1 + \sum_{n=1}^{m} \alpha_m \right) =$$

$$= \lim_{\substack{m\to\infty \\ m\in\mathbb{N}_1}} \frac{m}{m+\mathcal{R}_{x_0}(m)} \left(\frac{\mathcal{R}_{x_0}(m)}{m} + \frac{1}{2} \right).$$

Hence we get

$$\lim_{\substack{m\to\infty \\ m\in\mathbb{N}_1}} \frac{\mathcal{R}_{x_0}(m)}{m} = 0.$$

A similar calculation yields

$$\lim_{\substack{m\to\infty \\ m\in\mathbb{N}_0}} \frac{\mathcal{R}_{x_0}(m)}{m} = 0.$$

Since $\mathbb{N} = \mathbb{N}_0 \cup \mathbb{N}_1$ we have (3.20). ■

3.21. Proposition. *Let m_1 be the Lebesgue measure on $\mathbb{R}$, and let $s = \frac{\lg 2}{\lg 3}$. Then*

$$m_1(G(A)) = \mathcal{H}^s(A)$$

for every $A \subseteq C$.

Proof. Let $\varphi_0, \varphi_1 : \mathbb{R} \to \mathbb{R}$ are similarity contractions defined by

$$\varphi_0(x) = \frac{1}{3}x \quad \text{and} \quad \varphi_1(x) = \frac{1}{3}x + \frac{2}{3}.$$

It is known that there is the unique Borel regular probability measure μ such that

$$\mu(A) = \frac{1}{2}\mu(\varphi_0^{-1}(A)) + \frac{1}{2}\mu(\varphi_1^{-1}(A)) \tag{3.22}$$

for every Borel set $A \subseteq \mathbb{R}$ [Fa]. Furthemore, if $s = \frac{\lg 2}{\lg 3}$ and $\mu(A) := \mathcal{H}^s(A \cap C)$ then μ satisfies (3.22) [Fa]. Now set

$$\mu(A) = m_1(G(A \cap C)).$$

It suffices to show that μ is a Borel regular probability measure which fulfils (3.22) for every Borel set $A \subseteq \mathbb{R}$.

Since C^1 is countable, $m_1(G(C^1)) = 0$. Hence $\mu(A) = m_1(G(C^\circ \cap A))$ for $A \subseteq \mathbb{R}$. The restriction $G|_{C^\circ} : C^\circ \to G(C^\circ)$ is a homeomorphism and $\mu(\mathbb{R}) = m_1(G(C^\circ)) = 1$. Hence μ is a Borel regular probability measure.

To prove (3.22) we will use the following functional equations

$$G\left(\frac{x}{3}\right) = \frac{1}{2}G(x), \quad 0 \le x \le 1, \tag{3.23}$$

$$G(x) = \frac{1}{2} + G\left(x - \frac{2}{3}\right), \quad \frac{2}{3} \le x \le 1, \tag{3.24}$$

see [Ev].

Let A be a Borel subset of $\mathbb{R}$. Put

$$A_0 := A \cap \left[0, \frac{1}{3}\right], \quad A_1 := A \cap \left[\frac{2}{3}, 1\right].$$

Since $\varphi_0^{-1}(x) = 3x$ and $\varphi_1^{-1}(x) = 3\left(x - \frac{2}{3}\right)$, we have

$$\mu(\varphi_i^{-1}(A)) = \mu(\varphi_i^{-1}(A_i))$$

for $i = 1, 2$. It follows from (3.23) that

$$2G(x) = G(\varphi_0^{-1}(x)),$$

for $x \in \left[0, \frac{1}{3}\right]$. Hence

$$\frac{1}{2}\mu(\varphi_0^{-1}(A_0)) = \frac{1}{2}m_1(G(\varphi_0^{-1}(A_0))) = \frac{1}{2}m_1(2G(A_0)) = m_1(G(A_0)).$$

Similarly, (3.23) implies that

$$2G\left(x - \frac{2}{3}\right) = G(\varphi_1^{-1}(x))$$

for $x \in \left[\frac{2}{3}, 1\right]$ and we have

$$\frac{1}{2}\mu(\varphi_1^{-1}(A_1)) = \frac{1}{2}m_1(G(\varphi_1^{-1}(A_1))) = \frac{1}{2}m_1\left(2G\left(A_1 - \frac{2}{3}\right)\right) = m_1\left(G\left(A_1 - \frac{2}{3}\right)\right).$$

Now using (3.24) we get

$$m_1\left(G\left(A_1 - \frac{2}{3}\right)\right) = m_1\left(-\frac{1}{2} + G(A_1)\right) = m(G(A_1)).$$

The set $G(A_1) \cap G(A_0)$ is empty or contains only the point $\frac{1}{2}$. Hence we have

$$m_1(G(A_0)) + m_1(G(A_1)) = m_1(G(A_0 \cup A_1)) = \mu(A).$$

Therefore $\mu(A)$ satisfies (3.22). $\blacksquare$

3.25. Proof of Theorem 1.16. As in Example 3.12 set

$$M = \left\{x \in C : \lim_{\substack{y \to x \\ y \in C}} K_G(y, x) = \frac{\lg 2}{\lg 3}\right\}.$$

We claim that

$$\dim(G(A)) = \frac{\lg 3}{\lg 2} \dim(A) \tag{3.26}$$

for every $A \subseteq M$, and

$$\mathcal{H}^s(M) = 1 \tag{3.27}$$

for $s = \frac{\lg 2}{\lg 3}$.

In order to prove (3.26), we can apply Corollary 2.22 with $X = M$, $X^\circ = C^1$, $Y = [0,1]$. Observe that C^1 is countable and hence we have (2.23). The restriction

$$G|_{M \cap C^\circ} : M \cap C^\circ \to G(M \cap C^\circ)$$

is strictly increasing and continuous, so that it is a homeomorphism.

It remains to verify (3.27). Let N^- be the set of numbers which are not simply normal to base 2. It is known [Ni, p.103] that $m_1(N^-) = 0$. By Proposition 3.17 I° is the superset of the set of all simply normal to base 2 numbers. Consequently, $m_1(I^\circ) = 1$, and by Proposition 3.21 we have (3.27). ■

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