Linear bilipschitz extension property

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Abstract

We give a sufficient quantitative geometric condition for a subset A of R^n to have the following property for a given $C \geq 1$: There is $\delta > 0$ such that for $0 \leq \varepsilon \leq \delta$, each $(1+\varepsilon)$-bilipschitz map $f\colon A \to \mathsf{R}^n$ has an extension to a $(1+C\varepsilon)$-bilipschitz map $F\colon \mathsf{R}^n \to \mathsf{R}^n$.

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1 Introduction

Let A be a subset of the euclidean n-space R^n. A map $f\colon A \to \mathsf{R}^n$ is *bilipschitz* if there is $L \geq 1$ such that

$$|x-y|/L \leq |fx - fy| \leq L|x-y|$$

for all $x, y \in A$. More precisely, such a map is called *L-bilipschitz*. The map f is *locally L-bilipschitz* if each point $a \in A$ has a neighborhood U such that $f|A \cap U$ is L-bilipschitz.

A bilipschitz map is always an embedding, and a bilipschitz map $f\colon \mathsf{R}^n \to \mathsf{R}^n$ is a homeomorphism onto R^n. In general, an L-bilipschitz map $f\colon A \to \mathsf{R}^n$ cannot be extended to a bilipschitz map $F\colon \mathsf{R}^n \to \mathsf{R}^n$, not even to a homeomorphism. However, this is often possible if the bilipschitz constant L is sufficiently close to 1. We say (as in [Vä]) that the set $A \subset \mathsf{R}^n$ has the *bilipschitz extension property* if there is a number $\delta > 0$ such that for $0 \leq \varepsilon \leq \delta$, every $(1+\varepsilon)$-bilipschitz map $f\colon A \to \mathsf{R}^n$ has an extension to a $(1+\varepsilon')$-bilipschitz map $F\colon \mathsf{R}^n \to \mathsf{R}^n$, where $\varepsilon' = \varepsilon'(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Since an L-bilipschitz map $f\colon A \to \mathsf{R}^n$ has always an L-bilipschitz extension to the closure $\bar{A}$ of A, it makes no loss of generality to consider only the case where A is closed.

It is known that several sufficiently regular sets have this property, for example, linear subspaces [TuV] and compact polyhedra [PV]. For examples of sets without this property, see [Vä, Section 7].

In this paper we give a sufficient condition for a set $A \subset \mathsf{R}^n$ to have this property with the linear bound $\varepsilon' = C\varepsilon$. We then say that A has the *C-linear bilipschitz extension property* or, more precisely, the *(C,δ)-linear bilipschitz extension property.* The number δ must usually be chosen very small, and we shall always assume that $\delta \leq 1$.

For example, a line in R^2 has the bilipschitz extension property with $\varepsilon' = 100\sqrt{\varepsilon}$ but not the C-linear bilipschitz extension property for any C; see [TuV, 5.12].

The condition of our main Theorem 2.8 is satisfied, for example, by bilipschitz images of balls, by several fractal sets, and also by sufficiently regular discrete sets, like Z^n.

We formulate the main result in Section 2 and prove it in Section 4 after giving some auxiliary results in Section 3. The extension method is rather similar to that in earlier papers, but we make use of the results in [ATV] to get better estimates for the bilipschitz constants.

2 Formulation of results

2.1. *Notation.* We let $|x|$ denote the norm of a vector $x \in \mathsf{R}^n$. Open and closed balls with center x and radius r are written as $B(x, r)$ and $\bar{B}(x, r)$, respectively, and we abbreviate $B(r) = B(0, r)$, $\bar{B}(r) = \bar{B}(0, r)$. The diameter of a set $A \subset \mathsf{R}^n$ is $d(A)$, and the distance between nonempty sets $A, B \subset \mathsf{R}^n$ is $d(A, B)$. If f and g are maps into R^n, defined in a set X, we write $\|f - g\|_X = \sup\{|fx - gx| \colon x \in X\}$. To simplify expressions we often omit parentheses writing $fx = f(x)$ etc. By an expression like ab/cde we mean $(ab)/(cde)$. For real numbers a, b we write $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$. The number of elements of a set A is $\#A$.

2.2. *Convention.* To avoid trivialities, we always assume that the set $A \subset \mathsf{R}^n$ contains at least two points.

2.3. *Thickness.* For each unit vector $e \in \mathsf{R}^n$ we define the projection $\pi_e \colon \mathsf{R}^n \to \mathsf{R}$ by $\pi_e x = x \cdot e$. Let $A \neq \varnothing$ be a bounded set in R^n. As in [ATV] we define the *thickness* of A as the number

$$\theta(A) = \inf\{d(\pi_e A) \colon |e| = 1\}.$$

Alternatively, $\theta(A)$ is the infimum of all $t > 0$ such that A lies between two parallel hyperplanes F, F' with $d(F, F') = t$. We have always $0 \leq \theta(A) \leq d(A)$.

If $x \in A \subset \mathsf{R}^n$ and $r > 0$, we set

$$A(a, r) = A \cap \bar{B}(a, r).$$

For $c \geq 1$, we say that a set $A \subset \mathsf{R}^n$ is *uniformly c-thick* if $\theta(A(a, r)) \geq 2r/c$ whenever $a \in A$ and $A \not\subset B(a, r)$. In fact, we have $c \geq 2$ except in the case $\bar{A} = \mathsf{R}^n$.

Uniformly thick sets were called n-thick in [VVW], and bounded uniformly thick sets were called thick in [Vä].

2.4. *Examples.* A ball is uniformly 2-thick. More generally, if A is convex and $B(a, R_1) \subset A \subset \bar{B}(a, R_2)$, then A is uniformly c-thick with $c = 2R_2/R_1$. For the Cantor middle third set C, the power $C^n = C \times \cdots \times C$ is uniformly $c(n)$-thick in R^n with $c(2) = 4$.

2.5. *Sturdiness.* A uniformly thick set cannot contain isolated points. We next generalize the concept by allowing isolated points. Let $A \subset \mathsf{R}^n$. For $a \in A$ we set $s(a) = s_A(a) = d(a, A \setminus \{a\})$. Then $s(a) > 0$ if and only if a is isolated in A.

Let $c \geq 1$. We say that the set $A \subset \mathsf{R}^n$ is *c-sturdy* if

(1) $\theta(A(a,r)) \geq 2r/c$ whenever $a \in A,\ r \geq cs(a),\ A \not\subset B(a,r)$,

(2) $\theta(A) \geq d(A)/c$.

If A is unbounded, we can omit (2) and the condition $A \not\subset B(a,r)$ of (1).

2.6. *Examples.* A uniformly c-thick set is trivially c-sturdy. A finite set is c-sturdy for some c if and only if it is not contained in a hyperplane. The set Z^n is $c(n)$-sturdy with $c(2) = 2$. More generally, we say that a set $A \subset \mathsf{R}^n$ is an *(α, β)-net* if

(1) $|x - y| \geq \alpha$ for all $x, y \in A$, $x \neq y$,

(2) $d(x, A) \leq \beta$ for all $x \in \mathsf{R}^n$.

2.7. Proposition. *If $A \subset \mathsf{R}^n$ is an (α, β)-net, then A is c-sturdy with $c = 1 + 2\beta/\alpha$.*

Proof. Let $a \in A,\ r \geq cs(a),\ E = A(a,r)$ and $t > \theta(E)$. It suffices to show that $t \geq 2r/c$.

There is a unit vector e such that $d(\pi_e E) \leq t$. Hence $\pi_e E$ is contained in an interval J of length t. Since $I = \pi_e \bar{B}(a,r)$ is an interval of length $2r$, there is an interval $]\lambda - \varrho, \lambda + \varrho[\, \subset I \setminus J$ where $\varrho = r/2 - t/4$. Hence $B(a+\lambda e, \varrho) \cap A = \varnothing$, which implies that $r/2 - t/4 \leq d(a+\lambda e, A) \leq \beta$, and we obtain $t \geq 2r - 4\beta$. Since $r \geq cs(a) \geq c\alpha$, we get $2r - 2r/c \geq 2c\alpha - 2\alpha = 4\beta$, and hence $t \geq 2r/c$ as desired. $\square$

We are ready to state our main result.

2.8. Theorem. *If $A \subset \mathsf{R}^n$ is c-sturdy, then A has the (C, δ)-linear bilipschitz extension property with $C = C(c,n),\ \delta = \delta(c,n)$. In particular, this is true for uniformly c-thick sets A.*

The proof is given in Section 4. The estimates for $C(c,n)$ and $\delta(c,n)$ are explicit but presumably far from the best possible. Some numerical estimates are given in 4.13.

A two-point set in the plane has the 1-linear bilipschitz extension property, but it is not sturdy. Some open problems are given in 4.14.

3 Preliminary results

In this section we give various auxiliary results needed in the proof of the main theorem.

3.1. *Degree.* Let $D \subset \mathsf{R}^n$ be a bounded domain, and let $f\colon \bar{D} \to \mathsf{R}^n$ be continuous. For $y \in \mathsf{R}^n \setminus f\partial D$, we let $\mu(y, f, D)$ denote the *topological degree*, defined, for example, in [Do, IV.5] and in [RR, II.2]. If $x \in D$ is an isolated point of $f^{-1}\{fx\}$, we can define the *local index* $i(x, f) = \mu(fx, f, U)$ where U is any connected neighborhood of x in D with $U \cap f^{-1}\{fx\} = \{x\}$. If f is a nonsingular linear map, then $i(x, f) = \operatorname{sgn} \det f$.

Let $G \subset \mathsf{R}^n$ be an open set. A continuous map $f\colon G \to \mathsf{R}^n$ is *sense-preserving* if $\mu(y, f, D) > 0$ whenever D is a bounded domain with $\bar{D} \subset G$ and $y \in fD \setminus f\partial D$. A linear map $T\colon \mathsf{R}^n \to \mathsf{R}^n$ is sense-preserving if and only if $\det T > 0$.

3.2. *Simplexes.* Suppose that $\Delta \subset \mathsf{R}^n$ is an n-simplex with vertices $v_0, \dots, v_n$. We write $\Delta^0 = \{v_0, \dots, v_n\}$ and $\Delta = [v_0, \dots, v_n]$. Furthermore, we let $b(\Delta)$ denote the smallest height of Δ, and the number $\varrho(\Delta) = d(\Delta)/b(\Delta)$ is called the *flatness* of Δ. A map $f\colon \Delta^0 \to \mathsf{R}^n$ is said to be sense-preserving if its unique affine extension $F\colon \mathsf{R}^n \to \mathsf{R}^n$ is sense-preserving.

3.3. *Terminology.* Let $M \geq 1$, and let Δ_1, Δ_2 be n-simplexes in R^n. We say that Δ_1 is *directly M-related to* Δ_2 if

(1) $\varrho(\Delta_j) \leq M$ for $j = 1, 2$.
(2) $d(\Delta_1)/M \leq d(\Delta_2) \leq M d(\Delta_1)$,
(3) $d(\Delta_1, \Delta_2) \leq M \max\{d(\Delta_1), d(\Delta_2)\}$.

If $A \subset \mathsf{R}^n$ and if Δ is an n-simplex with $\Delta^0 \subset A$, we say that Δ is a *simplex of A*. Let Δ and Δ' be simplexes of A and let $M \geq 1$. We say that Δ is *M-related to Δ' in A* if there is a finite sequence $\Delta = \Delta_0, \dots, \Delta_N = \Delta'$ of simplexes of A such that Δ_{j-1} is directly M-related to Δ_j for all $1 \leq j \leq N$. This relation is an equivalence.

3.4. Lemma. *Suppose that $A \subset \mathsf{R}^n$ is a compact set with $\theta(A) > 0$. Then there is an n-simplex Δ of A with $b(\Delta) \geq \theta(A)/2$.*

Proof. Let $\Delta = [v_0, \dots, v_n]$ be a simplex of A with maximal volume and choose the notation so that $b(\Delta)$ is the height of Δ measured from v_n. Let E be the hyperplane containing the vertices $v_0, \dots, v_{n-1}$. Then A lies in $E + \bar{B}(b(\Delta))$, since otherwise it easy to find a simplex of A larger than Δ. Hence $\theta(A) \leq 2b(\Delta)$. □

3.5. *Special simplexes.* Suppose that $A \subset \mathsf{R}^n$, $a \in A$, $r > 0$ and $c \geq 1$. We say that an n-simplex Δ is *c-special for (A, a, r)* or briefly a *c-special simplex of A* if

(1) $\Delta^0 \subset A(a, r)$,
(2) $b(\Delta) \geq r/c$.

Observe that since $d(\Delta) \leq 2r$, we have

$$\varrho(\Delta) \leq 2c \tag{3.6}$$

for every c-special simplex Δ of A.

The following simple results on special simplexes are formulated for unbounded sets, because they are only needed in that case.

3.7. Lemma. *Let $A \subset \mathsf{R}^n$ be closed, unbounded and c-sturdy. If $a \in A$ and $r \geq cs(a)$, then there is a c-special simplex for (A, a, r).*

Proof. Since $\theta(A(a,r)) \geq 2r/c$ by c-sturdiness, the lemma follows from 3.4. □

3.8. Lemma. *Suppose that $A \subset \mathsf{R}^n$ is closed, unbounded and c-sturdy. Let Δ, Δ' be c-special simplexes of A. Then Δ is M-related to Δ' in A with $M = 4c$.*

Proof. Let Δ and Δ' be c-special for (A, a, r) and (A, a', r'), respectively. Suppose first that $a = a'$ and $r \leq r' \leq 2r$. By (3.6) we have $\varrho(\Delta), \varrho(\Delta') \leq 2c < M$. Next,

$$\begin{aligned} d(\Delta) &\leq 2r \leq 2r' \leq 2cb(\Delta') \leq Md(\Delta'), \\ d(\Delta') &\leq 2r' \leq 4r \leq 4cb(\Delta) \leq Md(\Delta). \end{aligned}$$

Finally, since $d(\Delta) \vee d(\Delta') \geq b(\Delta) \geq r/c$ and $d(\Delta, \Delta') \leq r + r' \leq 3r$, condition (3) of 3.3 holds, and hence Δ is directly M-related to Δ'.

Iteration of the result proved above shows that the lemma is true in the case $a = a'$. Suppose that $|a - a'| = t > 0$. Then $t \geq s(a) \vee s(a')$. By 3.7 we find c-special simplexes Δ_1 for (A, a, ct) and Δ_1' for (A, a', ct). Since Δ and Δ' are M-related to Δ_1 and Δ_1', respectively, by the first part of the proof, it suffices to show that Δ_1 is directly M-related to Δ_1'.

We have $b(\Delta_1), b(\Delta_1') \geq t$. Hence $d(\Delta_1) \leq 2ct \leq 2cb(\Delta_1') \leq Md(\Delta_1')$ and similarly $d(\Delta_1') \leq Md(\Delta_1)$. Moreover,

$$d(\Delta_1, \Delta_1') \leq (2c+1)t \leq 3ct \leq M \max\{b(\Delta_1), b(\Delta_1')\},$$

and the lemma is proved. □

We recall from [ATV] that a map $f\colon A \to \mathsf{R}^n$ is an *ε-nearisometry* if $|x - y| - \varepsilon \leq |fx - fy| \leq |x - y| + \varepsilon$ for all $x, y \in A$. If A is bounded and if $f\colon A \to \mathsf{R}^n$ is $(1+\varepsilon)$-bilipschitz, then f is an $\varepsilon d(A)$-nearisometry. We next recall the approximation theorem [ATV, 3.3]:

3.9. Theorem. *Suppose that $c \geq 1$ and that $A \subset \mathsf{R}^n$ is a compact set with $\theta(A) \geq d(A)/c$. Let $f\colon A \to \mathsf{R}^n$ be an ε-nearisometry. Then there is an isometry $S\colon \mathsf{R}^n \to \mathsf{R}^n$ such that $\|S - f\|_A \leq cc_n\varepsilon$, where c_n depends only on n.*

In the rest of the paper, we let c_n denote the constant given by 3.9.

3.10. Corollary. *Let $A \subset \mathsf{R}^n$ be a closed set and let $f\colon A \to \mathsf{R}^n$ be $(1+\varepsilon)$-bilipschitz. Assume that $a \in A$ and $r > 0$ are such that $\theta(A(a,r)) \geq 2r/c$ for some constant $c \geq 1$. Then there is an isometry $S\colon \mathsf{R}^n \to \mathsf{R}^n$ such that $Sa = fa$ and $\|S - f\|_{A(a,r)} \leq 4cc_n\varepsilon r$.*

In particular, S exists if A is c-sturdy and $r \geq cs(a)$, $A \not\subset B(a,r)$.

Proof. Since $d(A(a,r)) \leq 2r$, the restriction $f|A(a,r)$ is a $2\varepsilon r$-near-isometry. As $\theta(A(a,r)) \geq 2r/c \geq d(A(a,r))/c$, Theorem 3.9 gives an isometry $S_1\colon \mathsf{R}^n \to \mathsf{R}^n$ such that $\|S_1 - f\|_{A(a,r)} \leq 2cc_n\varepsilon r$. Then $|S_1a - fa| \leq 2cc_n\varepsilon r$, and thus $S = S_1 - S_1a + fa$ is the desired isometry. □

3.11. *Special isometries.* We say that the isometry S given by Corollary 3.10 is a *special isometry* for (f,a,r).

Our next lemma is an explicit version of [Vä, 5.4].

3.12. Lemma. *Suppose that Δ_1 is directly M-related to Δ_2 and that $f\colon \Delta_1^0 \cup \Delta_2^0 \to \mathsf{R}^n$ is an $(1+\varepsilon)$-bilipschitz map with $\varepsilon \leq \delta$, where $1/\delta = (n+1)^2M^3(M+2)^2c_n$ and c_n is the constant of 3.9. If $f|\Delta_1^0$ is sense-preserving, then $f|\Delta_2^0$ is sense-preserving.*

Proof. We may assume that $d(\Delta_1) \leq d(\Delta_2)$. Set $F = \Delta_1^0 \cup \Delta_2^0$. By [TuV, 3.3] it suffices to find an isometry $S\colon \mathsf{R}^n \to \mathsf{R}^n$ such that

$$\|S - f\|_F \leq b(\Delta_j)/2(n+1) \tag{3.13}$$

for $j = 1,2$.

By [ATV, 5.3] we have $b(\Delta_2) \leq (n+1)\theta(\Delta_2)/2$. Since $b(\Delta_2) = d(\Delta_2)/\varrho(\Delta_2) \geq d(\Delta_2)/M$, we obtain $\theta(F) \geq \theta(\Delta_2) \geq 2d(\Delta_2)/(n+1)M$. On the other hand,

$$d(F) \leq d(\Delta_1) + d(\Delta_2) + d(\Delta_1,\Delta_2) \leq 2d(\Delta_2) + Md(\Delta_2), \tag{3.14}$$

and this implies that $d(F)/\theta(F) \leq (n+1)M(M+2)/2$. Since f is an $\varepsilon d(F)$-nearisometry, Theorem 3.9 gives an isometry S satisfying $\|S - f\|_F \leq c_n\varepsilon d(F)(n+1)M(M+2)/2$. Since $d(\Delta_2) \leq Md(\Delta_j) \leq M^2b(\Delta_j)$ for $j = 1,2$, this and (3.14) yield (3.13). □

3.15. Lemma. *Let $A \subset \mathsf{R}^n$ be closed, unbounded and c-sturdy, and let $f\colon A \to \mathsf{R}^n$ be $(1+\varepsilon)$-bilipschitz. Suppose that $a,b \in A$ and $r_1 \geq cs(a), r_2 \geq cs(b)$. If S and T are special isometries for (f,a,r_1) and (f,b,r_2), respectively, then they have the same orientation, provided that $0 < \varepsilon \leq \delta(c,n)$.*

Proof. By 3.7, there are c-special simplexes Δ_a and Δ_b for (A,a,r_1) and (A,b,r_2), respectively, and by 3.8, these simplexes are $4c$-related in A. Let $\delta = \delta(c,n)$ be the number given by 3.12 for $M = 4c$. By repeated applications of 3.12 we see that the maps $f|\Delta_a^0$ and $f|\Delta_b^0$ have the same orientation if $0 \le \varepsilon \le \delta$. Since $b(\Delta_a) \ge r_1/c$, we have

$$\|S - f\|_{\Delta_a^0} \le 4cc_n\varepsilon r_1 \le 4c^2 c_n \varepsilon b(\Delta_a) \le b(\Delta_a)/2(n+1)$$

for $\varepsilon \le \delta'$, where $\delta' = 1/8(n+1)c^2 c_n > \delta$. Assume that $\varepsilon \le \delta$. Then [TuV, 3.3] implies that S has the same orientation as the affine extension of $f|\Delta_a^0$. In a similar way, we see that T has the same orientation as $f|\Delta_b^0$, and the lemma follows. □

Finally, we need the following auxiliary results.

3.16. Lemma. *Let $\Delta \subset \mathsf{R}^n$ be an n-simplex, and let $S, T\colon \Delta \to \mathsf{R}^n$ be isometries such that $\|S - T\|_{\Delta^0} \le \eta$. Then*

$$|Sx - Tx| \le \eta(1 + M|x - z|/d(\Delta))$$

for all $x \in \mathsf{R}^n$ and $z \in \Delta^0$, where $M = 4 + 6n\varrho(\Delta)(1 + \varrho(\Delta))^{n-1}$.

Proof. This is a special case of [Vä, 2.11]. □

3.17. Lemma. *Suppose that $A \subset \mathsf{R}^n$ is closed, $F\colon \mathsf{R}^n \to \mathsf{R}^n$ is continuous, $F|A$ is L-Lipschitz and $F|\mathsf{R}^n \setminus A$ is locally L-Lipschitz. Then F is L-Lipschitz.*

Proof. Let $a, b \in \mathsf{R}^n$ and set $J = [a, b]$. If $J \cap A = \varnothing$, then

$$|Fa - Fb| \le L|a - b|. \tag{3.18}$$

By continuity, (3.18) holds if $J \cap A \subset \{a, b\}$. If A contains an interior point of J, we choose points $u, v \in J \cap A$ such that $[a, u)$ and $[b, v)$ lie in $\mathsf{R}^n \setminus A$, and then (3.18) follows by the triangle inequality. □

3.19. Lemma. *Let $G \subset \mathsf{R}^n$ be a domain and let $F\colon G \to \mathsf{R}^n$ be continuous, discrete, and sense-preserving. Then F is open.*

Proof. See [TY, p. 333] for the proof of an even more general result. □

4 Proof of Theorem 2.8

We first reduce the proof to the case where the set A is unbounded.

4.1. Lemma. *Assume that every unbounded c-sturdy set $A \subset \mathsf{R}^n$ has the (C, δ)-linear bilipschitz extension property with $C = C(c, n)$, $\delta = \delta(c, n)$. Then all c-sturdy sets have the (C', δ')-linear bilipschitz extension property with*

$$C' = C'(c, n) = 2cc_n C(6c, n),\ \delta' = \delta'(c, n) = \delta(6c, n)/2cc_n.$$

Proof. Suppose that A is bounded and c-sturdy. Let $\varepsilon \le \delta'(c,n)$ and let $f\colon A \to \mathsf{R}^n$ be $(1+\varepsilon)$-bilipschitz. Let $R = d(A)$. Then $\theta(A) \ge R/c$ by the definition of sturdiness. Since f is an εR-nearisometry, there is by 3.9 an isometry $S\colon \mathsf{R}^n \to \mathsf{R}^n$ satisfying $\|S - f\| \le cc_n\varepsilon R$.

We normalize $S = \mathrm{id}$ and $0 \in A$, so that $A \subset \bar{B}(R)$. Let $A_1 = A \cup (\mathsf{R}^n \setminus B(2R))$.

We start by showing that A_1 is $6c$-sturdy. Thus let $a \in A_1$ and suppose that $r \ge 6cs(a)$. We must show that $\theta(A_1(a,r)) \ge 2r/6c = r/3c$. We consider three cases.

Case 1. $a \in A$, $r \le 3R$. Since $r/6 \le R/2$, we have $A \not\subset B(a, r/6)$, and also $r/6 \ge cs(a)$. Since A is c-sturdy, this implies that $\theta(A_1(a,r)) \ge \theta(A(a,r/6)) \ge 2(r/6)/c = r/3c$.

Case 2. $a \in A$, $r \ge 3R$. Now $\bar{B}(a,r) \cap A_1$ contains a half ball H of radius $r - 2R \ge r/3$, so that

$$\theta(A_1(a,r)) \ge \theta(H) \ge r/3 \ge r/3c.$$

Case 3. $|a| \ge 2R$. In this case $\bar{B}(a,r) \cap A_1$ contains a half ball H of radius r, and thus $\theta(A_1(a,r)) \ge \theta(H) \ge r$.

This completes the proof that A_1 is $6c$-sturdy.

We now extend f to a map $f_1\colon A_1 \to \mathsf{R}^n$ by setting $f_1 x = x$ for $|x| \ge 2R$.

We claim that f_1 is $(1 + 2cc_n\varepsilon)$-bilipschitz if $\varepsilon \le 1/2cc_n$. To prove this, let $x \in A$ and $|y| \ge 2R$, this being the only non-trivial case. Then

$$\frac{|f_1x - f_1y|}{|x-y|} = \frac{|fx - y|}{|x-y|} \le \frac{|fx - x| + |x - y|}{|x-y|} \le 1 + \frac{cc_n\varepsilon R}{R} = 1 + cc_n\varepsilon,$$

and in the other direction,

$$\frac{|f_1x - f_1y|}{|x-y|} \ge \frac{|x-y| - |fx - x|}{|x-y|} \ge 1 - \frac{cc_n\varepsilon R}{R} = 1 - cc_n\varepsilon \ge \frac{1}{1 + 2cc_n\varepsilon},$$

because $cc_n\varepsilon \le cc_n\delta' \le \delta/2 \le 1/2$.

Since $2cc_n\varepsilon \le \delta(6c,n)$, the map f_1 has an extension $F\colon \mathsf{R}^n \to \mathsf{R}^n$, which is bilipschitz with the constant $1 + 2C(6c,n)cc_n\varepsilon = 1 + C'(c,n)\varepsilon$. Hence F is the desired extension of f. $\square$

¿From now on, we assume that A is unbounded, closed and c-sturdy. Then $\theta(A(a,r)) \ge 2r/c$ whenever $a \in A$ and $r \ge cs(a)$.

To prove Theorem 2.8, assume that $f\colon A \to \mathsf{R}^n$ is $(1+\varepsilon)$-bilipschitz. We shall define an extension $F\colon \mathsf{R}^n \to \mathsf{R}^n$ of f and prove that F is $(1 + C\varepsilon)$-bilipschitz for small ε with $C = C(c,n)$.

In the proof, we let $C_0, C_1, \ldots$ denote constants $C_i \ge 0$ depending only on c and n.

Let K be a decomposition of $G = \mathsf{R}^n \setminus A$ into Whitney cubes Q such that

(1) $1 \le d(Q,A)/d(Q) < 3$,

(2) $1/2 \le d(Q)/d(Q') \le 2$ if $Q, Q' \in K$ and $Q \cap Q' \ne \varnothing$;

see [Gu, p. 10] or [Re, p. 165]. We first define the extension F on the set K^0 of the vertices of the cubes $Q \in K$.

Let $v \in K^0$ and choose $a_v \in A$ such that $r_v = |v - a_v| = d(v, A)$. Set

$$t_v = s(a_v) \vee 8r_v.$$

By 3.10 there is a special isometry S_v for (f, a_v, ct_v) satisfying

$$S_v a_v = f a_v,\ \|S_v - f\|_{A(a_v, ct_v)} \le C_0 \varepsilon t_v, \tag{4.4}$$

where $C_0 = 4c^2 c_n$. These isometries can be chosen so that

$$S_u = S_v \tag{4.5}$$

whenever $a_u = a_v = a$ and $u, v \in \bar{B}(a, s(a)/8)$.

We define $Fv = S_v v$.

By 3.15, all special isometries of A have the same orientation, provided that $\varepsilon \le \delta_1(c,n)$. Thus we assume from now on that $\varepsilon \le \delta_1$ and that all isometries S_v are sense-preserving.

In order to extend the function $F\colon K^0 \to \mathsf{R}^n$ to G, we use the method in [Tr]. (Alternatively, one can make use of the less economical Whitney triangulation given in [Vä, 5.1]) We recall the construction. Let Q be a cube in K with side 2λ. We divide Q into 2^n subcubes with side λ bisecting the sides of Q. Each subcube is divided into mutually congruent n-simplexes Δ without new vertices such that

$$d(\Delta) = \lambda\sqrt{n},\ b(\Delta) = \lambda/\sqrt{2},\ \varrho(\Delta) = \sqrt{2n}, \tag{4.6}$$

Let W be the collection of all these simplexes, i.e. $W = \{\Delta \subset Q\colon Q \in K\}$. The extension F is defined on the set W^0 of vertices of W as follows. If $u \in K^0$, then Fu was defined above. If $u \in W^0 \setminus K^0$, then $u = \frac{1}{2}(u' + u'')$ for some vertices u', u'' of Q such that $u'_i \le u_i \le u''_i$ for all $1 \le i \le n$. The vertices u' and u'' are uniquely determined by u, and we define $Fu = \frac{1}{2}(Fu' + Fu'')$. Since W consists of n-simplexes, we can extend F affinely into each $\Delta \in W$ to get a map $F\colon G \to \mathsf{R}^n$.

Finally, we define $F|A = f$ and obtain an extension $F\colon \mathsf{R}^n \to \mathsf{R}^n$ of f. We claim that F is the desired bilipschitz extension.

The proof will be carried through by stating and proving eight facts, and drawing these together in the final step 4.12.

Fact 1. Let $Q \in K$ and let $u, v \in Q \cap K^0$. Then

(1) $|a_u - a_v| \le 3(r_u \vee r_v)$,

(2) $r_v \le 2r_u$,

(3) $\|S_u - S_v\|_Q \le C_1(r_u \vee r_v)\varepsilon$.

Proof. Since $|u-v| \le d(Q) \le d(Q,A) \le r_u \wedge r_v$, we obtain (1) and (2) as follows:

$$|a_u - a_v| \le r_u + |u-v| + r_v \le 3(r_u \vee r_v),$$
$$r_v \le |a_u - v| \le r_u + |u-v| \le 2r_u.$$

To prove (3) we may assume that $r_u \le r_v$. Assume first that $t_v = s(a_v) \ge 8r_v$. Now a_v is isolated in A. If $a_u \ne a_v$, then (1) and (2) yield

$$s(a_v) \le |a_u - a_v| \le 3r_v \le 3s(a_v)/8,$$

a contradiction. Thus $a_u = a_v$. Since $r_u \le r_v \le s(a_v)/8$, we have $S_u = S_v$ by (4.5), and (3) holds with $C_1 = 0$.

Next assume that $t_v = 8r_v \ge s(a_v)$. We consider two cases.

Case 1. $r_u \ge cs(a_u)$. Now $t_u = 8r_u$ and $r_v \ge r_u \ge cs(a_u)$. By 3.7 there is a c-special simplex Δ_u for (A, a_u, r_v) satisfying

$$\Delta_u^0 \subset A(a_u, r_v),\ b(\Delta_u) \ge r_v/c,\ \varrho(\Delta_u) \le 2c.$$

Since $r_v \le 2r_u < t_u$ by (2), we obtain by (4.4) the inequality $\|S_u - f\|_{\Delta_u^0} \le C_0\varepsilon t_u \le C_0\varepsilon t_v$. By (1) we also have $\bar{B}(a_u, r_v) \subset \bar{B}(a_v, 4r_v)$, and thus (4.4) implies that $\|S_v - f\|_{\Delta_u^0} \le C_0\varepsilon t_v$. Consequently,

$$\|S_u - S_v\|_{\Delta_u^0} \le 2C_0\varepsilon t_v.$$

Let $x \in Q$. Choose a vertex $z \in \Delta_u^0$ and apply 3.16 to get

$$|S_u x - S_v x| \le 2C_0\varepsilon t_v(1 + M|x-z|/d(\Delta_u)),$$

where

$$M = 4 + 6n\varrho(\Delta_u)(1+\varrho(\Delta_u))^{n-1} \le 4 + 12cn(1+2c)^{n-1} = M_1(c,n),$$
$$|x-z| \le |x-u| + |u-a_u| + |a_u - z| \le d(Q) + r_u + r_v \le 3r_v,$$
$$d(\Delta_u) \ge b(\Delta_u) \ge r_v/c.$$

Since $t_v = 8r_v$, these estimates imply (3) with $C_1 = 16C_0(1+3cM_1)$.

Case 2. $r_u \le cs(a_u)$. By 3.7 we can choose a c-special simplex Δ_u for $(A, a_u, cs(a_u))$. Since $s(a_u) \le t_u$, we have $\|S_u - f\|_{\Delta_u^0} \le C_0\varepsilon t_u$ by (4.4). We next show that

$$s(a_u) \le 8r_v, \quad \Delta_u^0 \subset A(a_v, 8cr_v). \tag{4.7}$$

Let $z \in \Delta_u^0$. If $a_u \ne a_v$, then $s(a_u) \le |a_u - a_v| \le 3r_v$ by (3), and

$$|z - a_v| \le cs(a_u) + |a_u - a_v| \le 3cr_v + 3r_v \le 6cr_v.$$

If $a_u = a_v$, then $s(a_u) = s(a_v) \le 8r_v$ and

$$|z - a_v| = |z - a_u| \le cs(a_u) = cs(a_v) \le 8cr_v,$$

and (4.7) is proved.

Since $t_v = 8r_v$, it follows from (4.7) that $t_u \le t_v$ and $\|S_v - f\|_{\Delta_u^0} \le 8C_0\varepsilon r_v$. Consequently, $\|S_u - S_v\|_{\Delta_u^0} \le 16C_0\varepsilon r_v$.

As in Case 1, we choose a vertex z of Δ_u and apply 3.16. For each $x \in Q$ we have

$$|x - z| \le d(Q) + r_u + cs(a_u) \le 3cs(a_u)$$

and $d(\Delta_u) \ge b(\Delta_u) \ge s(a_u)$. We again obtain (3) with $C_1 = 16C_0(1+3cM_1)$. $\square$

Fact 2. There is a number $\delta_2(c,n) > 0$ such that if $\varepsilon \le \delta_2$ and $\Delta \in W$, then $F|\Delta$ is sense-preserving and $(1 + C_2\varepsilon)$-bilipschitz.

f*Proof.* Let $Q \in K$ be the cube containing Δ and let v be a point in $Q \cap K^0$ such that $r_v = |v - a_v| = d(v, A)$ is maximal. From Fact 1(3) and from the construction of F it follows that $\|S_v - F\|_\Delta \le C_1 r_v \varepsilon$. By (4.6) we obtain

$$r_v \le d(Q, A) + d(Q) \le 4d(Q) = 8\lambda\sqrt{n} \le 8b(\Delta)\sqrt{2n}.$$

Hence $\|S_v - F\|_\Delta \le \alpha b(\Delta)/(n+1)$, where $\alpha = 8C_1\sqrt{2n}(n+1)\varepsilon \le 1/2$ if

$$\varepsilon \le \delta_2(c,n) = (16\sqrt{2n}(n+1)C_1)^{-1}.$$

For $\varepsilon \le \delta_2$ it follows from [TuV, 3.3] that $F|\Delta$ is sense-preserving and $(1 + C_2\varepsilon)$-bilipschitz with $C_2 = 2\alpha/\varepsilon = 16C_1\sqrt{2n}(n+1)$. $\square$

¿From now on, we assume that $\varepsilon \le \delta_1 \wedge \delta_2$. We shall show that Theorem 2.8 holds with $C = C_2$, but we need an additional bound for ε.

Fact 3. For each $a \in A$ and $r > 0$ there is a sense-preserving isometry T of R^n such that $Ta = fa$ and $\|T - F\|_{\bar{B}(a,r)} \le C_3\varepsilon r$.

Proof. If $s(a) > 0$, it follows from (4.5) that the map $F|K^0$ agrees with an isometry T in $\bar{B}(a, s(a)/8)$. This implies that $F = T$ in $\bar{B}(s(a)/16)$. We may thus assume that

$$s(a) \le 16r. \tag{4.8}$$

By 3.10 there is a special isometry T for $(f, a, 20cr)$. The map T is sense-preserving and $Ta = fa$. Let $x \in \bar{B}(a,r)$. We must find an estimate $|Tx - Fx| \le C_3\varepsilon r$. Since

$$\|T - f\|_{A(a,20cr)} \le 20C_0\varepsilon r \tag{4.9}$$

with $C_0 = 4c^2c_n$ as before, we may assume that $x \in G$.

Let $Q \in K$ be a cube containing x and let $v \in Q \cap K^0$. It clearly suffices to find an estimate

$$|Tv - Fv| \le C_3\varepsilon r. \tag{4.10}$$

Recall from (4.4) that $Fv = S_v v$ with

$$\|S_v - f\|_{A(a_v, ct_v)} \le C_0\varepsilon t_v, \tag{4.11}$$

where $t_v = s(a_v) \vee 8r_v$. Since $ct_v \geq cs(a_v)$, there is a c-special simplex Δ_v for (A, a_v, ct_v). As $d(Q) \leq d(Q, A) \leq r$, we have $r_v \leq |v - a| \leq |v - x| + |x - a| \leq 2r$. If $a_v \neq a$, this yields $s(a_v) \leq |a_v - a| \leq 4r$. If $a_v = a$, then $s(a_v) = s(a) \leq 16r$ by (4.8). Hence $t_v \leq 16r \vee 8r_v = 16r$. It follows that $\Delta_v^0 \subset A(a, 20cr)$. By (4.9) and (4.11) we obtain

$$\|S_v - T\|_{\Delta_v^0} \leq 20C_0\varepsilon r + C_0\varepsilon t_v \leq 36C_0\varepsilon r.$$

Fix a point $z \in \Delta_v^0$. By 3.16 we obtain

$$|Fv - Tv| \leq 36C_0\varepsilon r(1 + M_1|v - z|/d(\Delta_v)),$$

where $M_1 = M_1(c, n)$ is as in the proof of Fact 1, and $|v - z| \leq r_v + ct_v$, $d(\Delta_v) \geq b(\Delta_v) \geq t_v$. Since $r_v \leq t_v/8$, we obtain (4.10) with $C_3 = 36C_0(1 + 2cM_1)$. □

We set $\delta_3 = 1/3C_3$ and $\delta = \delta(c, n) = \delta_1 \wedge \delta_2 \wedge \delta_3$. From now on we assume that $\varepsilon \leq \delta$.

Fact 4. We have $FA \cap FG = \varnothing$.

Proof. Let $a \in A$, $x \in G$ and write $r = |a - x|$. Let T be the isometry given by Fact 3 for these a and r. Then

$$|Fa - Fx| \geq |Ta - Tx| - 2C_3\varepsilon r \geq r - C_3\delta_3 r = 2r/3 > 0. \ \square$$

Fact 5. The extension F is $(1 + C_2\varepsilon)$-Lipschitz.

Proof. By Fact 2, the restriction $F|G$ is locally $(1 + C_2\varepsilon)$-Lipschitz, and $F|A = f$ is $(1+\varepsilon)$-Lipschitz. By 3.17, it suffices to show that F is continuous at an arbitrary point $a \in A$.

Let $x \in \mathsf{R}^n$ and let T be the isometry given by Fact 3 for a and $r = |x - a|$. Then

$$|Fx - Fa| \leq |Fx - Tx| + |Tx - Ta| \leq r/3 + r.$$

Hence F is continuous at a. □

Fact 6. The restriction $F|G$ is sense-preserving, discrete and open.

Proof. If $y \in FG$, then $F^{-1}y \subset G$ is a discrete set, since $F|\Delta$ is injective for each $\Delta \in W$. Thus F is discrete.

By 3.19 it remains to prove that $F|G$ is sense-preserving. Let D be a domain with $\bar{D} \subset G$ compact. Let $y \in FD \setminus F\partial D$ and let V be the y-component of $\mathsf{R}^n \setminus F\partial D$. Then $z \mapsto \mu(z, F, D)$ is constant in V. Set $B = \cup\{\partial\Delta \colon \Delta \in W\}$. Since $\operatorname{int} FB = \varnothing$, there is a point $z \in V \cap FD \setminus FB$. Then $D \cap F^{-1}\{z\}$ is a nonempty finite set whose elements are interior points of certain simplexes $\Delta \in W$. For each Δ, the restriction $F|\Delta$ is affine and sense-preserving by Fact 2. This implies that

$$\begin{aligned} \mu(y, F, D) = \mu(z, F, D) &= \sum\{i(x, F) \colon x \in D \cap F^{-1}\{z\}\} \\ &= \#(D \cap F^{-1}\{z\}) > 0. \end{aligned}$$

This completes the proof of Fact 6. □

Before proceeding, we make the normalization $0 \in A$ and $f(0) = 0$.

Fact 7. We have $\mu(y, F, B(r)) = 1$ for each $r > 0$ and $y \in B(r/2)$.

Proof. Let T be the isometry given by Fact 3 for $a = 0$. If $|x| \le r$, then $|Tx - Fx| \le C_3 \varepsilon r \le r/3$ so that $z \mapsto \mu(z, F, B(r))$ is constant for $z \in B(r/2)$. Moreover, the segmental homotopy $h_t \colon T \sim F,\ h_t(x) = (1-t)Tx + tFx$ has the property $h_t(0) = 0 \notin h_t S(r)$. By a basic property of degree, this implies that $\mu(0, F, B(r)) = \mu(0, T, B(r)) = 1$, since T is affine and sense-preserving. □

Fact 8. The extension $F \colon \mathsf{R}^n \to \mathsf{R}^n$ is a homeomorphism.

Proof. Fact 7 implies that $B(r/2) \subset F\mathsf{R}^n$ for all $r > 0$, and thus $F\mathsf{R}^n = \mathsf{R}^n$.

To show that F is injective, let $y \in \mathsf{R}^n$. We show that $\# F^{-1} y = 1$ by considering three cases. Recall that $B = \cup\{\partial\Delta \colon \Delta \in W\}$.

Case 1. $y \in FA = fA$. The claim is obvious by Fact 4.

Case 2. $y \in FG \setminus FB$. Let $r > 2|y|$. By Fact 7, we have $\mu(y, F, B(r)) = 1$. Therefore, the set $B(r) \cap F^{-1}y$ is nonempty, and since $F|G$ is discrete by Fact 6, this set is finite. In a neighborhood of each point of this set, the map F is affine and sense-preserving by Fact 2. Thus

$$1 = \mu(y, F, B(r)) = \sum\{i(x, f) \colon x \in B(r) \cap F^{-1}y\} = \#(B(r) \cap F^{-1}y).$$

Since $r > 2|y|$ was arbitrary, the claim follows from this.

Case 3. $y \in FB$. Assume that $x_1, x_2 \in F^{-1}y$ and $x_1 \ne x_2$. Choose disjoint neighborhoods U_1, U_2 of x_1, x_2 in G. By Fact 8, the sets FU_1 and FU_2 are open. Since FB has no interior, there is a point $z \in FU_1 \cap FU_2 \setminus FB$, and the previous cases imply that $\# F^{-1}z = 1$. This is a contradiction, since $F^{-1}z$ meets both U_1 and U_2.

This completes the proof of injectivity of F, and Fact 8 is proved. □

4.12. *Final Step.* By Facts 5 and 8, the extension $F \colon \mathsf{R}^n \to \mathsf{R}^n$ is a $(1 + C_2\varepsilon)$-Lipschitz homeomorphism provided that $\varepsilon \le \delta(c, n)$. It remains to show that F^{-1} is $(1 + C_2\varepsilon)$-Lipschitz as well.

Assume that $[y, z] \subset FG$. Then $[y, z]$ can be divided into a finite number of subintervals J such that each J is contained in $F\Delta$ for some $\Delta \in W$. By Fact 2, the restriction $F^{-1}|J$ is $(1 + C_2\varepsilon)$-Lipschitz. This implies that $F^{-1}|FG$ is locally $(1 + C_2\varepsilon)$-Lipschitz, and thus, by 3.17, the inverse F^{-1} is $(1 + C_2\varepsilon)$-Lipschitz.

We have proved that F is $(1 + C_2\varepsilon)$-bilipschitz. Hence Theorem 2.8 holds with $C = C_2$ and $\delta = \delta_1 \wedge \delta_2 \wedge \delta_3$. □

4.13. *Numerical estimates.* Using the numerical bounds in [ATV, 3.8] one can show that Theorem 2.8 holds with $C(c, 2) = 3{\cdot}10^{14} c^6$, $C(c, 3) = 8{\cdot}10^{24} c^7$, and with $\delta(c, 2) = 1/C(c, 2)$, $\delta(c, 3) = 1/C(c, 3)$.

4.14. *Open problems.* We close this article with some open problems related to our results.

(i) Sturdiness is a sufficient condition for c-linear bilipschitz extension property; is it also necessary under some additional conditions? For example, is it true that if A is c-relatively connected in the sense of [TrV], then A has the c_1-linear bilipschitz extension property if and only if it is c_2-sturdy?

(ii) In general, bilipschitz maps $f\colon A \to \mathsf{R}^n$ cannot be extended to R^n. In some special cases this may be possible, so we ask the following questions. Let $L \geq 1$ and let $f\colon A \to \mathsf{R}^n$ be L-bilipschitz. Does f have an L_1-bilipschitz extension to R^n with $L_1 = L_1(L, n)$ if

(a) $A = \mathsf{Z}^n$ or, more generally, if

(b) $A \subset \mathsf{R}^n$ is an (α, β)-net?

If the answer is affirmative in the latter case, then the constant L_1 must also depend on α, β. Note that (a) and (b) are not as closely related as it may seem, since for every $n \geq 2$ there are nets $A \subset \mathsf{R}^n$, which are not bilipschitz equivalent to Z^n; see [BK, Theorem 1.1 and Remark 1] and [Mc].

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