Inverse Schrödinger scattering problem for random potentials in plane

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Abstract: We study an inverse problem for the two-dimensional random Schrödinger equation $(\Delta + q + k^2)u(x, y, k) = 0$. The potential $q(x)$ is assumed to be a Gaussian random function that defines a Markov field. We show that the back scattered field, obtained from a single realization of the random potential q, determines uniquely the principal symbol of the covariance operator. The analysis is carried out by combining methods of harmonic and microlocal analysis to stochastic methods, in particular, to the theory of ergodic processes and properties of the Wiener chaos decomposition.

Keywords: Inverse scattering, Random potential, Schrödinger equation, Backscattering, Markov fields.

AMS classification: 35R30, 35P25, 82D30, 60J25.

1 Introduction

1.1 Background

Consider scattering of waves from a perturbed half-space, like the surface of earth, where the macro-scale structure of the perturbation is smooth but in very small scale, the surface is rough. If one forgets the micro-scale roughness, and approximate the surface with a smooth surface, on the high frequencies waves scatter according to Kirchoff law, that is, like from a mirror. This contradicts many everyday experiences, like walking with a torch in dark field. For a valid model to describe the scattering as above one needs to deal with the micro-structure of the object. The inverse problem here is not so much to recover the exact micro-structure of an object but merely to determine the parameters or functions describing the characteristic properties of the micro-structure. One example of such a parameter is the correlation length of the medium that is related to the typical size of the "particles" in the scatterer or the material of the medium.

A stochastic model serves best to formulate the above problem in rigorous terms. One explanation to this is that very general independence assumptions lead to non-smooth structures. For example, the paths of a Brownian motion are nowhere differentiable.

In this work we consider scattering from a random medium and the inverse problem for it. More precisely, we study as a basic model case the Schrödinger equation

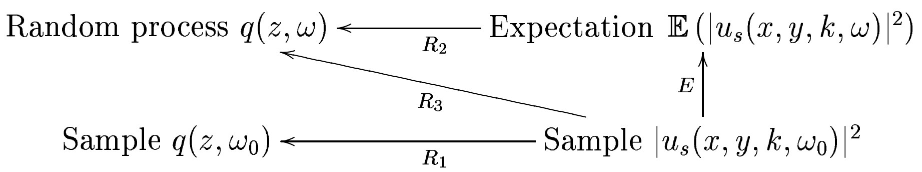
$$(\Delta + k^2 + q)u = \delta_y,$$

where q is a random, compactly supported potential and y is a point outside the support of q. Throughout the paper we assume that the deterministic part of q vanish, i.e., $\mathbb{E}\, q = 0$. The field u can be decomposed as

$$u = u_0(x, k) + u_s(x, k)$$

where $u_0(x, k) = u_0(x, y, k)$ is the incident field in the homogeneous space corresponding a point source at y and $u_s(x, k) = u_s(x, y, k)$ is the scattered field induced by q. We adopt a terminology where the parameter k is referred to as frequency.

By definition, a random potential means a measurable mapping from a probability space to some function space: $\omega \mapsto q(\cdot, x)$. In the realm of inverse problems the existing deterministic theory would answer to the question: If we have measurements from a single realization of the potential $q(\cdot\,, \omega_0)$, can we determine the potential uniquely. In the below diagram this correspond the existence of the map R_1.



Typical features of deterministic theory of inverse problems is that the theory is mathematically rigorous, i.e. requires no approximations and applies harmonic and microlocal analysis, see [8], [32], [33], [51]. An excellent review for deterministic inverse problems has been given in [53]. For scattering from non-smooth deterministic structures see, [7], [38] and [39].

In the random side the first question to answer is: What is meant by measurements from random a object? To enlight the problem, consider an example of radar measurements done far from the surface of the Earth. A common model is to assume that the surface is random and when several measurements are done, we may compute e.g. the covariance of the scattered signal by taking averages over sequences of measurements. The corresponding inverse problem is to find the stochastical characteristics of the process $q(z, \omega)$ from the stochastical characteristics of the measured signal. In the above diagram this corresponds to the map R_2. In this interpretation one considers the surface as a different sample of the "random scatterer" for each measurement. In reality, the object that we are measuring is all the time the same. For instance the trees and stones on the surface of the Earth have not moved their locations between times of different measurements. Mathematically speaking this means that we have done measurements from one fixed realization of the random scatterer. The reason why the traditional approach might work is the underlying assumption (that has not often been explicitly formulated in applied literature) that the measured signal is ergodic in the sense that one realization determines a.s. stochastical characteristics of the process. In the above diagram this corresponds to the existence of the map E.

In traditional applied and engineering literature concerning inverse scattering from random media the problem has been formulated as the existence of the map R_2 in the above diagram: Does the expectation of the scattered signal $\mathbb{E}\left(|u_s(x, y, \omega)|^2\right)$, determine e.g. the covariance function $\mathbb{E}\left(q(x, \omega) q(y, \omega)\right)$. Typical "ad hoc" model for covariance function used in 3-dimensional acoustic problems is

$$\mathbb{E}\left(q(x, \omega) q(y, \omega)\right) = \exp\left(-\frac{|x-y|}{l}\right) \tag{1}$$

where parameter l is called the correlation length. This parameter l describes how distant points can still have a correlation above some given bound. The correlation length can be considered as a spatially varying parameter that, at the end of the day, one wants to determine from the measurements. Moreover, the multiple scattering is often omitted, that is, the data is thought to be the covariance of the first order term in Born series, $\mathbb{E}\,|u_1(x, y, k, \omega)|^2$. This leads to a linearization of the inverse the problem and can be justified when

q is small, see classical books [19] and [36] and in general, Journal *Waves in Random Media*. Also, an extensively studied question is the scattering from a random surface having small amplitude, see e.g. [47], [48].

A related approach for the scattering from a random media is the study of the multi-scale asymptotics. In this case the made approximations can be justified when the frequency k and the spatial frequency of the scatterer have appropriate magnitudes. Indeed, if measurements are done on frequencies $k \in [K_1, K_2]$ that are much smaller than the spatial frequencies of the scatterer, done approximations coincide with measurements. For this type of approach, consider the case of Schrödinger equation assuming that there is a small parameter $\epsilon > 0$ such that $k = k_0\epsilon^{-1}$ and $q = q_0(\frac{x}{\varepsilon^2}, \omega)$, where q_0 is a random function. Then one has an asymptotic expansion $u_s(x, y, k) = \varepsilon^2 u_{s,0}(x, y, k_0) + \mathcal{O}(\varepsilon^3)$ and properties of random function q_0 can be derived from leading order term $u_{s,0}(x, y, k_0)$ of the measurement when $\varepsilon \to 0$. This type of asymptotic analysis has been studied by Papanicolaou's school in various cases, see e.g. [3], [46], [42], [23], [41].

In this paper we use as the measured data mean values (over the frequency k) of the energy $|u_s(x, y, k, \omega_0)|^2$, obtained from a single realization $q(z, \omega_0)$. This corresponds to studying the existence of the map R_3 in the above diagram, which is a more realistic point of departure since it is often impossible to measure the averaged signal $\mathbb{E}\,|u_s(x, y, k, \omega)|^2$. See Sections 1.3 and 1.4 for the exact statement of our results.

This type of approach, i.e., study of properties of random Schrödinger operator that are valid almost surely have been used successfully in spectral theory of random Schrödinger operators. For instance, the structure of the spectrum of random Schrödinger operators has been studied in celebrated papers of Kotani, [25],[26], that has been extensively generalized by B. Simon and others (cf. [43],[12],[50],[49],[29],[11]). These results show that Schrödinger operators having Anderson model-type random potentials or potentials that are ergodic in translations $x \mapsto x + a$ have almost surely no absolutely continuous spectrum. One-dimensional inverse problems for potentials that are ergodic in translations have been studied, see e.g. [24]. For physical applications, see [27], [28]. Intuitively, the absence of absolutely continuous spectrum means that all propagating waves get a.s. trapped in the random potential in $\mathbb{R}^n$. Analogous intuitive interpretation for our result is that near each point the potential a.s. has wave front set to all directions. This explains why the micro-structure of q can be reconstructed from backscattered wave.

1.2 Mathematical model for the random potential

Fix a bounded domain $D \subset \mathbb{R}^2$. Assume that the potential q is a localization of a generalized Gaussian field. Thus, we may write $q = \chi Q$, where $\chi \in C_0^\infty(D)$ and Q is a centered (i.e., $\mathbb{E}\, Q = 0$) generalized Gaussian field on $\mathbb{R}^2$. Recall, that this means that Q is a measurable map from the probability space Ω to the space of distributions $\mathcal{D}'(\mathbb{R}^2)$ such that for all $\phi \in C_0^\infty(\mathbb{R}^2)$ the mapping

$$\Omega \ni \omega \mapsto \langle Q(\omega), \phi \rangle$$

is a centered Gaussian random variable. We will assume that the probability measure space $(\Omega, \mathcal{F}, \mathcal{P})$ is complete. The reason for introducing the cutoff χ is to avoid the possible effects arising from discontinuity at the boundary.

To get a more concrete structure we will assume further that Q has additionally a Markov structure. Below we will follow [45] and give a definition and some basic properties of a *generalized Markov field*; these are discussed in more detail in Appendix 1. The definition that we use for Markov fields mimics the situation where physical particles in a lattice have no long-term interaction, i.e., only neighboring particles have direct interaction. Assume that $S_1 \subset D$ is an open set. We set $S_2 = D \setminus \overline{S}_1$ and $S_\epsilon = \{x \in D : d(x, \partial S_1) \le \varepsilon\}$, $\varepsilon > 0$, a collar neighborhood of the boundary ∂S_1 in S_1. Intuitively the Markov property means that the influence from the inside to the outside must pass through the collar. This leads to

Definition 1.1 *A generalized random field Q on $\mathbb{R}^2$ satisfies the Markov property if for any S_1, S_2 and S_ε as described above, and $\varepsilon > 0$ small enough, the conditional expectations satisfy*

$$\mathbb{E}\left(h \circ Q(\psi) | \mathcal{B}(S_\varepsilon)\right) = \mathbb{E}\left(h \circ Q(\psi) | \mathcal{B}(S_\varepsilon \cup S_1)\right)$$

for any complex polynomial h and for any test function $\psi \in C_0^\infty(S_2)$.

Here $\mathcal{B}(S_j)$ is the σ-algebra generated by the random variables $Q(\phi)$, $\phi \in C_0^\infty(S_j)$, $j = 1, 2$, and $\mathcal{B}(S_\varepsilon)$ is defined respectively.

The Markov property has dramatic implications to the structure of the field Q and especially to its covariance operator C_Q defined by

$$\begin{aligned} &C_Q : C_0^\infty(\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2), \\ &\langle \psi_1, C_Q \psi_2 \rangle = \mathbb{E}\left(Q(\psi_1) Q(\psi_2)\right). \end{aligned} \qquad (2)$$

It turns out, that under minor additional conditions, the inverse operator $(C_Q)^{-1}$ (assuming that it exists in a suitable sense) must be a local operator: it cannot increase the support of a test function. By a well-known theorem of J. Peetre [44] $(C_Q)^{-1}$ must be a linear partial differential operator, which we shall henceforth assume.

We shall next further reduce the complexity of the structure by making three additional assumptions on $(C_Q)^{-1}$ (the discussion in rest of this subsection is still somewhat descriptive; see Definition 1.3 and Subsection 1.4 below for the definitive precise statement). The first assumption is that $(C_Q)^{-1}$ is a non-degenerate operator, i.e. its principal symbol does not vanish in $\mathbb{R}^2 \setminus \{0\}$. Since $(C_Q)^{-1}$ is positive, it corresponds to an elliptic partial differential operator, of even order.

Our second assumption is that the order of $(C_Q)^{-1}$ is two. Otherwise, in the case where $(C_Q)^{-1}$ is of fourth order or higher, with smooth coefficients, one could easily verify (c.f. the proof of Theorem 2.3) that the realizations of q are in the Sobolev class $W^{s,p}_{comp}(\mathbb{R}^2)$ for all $s < 1$ and $1 < p < \infty$. Recall that the aim was to consider the case of non-smooth potentials, that is the most interesting case in view of many applications. Since $(C_Q)^{-1}$ is a symmetric operator, it can be written in the form

$$(C_Q)^{-1} = P(z, D_z) = -\sum_{j,k=1}^{2} \frac{\partial}{\partial z^j} a_{jk}(z) \frac{\partial}{\partial z^k} + b(z) \tag{3}$$

where

$$[a_{jk}(z)] \geq c(z)I, \quad c(z) > 0$$

is a symmetric matrix.

Our last simplifying assumption is that the random structure of the potential is (micro-)locally isotropic with smooth coefficients structure. In terms of the coefficients, we thus assume that

$$a_{jk}(z) = a(z)\delta_{ij}. \tag{4}$$

and that $a(z) > 0$ and $b(z)$ are smooth functions.

Summarizing: we assume that $(C_Q)^{-1}$ is a non-degenerate, is of 2nd order, has smooth coefficients, and finally its principal part is positive and homogeneous. We call such a field Q *micro-locally isotropic*, as then C_Q is a pseudodifferential operator with an isotropic principal symbol. Namely, in our situation C_Q is (some) parametrix of $(C_Q)^{-1}$ and thus its Schwartz kernel $k_Q(z,z') = G(z,z')$ is a Green's function of $P(z, D_z)$ satisfying

$$P(z, D_z)G(z,z') = \delta(z - z'),$$

An important example of such random fields of this type is obtained by the free Gaussian fields, which appears in two dimension quantum field theory (c.f. e.g. [14]). The free Gaussian field on the bounded domain D, corresponding to Dirichlet boundary values, has the (Dirichlet-)Green's function G_D as the kernel of its covariance operator. This correspond to choices $a(z) = 1$, $b(z) = 0$. More complicated examples can be constructed easily.

Starting from the fact that our covariance operator corresponds locally (modulo smooth error) to a Green's function of the operator P, one may verify (c.f. Proposition 2.2 below), that C_Q has locally integrable kernel k_Q that has for fixed z_2 the asymptotics

$$k_Q(z_1, z_2) = -\frac{1}{a(z_2)} \log|z_1 - z_2| + f(z_1, z_2)$$

where f is locally bounded. Hence the function $a(\cdot)$ describes the strength of the singularity of k_Q near the diagonal, and is approximately proportional to the radius of the set $\{z_1 : \ k_Q(z_1, z_2) > M\}$ with a given large bound M. Because of the analogy to parameter l in formula (1) we call $a(z_2)^{-1}$ *the micro-correlation length function.*

Let $C = C_q$ be the covariance operator of the potential q, and let us denote also its kernel by the symbol C. We obtain that $C(z_1, z_2) = k_q(z_1, z_2) = \chi(z_1) k_Q(z_1, z_2) \chi(z_2)$. This implies for $C(z_1, z_2)$ the form

$$C(z_1, z_2) = -\mu(z_2) \log|z_1 - z_2| + F(z_1, z_2),$$

where $F \in L^\infty_{loc}$, and

$$\mu(z) := \frac{\chi(z)^2}{a(z)}. \tag{5}$$

We call $\mu(x)$ *the micro-correlation length of* q.

1.3 Direct and inverse scattering problem and the measurement

We consider the Schrödinger equation with outgoing radiation condition

$$\begin{cases} (\Delta - q + k^2)u = \delta_y \\ \left(\frac{\partial}{\partial r} - ik\right) u(x) = o(|x|^{-1/2}) \end{cases} \tag{6}$$

where the potential q is compactly supported random generalized function, as described above. The support of q is hence contained in the domain D. In Section 3 we will show that this equation has a unique solution in the appropriate class of functions.

For a non-vanishing potential q, we decompose u to two parts

$$u = u_0(x, y, k) + u_s(x, y, k).$$

Here $u_s(x, y, k)$ is the scattered field and

$$u_0(x, y, k) = \Phi_k(x - y) = \frac{i}{4} H_0^{(1)}(k|x - y|)$$

is the incident field corresponding a point source at y. We shall assume that $y \in U$, where the domain $U \subset \mathbb{R}^2 \setminus \overline{D}$ is called the *measurement domain*, and we assume that U is bounded and convex. As $q = q(\omega)$ is random, also the scattered field is a random variable. We sometimes emphasize this by writing $u_s(x, y, k) = u_s(x, y, k, \omega)$.

Definition 1.2 *Given $\omega \in \Omega$, the measurement $m(x, y, \omega)$ is the weak limit*

$$m(x, y, \omega) = \lim_{K \to \infty} \frac{1}{K - 1} \int_1^K k^4 |u_s(x, y, k, \omega)|^2 dk. \tag{7}$$

Here the limit is defined in $\mathcal{D}'(U \times U)$, i.e., in the sense of distributions with respect to variables x and y.

The measurement is an average over all frequencies so that it is not sensitive to measurement errors. For example, the white noise error in the measurement, is filtered out with frequency averaging.

As later will be observed, the existence of the distribution limits $m(x, y, \omega)$ in Definition 7 is equivalent to the existence of limits

$$M(\phi, \psi, \omega) = \lim_{K \to \infty} \frac{1}{K - 1} \int_1^K \left(\int_U \int_U k^4 |u_s(x, y, k, \omega)|^2 \phi(x) \psi(y)\, dx dy \right) dk \tag{8}$$

for $\phi, \psi \in C_0^\infty(U)$.

It is highly non-trivial fact that the above definition gives a well-defined, finite and non-zero quantity. That this is so, is a part of our main result.

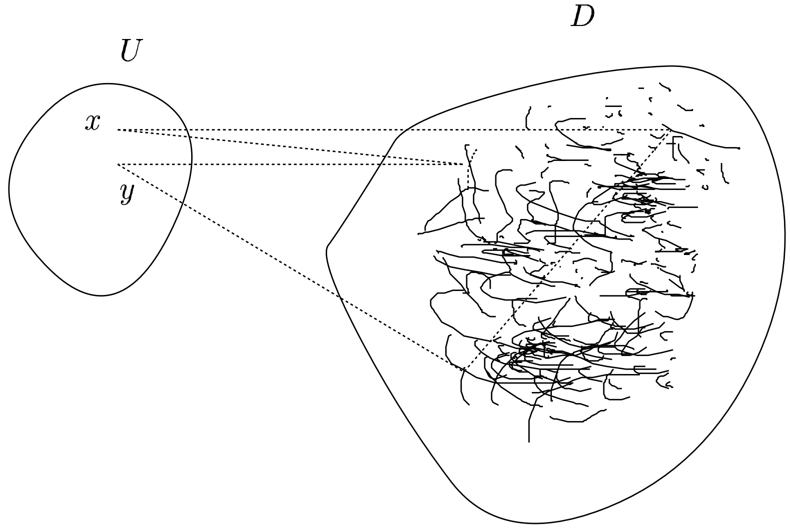


Figure 1: Measurements corresponding to a point source at $x \in U$. Lines corresponds to first and second order scattering observed at point y.

1.4 The result

Before stating the result, we collect together the definitions made in Section 1.2 in a mathematically precise, and slightly more general form:

Assumption 1.3 We assume that: $D \subset \mathbb{R}^2$ is a simply connected bounded domain, $\mu \in C_0^\infty(D)$, $\mu(x) \geq 0$. The potential q is a generalized centered Gaussian random field on $\mathbb{R}^2$, with realizations almost surely supported on the domain D, and whose covariance operator is a classical pseudodifferential operator having the principal symbol $\mu(x)|\xi|^{-2}$ for $|\xi| > 1$. The measurement domain $U \subset \mathbb{R}^2 \setminus \overline{D}$ is assumed to be bounded and convex.

Observe that our assumptions cover the case of Markov fields of the type considered in Section 1.2. Moreover, one could easily dispense with the assumption that q is centered. We discuss this in more detail in the remarks at the end of Section 7.

The main result of this paper is

Theorem 1.4 *Assume that the conditions stated in Assumption 1.3 hold. Then*

(i) *The measurement* $m(x, y, \omega)$ *exists almost surely as a limit in* $\mathcal{D}'(U \times U)$.

(ii) *Almost surely, the distribution* $m(x, y, \omega)$ *coincides with a deterministic function* $m_0(x, y)$ *which is continuous in* $x, y \in U$.

(iii) *The back-scattering data* $n_0(x) = m_0(x, x), \quad x \in U$ *uniquely determines the function* μ*, i.e. the micro-correlation length of the potential. Moreover, there is a linear operator* T *operating to* n_0 *such that*

$$T(n_0) = \mu \in C_0^\infty(D).$$

We stress that the above result allows us to determine the principal structure of the covariance from measurements from a single realization of the potential only! Property (ii) in Theorem 1.4 is sometimes called statistical stability, c.f. [6]. Observe that the needed data is essentially energy averages of the back-scattered field. We refer to the Remarks in the end of section for a more through discussion of the relation of the above result to its deterministic counterparts.

2 Stochastic properties of q

We start our proof by first considering the regularity of the covariance and the realizations of th potential q. It turns out that $q(\omega)$ is not a function (or even a measure); almost surely it is a proper distribution. This is not so surprising since in the special case where $a \equiv 1$ and $b \equiv 0$ our random field corresponds locally to a free Gaussian field. However, the potential just barely fails to be a function: almost every realization of the potential satisfies

$$q(\omega) \in W_0^{-\epsilon,p}(D) \quad \text{for all } \varepsilon > 0 \text{ and } 1 < p < \infty. \tag{9}$$

Above $W_0^{s,p}(D)$ is the space of $W^{s,p}(\mathbb{R}^2)$ functions supported in $\overline{D}$, and $W^{s,p}(\mathbb{R}^2) = \mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\mathcal{F}L^p(\mathbb{R}^2))$ is the standard Sobolev space, defined with Fourier transform $\mathcal{F}$. In this section we prove this fact, which is crucial for the success of the subsequent analysis of our problem. For example, it enables us to prove in the following chapter the uniqueness for the corresponding scattering problem, even though the uniqueness is known to fail for certain integrable potentials.

We start by recording a result which yields a criterion for realizations of a random field to lie in $\bigcap_{p>1} L^p(D)$.

Lemma 2.1 *Assume that the covariance operator K of a random field F on the open bounded set $D \subset \mathbb{R}^n$ has a uniformly bounded kernel (denoted also by) K:*

$$|K(x,y)| \leq C < \infty \quad \text{for every} \quad x, y \in D.$$

Then the realizations of F belong almost surely to $\bigcap_{p>1} L^p(D)$.

Proof. This is an immediate consequence of [5]. A simple direct proof of this result is can also be obtained by approximating first K by smooth covariances and observing that in that case $\mathbb{E}\,(\|E\|_p)^p = c_p \int_D |K(x,x)|^{p/2} dx$. □

We next analyze the singularity of the covariance operator

Proposition 2.2 *The Schwartz kernel of the covariance operator C is regular and may be decomposed as*

$$C(x,y) = c_0(x,y)\log|x-y| + r_1(x,y),$$

where $c_0 \in C_0^\infty(D \times D)$ and the term r_1 satisfies

$$\widehat{r}_1 \in L^1(\mathbb{R}^4), \tag{10}$$

and, consequently

$$\sup_{x,y\in D} |r_1(x,y)| < \infty. \tag{11}$$

Above $\widehat{r}_1$ denotes the Fourier transform of r_1 with respect to both variables x, y.

Proof. By definition, $C(x,y)$ is corresponds to a (compactly supported) classical pseudodifferential operator in the class $S^{-2}(\mathbb{R}^2 \times \mathbb{R}^2)$ with the (x,ξ)-kernel $a(x,\xi) = \mu(x)(1-\psi(\xi))|\xi|^{-2} + b(x,\xi)$, where the smooth cutoff $\psi \in$

$C_0^\infty(\mathbb{R}^2)$ equals 1 near the origin, and $b \in S^{-3}_{comp}$. We obtain

$$2\pi^2 C(x,y) = \mu(x) \int_{\mathbb{R}^2} e^{i(x-y)\cdot\xi}(1-\psi(\xi))|\xi|^{-2}d\xi + \int_{\mathbb{R}^2} e^{i(x-y)\cdot\xi}a(x,\xi)d\xi$$
$$= I(x,y) + r_2(x,y),$$

where the asymptotics of I is well known, and it may be clearly written in the desired form $c_0(x,y)\log|x-y| + r(x,y)$. In order to analyze the rest term $r_2(x,y)$, write $R(x,y) = r_2(x,x-y)$, and observe that (10) follows as soon as we show that $\widehat{R}$ is integrable. A simple computation (c.f. [18, p.69]) shows that $\widehat{R}$ is the Fourier transform of the symbol b with respect to x. Since the support of b is compact with respect to x, and we have the uniform estimates $|\partial_x^\alpha b(x,\xi)| \leq C_\alpha(1+|\xi|)^{-3}$, $\alpha \geq 0$, it follows that $|\mathcal{F}_x b(\eta,\xi)| \leq C(1+|x|)^3(1+|\xi|)^{-3}$. This verifies that $\widehat{R} \in L^1$. As (10) implies (11), the proof of is complete. □

Let us then prove

Theorem 2.3 *Almost surely $q(\omega) \in W^{-\epsilon,p}(D)$ for all $\varepsilon > 0$ and $1 < p < \infty$.*

Proof. Recall that for given $s \in \mathbb{R}$ the Bessel potential J^s provides an isomorphism $J^s : W^{t,p}(\mathbb{R}^2) \to W^{t+s,p}(\mathbb{R}^2)$ for all $t \in \mathbb{R}$ and $1 < p < \infty$. Moreover, J^s is a pseudodifferential operator, whence it preserves singular supports. Thus it is enough to verify that locally the covariance of $J^\varepsilon q$ has a uniformly bounded kernel for any small $\varepsilon > 0$. That is, by letting J^ε_{loc} stand for a suitable localization of J^ε we have to study the kernel $J^\varepsilon_{loc} C J^\varepsilon_{loc}$. It is well known that for small $\varepsilon > 0$ we have

$$J^\varepsilon(x,y) = \frac{c}{|x-y|^{2-\varepsilon}} + S(x,y),$$

where S has a lower order singularity. Now the claim follows by combining Proposition 2.2 and the fact

$$\int_{B(0,R)} \frac{|\log|x||}{|x|^{2-\varepsilon}} dx < \infty$$

for any radius $R > 0$. □

In Section 5 we will make use of the the fact that there are finite dimensional Gaussian variables that may be used to approximate the potential q in the norm.

Lemma 2.4 *Let $\varepsilon > 0$ and $p \in (1, \infty)$. Then there is a sequence of finite dimensional Gaussian random distributions q_n such that for almost every realization it holds that*

$$\lim_{n \to \infty} \|q - q_n\|_{W^{-\varepsilon,p}} = 0$$

Proof. The space $W^{-\varepsilon,p}(\mathbb{R}^2)$ has a Schauder basis since, by the action of J^ε, it is isomorphic to $L^p(\mathbb{R}^2)$. Thus there are finite dimensional projections $P_n : W^{-\varepsilon,p}(\mathbb{R}^2) \to W^{-\varepsilon,p}(\mathbb{R}^2)$ with uniformly bounded norm and such that $P_n f \to f$ in norm as $n \to \infty$ for all $f \in W^{-\varepsilon,p}(\mathbb{R}^2)$. The finite dimensional approximations are now obtained by the simply taking $q_n = P_n q$. $\square$

Remark. By e.g. applying Fejer summation in a local Fourier-development one obtains also approximations q_n that work simultaneously for all p, ε, but we do not need this.

3 Direct scattering from distributional potential.

3.1 Unique continuation for distributional potentials

We showed in the previous chapter that the random potential $q(\omega)$ belongs with probability one to the Sobolev space $W^{-\epsilon,p}(D)$ for all $1 \leq p < \infty$ and $\epsilon > 0$. Consequently, we need to study the existence and properties of the solution for the Schrödinger equation for such irregular potentials. In this section we accomplish this by considering scattering from a deterministic non-smooth potential $q_0 \in W^{-\epsilon,p}(D)$, and the obtained results have independent interest.

The direct scattering theory from a potential that is in a weighted L^2 space is classical 8c.f. [4],[2]). For the L^p scattering theory the key tool is the unique continuation of the solution. Jerison and Kenig shoved in [20] that the strong unique continuation principle for L^p-potentials in $\mathbb{R}^n$ holds for $p \geq n/2$ and fails for $p < n/2$ in dimensions $n > 2$. In dimension two the unique continuation holds in a space of functions that is close to L^1 [20]. For Sobolev space potentials, the selfadjointness of the operator has been studied for [35]. Below in Lemma 3.2 we show a positive result for negative index Sobolev spaces.

More precisely, we study the scattering problem

$$\begin{cases} (\Delta - q_0 + k^2)u = \delta_y \\ \left(\frac{\partial}{\partial r} - ik\right) u(x) = o(|x|^{-1/2}) \end{cases} \tag{12}$$

where the potential $q_0 \in W^{-\epsilon,p'}_{\text{comp}}(\mathbb{R}^2)$, $p^{-1} + (p')^{-1} = 1$, $1 < p < 2$. We claim that the problem (12) is equivalent to the Lippmann-Schwinger equation

$$u(x) = u_0(x) - \int_{\mathbb{R}^2} \Phi_k(x-y) q_0(y) u(y) dy. \tag{13}$$

In the proof we show that the pointwise product $q_0 u$ in the integrand of (13) is well defined and that the integral exist in the sense of distributions. We will then show that (13) has a unique solution $u \in W^{2p,\epsilon}_{loc}(\mathbb{R}^2)$. The starting point is the unique continuation principle. For completeness we formulate it in dimension n. Roughly speaking, it says that if u is a compactly supported solution of the Schrödinger equation with $q_0 \in W^{-\epsilon,r}$, $r > n/2$ and ϵ is small then u must vanish identically. It appears to the authors that this result could also be obtained as a special case of D. Tataru's and H. Koch's recent unique continuation results based on L^p Carleman estimates [22]. In our case, we present a direct and simple proof for unique continuation. We start by observing that known pointwise multiplication results allow us to define the product distribution $q_0 u$.

Lemma 3.1 *Assume that $u \in W^{\epsilon,2p}_{loc}(\mathbb{R}^n)$, $q_0 \in W^{-\epsilon,p'}_{comp}(\mathbb{R}^n)$, $1 < p < \infty$, $\epsilon > 0$. Then the product $q_0 u$ is well-defined as an element of $W^{-\epsilon,\tilde{p}}_{comp}(\mathbb{R}^n)$, where $\tilde{p} = \frac{2p}{2p-1}$ and*

$$||q_0 u||_{W^{-\epsilon,\tilde{p}}(D)} \leq c||q_0||_{W^{-\epsilon,p'}(D)}||u||_{W^{\epsilon,2p}(D)}. \tag{14}$$

Proof: Take $\phi \in C_0^\infty(\mathbb{R}^n)$ to be a test function. By duality, the product $q_0 u \in \mathcal{D}'(\mathbb{R}^2)$ is a well defined through

$$\langle q_0 u, \phi \rangle = \langle q_0, \phi u \rangle \tag{15}$$

when $q_0 \in W^{-\epsilon,p'}_{\text{comp}}(\mathbb{R}^2)$ and $u \in W^{\epsilon,p}_{\text{loc}}(\mathbb{R}^2)$. By using Bony's paraproducts one can verify the following pointwise multiplier estimate in Sobolev spaces ([52, pp. 105])

$$\|\phi u\|_{W^{\epsilon,p}(D)} \leq c\left(\|\phi\|_{L^{r_1}(D)}\|u\|_{W^{\epsilon,r_2}(D)} + \|u\|_{L^{r_1}(D)}\|\phi\|_{W^{\epsilon,r_2}(D)}\right) \tag{16}$$

for $1/p = 1/r_1 + 1/r_2$. From (15), and (16) with $r_1 = r_2 = 2p$ it readily follows by duality that $q_0 u \in W^{-\epsilon,\tilde{p}}_{\text{comp}}$ where $\tilde{p} = \frac{2p}{2p-1}$. $\square$

Lemma 3.2 (Unique continuation principle in to an interior domain) *Assume that $p' \in (n/2, \infty)$, together with $0 < \epsilon < \frac{n}{4}(\frac{2}{n} - \frac{1}{p'})$. Let $q_0 \in W^{-\epsilon,p'}_{comp}(\mathbb{R}^n)$. If $u \in W^{\epsilon,2p}_{loc}(\mathbb{R}^n)$ is compactly supported and satisfies the Schrödinger equation*

$$(\Delta - q_0 + k^2)u = 0 \tag{17}$$

in the weak sense, then $u = 0$ identically.

Proof: To prove the unique continuation we use the well-known techniques of exponentially growing solutions for the Schrödinger equation, cf. [51], [32], [33], [16]. To this end we write the equation

$$(\Delta + k^2)u = q_0 u$$

as

$$(\Delta + 2i\zeta \cdot \nabla)e^{-i\zeta\cdot x}u = e^{-i\zeta\cdot x}q_0 u,$$

where $\zeta \in \mathbf{C}^n$ is such that $\zeta \cdot \zeta = k^2$. Since u is supposed to have a compact support we have $v := e^{-i\zeta\cdot x}u \in W^{\epsilon,2p}_{\text{comp}}(\mathbb{R}^2)$. For v we obtain the equation

$$v = \mathcal{G}_\zeta(q_0 v) \tag{18}$$

where the Faddeev operator $\mathcal{G}_\zeta$ is defined as the Fourier multiplier

$$\mathcal{G}_\zeta(f)(x) = \mathcal{F}^{-1}(\frac{-1}{\xi^2 + 2\zeta\cdot\xi}\hat{f})(x).$$

It is well known (see for example the proof of Theorem 4.1 in [37]) that for $0 \le s \le \frac{1}{2}$

$$||\mathcal{G}_\zeta||_{H_0^{-s}(D)\to H^{+s}(D)} \le \frac{c}{|\zeta|^{1-2s}} \tag{19}$$

where $H^s(D) = W^{s,2}(D)$ and $H^s_0(D) = W^{s,2}_0(D)$ are L^2-based Sobolev spaces. By [21],

$$G_\zeta : L^r(D) \to L^{r'}(D), \tag{20}$$

for $r = \frac{2n}{n+2}$ if $n \ge 3$ and for $r > 1$ for $n = 2$. We continue first in the case $n \ge 3$. Interpolation of (19) and (20 yields

$$||G_\zeta||_{W_0^{-\epsilon,\tilde{p}}(D)\to W^{\epsilon,2p}(D)} \le c|\zeta|^{-(1-2s)\theta} \tag{21}$$

where $\epsilon = \theta s$ and $\theta = 1 - \frac{n}{2p'}$. Finally, (14),(18), and (21) show that

$$||v||_{W^{\epsilon,2p}(D)} \leq \frac{c}{|\zeta|^{(1-2s)\theta}} ||v||_{W^{\epsilon,2p}(D)} \tag{22}$$

Now if $p' > n/2$ and $0 < \epsilon < 1/2 - \frac{n}{4p'} = \frac{1}{2}$ we conclude that v and hence u must vanish identically. Finally in the case $n = 2$ we interpolate (19) and (20) for $r > 1$ and by letting $r \to 1$ the same conclusion follows. $\square$

3.2 Existence and uniqueness for solutions of the scattering problem

Theorem 3.3 *For $q_0 \in W^{-\epsilon,p'}_{comp}(\mathbb{R}^n)$, with $n \geq 2$, $p' \in (n/2, \infty)$, and $0 < \epsilon < \frac{n}{4}(\frac{2}{n} - \frac{1}{p'})$, the Lippmann-Schwinger-equation* (13) *has a unique solution $u \in W^{\epsilon,2p}_{loc}(\mathbb{R}^n)$.*

Proof: Let D be a bounded domain such that $\operatorname{supp}(q_0) \subset D$. Consider the equation (13) in $W^{\epsilon,2p}(D)$. Since the operator H_k,

$$H_k f = \Phi_k * f \tag{23}$$

defines a bounded operator $H_k : H_0^{-s}(D) \to H^s(D)$ for $s \leq 1$ we see from Sobolev embedding and Rellich's compact embedding theorem that

$$H_k : W_0^{-\epsilon,\tilde{p}}(D) \to W^{\epsilon,2p}(D)$$

compactly. This and Lemma 3.2 give that the operator $K : W_0^{\epsilon,2p}(D) \to W^{\epsilon,2p}(D)$, $Kf = H_k q_0 f$ is compact, as well.

Thus by Fredholm's alternative it is enough to show that in $W^{\epsilon,2p}(D)$ the homogeneous equation

$$u = H_k q_0 u \tag{24}$$

has only the trivial solution $u = 0$. If $u \in W^{\epsilon,2p}(D)$ satisfies (24) then u belongs to Schwartz class $\mathcal{S}'$ and by taking the Fourier transform we obtain that in the sense of distributions

$$(\Delta + k^2)u(x) = q_0 u \tag{25}$$

In particular u must be smooth in $\mathbb{R}^n \setminus \overline{D}$ and satisfy $(\Delta + k^2)u = 0$ there. Note that by (24) the values of u in D define u all over in $\mathbb{R}^n$.

By Rellich's lemma (cf. [9]) and unique continuation principle it is enough to show that the far field u_∞ of u, defined by

$$u(x) = \frac{e^{ik|x|}}{4\pi |x|^{(n-1)/2}} u_\infty \left(\frac{x}{|x|} \right) + o\left(|x|^{-(n-1)/2} \right)$$

as $|x| \to \infty$, vanishes for u.

Take $r > 0$ so large that $\overline{D} \subset B(0,r)$. By Green's formula for any $\phi \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{|x|=r} \phi \frac{\partial}{\partial \nu} \overline{u}\, ds = \int_{|x|\le r} (\nabla \phi \cdot \nabla \overline{u} + \overline{u} \Delta \phi)\, dx. \tag{26}$$

Note that

$$\Delta u = (q_0 - k^2) u \in W^{-\epsilon, \tilde{p}}(\mathbb{R}^2) + W^{\epsilon, 2p}_{loc}(\mathbb{R}^n). \tag{27}$$

This implies that $\nabla u \in L^2_{loc}$ and that u and Δu belong locally to spaces that are dual to each other. Thus by approximating u by smooth functions we get from (26) that

$$\begin{aligned} \operatorname{Im} \int_{|x|=r} u \frac{\partial}{\partial \nu} \overline{u}\, ds &= \operatorname{Im} \int_{|x|\le r} \left(|\nabla u|^2 + \overline{u} \Delta u \right) dx \\ &= \operatorname{Im} \int_{|x|\le r} \left(|\nabla u|^2 + (q_0 - k^2)|u|^2 \right) dx = 0. \end{aligned} \tag{28}$$

Thus

$$\int_{|x|=r} \left(\left| \frac{\partial}{\partial \nu} u \right|^2 + k^2 |u|^2 \right) ds = \int_{|x|=r} \left| \frac{\partial}{\partial \nu} u - iku \right|^2 ds \to 0$$

as $r \to \infty$. Especially,

$$\int_{|x|=r} |u|^2\, ds = \int_{S^{n-1}} |u_\infty(\theta)|^2\, ds(\theta) + o(r) \to 0$$

as $r \to \infty$. This is possible only if $u_\infty \equiv 0$. Thus the assertion is proven. □

Theorem 3.4 *For $q_0 \in W^{-\epsilon, p'}_{comp}(\mathbb{R}^n)$, with $n \geq 2$, $p' \in (n/2, \infty)$, and $0 < \epsilon < \frac{n}{4}(\frac{2}{n} - \frac{1}{p'})$, the scattering problem (12) is equivalent to the Lippmann-Schwinger-equation and has thus a unique solution $u \in W^{\epsilon, 2p}_{loc}(\mathbb{R}^n)$.*

Proof: As reasoned in the proof of the previous theorem a solution of Lippmann-Schwinger-equation satisfies (12). Suppose $u \in W^{\epsilon,2p}_{loc}(\mathbb{R}^n) \bigcap \mathcal{S}'$ is a solution of (12). We need to show that

$$u_s(x) = \int \Phi_k(x-y) q_0(y) u(y)\, dy. \tag{29}$$

Since $(\Delta + k^2)u_s = q_0 u \in W^{-\epsilon,\tilde{p}}_{\text{comp}}(\mathbb{R}^n)$ and $\Phi_k(x - \cdot) \in W^{\epsilon,2p}_{\text{loc}}(\mathbb{R}^n)$ and both functions are real-analytic outside a large ball we have from (12) in the sense of distributions that

$$\int_{|y|\le r} \Phi_k(x-y)(\Delta + k^2) u_s(y)\, dy = H_k(q_0 u). \tag{30}$$

Denote the operator that operates to u_s in the left hand side of (30) by T. Now for $\phi \in C^\infty(\mathbb{R}^n)$,

$$T\phi = \phi + \int_{|y|=r} \Phi_k(\cdot - y)\frac{\partial}{\partial r(y)}\phi(y)\, ds(y) - \int_{|y|=r} \frac{\partial}{\partial r(y)}\Phi_k(\cdot - y)\phi(y)\, ds(y).$$

Thus

$$\begin{aligned} u_s(x) + \int_{|y|=r} \Phi_k(x-y)\frac{\partial}{\partial r(y)}u_s(y)\, ds(y) - \\ - \int_{|y|=r} \frac{\partial}{\partial r(y)}\Phi_k(x-y) u_s(y)\, ds(y) = H_k(q_0 u). \end{aligned} \tag{31}$$

From the radiation condition it follows that the difference of the boundary integrals in (31) approaches to zero as $r \to \infty$. This proves (29) and hence the theorem $\square$

3.3 The Born series.

Next we return to case $n = 2$. Note that in this case the conditions $p' > n/2$ and $0 < \epsilon < \frac{1}{2} - \frac{n}{4p'}$ take the form

$$1 < p < \infty, \quad 0 < \epsilon < \frac{1}{2p}.$$

We are back to our basic situation (Assumption 1.3) and consider again random q for which the results of Section 2 may be specified to yield that a.s.

$q \in W^{-s,p'}(\mathbb{R}^2)$ for some $1 < p < \infty, 0 < s < \frac{1}{2p}$. Thus results of preceding subsection take place a.s.

By iterating the Lippmann-Schwinger equation, we can formally represent u as the Born series,

$$u(x,y,k) = u_0(x,y,k) + u_1(x,y,k) + u_2(x,y,k) + \dots \tag{32}$$

where

$$u_0(x,y,k) = \Phi_k(x-y), \tag{33}$$
$$u_{n+1} = (\Delta + k^2 + i0)^{-1}(qu_n). \tag{34}$$

For future purposes we show that almost surely the series (32) converges uniformly in the measurement domain U. Recall that $\mathrm{supp}(q) \subset D$, where $D \subset \mathbb{R}^2$ is bounded, and that $U \subset \mathbb{R}^2$ is a bounded domain with positive distance to D.

Theorem 3.5 *For any $\epsilon > 0$ there exist $C = C(\epsilon)$ such that*

$$\sum_{n=3}^{\infty} \sup_{x,y \in U} |u_n(x,y,k)| \le Ck^{-5/2+\epsilon}$$

holds with probability one.

The main difficulties are collected in

Lemma 3.6 *For any deterministic potential $q_0 \in W_0^{-s,p'}(D)$, $p > 1$, the $n^{\underline{th}}$ order term u_n in the Born series satisfies*

$$\sup_{x,y \in U} |u_n(x,y,k)| \le C^n k^{1/2-n+\epsilon_n}$$

where

$$\epsilon_n = s - 1 + 1/p + 2n(s + (1 - 1/p)).$$

Before we prove Lemma 3.6 we show how Theorem 3.5 follows from it. The strategy is to take $s > 0$ small and p close to one to keep ϵ_n small in Lemma 3.6.

Proof of Theorem 3.5. Since with probability one $q \in W_0^{-s,p'}(D)$, for any $s > 0$ and $1 < p < \infty$, we denote $s-1+\frac{1}{p} = \epsilon_1$ and $2(s+2(1-1/p)) = \epsilon_2$ and we can take $\epsilon_1 > 0$ and $\epsilon_2 > 0$ arbitrarily small. Thus by Lemma 3.6

$$||u_n(\cdot,\cdot,k)||_{L^\infty(U\times U)} \leq C^n k^{1/2+\epsilon_1-n(1-\epsilon_2)}$$

and consequently

$$\sum_{n=3}^{\infty} ||u_n(\cdot,\cdot,k)||_{L^\infty(U\times U)} \leq C^3 k^{-5/2+(\epsilon_1+3\epsilon_2)} \frac{1}{1-Ck^{\epsilon_2-1}}$$

proving the claim. □

For the proof of Lemma 3.6 we need two estimates for the convolution with the Hankel function. Define the operator H_k by

$$H_k f(x) = \int \Phi_k(x-y) f(y)\, dy$$

Then for $0 < s < 1$ and $1 \leq p \leq 2 \leq r$ we have

$$H_k : W_0^{-s,p}(D) \to W^{s,r}(D) \oplus L^\infty(U)$$

with norm estimates

$$||H_k||_{W_0^{-s,p}(D)\to W^{s,r}(D)} \leq Ck^{-1+2(s+(1/p-1/r))} \tag{35}$$

and

$$||H_k||_{W_0^{-s,p}(D)\to L^\infty(U)} \leq Ck^{-1+s+2/p} \tag{36}$$

These estimates follow from the proof of Theorem 3.1 in [37].

Finally, we consider the operator K that combines the multiplication operator with q to H_k i.e. $K_k f = H_k(qf)$. By (35) and Lemma 3.2 in the previous chapter we obtain for $p > 1$ that $K_k : W^{s,2p}(D) \to W^{s,2p}(D)$ and $K_k : W^{s,2p}(D) \to L^\infty(U)$ with

$$||K||_{W^{s,2p}\to W^{s,2p}} \leq C|k|^{-1+2(s+(1-1/p))}, \tag{37}$$

$$||K||_{W^{s,2p}\to L^\infty} \leq C|k|^{1+2s-1/p}. \tag{38}$$

Proof of Lemma 3.6. Assume $y \in U$ and define the function f_y by

$$f_y(z) = \Phi_k(z-y)$$

Recall that $\operatorname{supp}(q) \subset D$ and $\operatorname{dist}(U, D) =: d > 0$. We make use of the estimates for $1 \le p < \infty$

$$\sup_{y\in U} ||f_y||_{L^p(D)} \le Ck^{-1/2}, \tag{39}$$

$$\sup_{y\in U} ||\Delta f_y||_{L^p(D)} \le Ck^{1/2}. \tag{40}$$

These estimates interpolate for $0 \le s \le 1$ to

$$||f_y||_{W^{s,p}} \le Ck^{-1/2+s}. \tag{41}$$

To see that (39) holds recall that

$$|H_0^{(1)}(t)| \le \frac{C}{\sqrt{t}}, \quad |\frac{d}{dt}H_0^{(1)}(t)| \le \frac{C}{\sqrt{t}}. \tag{42}$$

Then by denoting $R = \sup\{|y-z| : \ y \in U, z \in D\}$ we have

$$\begin{aligned} \int_D |f_y(z)|^p dz \quad &\le \int_{d\le|u|\le R} \Phi_k(u)^p du, \\ &\le C_p k^{-p/2}(R^{1-p/2} - d^{1-p/2}) \ = \ Ck^{-p/2} \end{aligned}$$

where C is independent of y. Similarly we obtain (40) from (42). Finally, since

$$u_n(x, y, k) = (K^n f_y)(x)$$

we get from (37), (38) and (41) that

$$\begin{aligned} ||u_n(\cdot, y, k)||_{L^\infty(U)} &\le ||K||_{W^{s,2p}\to L^\infty} ||K||^{n-1}_{W^{s,2p}\to W^{s,2p}} ||f_y||_{W^{s,2p}} \\ &\le C^n k^{1+2s-1/p} k^{(n-1)(-1+2(s+(1-1/p)))} k^{-1/2+s}. \end{aligned}$$

Here the constant C is independent of y and thus the desired estimate follows. □

4 Microlocal analysis and the asymptotic independence of the first term

In the proof of our main result we need to establish asymptotic independence for the first terms in the Born series, corresponding to different values of k.

The verification of this fact and leads to estimation of certain oscillatory integrals, and needs a fairly involved computation. As a useful tool we apply the calculus of conormal distributions, and we compute the leading order of the (self-)covariance of the first term in Born series. The results of this section will be applied in later sections 6 and 7 below.

As the first term in Born series is

$$u_1(x,y,k) = \int_D \Phi_k(x-z)q(z)\Phi_k(z-y)\,dz,$$

we start with the asymptotics of $\Phi_k(z) = \frac{i}{4}H_0^{(1)}(k|z|)$, when $k \to \infty$. It is given by

$$\Phi_k(z) = \sqrt{\frac{1}{k|z|}}e^{i(k|z|-\pi/4)}F(\frac{1}{k|z|}), \quad F(t) = \sum_{j=0}^{\infty} f_j t^j, \quad t>0, \tag{43}$$

where $f_0 = \frac{i}{\sqrt{8\pi}}$ and f_j are constants which the actual values are not important for us in the sequel. The series (43) and its derivative have the property that for $N > 1$ (c.f. [1, formulae 9.1.27, 9.2.7–9.2.10])

$$|F(t) - \sum_{j=0}^{N} f_j t^j| \leq 2|f_{N+1}|t^{N+1} + 2|f_{N+2}|t^{N+2}, \quad t>0, \tag{44}$$

$$|\frac{d}{dt}(F(t) - \sum_{j=0}^{N} f_j t^j)| \leq 2(N+1)|f_{N+1}|t^N + 2(N+2)|f_{N+2}|t^{N+1}, \quad t>0.$$

Using first m terms in the asymptotics of Φ_k, we write

$$u_1(x,y,k) = u_1^m(x,y,k) + b^m(x,y,k) \tag{45}$$

where

$$u_1^m(x,y,k) = \int_D \Phi_k^m(x-z)q(z)\Phi_k^m(z-y)\,dz,$$

$$\Phi_k^m(z) = (k|z|)^{-\frac{1}{2}}e^{i(k|z|-\pi/4)}\sum_{j=0}^{m} f_j\,(k|z|)^{-j}.$$

Here, $|k|>1$, and for $k<0$ we define the square root by $k^{-\frac{1}{2}} = -i|k|^{-\frac{1}{2}}$.

We remark that the case $m = 2$ suffices for our purposes.

Next we denote by $\mathcal{O}((1+|k_1-k_2|)^{-n_1}(1+|k_1+k_2|)^{-n_2})$ functions $h(x,y,k_1,k_2)$ which satisfy an estimate of the form

$$|h(x,y,k_1,k_2)| \le C(1+|k_1-k_2|)^{-n_1}(1+|k_1+k_2|)^{-n_2}$$

for $x, y \in U$ and $k_1, k_2 \in \mathbb{R}$, $|k_1|, |k_2| > 1$ where C is independent of x, y, k_1, and k_2. Next we prove the asymptotic expansion for the covariance of u_1^m proving that the fields u_1^m with different frequencies are asymptotically independent. We emphasize that formula (48) below is crucial for construction of $\mu(z)$ in Section 7.

Proposition 4.1 *Let $m > 0$. In the decomposition (45) $u_1^m(x,y,k)$ satisfies*

$$|\mathbb{E}\,(u_1^m(x,y,k_1)\overline{u_1^m(x,y,k_2)})| \le C_n(1+|k_1|+|k_2|)^{-4}(1+|k_1-k_2|)^{-n}, \quad n>0 \tag{46}$$

for any $n \ge 1$, $|k_1|, |k_2| > 1$, and $x, y \in U$. Moreover, it satisfies the asymptotics

$$\begin{aligned}&\mathbb{E}\,(u_1^m(x,y,k_1)\overline{u_1^m(x,y,k_2)}) - R_m(x,y,k_1,k_2)\frac{1}{1+(k_1+k_2)^4} = \\ &= \mathcal{O}((1+|k_1-k_2|)^{-n}(1+|k_1+k_2|)^{-5}), \qquad n>0.\end{aligned} \tag{47}$$

Here $R_m(x,y,k_1,k_2)$ is a bounded smooth function with

$$R_m(x,x,k,k) = \frac{1}{2}\int_{\mathbb{R}^2}\frac{\mu(z)}{|z-x|^2}\,dz + \mathcal{O}((1+|k|)^{-1}). \tag{48}$$

Proof. We see that

$$\mathbb{E}\,(u_1^m(x,y,k_1)\overline{u_1^m(x,y,k_2)}) = \sum_{j_1,j_2,l_1,l_2=0}^{m} I_{j_1,j_2,l_1,l_2}(k_1,k_2,x,y) \tag{49}$$

where

$$\begin{aligned}&I_{j_1,j_2,l_1,l_2}(k_1,k_2,x,y) = \\ &= \frac{f_{j_1}f_{j_2}\overline{f}_{l_1}\overline{f}_{l_2}}{k_1^{1+j_1+j_2}k_2^{1+l_1+l_2}}\int_{\mathbb{R}^2} e^{ik_1(|x-z_1|+|z_1-y|)-ik_2(|x-z_2|+|z_2-y|)}\cdot \\ &\quad\cdot\frac{\mathbb{E}\,(q(z_1)q(z_2))}{|x-z_1|^{j_1+\frac12}|z_1-y|^{j_2+\frac12}|x-z_2|^{l_1+\frac12}|z_2-y|^{l_2+\frac12}}\,dz_1dz_2.\end{aligned} \tag{50}$$

Observe that the correctness of the above computation follows directly from the definition (2), by noting that $\mathbb{E}\left(q(z_1)q(z_2)\right)$ refers to the distribution kernel of the covariance operator of the field q, and the integral refers to the appropriate pairing. In our case, however, according to Proposition 2.2 the covariance operator has a regular kernel $C(z_1, z_2)$. Thus we may replace $\mathbb{E}\left(q(z_1)q(z_2)\right)$ by $C(z_1, z_2)$ in the expression (50) and do standard integration there.

Assumption 1.3 yields that $C(z_1, z_2)$ is the Schwartz kernel of is a pseudodifferential operator C with a classical symbol $c(x, \xi) \in S^{-2}_{1,0}(\mathbb{R}^2 \times \mathbb{R}^2)$, the principal symbol of C is $c^p(z, \xi) = \mu(z)(1+|\xi|^2)^{-1}$, and $C(z_1, z_2)$ is supported in $D \times D$. Whence we may write (c.f. [18])

$$C(z_1, z_2) = \int_{\mathbb{R}^2} e^{i(z_1 - z_2)\cdot\xi} c(z_1, \xi)\, d\xi. \tag{51}$$

Note that all symbols appearing below will be classical symbols [18].

In order to obtain uniform estimates with respect to variables x and y we introduce them as variables in the covariance in the following way. We define the function

$$C_1(z_1, z_2, x, y) = C(z_1, z_2)\theta(x)\theta(y) \tag{52}$$

where $\theta \in C_0^\infty(\mathbb{R}^2)$ equals to one in the domain U and has its support outside $\overline{D}$.

The formula (51) leads to

$$C_1(z_1, z_2, x, y) = \int_{\mathbb{R}^2} e^{i(z_1 - z_2)\cdot\xi} c_1(z_1, x, y, \xi)\, d\xi \tag{53}$$

where the symbol $c_1(z_1, x, y, \xi) \in S^{-2}_{1,0}(\mathbb{R}^6 \times \mathbb{R}^2)$ has the principal symbol

$$c_1^p(z_1, x, y, \xi) = \mu(z_1)\frac{1}{1 + |\xi|^2}\theta(x)\theta(y).$$

Observe that formally $c_1 \in S^{-2}_{1,0}((D \times \mathbb{R}^4) \times \mathbb{R}^2)$, but we consider it extended by zero to values $z_1 \notin D$. By definition, (53) means that $C_1(z_1, z_2, x, y)$ is a conormal distribution in $\mathbb{R}^8$ of Hörmander type having conormal singularity on the surface $S_1 = \{(z_1, z_2, x, y) \in \mathbb{R}^8 : \ z_1 - z_2 = 0\}$. Using notations of [18], when $X \subset \mathbb{R}^n$ and $S \subset X$ is a smooth submanifold of X, we denote by $I(X; S)$ the distributions in $\mathcal{D}'(X)$ that are smooth in $X \setminus S$ and has a conormal singularity near S. The set of distributions in $I(X;S)$ supported in a compact subset of X is denoted by $I_{comp}(X;S)$. Let $\mathbf{D} \subset \mathbb{R}^8$ be an open set containing $D \times D \times \text{supp }(\theta) \times \text{supp }(\theta)$ so that $C_1 \in I_{comp}(\mathbf{D}; S_1 \cap \mathbf{D})$.

We use the fact that conormal distributions are invariant in change of coordinates. Actually, we plan to consider several different coordinates.

The first set of coordinates that we consider are (V, W, x, y), defined as $V = z_1 - z_2$ and $W = z_1 + z_2$. Denote by η the change of coordinates $\eta : (V, W, x, y) \mapsto (z_1, z_2, x, y)$. We consider the pull-back $C_2 = \eta^*(C_1)$. Then the direct substitution shows that

$$\begin{aligned} C_2(V, W, x, y) &= \int_{\mathbb{R}^2} e^{iV\cdot\xi} c_2(V, W, x, y, \xi)\, d\xi, \\ c_2(V, W, x, y, \xi) &= c_1(z_1(V, W, x, y), x, y, \xi) \end{aligned} \tag{54}$$

which means that $C_2 \in I(\mathbb{R}^8; S_2)$ where $S_2 = \{(V, W, x, y) :\ V = 0\}$.

To find out how the symbol transforms in the change of coordinates, we have to represent $C_2(V, W, x, y)$ with a symbol that does not depend on V. Because of the special form of the surface $S_2 = \{V = 0\}$ we can use the representation theorem for conormal distribution [18, Lemma 18.2.1], and represent $C_2(V, W, x, y)$ with a symbol that is independent of V:

$$C_2(V, W, x, y) = \int_{\mathbb{R}^2} e^{iV\cdot\xi} c_3(W, x, y, \xi)\, d\xi, \tag{55}$$

where

$$c_3(W, x, y, \xi) \sim \sum_{l=0}^{\infty} \langle -iD_V, D_\xi \rangle^l c_2(V, W, x, y, \xi)|_{V=0} \in S^{-2}_{1,0}(\mathbb{R}^6 \times \mathbb{R}^2).$$

In particular, we see that $c_3(W, x, y, \xi)$ has the principal symbol

$$c_3^p(W, x, y, \xi) = \mu(z_1(V, W, x, y))(1 + |\xi|^2)^{-1}\theta(x)\theta(y)\big|_{V=0}. \tag{56}$$

The second set of coordinates that we consider are (v, w, x, y) defined below. For this, consider the phase of the oscillatory integrals (50) and denote

$$\phi(z, x, y) = |x - z| + |z - y|. \tag{57}$$

The idea is change the coordinates so that $\phi(z_1, x, y) - \phi(z_2, x, y)$ will be a coordinate. We will do this change of coordinates in two steps. First we change the coordinates (z_1, z_2, x, y) to (Z_1, Z_2, x, y), where $Z_j = Z_j(x, y, z_j) \in \mathbb{R}^2$, $j = 1, 2$ are related to ellipses having focal points in x and y. More precisely, we write

$$Z_j = (t_j, s_j) \in \mathbb{R}^2, \tag{58}$$
$$t_j = \frac{1}{2}\phi(z_j, x, y), \quad s_j = \frac{1}{2}\phi(z_j, x, y) \cdot \arcsin(e_1 \cdot \frac{\nabla_{z_j}\phi(z_j, x, y)}{||\nabla_{z_j}\phi(z_j, x, y)||}),$$

where $e_1 = (1, 0)$. In other words, here t_j correspond to the semi-major axis of the ellipse having focal points x and y and containing the point z_j. The variable s_j specifies the angle of the normal vector of the ellipse with the x-axis at the point z_j. Since domain U is convex and D is simply connected, our definition of the new coordinates is well-posed in a neighborhood of the domain D.

Second we change from (Z_1, Z_2, x, y), to coordinates (v, w, x, y) where $v = Z_1 - Z_2$, $w = Z_1 + Z_2$. Together, above steps define the coordinates (v, w, x, y) and the map $\tau : (v, w, x, y) \mapsto (z_1, z_2, x, y)$. Note that the first component of $v(z_1, z_2, x, y)$ equals to $\phi(z_1, x, y) - \phi(z_2, x, y)$.

To simplify the notations, we denote $X_1 = \mathbf{D}$, $X_2 = \eta^{-1}(\mathbf{D})$ and $X_3 = \tau^{-1}(\mathbf{D})$ so that $\tau : X_3 \to X_1$ and $\eta : X_2 \to X_1$. We are ready to represent the conormal distribution $C_1(z_1, z_2, x, y)$ in coordinates (v, w, x, y) as the pull-back distribution $C_4 = \tau^*(C_1) \in I(X_3; S_3 \cap X_3)$, $S_3 = \{(v, w, x, y) : \ v = 0\}$. By the invariance of conormal distributions under the change of variables we may write

$$\begin{array}{ccc} & I_{comp}(X_1; S_1 \cap X_1) & \\ \swarrow^{\eta^*} & & \searrow^{\tau^*} \\ I_{comp}(X_2; S_2 \cap X_2) & & I_{comp}(X_3; S_3 \cap X_3) \end{array}$$

To apply this diagram and the integral representation (55) of $C_2 \in I_{comp}(X_2; S_2 \cap X_2)$, consider the transformation $\kappa = \eta^{-1} \circ \tau$. We will below use [18, Theorem 18.2.9.], to provide a representation for pull-back $C_4 = \kappa^* C_2$. Since surfaces S_2 and S_3 have the special form $S_2 = \{V = 0\}$ and $S_3 = \{v = 0\}$, and κ maps $S_3 \cap X_3$ onto $S_2 \cap X_2$, we obtain

$$C_4(v, w, x, y) = \int_{\mathbb{R}^2} e^{iv \cdot \xi} c_4(w, x, y, \xi)\, d\xi, \tag{59}$$

where $c_4(w,x,y,\xi) \in S^{-2}_{1,0}(\mathbb{R}^6 \times \mathbb{R}^2)$ is a symbol satisfying

$$c_4(w,x,y,\xi) = \tag{60}$$
$$c_3(\kappa_2(v,w,x,y),((\kappa'_{11}(v,w,x,y))^{-1})^t\xi)|\det \kappa'_{11}(v,w,x,y)|^{-1}\big|_{v=0} + r(w,x,y,\xi).$$

Here, $r(w,x,y,\xi) \in S^{-3}_{1,0}(\mathbb{R}^6 \times \mathbb{R}^2)$ and the coordinate transform κ is decomposed to two parts, the $\mathbb{R}^2$-valued function $\kappa_1(v,w,x,y) = V(v,w,x,y)$ and the $\mathbb{R}^6$-valued function $\kappa_2(v,w,x,y) = (W(v,w,x,y),x,y)$. This yields for the differential κ' of κ the corresponding representation

$$\kappa' = \begin{pmatrix} \kappa'_{11} & \kappa'_{12} \\ \kappa'_{21} & \kappa'_{22} \end{pmatrix}.$$

We note that the transformation rule in κ^* in [18, Theorem 18.2.9] is presented for half-densities. However, formula (60) that is the transformation rule for distributions is obtained directly from the proof of [18, Theorem 18.2.9.]

Plugging the principal symbol of $c_3(x,\xi)$ given in (56) to formula (60), we see that the principal symbol of $c_4(w,x,y,\xi)$ is

$$\begin{aligned} c_4^p(w,x,y,\xi) &= \mu(z_1(v,w,x,y))(1+|((\kappa'_{11}(v,w,x,y))^{-1})^t\xi|^2)^{-1}\big|_{v=0} \\ &\quad \cdot \theta(x)\theta(y)J(w,x,y) \end{aligned}$$

where $J(w,x,y) = |\det \kappa'_{11}(0,w,x,y)|^{-1}$. Note that for $y = x$ the matrix $\kappa'_{11}(0,w,x,x)$ is unitary and its determinant $J(w,x,x)$ is equal to one.

We are ready to compute the asymptotics of $I_{j_1,j_2,l_1,l_2}(k_1,k_2,x,y)$. We denote $\vec{j} = (j_1,j_2,l_1,l_2)$. By writing the integral $I_{\vec{j}}$ in coordinates (v,w,x,y) we obtain

$$\begin{aligned} &I_{j_1,j_2,l_1,l_2}(k_1,k_2,x,y) = \\ &= k_1^{-(1+j_1+j_2)}k_2^{-(1+l_1+l_2)}\int_{\mathbb{R}^4} \exp(i\big((k_1+k_2)e_1\cdot v + (k_1-k_2)e_1\cdot w\big))\cdot \\ &\qquad \cdot C_4(v,w,x,y)H^{\vec{j}}(v,w,x,y)\,dvdw \end{aligned} \tag{61}$$

where $e_1 = (1,0)$ is the unit vector and

$$\begin{aligned} &H^{\vec{j}}(v,w,x,y) = \\ &= \frac{f_{j_1}f_{j_2}\overline{f}_{l_1}\overline{f}_{l_2}}{|x-z_1|^{j_1+\frac12}|z_1-y|^{j_2+\frac12}|x-z_2|^{l_1+\frac12}|z_2-y|^{l_2+\frac12}}\det(\tau'(v,w,x,y)) \end{aligned}$$

where $z_1 = z_1(v,w,x,y)$ and $z_2 = z_2(v,w,x,y)$.

Since $H^{\vec{j}}$ is smooth in X_3 in all variables and class $I(\mathbb{R}^8; S_3)$ is closed in multiplication with a smooth function, we have $C_4(v,w,x,y)\, H^{\vec{j}}(v,w,x,y) \in I(\mathbb{R}^8; S_3)$. To evaluate the oscillatory integrals (61) in a convenient way, we need to represent this conormal distribution with a symbol that does not depend on v. Again, by using the representation theorem of conormal distribution [18, Lemma 18.2.1], we obtain

$$C_4(v,w,x,y)\, H^{\vec{j}}(v,w,x,y) = \int_{\mathbb{R}^2} e^{iv\cdot\xi} c_5^{\vec{j}}(w,x,y,\xi)\, d\xi \tag{62}$$

where

$$c_5^{\vec{j}}(w,x,y,\xi) \sim \sum_{l=0}^{\infty} \langle -iD_v, D_\xi\rangle^l (c_4(w,x,y,\xi) H^{\vec{j}}(v,w,x,y))|_{v=0} \in S^{-2}_{1,0}(\mathbb{R}^6 \times \mathbb{R}^2).$$

In particular, we see that $c_5^{\vec{j}}(w,x,y,\xi)$ has the principal symbol

$$\begin{aligned} c_5^{\vec{j}p}(w,x,y,\xi) = \mu(z_1(v,w,x,y))(1 + |((\kappa_{11}'(v,w,x,y))^{-1})^t\xi|^2)^{-1}\cdot \\ \cdot\, \theta(x)\theta(y)\, J(w,x,y) H^{\vec{j}}(v,w,x,y)\Big|_{v=0}. \end{aligned} \tag{63}$$

Plugging (62) to (61) and using Fourier inversion formula we obtain

$$\begin{aligned} &I_{\vec{j}}(k_1,k_2,x,y) = \\ &(2\pi)^2 k_1^{-(1+j_1+j_2)} k_2^{-(1+l_1+l_2)} (\mathcal{F}_w c_5^{\vec{j}})((k_2-k_1)e_1, x, y, -(k_1+k_2)e_1) \end{aligned} \tag{64}$$

where $\mathcal{F}_w$ denotes the Fourier transform in w-variable,

$$\mathcal{F}_w c_5^{\vec{j}}(\eta,x,y,\xi) = \int_{\mathbb{R}^2} e^{-i\eta\cdot w} c_5^{\vec{j}}(w,x,y,\xi)\, dw$$

As the symbol $c_5^{\vec{j}}(w,x,y,\xi)$ is C^∞ smooth and compactly supported in (x,y,w)-variables, we see that

$$|D_w^\alpha c_5^{\vec{j}}(w,x,y,\xi)| \le C_\alpha (1+|\xi|)^{-2}$$

for all $|\alpha| \ge 0$, where C_α is independent of $(w,x,y) \in \mathbb{R}^6$. This implies after n integrations by parts

$$|I_{\vec{j}}(k_1,k_2,x,y)| \le C_n \frac{1}{1+|k_1+k_2|^2} k_1^{-j_1-j_2-1} k_2^{-\ell_1-\ell_2-1} (1+|k_1-k_2|)^{-n}$$

for all $n \geq 0$. By considering separately the cases $|k_1 - k_2| \leq |k_1 + k_2|/2$ and $|k_1 - k_2| \geq |k_1 + k_2|/2$ one deduces the estimate

$$|I_{\vec{j}}(k_1, k_2, x, y)| \leq C'_n(1 + |k_1 - k_2|)^{-n}(1 + |k_1 + k_2|)^{-4-j_1-j_2-l_1-l_2}, \quad n > 0. \tag{65}$$

This proves the estimate (46).

We denote $\mathbf{0} = (0, 0, 0, 0)$. Since for $\vec{j} \neq \mathbf{0}$

$$I_{\vec{j}}(k_1, k_2, x, y) = \mathcal{O}((1 + |k_1 + k_2|)^{-5}(1 + |k_1 - k_2|)^{-n}).$$

Thus, in order to establish (47) it is enough to consider $I_{\mathbf{0}}(k_1, k_2, x, y)$. To obtain the leading order asymptotics of $I_{\mathbf{0}}$, we consider the contributions of the principal symbol and the lower order remainder terms separately. Write

$$c_5^{\mathbf{0}}(w, x, y, \xi) = c_5^{\mathbf{0}p}(w, x, y, \xi) + c_r(w, x, y, \xi),$$

where $c_r(w, x, y, \xi) \in S^{-3}_{1,0}(\mathbb{R}^6 \times \mathbb{R}^2)$ is smooth and compactly supported in (w, x, y)-variables. Thus $|D^\alpha_w c_r(w, x, y, \xi)| \leq C_\alpha(1+|\xi|)^{-3}$ for all multi-indices α and we infer as above that

$$\begin{aligned} &|(\mathcal{F}_w c_r)((k_2 - k_1)e_1, x, y, -(k_1 + k_2)e_1)| \\ &\quad = \mathcal{O}((1 + |k_1 + k_2|)^{-3}(1 + |k_1 - k_2|)^{-n}), \quad n > 0. \end{aligned}$$

Thus the contribution of c_r to $I_{\mathbf{0}}$ is estimated by the right hand side of (47). Hence we need only to consider the principal part. To this end, we substitute the principal symbol (63) to formula (64) to get

$$\begin{aligned} &I_{\mathbf{0}}(k_1, k_2, x, y) = \qquad (66)\\ = \;& 4\pi^2 k_1^{-1} k_2^{-1} \theta(x)\theta(y) \cdot \\ &\cdot \mathcal{F}_w \left(\frac{\mu(z_1(0, w, x, y)) H^{\mathbf{0}}(0, w, x, y) J(w, x, y)}{1 + |k_1 + k_2|^2 |((\kappa'_{11}(0, w, x, y))^{-1})^t e_1|^2} \right) ((k_2 - k_1)e_1) + \\ &+ \mathcal{O}((1 + |k_1 + k_2|)^{-5}(1 + |k_1 - k_2|)^{-n}), \quad n > 0. \end{aligned}$$

Since for $a = |((\kappa'_{11}(0, w, x, y))^{-1})^t e_1|^2$ it holds for large $k_1 + k_2$ that

$$\frac{1}{1 + (k_1 + k_2)^2 a} = \frac{1}{(k_1 + k_2)^2} \sum_{j=0}^{\infty} (k_1 + k_2)^{-2j} (-a)^{-j},$$

we get after integrating by parts n times in w-variable in (66) the needed formula (47) with

$$R_m(x,y,k_1,k_2) = 4\pi^2 k_1^{-1}k_2^{-1}(1+|k_1+k_2|^2)\cdot \qquad (67)$$
$$\cdot \mathcal{F}_w\left(\frac{\mu(z_1(0,w,x,y))H^{\mathbf{0}}(0,w,x,y)J(w,x,y)}{|((\kappa'_{11}(0,w,x,y))^{-1})^t e_1|^2}\right)((k_2-k_1)e_1)$$

by considering separately the cases $|k_1-k_2|\leq |k_1+k_2|/2$ and $|k_1-k_2|\geq |k_1+k_2|/2$. Moreover, when $y=x$ and $k_1=k_2=k$ we have that κ'_{11} is unitary, $J(w,x,x)=1$, $\det(\tau'(v,w,x,y))=\frac{1}{4}$ and $\frac{dz_1}{dw}(v,w,x,x)|_{v=0}=2$. Thus, formula (66) implies that

$$R_m(x,x,k,k) = \frac{1}{2}\int_D \mu(z_1)\frac{1}{|z_1-x|^2}\,dz_1$$

which verifies (48). Thus lemma is proven. $\square$

Lemma 4.2 *In the decomposition (45) the random variable $b^m(x,y,k)$ satisfies a.s. the condition*

$$|b^m(x,y,k)|\leq C_m(1+|k|)^{-1-m}, \quad x,y\in U,\ k>0 \qquad (68)$$

where the constant C_m depends only on $W_0^{-1,1}(D)$-norm of $q(z,\omega)$ and m.

Proof. By (44), $||\Phi_k(\cdot-x)||_{W^{1,\infty}(D)}+||\Phi_k^m(\cdot-y)||_{W^{1,\infty}(D)}\leq ck^{1/2}$ for $k>1$, $x,y\in U$. This implies

$$\begin{aligned}|b^m(x,y,k)| \leq\ & ||q||_{W_0^{-1,1}(D)}\big(||\Phi_k(\cdot-x)-\Phi_k^m(\cdot-x)||_{W^{1,\infty}(D)}||\Phi_k(\cdot-y)||_{W^{1,\infty}(D)}+ \\ & +||\Phi_k^m(\cdot-x)||_{W^{1,\infty}(D)}||\Phi_k(\cdot-y)-\Phi_k^m(\cdot-x)||_{W^{1,\infty}(D)}\big) \\ & \leq C'_m||q||_{W_0^{-1,1}(D)}(1+|k|)^{-1-m}.\end{aligned}$$

Here $W_0^{-1,1}(D)$ denotes the closure of smooth functions in the corresponding norm. This proves the lemma. $\square$

The above results have the following corollary that plays an important role in sequel.

Corollary 4.3 *Assume that $k_1,k_2>1$ and $x,y\in U$. Then*

$$\mathbb{E}\,\big(|Re\,(k_1^2u_1^m(x,y,k_1))Re\,(k_2^2u_1^m(x,y,k_2)|)\big)\leq C_n(1+|k_1-k_2|)^{-n}, \quad n>0,$$

where C_n is independent of $x,y\in U$, and one may replace one or both of the real parts by imaginary parts.

Proof. Observe that $\overline{u_1^m(x,y,k_1)} = \widetilde{u}_1^m(x,y,-k_1)$ where $\widetilde{u}_1$ is exactly like u_1, the only difference being that the coefficients f_j have been replaced new coefficients having the same absolute values as f_j. Thus for $k_1, k_2 > 1$ the exact analogue of the proof of Lemma 4.1 yields for abritrary $n > 1$ that

$$\begin{aligned} |\mathbb{E}\,(k_1^2 u_1^m(x,y,k_1) k_2^2 u_1^m(x,y,k_2))| &\leq |\mathbb{E}\,(k_1^2 u_1^m(x,y,k_1)\overline{k_2^2 \widetilde{u}_1^m(x,y,-k_2)})| \\ &\leq C_n(1+|k_1+k_2|)^{-n}. \end{aligned}$$

Moreover, we obtain directly from Lemma 4.1

$$|\mathbb{E}\,(k_1^2 u_1^m(x,y,k_1)\overline{k_2^2 u_1^m(x,y,k_2)})| \leq C_n(1+|k_1-k_2|)^{-n}.$$

When $k_1, k_2 > 1$ we have that $|k_1+k_2|^{-n} \leq |k_1-k_2|^{-n}$. The claim now follows by simply observing that we may recover all the products ac, ad, bc and bd as linear combinations of real or imaginary parts of the numbers $(a+ib)(c \pm id) = (ac \mp bd) + i(bc \pm ad)$. This proves the claim. □

5 Behaviour of the second term

In this quite long section we consider the second term u_2 of the Born series for the solution u. We first derive formulae for the second moment of $u_2(x,y,k)$. They are rather obvious in the formal level but need to be justified in our situation. The second subsection is devoted to estimates, which show that u_2 is small compared to u_1 as $k \to \infty$, at least in the level of second moments. The analysis at this point turns out to be surprisingly involved, and we obtain only estimates in the mean with respect to x and y, which, however, suffice for our purposes.

Later we will need additional information on the tail behaviour of the distribution of u_2. This is the topic of the third subsection, which provides Wiener chaos type exponential tail estimates for suitable integral means related to the second term. Finally, Theorem 5.12 applies these results to provide the desired decay along the whole spectrum of frequencies: one has that almost surely $\lim_{k\to\infty} k^4 |u_2(x,y,k)|^2 = 0$, which holds true in the mean sense which respect to x and y.

5.1 Representing the covariance of the second term

Our first aim is to obtain formulas for $\mathbb{E}\,|u_2(x,y,k)|^2$. As u_2 is bilinear with respect to q we need to be able to compute expectations for products of four

Gaussian variables. The following Lemma is a special case of the well-know formulae but we include the simple proof in Appendix B for the readers convenience.

Lemma 5.1 *Assume that* (X_1, X_2, X_3, X_4) *is a real centered Gaussian random vector with the covariance structure* $\mathbb{E}\, X_i X_j = b_{ij}$ *for* $i, j \in \{1, 2, 3, 4\}$. *Then*

$$\mathbb{E}\, X_1 X_2 X_3 X_4 = b_{12}b_{34} + b_{13}b_{24} + b_{14}b_{23}.$$

In order to apply the above to u_2 we first consider $u_{2,\varepsilon}$ that is obtained by replacing q by the standard mollification $q_\varepsilon := q * \rho_\varepsilon$, where $\rho_\varepsilon(x) = \varepsilon^{-2}\rho(x/\varepsilon)$ and $\rho \in C_0^\infty(\mathbb{R}^2)$ is radially symmetric function that satisfies $\int \rho(x)dx = 1$. Let us denote the operator that corresponds to the mollification by $M_\varepsilon : f \mapsto f * \rho_\varepsilon$. The covariance operator of q_ε equals $C_\varepsilon = M_\varepsilon C M_\varepsilon$. This entails that C_ϵ has a smooth compactly supported kernel, and hence we may obviously write

$$\mathbb{E}\, |u_{2,\varepsilon}(x, y, k)|^2 = \int_D \cdots \int_D \Phi_k(x - z_1)\Phi_k(z_1 - z_2)\Phi_k(z_2 - y)\overline{\Phi_k(x - \tilde{z}_1)} \cdot$$
$$\cdot \overline{\Phi_k(\tilde{z}_1 - \tilde{z}_2)\Phi_k(\tilde{z}_2 - y)} E(q_\varepsilon(z_1)q_\varepsilon(z_2)q_\varepsilon(\tilde{z}_1)q_\varepsilon(\tilde{z}_2))\, dz_1\, dz_2\, d\tilde{z}_1\, d\tilde{z}_2$$

whence an application of the previous lemma yields that

$$\mathbb{E}\, |u_{2,\varepsilon}(x, y, k)|^2 =$$
$$\int_D \cdots \int_D \Phi_k(x - z_1)\Phi_k(z_1 - z_2)\Phi_k(z_2 - y)\overline{\Phi_k(x - \tilde{z}_1)\Phi_k(\tilde{z}_1 - \tilde{z}_2)\Phi_k(\tilde{z}_2 - y)} \cdot$$
$$\cdot \Big(C_\varepsilon(z_1, z_2)C_\varepsilon(\tilde{z}_1, \tilde{z}_2) + C_\varepsilon(z_1, \tilde{z}_2)C_\varepsilon(\tilde{z}_1, z_2) + C_\varepsilon(z_1, \tilde{z}_1)C_\varepsilon(z_2, \tilde{z}_2) \Big) \cdot$$
$$\cdot\, dz_1 dz_2\, d\tilde{z}_1 d\tilde{z}_2.$$

We aim to take the limit $\varepsilon \to 0$ inside the above integral, whose integrand we denote by $I_\varepsilon(z_1, z_2, \tilde{z}_1, \tilde{z}_2)$. The same integrand with C in place of C_ε is denoted by I. Clearly $\lim_{\varepsilon \to 0} C_\varepsilon(z_1, z_2) = C(z_1, z_2)$ for almost all (z_1, z_2). Moreover, clearly for any $p > 1$ the norm $\|C_\varepsilon\|_{L^p(D \times D)}$ is uniformly bounded with respect to $\varepsilon \in (0, 1)$. One also has that $\Phi_k(\cdot) \in L^p_{loc}$ for all $p \in (1, \infty)$. Hence, as D is bounded, we may apply Hölders inequality to deduce that $\|I_\varepsilon\|_{L^2(D^4)} \le C < \infty$ independently of ε. This shows that the functions I_ε are uniformly integrable, and an application of Vitali's convergence theorem yields that

$$\lim_{\varepsilon \to 0^+} \mathbb{E}\, |u_{2,\varepsilon}(x, y, k)|^2 = \int_{D^4} I(z_1, z_2, \tilde{z}_1, \tilde{z}_2)\, dz_1 dz_2\, d\tilde{z}_1 d\tilde{z}_2.$$

It remains to show that $\lim \mathbb{E}\,|u_{2,\varepsilon}(x,y,k)|^2 = \mathbb{E}\,|u_2(x,y,k)|^2$. To that end, note first that $\lim_{\varepsilon\to 0} u_{2,\varepsilon}(x,y,k) = u(x,y,k)$ almost surely, as we have $\lim_{\varepsilon\to 0}\|q_\varepsilon - q\|_{W^{-\delta,p}} = 0$ for any $\delta > 0$ and $p > 1$ almost surely by Theorem 2.3 (compare the proof of Lemma 5.8 below). Let us also observe that there is an analogous formula to $\mathbb{E}\,|u_{2,\varepsilon}(x,y,k)|^4$, which is similar to (69) and uses an analogue of Lemma 5.1 – there just appear fourfold products of C_ε-terms and longer products of Hankel functions. Just as before, an applications of Hölder's inequality shows that $\mathbb{E}\,|u_2(x,y,k)|^4 \le C < \infty$, uniformly with respect to ε. Hence we may apply uniform integrability again to obtain

$$\mathbb{E}\,|u_{2,\varepsilon}(x,y,k)|^2 = \lim_{\varepsilon\to 0^+} \mathbb{E}\,|u_{2,\varepsilon}(x,y,k)|^2 = \int_{D^4} I(z_1,z_2,\tilde z_1,\tilde z_2)\,dz_1 dz_2\,d\tilde z_1 d\tilde z_2.$$

5.2 The estimate for the second moment

Let $x, y \in U$. By the last formula of the previous subsection there is the equality

$$\begin{aligned}
&\mathbb{E}\,|u_2(x,y,k)|^2 = \qquad\qquad (69)\\
&\int_D \cdots \int_D \Phi_k(x-z_1)\Phi_k(z_1-z_2)\Phi_k(z_2-y)\overline{\Phi_k(x-\tilde z_1)\Phi_k(\tilde z_1-\tilde z_2)\Phi_k(\tilde z_2-y)}\\
&\Big(C(z_1,z_2)C(\tilde z_1,\tilde z_2) + C(z_1,\tilde z_2)C(\tilde z_1,z_2) + C(z_1,\tilde z_1)C(z_2,\tilde z_2)\Big)\,dz_1dz_2\,d\tilde z_1 d\tilde z_2\\
&=: J_1 + J_2 + J_2
\end{aligned}$$

where J_1 corresponds to the term involving the factor $C(z_1,z_2)C(\tilde z_1,\tilde z_2)$, and J_2, J_3 are defined accordingly. We will estimate the right hand side of the corresponding equality that is obtained from the above one by integrating both sides with respect to x and y over the measurement domain. This will be done separately for the terms J_i, $i = 1,2,3$. The main result of this subsection is Theorem 5.5 below.

We begin with J_1, since this is the easiest one, and it also gives an idea of the techniques used for the more unyielding terms J_2, J_3. For that end, we use in addition to the operator H_k defined in formula (23) the operator

$$Tu(x) = \int_D C(x,z)u(z)\,dz.$$

We also recall a localized version of the classical estimate of Agmon [2]:

$$\|H_k\|_{L^2(B(0,R))\to L^2(B(0,R))} \le c(R)k^{-1} \quad \text{for} \quad R > 0 \qquad (70)$$

(c.f. [37] for a simple proof of this result).

Lemma 5.2 *Assume that* $2 \le p < 4$. *For any ball* $B(0,R) \subset \mathbb{R}^2$ *there is* $c = c(R,p) < \infty$ *such that*

$$||\Phi_k||_{L^p(B(0,R))} \le ck^{-1/2} \quad \text{for all} \quad k \ge 1.$$

Proof. By using the asymptotics $|\Phi_k(s)| \le c(k|s|)^{-1/2}$ (which is fairly crude near the origin) we obtain

$$||\Phi_k||^p_{L^p(B(0,R))} \le \int_0^R |\Phi_k(r)|^p r\,dr \le c^p k^{-p/2} \int_0^R s^{1-p/2}\,ds.$$

□

The estimate for J_1 is contained in the following lemma. Observe, that it does not deal with fixed $x, y \in U$, but rather integral averages of the second moment over $(x,y) \in U \times U$.

Lemma 5.3 *There is a constant* $c > 0$ *such that for all* $k \ge 1$ *it holds that*

$$||J_1||_{L^1(U\times U)} \le ck^{-5}.$$

Proof. Denote $b(z_1,z_2) = C(z_1,z_2)\Phi_k(z_1 - z_2)$ for $z_1, z_2 \in D$ and let $\varepsilon \in (0,1/2)$. By Lemma 5.2 we have for suitable $R_0 > 0$ that

$$\int_D \int_D |\Phi_k(z_1 - z_2)|^{2+2\varepsilon} dz_1 dz_2 \le c\|\Phi_k(z)\|^{2+2\varepsilon}_{L^{2+2\varepsilon}(B(0,R_0))} \le c(1/\sqrt{k})^{2+2\varepsilon}.$$

Moreover, by Proposition 2.2 it it clear that $\|C(z_1,z_2)\|_{L^p(D\times D)} < \infty$ for all $p < \infty$. Let us denote by B the operator with the kernel b. By the Hölder inequality we can estimate for Hilbert-Schmidt norm of the operator $B : L^2(D) \to L^2(D)$ by

$$\begin{aligned} \|B\|_{HS} &= \left(\int_D \int_D |b(z_1,z_2)|^2 dz_1 dz_2\right)^{1/2} \\ &\le \left(\||C|^2\|_{L^{(1+\varepsilon)'}(D)} \||\Phi_k(\cdot - \cdot)|^2\|_{L^{1+\varepsilon}(D\times D)}\right)^{1/2} \le ck^{-1/2}. \end{aligned} \quad (71)$$

Finally, we denote the kernel of the operator $H_k B H_k$ by $a(x,y)$ for $x, y \in U$ and combine (70) together with (71) to estimate

$$\begin{aligned} &\int_U \int_U |J_1(x,y)|dxdy = \int_U \int_U |a(x,y)|^2 dxdy = \|H_k B H_k\|^2_{HS} \le \\ &\le (\|H_k\|_{L^2(U)\to L^2(D)})^2 (\|B\|_{HS})^2 (\|H_k\|_{L^2(D)\to L^2(U)})^2 \le Ck^{-5}. \end{aligned}$$

□

The above proof does not directly extend to estimate J_2 or J_3, since they contain cross-terms which do not allow us to write the integrand as square of a kernel. However, the same idea works after we make an asymptotic separation of the variables in the cross terms, as given by the following proposition.

Proposition 5.4 *Fix $\varepsilon > 0$. There in a positive constant $C(\varepsilon)$ such that for any $k \geq 1$ one may decompose the covariance in two parts*

$$C(z_1, z_2) = S_k(z_1, z_2) + R_k(z_1, z_2) \tag{72}$$

with supp(S_k), supp(R_k) $\subset D \times D$, and such that for $z_1, z_2 \in D$ one may write

$$S_k(z_1, z_2) = \int_{(\xi_1,\xi_2)\in\mathbb{R}^4} e^{i\xi_1\cdot z_1} e^{i\xi_2\cdot z_2} d\mu_k(\xi_1, \xi_2), \tag{73}$$

where the positive measure μ_k satisfies

$$\int_{(\xi_1,\xi_2)\in\mathbb{R}^4} d\mu_k(\xi_1, \xi_2) \leq c(\varepsilon) k^{\varepsilon}. \tag{74}$$

Moreover, given any $m, \ell \geq 1$ the remainder term R_k satisfies

$$\int_{D\times D} |R_k(z_1, z_2)|^m dz_1 dz_2 \leq c(\varepsilon, m, l) k^{-\ell}. \tag{75}$$

Proof. According to Proposition 2.2 we may write

$$C(z_1, z_2) = c_0(z_1, z_2) \log |z_1 - z_2| + r_1(z_1, z_2),$$

where $c_0 \in C_0^\infty(\mathbb{R}^4)$ and the Fourier transform of r_1 is integrable over $\mathbb{R}^4$. Hence, as soon as we prove the decomposition for the function $c_0(z_1, z_2) \log |z_1 - z_2|$, the term r_1 may be immersed to the term S_k, and we obtain the desired decomposition. We thus consider only the logarithmic summand.

Recall that (in dimension 2, see [17]) $\mathcal{F}(\log(1/|\cdot|)) = c_0|\xi|^{-2} + c_1\delta_0$, where the first term is understood in the sense of a Hadamard principal value. Let $\psi \in C_0^\infty(\mathbb{R}^2)$ satisfy $\psi(z) = 1$ on $B(0, 2 + 2\text{diam}(D))$. One obtains

$$\mathcal{F}(\psi \log(1/|\cdot|)) = \psi_1(\xi) + \psi_2(\xi)|\xi|^{-2}, \tag{76}$$

where ψ_1, ψ_2 are smooth, ψ_1 is compactly supported, ψ_2 is supported outside the origin and satisfies $\|\psi_2\|_\infty \leq C$. Let $(\phi_\delta)_{\delta>0}$ be a smooth approximation of delta-distribution so that $\int_{\mathbb{R}^2} \phi = 1$ and $\widehat{\phi}_\delta(\xi) = \widehat{\phi}(\delta\xi)$. We assume that ϕ is radially symmetric and decreasing with supp$(\phi)\subset B(0,1)$. Let us write

$$A(z) = \phi_\delta(z) * (\psi(z)\log(1/|z|)) \quad \text{and} \quad B(z) = \psi(z)\log(1/|z|) - A(z),$$

where δ has the value

$$\delta = \exp(-k^\varepsilon).$$

Finally, we choose

$$S_k(z_1, z_2) = c_0(z_1, z_2)A(z_1 - z_2), \tag{77}$$

which also automatically defines the remainder term R_k in the decomposition of $c_0(z_1, z_2)\log|z_1 - z_2|$.

We first verify that S_k satisfies (73). For $k \geq 1$ we have

$$\begin{aligned} \|\widehat{A}\|_{L^1(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |\psi_1(\xi) + \psi_2(\xi)||\xi|^{-2}||\widehat{\phi}(\exp(-k^\varepsilon)\xi)|d\xi \\ \leq \;& c + c\int_{|\xi|\geq 1} \frac{|\xi|^{-2}}{1 + |\exp(-k^\varepsilon)\xi|^2}d\xi \leq c + c2\pi\int_1^\infty \frac{dt}{t(1+\exp(-2k^\varepsilon)t^2)} \\ = \;& c + c\int_{\exp(-k^\varepsilon)}^\infty \frac{dt}{t(1+t^2)} \leq ck^\varepsilon. \end{aligned}$$

Hence the (4-dimensional) Fourier-transform of the function $A(z_1 - z_2)$ is a measure with total variation less than ck^ε. Thus the same estimate holds true (with possibly different c) for the function S_k, as the Fourier transform of $c_0(z_1, z_2)$ is obviously integrable. The inverse Fourier transform formula now yields the representation (73).

Let us next treat R_k. Since $\psi(z_1 - z_2) = 1$ for $z_1, z_2 \in D$ we have $R_k(z_1, z_2) = c_0(z_1, z_2)B(z_1 - z_2)$, whence it is enough to show for all $m, \ell \geq 1$ that

$$\int_{\mathbb{R}^2} |B(z)|^m dz = o(k^{-\ell}). \tag{78}$$

Denote $f(z) = \psi(z)\log(1/|z|)$ and $f_\delta(z) = (\phi_\delta * f)(z)$, so that

$$B(z) = f(z) - f_\delta(z).$$

In order to estimate $B(z)$ we use the fact that $\mathrm{supp}(\phi) \subset B(0,1)$ and assume first that $|z| > 2\sqrt{\delta}$. The very definition of convolution yields that

$$\begin{aligned} |f(z) - f_\delta(z)| &\leq \sup_{|w-z|\leq\delta} |f(w) - f(z)| \leq \delta \sup_{|w-z|\leq\delta} |Df(w)| \leq \delta \sup_{|w-z|\leq\delta} \frac{c}{|w|} \\ &\leq 2c\sqrt{\delta}. \end{aligned}$$

Hence (recall also that ψ has compact support)

$$\int_{|z|\geq 2\sqrt{\delta}} |B(z)|^m dz = c\delta^{m/2} = c\exp(-mk^\varepsilon/2) = o(k^{-\ell}) \tag{79}$$

for all $m, \ell \geq 1$.

Next, if $2\delta \leq |z| \leq 2\sqrt{\delta}$ we clearly have $|f_\delta(z)| \leq \log(1/|z/2|)$ and for $|z| \leq 2\delta$ it obviously holds that $|f_\delta(z)| \leq |f_\delta(0)|$. One has

$$|f_\delta(0)| \leq (C/\delta^2) \int_{|z|\leq\delta} \log(1/|z|) dz \leq c \log(1/\delta).$$

Moreover, $\int_{|z|\leq u} \log^m(1/|z|)dz \leq Cu^2 \log^m(1/u)$ for $u \leq 1/2$. By combining these facts with the crude estimate $|f(z) - f_\delta(z)| \leq |f(z)| + |f_\delta(z)|$ we see that

$$\int_{|z|\leq 2\sqrt{\delta}} |B(z)|^m dz \leq C\delta \log^m(1/\delta) + \delta^2 \log^m(1/\delta) = O(k^{-\ell}) \tag{80}$$

for all m, ℓ. □

We are now ready to extend Lemma 5.3 to cover the whole second moment.

Theorem 5.5 *For any $\varepsilon > 0$ there is the estimate*

$$||\mathbb{E}\, |u_2(x,y,k)|^2||_{L^1(U\times U)} \leq ck^{-5+\varepsilon} \quad \text{for} \quad k \geq 1.$$

Proof. By Lemma 5.3 it is enough to consider the term J_2 that corresponds to the cross term

$$C(z_1, \tilde{z}_2)C(\tilde{z}_1, z_2),$$

as the analysis of the second cross term J_3 is completely analogous.

Fix $\varepsilon \in (0, 1/8)$ and let $k \geq 1$ be arbitrary. According to the Proposition 5.4 we may write

$$C(z_1, \tilde{z}_2)C(\tilde{z}_1, z_2) = S_k(z_1, \tilde{z}_2)S_k(\tilde{z}_1, z_2) + h(z_1, \tilde{z}_1, z_2, \tilde{z}_2),$$

where

$$h(z_1,\tilde z_1,z_2,\tilde z_2) = R_k(z_1,\tilde z_2)S_k(\tilde z_1,z_2)+S_k(z_1,\tilde z_2)R_k(\tilde z_1,z_2)+R_k(z_1,\tilde z_2)R_k(\tilde z_1,z_2).$$

Let us first treat the contribution of the part that corresponds to h. By (73) and (74) one has $\|S_k\|_\infty \le Ck^\varepsilon$. Hence (75) and the Hölder inequality yield that

$$\|h\|_{L^p(D^4)} = o(k^{-\ell}) \tag{81}$$

for all $p,\ell \ge 1$. We shall denote by N the operator on $L^2(D\times D)$ with the kernel

$$n(z_1,\tilde z_1,z_2,\tilde z_2) := \Phi_k(z_1-z_2)\overline{\Phi_k(\tilde z_1-\tilde z_2)}h(z_1,\tilde z_1,z_2,\tilde z_2).$$

The obvious higher integrability bounds for the function $\Phi_k(z_1-z_2)\overline{\Phi_k(\tilde z_1-\tilde z_2)}$ and (81) enable us to estimate the Hilbert-Schmidt norm of the operator N:

$$\|N\|_{\mathrm{HS}} = \|n\|_{L_2(D^4)} = o(k^{-\ell}) \tag{82}$$

for all $\ell \ge 1$.

Define the operator $R: L^2(U) \to L^2(D\times D)$ that has the kernel $r(z_2,\tilde z_2,y)$, where

$$r(z_2,\tilde z_2,y) = \Phi_k(z_2-y)\overline{\Phi_k(\tilde z_2-y)}.$$

By e.g. estimating the Hilbert-Schmidt norm we see easily that

$$\|R\|_{L^2(U)\to L^2(D\times D)} = \mathrm{O}(1) \tag{83}$$

with respect to k (actually, by a more careful analysis one obtains the estimate $\mathrm{O}(k^{-3/2})$, but we shall not need it)

The term corresponding to h may be written in the form

$$w(x,y) = \int_D \cdots \int_D \Phi_k(x-z_1)\Phi_k(z_1-z_2)\Phi_k(z_2-y)\cdot$$
$$\cdot\, \overline{\Phi_k(x-\tilde z_1)\Phi_k(\tilde z_1-\tilde z_2)\Phi_k(\tilde z_2-y)}h(z_1,\tilde z_1,z_2,\tilde z_2)dz_1dz_2\,d\tilde z_1 d\tilde z_2,$$

whence we observe that w is the kernel of the operator $R'NR$, where R' is the transpose of the operator R. Hence

$$\|w\|_{L^2(U\times U)} = \|R'NR\|_{\mathrm{HS}} \le \|R'\|\|N\|_{HS}\|R\| = o(k^{-\ell}) \tag{84}$$

for all $\ell \geq 1$ by (82) and (83). A fortiori, as U is bounded we obtain for any $\ell \geq 1$

$$\|w\|_{L^1(U\times U)} = o(k^{-\ell}). \tag{85}$$

We consider the term containing the product $S_k(z_1, \tilde{z}_2)S_k(\tilde{z}_1, z_2)$. By (73) we may write

$$S_k(z_1,\tilde{z}_2)S_k(\tilde{z}_1,z_2) = \int_{(\xi_1,\dots,\xi_4)\in\mathbb{R}^8} e^{i\xi_1\cdot z_1}e^{i\xi_2\cdot\tilde{z}_2}e^{i\xi_3\cdot\tilde{z}_1}e^{i\xi_4\cdot z_2}d\mu_k(\xi_1,\xi_2)d\mu_k(\xi_3,\xi_4)$$

and the corresponding part of J_2 may be written in the form

$$W(x,y) = \int_{(\xi_1,\dots,\xi_4)\in\mathbb{R}^8} H(x,y,\xi_1,\dots,\xi_4)d\mu_k(\xi_1,\xi_2)d\mu_k(\xi_3,\xi_4),$$

where

$$\begin{aligned} & H(x,y,\xi_1,\dots,\xi_4) \\ = & \int_D\int_D \Phi_k(x-z_1)e^{i\xi_1\cdot z_1}\Phi_k(z_1-z_2)e^{i\xi_4\cdot z_2}\Phi_k(z_2-y)dz_1dz_2 \\ \times & \int_D\int_D \overline{\Phi_k(x-\tilde{z}_1)}e^{i\xi_3\cdot\tilde{z}_1}\overline{\Phi_k(\tilde{z}_1-\tilde{z}_2)}e^{i\xi_2\cdot\tilde{z}_2}\overline{\Phi_k(\tilde{z}_2-y)}d\tilde{z}_1d\tilde{z}_2. \end{aligned} \tag{86}$$

According to (74) we obtain

$$\|W\|_{L^1(U\times U)} \leq (c(\varepsilon))^2k^{2\varepsilon}\sup_{(\xi_1,\dots,\xi_4)\in\mathbb{R}^8}\|H(\cdot,\cdot,\xi_1,\dots,\xi_4)\|_{L^1(U\times U)}. \tag{87}$$

By definition one has $H(x,y,\xi_1,\dots,\xi_4) = H_1H_2$, where

$$\begin{aligned} & H_1(x,\cdot,y,\dots,\xi_4) \\ = & \int_D\int_D \Phi_k(x-z_1)e^{i\xi_1\cdot z_1}\Phi_k(z_1-z_2)e^{i\xi_4\cdot z_2}\Phi_k(z_2-y)dz_1dz_2 \end{aligned}$$

and H_2 is defined analogously from (86). As in the proof of Lemma 5.3 we see that the Hilbert-Schmidt norm of the kernel

$$t(z_1,z_2) = e^{i\xi_1\cdot z_1}\Phi_k(z_1-z_2)e^{i\xi_4\cdot z_2}$$

is bounded by $Ck^{\varepsilon-1/2}$, uniformly with respect to all ξ_i. Hence, again as in the proof of Lemma 5.3 we obtain the uniform (in the ξ_i) estimate

$$\|H_1(\cdot,\cdot,\xi_1,\ldots,\xi_4)\|_{L^2(U\times U)} \leq ck^{-5/2}.$$

Since a similar estimate is valid for H_2 we obtain by the Cauchy-Schwartz inequality that $\|H(\cdot,\cdot,\xi_1,\ldots,\xi_4)\|_{L^1(U\times U)} \leq ck^{-5}$, and combining this with (87) yields finally

$$\|W\|_{L^1(U\times U)} \leq Ck^{2\varepsilon-5}.$$

This proves the claim since we may take $\varepsilon > 0$ as small as we wish. □

5.3 Wiener chaos type estimates for the tails

Here we establish Wiener Chaos type tail estimates for the second term. This is needed in order to be able to provide estimates for the second term over the whole set of frequencies in Theorem 5.12 below, which is the key result of whole present section. In principle, these results can be deduced from the well-known estimates for random variables belonging to given levels in the Wiener Chaos decomposition (c.f. e.g. [30, Chapter 3.2]). However, to make all the details transparent, we give a self-contained treatment based on finite-dimensional approximation. Proofs of certain basicly well-known lemmata are presented in Appendix B.

Before we continue, it is perhaps of interest to note that there is an obvious temptation to consider u_2 as a bilinear form on a suitably chosen Sobolev space, and apply well-known tail estimates for the norms of Banach space valued Gaussian random variables. However, this approach seems to yield estimates that are far from what is needed; actually the averaging over $x, y \in U$ in the previous subsection does the trick.

For random variables Y we shall denote $\|Y\|_p = (\mathbb{E}\,|Y|^p)^{1/p}$. Let us start with the tail behaviour of a quadratic form of Gaussian variables.

Lemma 5.6 *Let $Y = \sum_{j,k=1}^n a_{jk}X_kX_j$, where the a_{jk} are complex constants and $(X_1,\ldots,X_n)$ is a centered Gaussian vector in $\mathbb{R}^n$. Then Y satisfies the distribution inequality*

$$\mathcal{P}(|Y| > \lambda) \leq 12 exp(\frac{-\lambda}{8\|Y\|_2}) \quad \text{for any } \lambda > 0, \tag{88}$$

where the right had side is interpreted as zero if $Y \equiv 0$ a.s.

Observe that the above estimate does not depend on n.

Proof. See Appendix B. □

Let us say that a (complex-valued) square-integrable random variable X belongs to the class $\mathcal{A}$ if X satisfies the distribution inequality of Lemma 5.6 (observe that according to our interpretation $X \in \mathcal{A}$ also in the case where X vanishes almost surely).

Lemma 5.7 *Let $X_k \in \mathcal{A}$ for $k = 1, 2, \ldots$ and assume that $X_k \to X$ almost surely as $k \to \infty$, where the limit random variable X is finite almost surely. Then $X \in \mathcal{A}$.*

Proof. See Appendix B. □

The term u_2 is definitely not a finite bilinear form of Gaussian variables, but it may be approximated by such ones.

Lemma 5.8 *For each $k > 0$ there is a sequence of finite bilinear forms Y_k of independent Gaussian variables such that almost surely $u_2(x, y, k) = \lim_{j\to\infty} Y_j$ all $x, y \in U$.*

Proof. Recall that

$$u_2(x, y, k) = \int_D \int_D \Phi_k(x - z_1) q(z_1) \Phi_k(z_1 - z_2) q(z_2) \Phi_k(z_2 - y)\, dz_1 dz_2$$

with the proper interpretation. In other words, we may express u_2 as the duality pairing

$$u_2(x, y, k) = \langle \lambda, g \rangle,$$

where

$$\lambda = \Phi_k(x - \cdot) q(\cdot) \qquad \text{and} \qquad g = H_k(q\Phi_k(\cdot - y)).$$

Above the operator H_k is defined as before. Let now $R > 0$ be such that $U \cup D \subset B(0, R)$. Now $H_k : W_0^{-1,2}(B(0, R)) \to W^{1,2}(B(0, R))$ directly from the Fourier-transform definition of H_k. Let us denote by $u_2^n(x, y, k)$, λ_n, and g_n the corresponding quantities that are obtained as we replace q by the finite-dimensional approximation q_n provided by Lemma 2.4 in the case $\varepsilon = 1$ and $p = 2$. Then $u_2^n(x, y, k)$ is a finite bilinear form of components of a Gaussian vector and $\|\lambda_n - \lambda\|_{W^{-1,2}} \to 0$ as $n \to \infty$. Moreover, we have $\|g_n - g\|_{W^{1,2}} \to 0$ and by duality it follows that $u_2^n(x, y, k) \to u_2(x, y, k)$ almost surely (independently of x, y). □

The combination of the last two lemmata give us

Corollary 5.9 *For each $k > 0$ and $x, y \in U$ we have $u_2(x, y, k) \in \mathcal{A}$.* $\square$

We shall denote for $k > 0$

$$Z(k) = k^4 \int_U \int_U |u_2(x, y, k)|^2 dx dy.$$

Our aim is to estimate the tail distribution of $Z(k)$ by establishing bounds for its moments. Write for that end

$$Z(k) = k^4 \int_U \int_U b(x, y) H^2(x, y) dx dy,$$

where $b(x, y) = \mathbb{E}\, |u_2(x, y, k)|^2 = \|u_2(x, y, k)\|_2^2$ and

$$H^2(x, y) = |u_2(x, y, k)| / \|u_2(x, y, k)\|_2$$

in case$\|u_2(x, y, k)\|_2 > 0$, and set $H(x, y) \equiv 0$ otherwise. Then we have for all $x, y \in U$ that

$$H(x, y) \in \mathcal{A} \quad \text{and} \quad \mathbb{E}\, |H(x, y)|^2 \leq 1.$$

Lemma 5.10 *If $Y \in \mathcal{A}$ and $\|Y\|_2 \leq 1$, then*

$$\|Y^2\|_n \leq 1000 \cdot ((2n)!)^{1/n} \quad \text{as} \quad n = 1, 2, \ldots.$$

Proof. Recall that we have $\mathcal{P}(|Y| > \lambda) \leq 12\exp(-\lambda/8)$. Hence

$$\begin{aligned} \mathbb{E}\left(|Y|^{2n}\right) &= 2n \int_0^\infty \lambda^{2n-1} \mathcal{P}(|Y| > \lambda) d\lambda \leq 24n \int_0^\infty \lambda^{2n-1} e^{-\lambda/8} d\lambda \\ &= 24n 8^{2n} \Gamma(2n) = 12 \cdot 8^{2n} (2n)!. \end{aligned}$$

Just observe that $12^{1/n} 64 \leq 1000$. $\square$

We are now prepared to bound the moments of $Z(k)$. An application of (an integral form) of Minkowsky's inequality yields that

$$\|Z(k)\|_n \leq k^4 \int_U \int_U |b(x, y)| \|H^2(x, y)\|_n dx dy \leq 1000\, A((2n)!)^{1/n},$$

where we wrote

$$A = k^4 \int_U \int_U |b(x, y)| dx dy.$$

In the above (and similar) computations we may use the Fubini theorem, as the needed joint measurability is a consequence of continuity of u_2 with respect to k. Assuming that A is finite we thus have

$$\mathbb{E}\,(Z(k)/A)^n \leq 1000^n(2n)!$$

so that

$$\mathbb{E}\,\cosh(\frac{1}{2}\sqrt{Z(k)/1000A}) \leq \sum_{n=0}^{\infty} 2^{-n} = 2.$$

Thus $\mathbb{E}\exp(\frac{1}{2}\sqrt{Z(k)/1000A}) \leq 4$ and $\mathcal{P}(\frac{1}{2}\sqrt{Z(k)/1000A} > \lambda) \leq 4e^{-\lambda}$. Hence we have proven the estimate

$$\mathcal{P}(Z(k) > \lambda) \leq 4\exp(-\frac{1}{70}\sqrt{\lambda/A}),$$

which is trivially true also in the case $A = \infty$.

An application of Theorem 5.5 from the previous subsection yields that $A \leq Ck^{-1/2}$ (actually the decay $k^{\varepsilon-1}$ is also true, but we do not need that here). By combining this with the just proven inequality we obtain

Proposition 5.11 *There is a constant C such that for all $k, \lambda > 0$ there is the estimate*

$$\mathcal{P}(k^4 \int_U \int_U |u_2(x,y,k)|^2 dxdy > \lambda) \leq 4exp(-C\lambda^{1/2}k^{1/4}).$$

We are finally able to settle the asymptotics of the second term.

Theorem 5.12 *Almost surely* $\lim_{k\to\infty} k^4 \int_U \int_U |u_2(x,y,k)|^2 dxdy = 0$.

Proof. We first practice uninteresting technicalities to verify that there is an index $m_1 \geq 0$ such that the simple estimate

$$|\frac{\partial}{\partial k} u_2(x,y,k)| \leq C(1+k)_1^m, \tag{89}$$

holds, where C is uniform over $x, y \in U$ and $k \geq 1$ for any fixed realization of the potential q. Towards (89), recall that $\frac{\partial}{\partial z} H_0^{(1)}(z) = -H_1^{(1)}(z)$, and we see that $\frac{\partial}{\partial k} H_0^{(1)}(k|z|) = |z| H_1^{(1)}(k|z|)$. ¿From this and the well known asymptotics of Hankel-functions and their derivatives is easily verified that

$$\|\frac{\partial}{\partial k}\Phi_k(x-\cdot)q(\cdot)\|_{H^{-1/2}(U)} \leq ck^{\ell_1}, \quad k > 1$$

with some ℓ_1. On the other hand, from the definition of H_k we see that $H_k = k^{-2}V_k H_1 V_{1/k}$, where V_k is the dilation operator $(V_k f)(x) = f(kx)$. Obviously $\frac{d}{dk}V_k = k^{-1}(x\cdot\nabla)V_k = k^{-1}v_k(x\cdot\nabla)$, and we may compute

$$\frac{\partial}{\partial k}H_k = \frac{1}{k}(-2H_k + (x\cdot\nabla)H_k - H_k(x\cdot\nabla)).$$

¿From asymptotics of Hankel functions, we see that $||H_k||_{H_0^s(B(0,R))\to H^s(B(0,R))}$ grows at most polynomially in k for any s and $R>0$. Hence we infer that

$$\|\frac{\partial}{\partial k}H_k\|_{H_0^s(B(0,R))\to H^{s+2}(B(0,R))} \le ck^{\ell_2}$$

with some ℓ_2. All in all, using the notation of Lemma 5.10, we have that both $\|(\frac{\partial}{\partial k})^j g\|_{H^{1/2}(U)}$ and $\|(\frac{\partial}{\partial k})^j \lambda\|_{H_0^{-1/2}(U)}$ grow at most polynomially for $j=0,1$. As $u_2 = \langle \lambda, g\rangle$ we obtain (89).

Obviously we also have $Z(k) \le C(\omega)(1+k)^{m_2}$ for some $m_2 \ge 0$, whence the boundedness of the domain U and the Cauchy-Schwartz inequality yield an integer $\ell > 0$ so that

$$|\frac{d}{dk}Z(k)| \le C(1+k)^{\ell}.$$

Choose $k_j = j^{1/(\ell+2)}$. The Proposition 5.11 yields for every $\varepsilon > 0$ that

$$\sum_{j=1}^{\infty}\mathcal{P}(Z(k_j) > \varepsilon) \le \sum_{j=1}^{\infty} 2\exp(-C\varepsilon^{1/2}j^{1/(4(\ell+2))}) < \infty.$$

The Borel-Cantell lemma then shows that

$$\lim_{j\to\infty} Z(k_j) = 0 \quad \text{almost surely.}$$

On the other hand, if $k_j < k < k_{j+1}$ we have

$$\begin{aligned} |Z(k) - Z(k_j)| &\le C(1+k_{j+1})^{\ell}(k_{j+1}-k_j) \\ &\le C' j^{\ell/(\ell+2)}((j+1)^{1/(\ell+2)} - j^{1/(\ell+2)}) \\ &\le C'' j^{\ell/(\ell+2)} j^{-1+1/(\ell+2)} = C'' j^{-1/(\ell+2)} \to 0 \end{aligned}$$

as $j\to\infty$, which finishes the proof of the Theorem. □

6 Convergence of ergodic averages of data

Now we are ready to analyze the measurement $m(x, y, \omega)$. First, since the function $\frac{1}{K-1}\int_1^K k^4|u_s(x, y, k, \omega)|^2$ in Definition 7 is non-negative, the existence of the distribution limit (7) is equivalent to the existence of limits

$$M(\phi, \psi, \omega) = \lim_{K\to\infty} \frac{1}{K-1}\int_1^K \left(\int_U\int_U k^4|u_s(x, y, k, \omega)|^2\phi(x)\psi(y)\, dxdy\right) dk \tag{90}$$

for $\phi, \psi \in C_0^\infty(U)$. Moreover, if these limits exists, knowledge of measurement $m(x, y, \omega)$ is equivalent to knowing $M(\phi, \psi, \omega)$ for all $\phi, \psi \in C_0^\infty(U)$. Observe that

$$M(\phi, \psi, \omega) = \langle m(\cdot, \cdot, \omega), \phi \otimes \psi\rangle.$$

In this section we will finally identify the outcome of the measurement in a form that contains explicitly the unknown parameter $\mu(x)$. At the same time this will show that the measurement is well-defined. All this is contained in the following Theorem.

Theorem 6.1 *Almost surely the limit* (90) *exist for every* $\phi, \psi \in C_0^\infty(U)$ *and has the value*

$$M(\phi, \psi, \omega) = \int_U\int_U R_2(x, y)\phi(x)\psi(y)dxdy, \tag{91}$$

where $R_2(x, y)$ *is a smooth function on* $U\times U$*, given by the formula (47) with* $m = 2$*.*

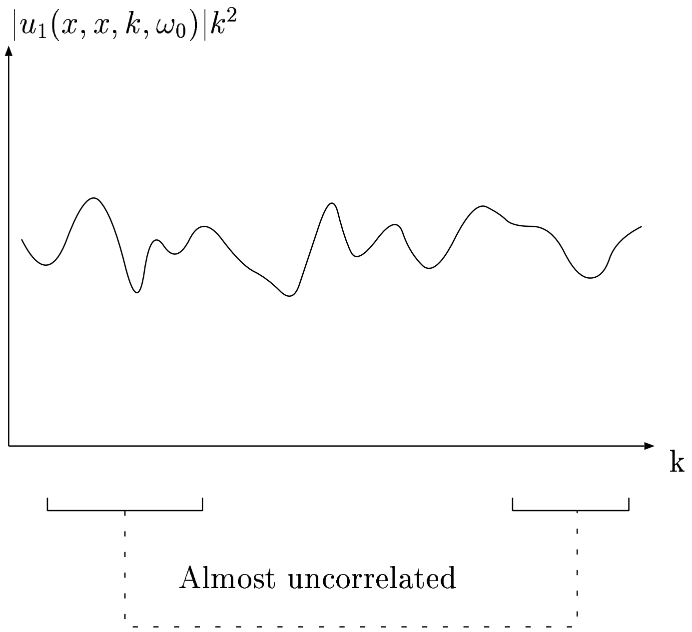
Recall that we have the relation $R_2(x, y) = \lim_{k\to\infty} k^4R_2(x, y, k, k)$, where $R_2(x, y, k, k)$ is defined by the formula (67) in Section 4. The values $R_2(x, y)$ can be expressed in terms of the function μ, which shows that the above result reduces the determination of μ to a deterministic problem.

Before giving the proof of we first describe the philosophy behind Theorem 6.1 and establish some auxiliary results. Using decomposition (45) with $m = 2$ we write

$$u_s(x, y, k) = u_1^2(x, y, k) + u_2(x, y, k) + u_R(x, y, k),$$

where $u_R = b^2 + (u_3 + u_4 + \ldots)$ stands for the rest term. The Fourier and stochastic analysis of the previous section actually shows that the second term is negligible. The contribution of the rest term u_R is likewise zero, which is is a consequence of Theorem 3.5 and Lemma 4.2, which actually were results of purely deterministic analysis. Finally we shall establish below an ergodic behaviour for the first term u_1^m with respect to k, that is based on Lemma 4.1 from Section 4, which in turn were obtained by a microlocal analysis of the covariance operator.

To be more specific, the analytic estimates of Section 4 imply that the expectation $\mathbb{E}\, k^4 |u_1^m(x,y,k)|^2$ tends uniformly over $x, y \in U$ to a limit as $k \to \infty$. In addition, the same estimates also show that the terms $k_1^2 u_1^m(x,y,k_1)$ and $k_2^2 u_1^m(x,y,k_2)$ become asymptotically independent as k_2 grows towards infinity (see the figure below). This makes us to expect that one might recover $\lim_{k\to\infty} \mathbb{E}\, |k^4 u_1^m(x,y,k)|^2$ as a suitable ergodic average, in view of the strong law of large numbers. In what follows we will make this precise.



We start by recording a simple lemma.

Lemma 6.2 *Let X and Y be zero-mean Gaussian random variables. Then*

$$\mathbb{E}\left((X^2 - \mathbb{E}\, X^2)(Y^2 - \mathbb{E}\, Y^2)\right) = 2(\mathbb{E}\, XY)^2.$$

Proof. By scaling one may obviously assume that $\mathbb{E}\,X^2 = \mathbb{E}\,Y^2 = 1$. Denote $\mathbb{E}\,XY = \cos\alpha \in [-1,1]$. Then $(X,Y) \sim (X, \cos(\alpha)X + \sin(\alpha)Y')$ holds in distribution, where Y' is an independent copy of X. The result follows now by a straightforward computation. □

We recall the law of large numbers in a form that is suitable our purposes. The following is obtained e.g. as an immediate corollary of [10].

Theorem 6.3 *Let X_t, $t \geq 0$ be a real valued stochastic process with continuous paths. Assume that for some positive constants $c, \varepsilon > 0$ the condition*

$$|\mathbb{E}\,X_t X_{t+r}| \leq c(1+r)^{-\varepsilon}$$

holds for all $t, r \geq 0$. Then almost surely

$$\frac{1}{K}\int_1^K X_t dt \to 0 \quad \text{as} \quad K \to \infty.$$

However, we are not able to apply this directly since we do not possess good enough estimates for covariances $\mathbb{E}\,u_1^m(x_1,y_1,k_1)u_1^m(x_2,y_2,k_2)$ for arbitrary $x_1, x_2,\ y_1,\ y_2 \in U$. For this reason, in the following Proposition we show that the classical argument can be modified to yield the ergodicity in our case. For reader's convenience, the details of the modification are presented.

Proposition 6.4 *Almost surely one has for all $\phi, \psi \in C_0^\infty(U)$ that*

$$\lim_{K\to\infty} \frac{1}{K-1}\int_1^K \left(\int_U\int_U k^4 |u_1^m(x,y,k)|^2 \phi(x)\psi(y)\,dxdy\right) dk$$
$$= \int_U\int_U R_2(x,y)\phi(x)\psi(y)\,dxdy.$$

Proof. Recall that according to Lemma 4.1 we have

$$R_2(x,y) = \lim_{k\to\infty} \mathbb{E}\,(k^4|u_1^m(x,y,k)|^2).$$

Let us define for $k > 0$ the random variable $X(k)$ by setting

$$X(k) \ := \ k^4 \int_U\int_U (|u_1^m(x,y,k)|^2 - \mathbb{E}\,|u_1^m(x,y,k)|^2)\phi(x)\psi(y)dxdy.$$

The convergence of $k^4\mathbb{E}\,|u_1^m(x,y,k)|^2 \to R_m(x,y)$ is uniform with respect to x and y, according to Lemma 4.1. Hence it is clear that the claim follows as we show that $\lim_{K\to\infty} \frac{1}{K-1}\int_1^K X(k)dk = 0$.

For that end we write

$$\begin{aligned} & Y(x,y,k) := k^4(|u_1^m(x,y,k)|^2 - \mathbb{E}\,|u_1^m(x,y,k)|^2) \\ = \;& k^4\left((\mathrm{Re}\,u_1^m(x,y,k))^2 - \mathbb{E}\,(\mathrm{Re}\,u_1^m(x,y,k))^2\right)+ \\ & +(\mathrm{Im}\,u_1^m(x,y,k))^2 - \mathbb{E}\,(\mathrm{Im}\,u_1^m(x,y,k))^2\big)\,. \end{aligned}$$

Now Corollary 4.3 together with Lemma 6.2 yields that

$$E|Y(x,y,k_1)Y(x,y,k_2)|^2 \le \frac{C}{1+|k_1-k_2|^2},$$

which estimate is uniform over $x,y \in U$. Given any positive integer $m>1$ we thus have

$$\begin{aligned} & \mathbb{E}\left|\frac{1}{m^2-1}\int_1^{m^2} Y(x,y,k)dk\right|^2 \\ & = \frac{1}{(m^2-1)^2}\int_1^{m^2}\int_1^{m^2} \mathbb{E}\,(Y(x,y,k_1)Y(x,y,k_2))dk_1dk_2 \\ \le \;& \frac{1}{(m^2-1)^2}\int_1^{m^2}\int_1^{m^2} \frac{C}{|k_1-k_2|^2+1}dk_1dk_2 \le \frac{C}{m^2-1}, \end{aligned}$$

where the constants are uniform with respect to $x,y\in U$. In other words, by denoting by Ω the underlying probability space we have

$$\|\frac{1}{m^2-1}\int_1^{m^2} Y(x,y,k)dk\|_{L^2(\Omega)} \le C(m^2-1)^{-1/2}.$$

Hence an application of the Minkowski inequality (and the boundedness of the functions ϕ and ψ) shows that

$$\|\frac{1}{m^2-1}\int_1^{m^2} X(k)dk\|_{L^2(\Omega)} \le C(m^2-1)^{-1/2}.$$

A fortiori,

$$\mathbb{E}\left(\sum_{m=2}^{\infty}|\frac{1}{m^2-1}\int_1^{m^2} X(k)dk|^2\right) < \infty,$$

which immediately yields the partial result

$$\lim_{m\to\infty}\frac{1}{m^2-1}\int_1^{m^2} X(k)dk = 0 \quad \text{almost surely;} \tag{92}$$

that is, convergence for an increasing sequence of K:s. In order to obtain the full result we assume that $m^2 \leq K \leq (m+1)^2$ (now $m = m(K)$) and argue in a standard manner

$$
\begin{aligned}
& |\frac{1}{K-1}\int_1^K X(k)dk - \frac{1}{m^2-1}\int_1^{m^2} X(k)dk| \\
\leq\ & \frac{2m+1}{(K-1)}\frac{1}{m^2-1}|\int_1^{m^2} X(k)dk| + \frac{1}{K-1}|\int_{m^2}^K X(k)dk|.
\end{aligned}
$$

As $K \to \infty$ the first term tends to zero by (92). The second term is bounded by the random variable

$$
Z(m) = \frac{1}{m^2-1}\int_{m^2}^{(m+1)^2} |X(k)|dk.
$$

In order to estimate $Z(m)$ we observe first that $\|Y(x,y,k)\|_{L^2(\Omega)}$ is bounded by a finite constant, let us call it A, that is independent of x, y, k. Hence the Minkowski inequality yields

$$
\|Z(m)\|_{L^2(\Omega)} \leq \frac{1}{m^2-1}\int_{m^2}^{(m+1)^2}\int_U\int_U CAdxdydk \leq \frac{C}{m-1}.
$$

This forces $\mathbb{E}\sum_{m=2}^{\infty}|Z(k)|^2 < \infty$, whence $\lim_{m\to\infty} Z(m) = 0$ almost surely.

Together with the previous estimates this proves the claim and finishes the proof of our Theorem for fixed ψ and ϕ. That the claim holds almost surely for arbitrary ψ and ϕ is easily deduced by first considering a dense denumerable set G of functions in $C_0^\infty(U)$. In this context it is useful to observe that we actually get a variant of the statement of the Proposition, where ψ, ϕ are replaced by $\|\psi\|_\infty\|\phi\|_\infty$ on the right hand side, and lim by lim sup on the left hand side. $\square$

We are ready for

Proof of Theorem 6.1. We simply collect our previous estimates for the terms in the decomposition

$$
u_s(x,y,k) = u_1^m(x,y,k) + u_2(x,y,k) + u_R(x,y,k).
$$

Let $\phi, \psi \in C_0^\infty(U)$ be given. As ψ and ϕ are bounded, Theorem 5.12 verifies that almost surely

$$
\lim_{k\to\infty}\int_U\int_U k^4|u_2(x,y,k)|^2\phi(x)\psi(y)dxdy = 0. \tag{93}
$$

In a similar vain, Theorem 3.5 and Lemma 4.2 show that (93) remains true if u_2 is replaced by the residual term u_R. These estimates can be of course be averaged over frequencies yielding

$$\lim_{K\to\infty} \frac{1}{K-1}\int_1^K \Big(\int_U\int_U (k^4|u_2(x,y,k)|^2 + \\ +k^4|u_R(x,y,k)|^2)\phi(x)\psi(y)\,dxdy\Big)dk = 0. \tag{94}$$

The desired statement now follows directly by combining (94) and Proposition 6.4, as the possible cross terms may be estimated with the aid of the Cauchy-Schwartz inequality in the space $[1,K]\times U\times U$ equipped with the weight $\phi(x)\psi(y)/(K-1)$ if $\phi,\psi\geq 0$, and the general case follows immediately form this. □

7 Conclusion: proof of Theorem 1.4

The results obtained so far (Theorem 6.1 from the previous section) prove directly parts (i) and (ii) of our main result, Theorem 1.4: the measurements are almost surely well defined and can be expressed in the form

$$M(\phi,\psi,\omega) = \int_U\int_U R_2(x,y)\phi(x)\psi(y)dxdy \quad \text{for all } \ \phi,\psi\in C_0^\infty(U).$$

It remains to prove part (iii) of the Theorem, which deals with the recovery of μ from the measurements. For that end, recall first that the results of Section 4 imply that R_2 is continuous on $U\times U$. We fix $x_0\in U$ and a standard bumb function $\phi_0\in C_0(\mathbf{R}^2)$, and choose in the above formula $\phi(x)\psi(y) = n^4\phi_0(n(x-x_0))\phi_0(n(y-y_0))$, with $n\geq 1$. Since $n^4\phi_0(n(x-x_0))\phi_0(n(y-y_0))$ tends weak* (in the sense of measures) to $\delta_{x_0}(x)\delta_{x_0}(y)$ as $n\to\infty$, we see that

Lemma 7.1 *The measurements $M(\phi,\phi,\omega)$ for $\phi,\phi\in C_0^\infty(U)$ determine almost surely the backscattering coefficient $R_2(x,x) = m_0(x,x)$.*

Observe that also the measurements in the above Lemma correspond, in a sense, to backscattering data.

According to formula (48) in Section 4 the following equality holds

$$R_2(x,x) = \frac{1}{2}\int_D \frac{1}{|x-z|^2}\mu(z)\,dz$$

Hence, in order to prove part (iii) of our main result, we are left with a deconvolution problem: the values of the convolution

$$K_2(x) = (h * \mu)(z), \quad h(z) = \frac{1}{2|z|^2}$$

are known in a open set U that has a positive distance to the support of $\mu \in C_0^\infty(\mathbb{R}^2)$, and we are to show that this knowledge is enough to recover μ. For that end, observe first that

$$\Delta_z \frac{1}{|z|^{2p}} = \frac{4p^2}{|z|^{2p+2}}.$$

Thus our data determines also the convolutions

$$c_p \Delta_x^p K_2(x) \;\; = \;\; \int_D \frac{1}{|x - z|^{2p}} \mu(z)\, dz$$

for $p > 1$ and $x \in U$. Let us denote

$$S(x,r) \;\; = \;\; c \int_{|z-x|=r} \mu(z)\, d|z|,$$

which correspond to the Radon transform along circles. Fix any $x \in U$. It follows that we are able to recover the integrals

$$\int_{\mathbb{R}_+} \frac{S(x,r)}{r^2} Q(\frac{1}{r^2})\, dr,$$

where $Q(t) = \sum_{j=0}^p a_j t^j$, $p \geq 0$. The support of the continuous function $r \mapsto S(x,r)$ lies in a finite interval $[a,b]$ with $a,b > 0$, and obviously the functions of the form $Q(1/r^2)$ are dense in $C([a,b])$. Thus the function $S(x,r)$ is uniquely determined for all $r > 0$.

The observation that we just made can be stated in another form: the data yields the knowledge of integrals of μ over all circles that are centered in the open set U. This is a classical problem of integral geometry, of the Radon type, which can be solved by a simple manner. Namely, let $g(z) = \exp(-|z|^2/2)$ for $z \in \mathbb{R}^2$, and observe that knowing the integrals over the above mentioned circles we may compute the values convolution $g * \mu$ in the set U. However, $g*\mu$ is clearly real analytic and the set U is open, whence we know $g*\mu$ everywhere. As the Fourier transform of g is smooth and non-zero all over $\mathbb{R}^2$, it follows that we can recover μ uniquely. This completes the proof of our main result. $\square$

Remark 1. We have assumed that the potential is Gaussian and centered. It is possible to dispense with the assumption that $\mathbb{E}\, q = 0$ in Theorem 1.4. Namely, assume that $\mathbb{E}\, q = p \in C_0^\infty(D)$ and denote by $q_0 = q - b$ the random potential with zero expectation. Then

$$\mathbb{E}\,(q(z_1)q(z_2)) = \mathbb{E}\,(q_0(z_1)q_0(z_2)) + p(z_1)p(z_2). \tag{95}$$

Next we analyze how the above proof should be modified for this case. Clearly, the fact that $q \in W_0^{-\varepsilon,p}(D)$ a.s. given in Section 2 is valid. Thus the results for the direct scattering problem given in Section 3 are valid without any change, and we see in particular that the higher order Born terms $u_3 + u_4 + \dots$ do not contribute to measurement (7).

In Section 4, where we analyze the first order Born term, we can obtain the essential formula (64) just like before. When the term $p(z_1)p(z_2)$ in formula (95) is added to the covariance operator, we see that this causes only a $S_{1,0}^{-\infty}$ perturbation for the symbol of the covariance operator C_q and thus also for the symbol $c_5^{\vec{j}}(w, x, y, \xi)$. Hence, continuing after formula (64) just as in the proof of Proposition 4.1, we see that proposition 4.1 is valid without any change.

Next we analyze 2nd order Born term u_2. When the term $p(z_1)p(z_2)$ in formula (95) is added to the covariance operator we see that $\mathbb{E}\,(q(z_1)q(z_2))$ has the same form as given before in Proposition 2.2. Using this, we see that all conclusions given for u_2 in Section 5 are valid.

Since the behaviour of all Born terms $u_1, u_2, u_3, \dots$ are same as in the case where $\mathbb{E}\, q = 0$ and since by the stationary phase method $\mathbb{E}\, u_1^m(x, y, k) = o(k^{-\infty})$, we obtain Theorem 1.4 by finishing the proof as in Sections 6 and 7.

Remark 2. It is interesting to compare (non-) stability of deterministic and stochastical inverse problems. In Theorem 1.4 the operator T is linear and thus the reconstruction of μ requires solving of a linear ill-posed inverse problem. More precisely, by the observations in the present section, T corresponds to a Radon transform over circles, which gives a pretty clear picture of the ill-posedness. This is markedly different to the corresponding deterministic problems: Let us assume that in addition to the amplitude $|u_s(x, y, k)|$ we are also given the phase. Then one can in principle find from the extended data $u_s(x, y, k)$, $x, y \in U$, $k \in \mathbb{R}_+$ the solution $u_s(x, y, k)$ for

$x, y \in \mathbb{R}^2 \setminus D$, $k \in \mathbb{R}_+$ by using analytic continuation. After this, by taking a suitable limit $|x|, |y| \to \infty$ with $x = |x|\omega$ and $y = |y|\omega$ one can obtain from the asymptotics of the solution the far field data $u_\infty(\theta, \omega, k)$. As a final step the Fourier transform of $\hat{q}(\xi)$ is obtained from $u_\infty(\theta, \omega, k)$ by letting $k \to \infty$ and keeping (θ, ω, k) on the manifold $k(\theta - \omega) = \xi$ (see e.g. [40, pp. x-xi], or [51] for an alternative solution based on exponentially growing plane waves). The procedure just described is remarkably instable and presumes considerably more information on data.

Appendix A: Markov random fields

We follow here the monograph of Rozanov [45] and recall in more detail the relation of local operators Markov random fields.

As in Section 1.2 we consider a bounded domain $D \subset \mathbb{R}$ and a Gaussian centered random variable q having values in $\mathcal{D}'(D)$. We simply call q a field on D. The Markov property of q was defined previously in Definition 1.1. One can also characterize the Markov property in terms of conditional independence. Recall that of three sub-σ-algebras Σ_i, $i = 0, 1, 2$ of Σ the σ-algebras Σ_1 and Σ_2 are called *conditionally independent with respect to* Σ_0 if the conditional probabilities satisfy

$$P(A_1 \cap A_2 | \Sigma_0) = P(A_1|\Sigma_0) P(A_2|\Sigma_0)$$

for any $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$. Let S_1, S_2, S_ε be as in Section 1.2. Thus, $S_1 \subset D$ is an arbitrary open subset, $S_2 = D \setminus \overline{S_1}$, and $S_\varepsilon = \{x \in D | d(x, \partial S_1) < \varepsilon\}$. We have

Theorem 7.2 *A generalized random field q is Markov, if and only if the σ-algebras $\mathcal{B}(S_1)$ and $\mathcal{B}(S_2)$ are conditionally independent with respect to $\mathcal{B}(S_\varepsilon)$, for any S_1, S_2 and S_ε as above.*

The above claim follows readily from the definition and basic properties of the the conditional probability and generalized random variable. We refer to [45, pp.54, 56, 97] for the proof and the definition of a Gaussian Markov fields.

In order to be able to express the Markov property in terms of the covariance operator we need to introduce the notion of a biorthogonal field p. Thus, a centered Gaussian field p on D is biorthogonal to the field q if

$$\mathbb{E}\left(\langle q, \psi_1 \rangle \langle p, \psi_2 \rangle\right) = (\psi_1, \psi_2), \quad \psi_i \in C_0^\infty(D)\ , i = 1, 2, \tag{96}$$

and, in addition there is the equality $H_q(D) = H_p(D)$. Here, given a field η on D and open subset $S \subset D$ we denote

$$H_\eta(S) = \overline{\text{span}}\{\langle \eta, \phi \rangle | \phi \in C_0^\infty(S)\},$$

where the closure in taken in $L^2(\Omega)$, i.e. in the space of square integrable random variables. By the definition, the biorthogonal function is unique, and the covariance operator C_p can be thought as a (partial) inverse operator of the covariance operator C_q of our original field. We have the following:

Theorem 7.3 *If the field q is Markov, then the covariance operator C_p of the biorthogonal field is* <u>*local*</u> *in the sense that $\langle C_p \psi_1, \psi_2 \rangle = 0$ if $\psi_1, \psi_2 \in C_0^\infty(D)$ have disjoint supports.*

See [45, pp.112-113] for this fact.

In order to obtain a converse statement we must assume slightly more than just biorthogonality from the relation between q and p. That is, p must be *dual* to q. Let us define $H_\eta^+(S) = \bigcap_{\varepsilon>0} H_\eta(S_\varepsilon)$, where the intersection is taken over all ε-neighborhoods S_ε of the subset $S \subset D$. The biorthogonal field p is *the dual field* of q if the equality

$$H_p(S) = H_q^+(D \setminus S)^\perp \tag{97}$$

holds for all open subsets $S \subset D$. There are useful sufficient conditions ([45, Lemma 1-2, pp. 108-109]) for the duality to hold. We then have (c.f. [45, Theorem on p. 112]) the converse result

Theorem 7.4 *Assume that the field p is dual to the field q (i.e. (97) holds). Then q is Markov if the covariance operator C_p is local.*

Put together, if the duality holds, then the Markov property is equivalent to the locality of the dual field.

By using above results, one easily verifies that there are always Markov fields q on D such that the inverse of the covariance operator $C_q^{-1} = C_p$ has the principal part $-a(x)\Delta$, assuming that $a \in C^\infty(D)$ satisfies $\inf_D a(x) > 0$. Actually, by considering the operator $-\nabla \cdot a(x)\nabla$, we obtain this statement easily from the following result in [45].

Theorem 7.5 *Assume that the covariance operator C_p of the biorthogonal field p is a partial differential operator of order $2\ell \geq 1$ with smooth coefficients, and it satisfies the coercivity inequality* $(c > 0)$

$$\langle Q_p \phi, \phi \rangle \geq c \sum_{|\alpha| \leq \ell} \|D^\alpha \phi\|^2_{L^2(D)}, \quad \phi \in C_0^\infty(D).$$

Then the field q is Markov.

See [45, Theorem 3 p.129] for this statement.

Appendix B: Lemmata on bilinear forms of Gaussian variables.

Proof of Lemma 5.1. Since q is Gaussian, we know that

$$E(X_1 \dots X_4) \;=\; c \int_{\mathbb{R}^4} x_1 x_2 x_3 x_4 f(x) dx,$$

where $f(q) = e^{-\langle B^{-1}x, x\rangle/2}$ and $B_{i,j} = b_{ij}$. By taking Fourier transform we see that

$$E(X_1 \dots X_4) \;=\; c_1 \frac{\partial}{\partial \xi_1} \dots \frac{\partial}{\partial \xi_4} \widehat{f}(0).$$

As

$$\widehat{f}(\xi) = c_2 e^{-\langle B\xi, \xi\rangle/2}, \quad \frac{\partial}{\partial \xi_i} e^{-\sum_{i,j} b_{ij}\xi_i \xi_j/2} = -\sum_j \frac{b_{ij}\xi_j}{2} e^{-b_{ij}\xi_i\xi_j/2},$$

we may compute derivatives of $\widehat{f}$ at zero and observe that only those terms are non-zero where every second derivative 'hits' to the exponential function and every second to the factors ξ_j. By this manner one deduces the formula

$$E(X_1 \dots X_4) = c_3(b_{12}b_{34} + b_{13}b_{24} + b_{14}b_{23}),$$

and the constant c_3 is easily evaluated by choosing $(X_1, \dots, X_4)$ to be the standard normal distribution in $\mathbb{R}^4$. $\square$

Proof of Lemma 5.6. By Gram-Schmidt orthogonalization procedure we can rewrite Y as a bilinear form of orthonormal, and hence independent

Gaussian variables. We may thus assume that the X_i:s are i.i.d. standard Gaussians. Let us first consider the case where the coefficients a_{jk} are real. Then we may assume that $a_{jk} = a_{kj}$, whence the corresponding bilinear form may be diagonalized:

$$Y = \sum_{k=1}^{n} \lambda_k (\sum_{j=1}^{n} b_{kj} X_j)^2,$$

where $\lambda_k \in \mathbb{R}$ for all k and $[b_{kj}]$ is an orthogonal matrix. Hence, by writing

$$\sum_{j=1}^{n} b_{kj} X_j = Y_k$$

we have

$$Y = \sum_{k=1}^{n} \lambda_k (Y_k)^2,$$

where the Y_k are i.i.d. Gaussian random variables.

Observe that $(Y_k)^2$ has the density function $(2\pi)^{-1/2} e^{-x/2} x^{-1/2} \chi_{[0,\infty)}(x)$. Let $a < 1/2$. We may compute

$$\mathbb{E}\, e^{aY_k^2} = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(a-1/2)x} x^{-1/2} dx = (1-2a)^{-1/2}.$$

This was evaluated by the substitution $x = u/(1/2 - a)$. Hence, for $|a| < (1/2) \max(|\lambda_1|^{-1}, \ldots, |\lambda_n|^{-1})$ we obtain by independence

$$\mathbb{E}\,(e^{aY} + e^{-aY}) = \prod_{k=1}^{n} (1 - 2\lambda_k a)^{-1/2} + \prod_{k=1}^{n} (1 + 2\lambda_k a)^{-1/2}. \tag{98}$$

Observe that

$$\mathbb{E}\, Y^2 = \mathbb{E}\,(\sum \lambda_k (Y_k^2 - 1) + \sum \lambda_k)^2 = 2 \sum \lambda_k^2 + (\sum \lambda_k)^2$$

where we used the facts $\mathbb{E}\, Y_k^2 = 1$ and $\mathbb{E}\, Y_k^4 = 3$.

By scaling it is enough to consider the case $\|Y\|_2 = 1$. The last computation we made shows that it is enough to prove a uniform tail estimate under the assumption

$$\sum_{k=1}^{n} \lambda_k^2 + |\sum_{k=1}^{n} \lambda_k| \leq 1. \tag{99}$$

Especially, we then have $|\lambda_k| \leq 1$ for all k.

For $|x| \leq 1/2$ it holds that $\log(1+x) \geq x - x^2$, or in other words $-\frac{1}{2}\log(1-x) \leq \frac{1}{2}(x+x^2)$. Hence, by choosing $a = 1/4$ in (98) we obtain

$$\prod_{k=1}^{n}(1-\lambda_k/2)^{-1/2} = \exp(\sum_{k=1}^{n} -\frac{1}{2}\log(1-\lambda_k/2)) \leq \exp(\sum_{k=1}^{n}(\lambda_k+\lambda_k^2)) \leq e.$$

The same computation yields that $\prod_{k=1}^{n}(1+\lambda_k/2)^{-1/2} \leq e$. Hence

$$\mathbb{E}\,(e^{Y/4} + e^{-Y/4}) \leq 2e.$$

This shows that $\mathcal{P}(|Y| > \lambda) \leq 2ee^{-\lambda/4}$. By considering $Y/\|Y\|_2$ we obtain (without assuming that $\|Y\|_2 = 1$) the inequality

$$\mathcal{P}(|Y| > \lambda) \leq (2e)e^{-\lambda/4\|Y\|_2}$$

(the case $\|Y\|_2 = 0$ being trivial). In order to finally obtain the statement of the Lemma we apply this separately to the imaginary and real parts of Y by using the inequality $\mathcal{P}(|Y| > \lambda) \leq \mathcal{P}(|\mathrm{Re}\,Y| > \lambda/2) + \mathcal{P}(|\mathrm{Im}\,Y| > \lambda/2)$. □

Proof of Lemma 5.7. We consider separately two cases.

1. $\sup_k \|X_k\|_2 = b < \infty$.

Under this assumption the variables X_k are uniformly integrable; we have for all k and $\lambda > 0$

$$\mathcal{P}(|X_k| > \lambda) \leq 12\exp(-\lambda/8b)$$

(we may clearly assume that $b > 0$). Hence we infer that

$$\|X_k\|_2 \to \|X\|_2 \quad \text{as } k \to \infty, \tag{100}$$

in particular $X \in L^2$. Moreover, Fatou's lemma yields for any $\varepsilon > 0$ that

$$\begin{aligned}
\mathcal{P}(|X| > \lambda) &= \mathbb{E}\,(\chi_{[-\lambda,\lambda]^c} \circ |X|) \leq \mathbb{E}\,(\liminf_{k\to\infty} \chi_{[-\lambda+\varepsilon,\lambda-\varepsilon]^c} \circ |X_k|) \\
&= \lim_{k\to\infty} \mathcal{P}(|X_k| > \lambda - \varepsilon) \leq \lim_{k\to\infty} 12\exp(\frac{\varepsilon-\lambda}{8\|X_k\|_2}) \\
&= 12\exp(\frac{\varepsilon-\lambda}{8\|X_k\|_2}).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we obtain the claim and case 1 is settled.

2. $\sup_k \|X_k\|_2 = \infty$.

In this case we may select a subsequence, that we still denote by (X_k), such that $\|X_k\|_2 \geq k$. Write $Y_k = X_k/\|X_k\|_2$. Then $Y_k \in \mathcal{A}$ and $\|Y_k\|_2 = 1$ for all k, whence be are back in case 1. Combined with (100) this shows that the square integral of the pointwise limit of Y_k equals 1. However, this is a contradiction, since this limit is zero a.s. by the assumption. Thus case 2 cannot occur at all. □

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