ON LARGE DEVIATIONS OF FIXED POINTS

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Abstract. Fixed point theorems have a decisive role in economic equilibrium theory and game theory. In this paper, random versions of two important fixed point theorems, Brouwer's and Kakutani's, are presented. In addition, we prove results concerning with large devitions of the fixed points.

1. Introduction

In the early 20th century Brouwer proved his famous result [Bro]. It can be stated as follows: if D is a convex, compact and nonempty subset of $\mathbb{R}^l$, then a continuous map $f : D \to D$ has a fixed point, that is, there exists such $x^* \in D$ that $f(x^*) = x^*$. In 1941 Kakutani proved his extension of Brouwer's fixed point theorem to correspondences (set-valued maps) [Kak]. We present the exact statement later.

In this paper, we will prove a stochastic version of these fixed point theorems for a sequence of random maps. Furthermore, we prove large deviation results concerning with the sequence of random fixed point sets. The paper is organized so that first we handle the Brouwer's fixed point theorem case and then Kakutani's.

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2. Preliminaries

Let us consider a family of random variables $F_n(p, \omega) \in \mathbb{R}^l$, $n \in \mathbb{N}$, $p \in D$, defined on some propabability space $(\Omega, \mathcal{F}, P)$. The set D is in this paper a nonempty, compact and convex subset of $\mathbb{R}^l$.

Let us denote $\pi_n^*(\omega) = \{p \in \mathbb{R}^l \mid F_n(p, \omega) = p\}$. This means that $\pi_n^*(\omega)$ is the set of fixed points for the realization $\omega \in \Omega$. We say, that π_n^* is nonempty eventually (a.s.), if there is a finite (a.s.) map $m : \Omega \to \mathbb{N}$ such that $\pi_n^*(\omega) \neq \emptyset$ for all $n \geq m(\omega)$. One aim in this

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paper is to prove, that π_n^* is nonempty eventually (a.s.) under certain conditions.

Let us denote by $C_n : \mathbb{R}^l \times D \to \mathbb{R}$ the logarithm of the Laplace transform of $F_n(p)$,

$$C_n(\beta, p) \doteq \log E e^{\beta \cdot F_n(p)}.$$

The so called free energy function associated with the sequence $(F_n(p))$ is defined as

$$c(\beta, p) \doteq \limsup_{n\to\infty} n^{-1} C_n(\beta, p).$$

We adopt the standard terminology of large deviations theory, see e.g. [DZ].

We denote by $\mathcal{N}(p)$ the set of outward unit normals at the point p, where p is on the boundary of the set D.

3. A Random Brouwer's Fixed Point Theorem

The following theorem gives us a random Brouwer's fixed point theorem. First we prove a modified version of the standard Brouwer's fixed point theorem in the spirit of [Sma, page 337] and [Num1, page 271] and then we use this deterministic result in proving the stochastic version of it.

Theorem 3.1. *Let $D \subset \mathbb{R}^l$ be a convex, compact and nonempty set. If $f : D \to R^l$ is a continuous map satisfying the condition $f(p) \cdot u < p \cdot u + |f(p) - p|$ for all $p \in \partial D, u \in \mathcal{N}(p)$, then there exists $p^* \in D$ such that $f(p^*) = p^*$. Furthermore, suppose that the maps $F_n(\cdot, \omega) : D \to \mathbb{R}^l$ satisfy following conditions:*

- (B1) *(a.s.) there exists $\lim_{n\to\infty} n^{-1} F_n(p) = f(p)$ for all $p \in \partial D$,*
- (B2) *(a.s.) $F_n(p)$ is differentiable on ∂D for all n and* $$\sup_{p\in\partial D, n\in\mathbb{N}} |n^{-1} F_n'(p)| < \infty,$$
- (B3) *(a.s.) $F_n(p)$ is continuous on D for all n.*

Then $\pi_n^* \neq \emptyset$ eventually (a.s.).

Proof. The set fD is compact because D is. Thus we can find such $M \geq 0$ that $fD \subset D + \overline{B}(0, M)$. Let us define $\hat{D} \doteq D + \overline{B}(0, M)$. Now $\hat{D}$ is compact and convex (see e.g. [Web, pages 42,51]). When $q \in \hat{D}$, there is $p \in D$ such that $q = p + \alpha v$ for some $0 \leq \alpha \leq 1$ where v is a normal in p of length M. Notice that, for $q \in D$, $\alpha = 0$ and if $q \notin D$, then $p \in \partial D$.

Let us define the map $\hat{f} : \hat{D} \to \hat{D}$ by

$$\hat{f}(p + \alpha v) = (1 - \alpha)(f(p) - p) + p.$$

This map is clearly continuous so by Brouwer's fixed point theorem there is a fixed point $q^* \in \hat{D}$. It is also clear, that $\hat{f}|D = f$. If $q^* \in \hat{D} \setminus D$, it follows that

$$(1-\alpha^*)(f(p^*) - p^*) + p^* = p^* + \alpha^* v$$

for some $0 < \alpha^* < 1$ and by simplifying we get

$$f(p^*) - p^* = \frac{\alpha^*}{1-\alpha^*} v.$$

Now

$$|f(p^*) - p^*| = \frac{\alpha^*}{1-\alpha^*}|v| \quad \text{and} \quad f(p^*) \cdot v - p^* \cdot v = \frac{\alpha^*}{1-\alpha^*}|v|^2.$$

By using the assumption we get $|v| < M$ which is a contradiction. So there must be $p^* \in D$ such that $f(p^*) = p^*$.

Due to the compactness of ∂D and $\mathcal{N}(p)$, there is $\delta > 0$ such that

$$f(p) \cdot u + \delta \leq p \cdot u + |f(p) - p|$$

for all $p \in \partial D, u \in \mathcal{N}(p)$. From the compactness of ∂D and assumptions (B1) and (B2) it follows that (a.s.)

$$n^{-1} F_n(p) \to f(p)$$

uniformly on ∂D. Now we can choose δ small enough to satisfy also

$$|n^{-1} F_n(p) - f(p)| < \frac{\delta}{3}$$

and

$$(n^{-1} F_n(p)) \cdot u \leq f(p) \cdot u + \frac{\delta}{3}$$

eventually for all $p \in \partial D, u \in \mathcal{N}(p)$. Because

$$|f(p) - p| \leq |f(p) - n^{-1} F_n(p)| + |n^{-1} F_n(p) - p| < \frac{\delta}{3} + |n^{-1} F_n(p) - p|$$

it follows that

$$(n^{-1} F_n(p)) \cdot u + \frac{2\delta}{3} \leq f(p) \cdot u + \delta < p \cdot u + \frac{\delta}{3} + |n^{-1} F_n(p) - p|$$

(a.s.) for all $p \in \partial D, u \in \mathcal{N}(p)$ eventually. Now, since δ can be arbitrarily small, we can use the first part of the theorem and we can conclude that $\pi_n^* \neq \emptyset$ eventually (a.s.). $\square$

4. LD Upper Bound for Brouwer

In order to prove a large deviations upper bound result for the random fixed point sets π_n^*, we have to make more assumptions. We can transfer the problem to concern the zeroes of the random vector field $Z_n(p) \doteq F_n(p) - p$ and utilize the earlier result in [Kuu].

If $c(\cdot, p)$ is the free energy function associated with the sequence $(F_n(p))$, its convex conjugate function $\hat{c} : \mathbb{R}^l \times D \to \mathbb{R}_+$ is defined as Legendre transform:

$$\hat{c}(v, \beta) \doteq \sup_{\beta \in \mathbb{R}^l} \{\beta \cdot v - c(\beta, p)\}, \quad \text{see e.g. [Roc].}$$

In the special case when $v = 0$, we denote

$$I(p) \doteq \hat{c}(0, p) = - \inf_{\beta \in \mathbb{R}^l} c(\beta, p).$$

Moreover, if E is a subset of R^l we denote

$$I(E) = \inf_{p \in E} I(p).$$

We assume that the free energy function c satisfies the following assumption:

(B4) the (positive definite) derivative $(\partial^2 c/\partial \beta^2)(\beta, p)$ is strictly positive definite for all β and p.

We denote the range of the derivative map by

$$\mathcal{R}((\partial c/\partial \beta)(\beta, p)) \doteq \left\{ \frac{\partial c}{\partial \beta}(\beta, p) \mid \beta \in \mathbb{R} \right\}.$$

We assume that

(B5) $0 \in \mathcal{R}((\partial c/\partial \beta)(\beta, p))$ for all $p \in D$.

Let us consider the random variable $|F_n(p) - F_n(q)|$ where $p, q \in \partial D$. Under assumption (B3), there exists a function $\tilde{\delta}_n(h, \omega)$, $h \geq 0$, such that

$$|F_n(p, \omega) - F_n(q, \omega)| \leq \tilde{\delta}_n(|p - q|, \omega)$$

when $p, q \in \partial D$ by uniform continuity. We shall refer to the random function $\tilde{\delta}_n(h)$ as the random modulus of continuity (r.m.c.) for $F_n(p)$. Notice that $\tilde{\delta}_n(h)$ is in fact a random variable (viz. measurable), because it can be defined as a supremum of $|F_n(p) - F_n(q)|$ over a countable dense subset of ∂D. We denote the free energy function associated with the sequence $(\tilde{\delta}_n(h))$ by $c_{\tilde{\delta}}(t, h)$.

To shorten notations we denote by $\partial_1 f$ the first derivative of a function f with respect to the first argument and by $\partial_1^2 f$ the second derivative taken twice with respect to the first argument. We make the following two assumptions about the function $c_{\tilde{\delta}}$.

(B6) The function $c_{\tilde{\delta}}$ is twice differentiable with respect to t for every $h \leq h_0$ for some $h_0 > 0$ and $\sup_{t>0} \partial_1^2 c_{\tilde{\delta}}(t,h) \to 0$ when $h \to 0$.

(B7) The function $d(h) \doteq \partial_1 c_{\tilde{\delta}}(0,h)$ is right continuous at 0.

Let us now consider random variables $Z_n(p) \doteq F_n(p) - p$. It is clear that if assumption (B3) (continuity) holds for F_n, it holds for Z_n, too. Furthermore, it is easy to see that if $c(\beta, p)$ is the free energy function of the sequence $(F_n(p))$, then $c(\beta, p)$ is also the free energy function of the sequence $(Z_n(p))$. Consequently, the functions $\hat{c}$ and I are the same. Finally, because

$$|Z_n(p,\omega) - Z_n(q,\omega)| = |F_n(p,\omega) - F_n(q,\omega)|$$

the $(\tilde{\delta}_n(h))$ sequence is the same also. Thus, if we assume (B3)-(B7) for functions F_n, the same assumptions hold for functions Z_n.

Let us denote $\pi_n^0(Z) \doteq \{\omega \in \Omega \mid Z_n(p,\omega) = 0\}$. Because π_n^* equals the set $\pi_n^0(Z)$, we get immediately the following upper bound result for random fixed points in view of [Kuu, Theorem 5.2].

Theorem 4.1. *Suppose that conditions (B3)-(B7) hold true. Then*

$$\limsup_{n\to\infty} n^{-1} \log P\{\pi_n^* \cap \{I \geq \alpha\} \neq \emptyset\} \leq -\alpha \text{ for all } \alpha > 0.$$

Remark 4.2. The assumption (A2) in [Kuu], which corresponds the assumption (B3), is not exactly the same. However, the same proof applies to this case also because in the proof of [Kuu, Theorem 5.2] we need in fact just the continuity of $Z_n(p)$.

Remark 4.3. Note that if

$$\limsup_{n\to\infty} n^{-1} \log P\{\pi_n^* \cap \{I \geq \alpha\} \neq \emptyset\} \leq -\alpha$$

for all $\alpha > 0$, then

$$\limsup_{n\to\infty} n^{-1} \log P\{\pi_n^* \cap E \neq \emptyset\} \leq -I(E)$$

for all closed sets $E \subset \mathbb{R}^l$ such that $E \cap D \neq \emptyset$ by choosing $\alpha = I(E)$ (see [Num2, page 242]).

5. Lower bound and related results for Brouwer

In order to prove a large deviations lower bound result for random equilibria, we use the same kind of approach as in the previous section: we have a lower bound result for $\pi_n^0(Z)$ and that is in fact a result for π_n^*. Notice that the derivative of $Z_n(p)$ is $F_n'(p) - I$.

Theorem 5.1. *Suppose that for each $p \in D$ the derivatives $n^{-1}(F_n'(p) - I)$, $n \in N$, exist and are uniformly non-singular almost surely; that means, there is a constant $M(p) < \infty$ such that*

(B8) $|n(F_n'(p) - I)^{-1}| \leq M(p)$ *for all $n \in N$, (a.s.).*

Moreover, suppose that the sequence $n^{-1}F_n'(p)$ is uniformly equicontinuous on D (a.s.):

(B9) $|n^{-1}F_n'(p) - n^{-1}F_n'(q)| \leq \varepsilon(|p-q|)$ *for all n, p and q, (a.s.),*

where $\lim_{h\to 0}\varepsilon(h) = 0$. Then

$$\liminf_{n\to\infty} n^{-1} \log P\{\pi_n^* \cap G \neq \emptyset\} \geq -I(G)$$

for all relatively open (w.r.t. D) sets $G \subset \mathbb{R}^l$.

The proof can be found in [Num2, page 247].

Because $I(\overline{B}) = I(B) = I(\mathring{B})$ for a convex subset B of R^l satisfying $\mathring{B} \cap D \neq \emptyset$ (see [Num2, Theorem 3]), we have the following result ([Num2, Corollary 4.1]), which follows from Corollary 4.1 and Theorem 5.1.

Corollary 5.2. *Suppose that $B \subset \mathbb{R}^l$ is a convex set such that $\mathring{B} \cap D \neq \emptyset$. If conditions (B3)-(B9) hold, then there exists a large deviation limit*

$$\lim_{n\to\infty} n^{-1} \log P\{\pi_n^* \cap B \neq \emptyset\} = -I(B).$$

Let us denote the entropy minimizing set of a convex set $B \subset \mathbb{R}^l$ by π_B^*, that is

$$\pi_B^* \doteq \{q \in \overline{B} \mid I(q) = I(B)\}.$$

Note that the set π_B^* is again the same for functions F_n and Z_n. We present now the conditional law of large numbers for the random equilibria π_n^* (see [Num2, Corollary 4.2]). The proof can be found in [Kuu].

Corollary 5.3. *Let $B \subset \mathbb{R}^l$ be a convex set such that $\mathring{B} \cap D \neq \emptyset$. Suppose that the conditions (B3)-(B9) hold true. Then*

$$\lim_{n\to\infty} P\{\pi_n^* \cap B \subset U(\pi_B^*, \varepsilon) \mid \pi_n^* \cap B \neq \emptyset\} = 1$$

for all $\varepsilon > 0$.

6. About correspondences

A correspondence $g : A \to B$ is a set-valued map $g : A \to \mathcal{P}(B)$. We assume that A and B are subsets of some Euclidean space and $g(x)$ is nonempty for all $x \in A$.

Definition 6.1. *A correspondence $g : A \to B$ is said to be upper hemi-continuous (u.h.c.) at x if for any neighborhood U of $g(x)$, there is a neighborhood V of x such that $fV \subset U$ ($fV = \cup_{v \in V} f(v)$). A correspondence $g : A \to B$ is u.h.c. if it is u.h.c. at every $x \in A$.*

Definition 6.2. *A correspondence $g : A \to B$ is said to be closed at x if $x_m \to x$, $y_m \to y$ and $y_m \in g(x_m)$ imply $y \in g(x)$. A correspondence $g : A \to B$ is closed if it is closed at every $x \in A$.*

Note that upper hemi-continuity of a singleton-valued correspondence is equivalent to continuity. It is also useful to observe that if a correspondence $g : A \to B$ is u.h.c. and closed-valued (that is, each image set is closed), then g is closed (see e.g. [Nik, Lemma 4.4]).

The classical Kakutani's fixed point theorem is as follows (for a proof see e.g. [Nik, pages 67-69]):

Theorem 6.3. *Let A be a nonempty compact convex set in $\mathbb{R}^l$. If $g : A \to A$ is a closed correspondence with nonempty convex images, then g has a fixed point (there is $x^* \in A$ such that $x^* \in g(x^*)$).*

We are now ready to prove the same kind of results as in previous sections. In fact, even the proofs are very similar to the earlier presentation.

7. Random Kakutani's Fixed Point Theorem

Let us define $B(C, \varepsilon) \doteq C + B(0, \varepsilon)$, that is, $B(C, \varepsilon)$ is the ε- neighborhood of the set C. We say that the sequence (C_n) of nonempty sets converges into a set C if for every $\varepsilon > 0$ there is n_ε such that $C_n \subset B(C, \varepsilon)$ for all $n \geq n_\varepsilon$. We will use the notation $C_n \to C$.

Let γ_n and γ be correspondences from E to $\mathbb{R}^k$. We say that γ_n converges to γ uniformly on E if for every $\varepsilon > 0$ there is n_ε such that $\gamma_n(x) \subset B(\gamma(x), \varepsilon)$ for all $n \geq n_\varepsilon$, $x \in E$.

Theorem 7.1. *Let $D \subset \mathbb{R}^l$ be a convex, compact and nonempty set. Let $\phi : D \to R^l$ be an upper hemi-continuous correspondence with convex, compact and nonempty values. If the condition $r \cdot u < p \cdot u + |r - p|$ for all $u \in \mathcal{N}(p), r \in \phi(p)$, holds for every $p \in \partial D$, then there exists $p^* \in D$ such that $p^* \in \phi(p^*)$. Furthermore, suppose that the correspondences $\Phi_n(\cdot, \omega) : D \to \mathbb{R}^l$ satisfy the following conditions:*

(K1) *(a.s.)* $\Phi_n \to \phi$ *uniformly on* ∂D,

(K2) *(a.s.)* $\Phi_n(p)$ *is upper hemi-continuous and has convex, compact and nonempty values on* D *for all* n.

Then $\pi_n^*(\Phi) \neq \emptyset$ *eventually (a.s.) where* $\pi_n^*(\Phi;\omega) \doteq \{p \in \mathbb{R}^l \mid p \in \Phi_n(p,\omega)\}$.

Proof. The set ϕD is compact because D is (see [Bor, Proposition 11.16]). Thus we can find such $M \geq 0$ that $fD \subset D + \overline{B}(0,M)$. Let us define $\hat{D} \doteq D + \overline{B}(0,M)$. Now $\hat{D}$ is compact and convex (see e.g. [Web, pages 42,51]). When $q \in \hat{D}$, there is $p \in D$ such that $q = p + \alpha v$ for some $0 \leq \alpha \leq 1$ where v is a normal in p of length M. Notice that, for $q \in D$, $\alpha = 0$ and if $q \notin D$, then $p \in \partial D$.

Let us define the set $\phi(p;\alpha)$ by

$$\phi(p;\alpha) \doteq \{z \in \mathbb{R}^l \mid z = p + (1-\alpha)(y-p), y \in \phi(p)\}.$$

Now we can define the correspondence $\hat{\phi} : \hat{D} \to \hat{D}$, $p + \alpha v \mapsto \phi(p;\alpha)$. It is easy to see that $\hat{\phi}(q)$ is nonempty and convex for all $q \in \hat{D}$ and that $\hat{\phi}|D = \phi$. Furthermore, $\hat{\phi}(q)$ is compact because it is the image of the compact set $\phi(p)$ under the continuous map $g(y;p,\alpha) = p + (1-\alpha)(y-p)$ for all $q \in \hat{D}$. Upper hemi-continuity follows from the fact that $\hat{\phi}(p;\alpha) = (g \circ \phi)(p;\alpha)$ and g and ϕ are both upper hemi-continuous (see [Bor, Proposition 11.23]). Thus by Kakutani's fixed point theorem there is a $q^* = p^* + \alpha^* v \in \hat{D}$ such that $q^* \in \hat{\phi}(q^*)$.

If $q^* \in \hat{D} \setminus D$, it follows that

$$p^* + \alpha^* v = (1-\alpha^*)(y^* - p^*) + p^*$$

for some $0 < \alpha^* < 1$, $y^* \in \phi(p^*)$ and by simplifying we get

$$y^* - p^* = \frac{\alpha^*}{1-\alpha^*} v.$$

Now

$$|y^* - p^*| = \frac{\alpha^*}{1-\alpha^*}|v| \quad \text{and} \quad y^* \cdot v - p^* \cdot v = \frac{\alpha^*}{1-\alpha^*}|v|^2.$$

By using the assumption we get $|v| < M$ which is a contradiction. So there must be $p^* \in D$ such that $p^* \in \phi(p^*)$.

Due to the compactness of ∂D, $\mathcal{N}(p)$ and $\phi(p)$, there is $\delta > 0$ such that

$$y \cdot u + \delta \leq p \cdot u + |y - p| \tag{1}$$

for all $p \in \partial D, u \in \mathcal{N}(p)$ and $y \in \phi(p)$. By (K1) we can choose $y_n(z_n) \in \phi(p)$ to satisfy

$$|z_n - y_n(z_n)| < \frac{\delta}{3}$$

for all $p \in \partial D, u \in \mathcal{N}(p)$ and $z_n \in \Phi_n(p)$ eventually and hence

$$z_n \cdot u \leq y_n(z_n) \cdot u + \frac{\delta}{3}.$$

Because

$$|y_n(z_n) - p| \leq |y_n(z_n) - z_n| + |z_n - p| < \frac{\delta}{3} + |z_n - p| \tag{2}$$

it follows by (1) and (2) that

$$z_n \cdot u + \frac{2\delta}{3} \leq y_n(z_n) \cdot u + \delta < p \cdot u + \frac{\delta}{3} + |z_n - p|$$

(a.s.) for all $p \in \partial D, u \in \mathcal{N}(p)$ and $z_n \in \Phi_n(p)$ eventually. Since δ can be arbitrarily small we can use the first part of the theorem and we can conclude that $\pi_n^*(\Phi) \neq \emptyset$ eventually (a.s.). □

8. LD Upper Bound for Kakutani

Suppose that $\Phi_n(\cdot, \omega) : D \to \mathbb{R}^l$ is a compact-valued correspondence for all $n \in N$ (a.s.). Let us define $\Psi_n(p, \omega) \doteq \Phi_n(p, \omega) - p$. Clearly Ψ_n is also compact valued (a.s.). We assume that

(K3) there is a continuous function $Z_n(\cdot, \omega) : D \to \mathbb{R}^l$ such that

$$Z_n(p, \omega) \in \{y \in \Psi_n(p, \omega) \mid |y| = \min_{y_1 \in \Psi_n(p,\omega)} |y_1|\} \text{ (a.s.)}.$$

Since D is compact, we have by uniform continuity that

$$|Z_n(p, \omega) - Z_n(q, \omega)| \leq \tilde{\delta}_n(|p - q|, \omega)$$

when $p, q \in D$ (a.s.). We assume also that $Z_n(p, \cdot)$ is a random variable (viz. measurable) for all $p \in D$. The random variable $\tilde{\delta}_n(h)$ is called the r.m.c. for $Z_n(p)$ in D.

We refer to the free energy function of the sequence $(Z_n(p))$ by $c(\cdot, p)$ and $c_{\tilde{\delta}}(\cdot, h)$ refers to the free energy function of $(\tilde{\delta}_n(h))$. Furthermore,

$$\hat{c}(v, \beta) \doteq \sup_{\beta \in \mathbb{R}^l} \{\beta \cdot v - c(\beta, p)\}$$

and

$$I(p) \doteq \hat{c}(0, p) = - \inf_{\beta \in \mathbb{R}^l} c(\beta, p).$$

We denote

$$\pi_n^0(\Psi;\omega) \doteq \{p \in \mathbb{R}^l \mid 0 \in \Psi_n(p,\omega)\}.$$

Because $\pi_n^0(\Psi;\omega) = \pi_n^*(\Phi;\omega)$ for all ω, we can study the set $\pi_n^0(\Psi)$ in order to find the fixed points of $\Phi_n(p)$.

We make the following assumptions:

(K4) the (positive definite) derivative $(\partial^2 c/\partial\beta^2)(\beta,p)$ is strictly positive definite for all β and p

(K5) $0 \in \mathcal{R}((\partial c/\partial\beta)(\beta,p))$ for all $p \in D$

(K6) $c_{\tilde{\delta}}$ is twice differentiable with respect to t for every $h \le h_0$ for some $h_0 > 0$ and $\sup_{t>0} \partial_1^2 c_{\tilde{\delta}}(t,h) \to 0$ when $h \to 0$.

(K7) the function $d(h) \doteq \partial_1 c_{\tilde{\delta}}(0,h)$ is right continuous at 0

Theorem 8.1. *Suppose that conditions (K3)-(K7) hold true. Then*

$$\limsup_{n\to\infty} n^{-1} \log P\{\pi_n^0(\Psi) \cap \{I \ge \alpha\} \ne \emptyset\} \le -\alpha \text{ for all } \alpha > 0.$$

Proof. Fix $\alpha > 0$ and $\varepsilon > 0$. Let us choose $h > 0$ small enough to satisfy $d(h) < \varepsilon/3$ and $\sup_{t>0} \partial_1^2 c_{\tilde{\delta}}(t,h) \le \varepsilon^2/(18(\alpha+1))$. Let $\{q_1, \ldots, q_K\} \subset \{I \ge \alpha\} \cap D$ be a finite set such that for each $p \in \{I \ge \alpha\} \cap D$ there is q_i satisfying $|p - q_i| < h$. Now we can estimate as follows:

$$\begin{aligned}
P\{\pi_n^0(\Psi) \cap \{I \ge \alpha\} \ne \emptyset\} &= P\{0 \in \Psi_n(p,\omega) \text{ for some } p \in \{I \ge \alpha\}\} \\
&= P\{\min_{y_1 \in \Psi_n(p)} |n^{-1} y_1| = 0 \text{ for some } p \in \{I \ge \alpha\}\} \\
&= P\{|n^{-1} Z_n(p)| = 0 \text{ for some } p \in \{I \ge \alpha\}\} \\
&\le P\{|n^{-1} Z_n(q_i)| \le n^{-1}\tilde{\delta}_n(h) \text{ for some } q_i\} \\
&\le P\{|n^{-1} Z_n(q_i)| < 2\varepsilon/3\} \\
&\quad + P\{|n^{-1}\tilde{\delta}_n(h) - d(h)| \ge \varepsilon/3\}. \\
&\le K \max_i P\{|n^{-1} Z_n(q_i)| < 2\varepsilon/3\} \\
&\quad + P\{|n^{-1}\tilde{\delta}_n(h) - d(h)| \ge \varepsilon/3\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\limsup_{n\to\infty} n^{-1} \log P\{\pi_n^0(\Psi) \cap \{I \ge \alpha\} \ne \emptyset\} \\
\le\ &\max\{\limsup_{n\to\infty} n^{-1} \log K \max_i P\{|n^{-1} Z_n(q_i)| < 2\varepsilon/3\}, \\
&\limsup_{n\to\infty} n^{-1} \log P\{|n^{-1}\tilde{\delta}_n(h) - d(h)| \ge \varepsilon/3\}\}.
\end{aligned}$$

The rest of the proof follows the earlier presentation in [Kuu, Proof of Theorem 4.2].

By Taylor expansion

$$\hat{c}_{\tilde{\delta}}(v,h) = \hat{c}_{\tilde{\delta}}(d(h),h) + \partial_1 \hat{c}_{\tilde{\delta}}(d(h),h)(v-d(h)) + \frac{1}{2}\partial_1^2 \hat{c}_{\tilde{\delta}}(\xi(h),h)(v-d(h))^2,$$

where $\xi(h)$ is between $d(h)$ and v (the derivatives exist by the implicit function theorem). The first two terms in this sum are zeroes. By using the fact that

$$\partial_1^2 \hat{c}_{\tilde{\delta}}(\xi(h),h) = \frac{1}{\partial_1^2 c_{\tilde{\delta}}(t(h),h)}$$

where $t(h)$ is defined through $\partial_1 c_{\tilde{\delta}}(t(h),h) = \xi(h)$, we get

$$\hat{c}_{\tilde{\delta}}(v,h) = \frac{(v-d(h))^2}{2\partial_1^2 c_{\tilde{\delta}}(t(h),h)}.$$

Now

$$\inf_{|v-d(h)|\geq \varepsilon/3} \hat{c}_{\tilde{\delta}}(v,h) = \inf_{|v-d(h)|\geq \varepsilon/3} \frac{(v-d(h))^2}{2\partial_1^2 c_{\tilde{\delta}}(t(h),h)} = \frac{\varepsilon^2}{18\partial_1^2 c_{\tilde{\delta}}(t(h),h)} \geq \alpha+1$$

and thus

$$\limsup_{n\to\infty} n^{-1} \log P\{|n^{-1}\tilde{\delta}_n(h) - d(h)| \geq \varepsilon/3\} \leq -\alpha - 1.$$

On the other hand

$$\begin{aligned}
& \limsup_{n\to\infty} n^{-1} \log K \max_i P\{|n^{-1} Z_n(q_i)| < 2\varepsilon/3\} \\
= \;& \max_i \limsup_{n\to\infty} n^{-1} \log P\{|n^{-1} Z_n(q_i)| < 2\varepsilon/3\} \\
\leq \;& -\min_i \inf_{|v|\leq 2\varepsilon/3} \hat{c}(v,q_i) \\
\leq \;& -\min_i I(q_i) + \eta(\varepsilon) \\
\leq \;& -\alpha + \eta(\varepsilon).
\end{aligned}$$

In conclusion,

$$\limsup_{n\to\infty} n^{-1} \log P\{\pi_n^0(\Psi) \cap \{I \geq \alpha\} \neq \emptyset\} \leq -\alpha + \eta(\varepsilon)$$

when ε is small enough and the assertion follows after letting $\varepsilon \to 0$. $\square$

9. LD Lower Bound for Kakutani

The basic structure of the proof of large deviations lower bound follows the presentation in [Num2, Theorem 9]. The main new ingredient is the use of an approximation scheme, see hypothesis (K8) and inequalities (4)-(5).

Theorem 9.1. *Let the correspondence $\Psi_n(p)$ have a function $Z_n(p)$ as defined in the previous section for all n and p. We assume that*

(K8) *for every n and $\eta > 0$ there exists a r.v. $Z_{n,\eta}(\cdot,\omega) : D \to \mathbb{R}^l$, $Z_{n,\eta}(p,\omega) \in \Psi_n(p,\omega)$, which satisfies*
$$|n^{-1}Z_n(p,\omega) - n^{-1}Z_{n,\eta}(p,\omega)| < \eta \text{ for all } p \in D \text{ (a.s.)}.$$

Suppose that for each $p \in D$ the derivatives $n^{-1}Z'_{n,\eta}(p)$, $n \in N$, $\eta > 0$, exist and are uniformly non-singular almost surely; that means, there is a constant $M(p) < \infty$ such that

(K9) $|nZ'_{n,\eta}(p,\omega)^{-1}| \leq M(p)$ *for all n, η (a.s.).*

Moreover, suppose that the following condition is also satisfied (a.s.):

(K10) $|n^{-1}Z'_{n,\eta}(p,\omega) - n^{-1}Z'_{n,\eta}(q,\omega)| \leq \varepsilon(|p-q|)$ *for all n, p, q and η,*

where $\lim_{h\to 0}\varepsilon(h) = 0$. Then
$$\liminf_{n\to\infty} n^{-1}\log P\{\pi_n^0(\Psi) \cap G \neq \emptyset\} \geq -I(G)$$
for all open (w.r.t. D) sets $G \subset \mathbb{R}^l$.

Remark 9.2. If $Z_n(p)$ satisfies the conditions (K9) and (K10), then we can choose $Z_{n,\eta} = Z_n$ for all $\eta > 0$. However, since $Z_n(p)$ is not necessarily smooth, we need the condition (K8).

Let us define
$$\operatorname{osc}\{f(z) \mid z \in A\} \doteq \sup\{|f(x) - f(y)| \mid x, y \in A\}.$$

Lemma 9.3. *Let $\varepsilon > 0$, $U \subset \mathbb{R}^d$ be an open set, $x_0 \in U$, and let $f : U \to \mathbb{R}^d$ be a continuously differentiable map such that f' is non-singular at x_0, i.e.,*
$$M \doteq |f'(x_0)^{-1}| < \infty.$$
Let ε be so small that
$$\overline{U}(x_0, 2M\varepsilon) \subset U$$
and
$$osc\{f'(z) \mid z \in \overline{U}(x_0, 2M\varepsilon)\} \leq (2M)^{-1}.$$
Then for each $y \in U(f(x_0), \varepsilon)$ there is a unique $x \in U(x_0, 2M\varepsilon)$ such that $f(x) = y$.

This corollary of a quantitative impicit function theorem presented in [Lan] can be found in [Num2, page 247].

Proof. (Proof of Theorem 9.1). Let $p_0 \in G$ be arbitrary. We have r.v.'s $Z_{n,1}(p)$ corresponding $\eta = 1$. Let $M \doteq M(p_0)$ be as in (K9). By condition (K10) there is $\varepsilon > 0$ such that $\overline{U}(p_0, 2M\varepsilon) \subset G$ (in relative topology) and
$$\operatorname{osc}\{n^{-1}Z'_{n,\eta}(p) \mid p \in \overline{U}(p_0, 2M\varepsilon)\} \leq (2M)^{-1} \text{ (a.s.)}.$$

Let us change η to be $\varepsilon/2$. The previous conclusions hold for this η too, because assumptions (K9) and (K10) do not depend on η. Now by LDP and the previous lemma

$$\begin{aligned}
&\liminf_{n\to\infty} n^{-1}\log P\{\pi_n^0 \cap G \neq \emptyset\} && (3)\\
\geq\ &\liminf_{n\to\infty} n^{-1}\log P\{|n^{-1}Z_{n,\eta}(p_0)| < \varepsilon\} && (4)\\
\geq\ &\liminf_{n\to\infty} n^{-1}\log P\{|n^{-1}Z_{n}(p_0)| < \varepsilon/2\} && (5)\\
\geq\ &-\inf_{|v|\leq\varepsilon/2} \hat{c}(v,p_0) && (6)\\
\geq\ &-I(p_0). && (7)
\end{aligned}$$

The assertion follows since $p_0 \in G$ was arbitrary. □

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