MINIMAL L^∞-TYPE SPACES ON STRICTLY PSEUDOCONVEX DOMAINS ON WHICH THE BERGMAN PROJECTION IS CONTINUOUS

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Abstract. We describe the space of functions on a smoothly bounded strictly pseudoconvex domain such that (i) the Bergman projection is continuous on it; (ii) its topology is given by a family of weighted L^∞-norms, with weights depending only on a given defining function; (iii) it contains L^∞; and (iv) it is contained continuously into any other function space satisfying (i)–(iii). This generalizes the results obtained by the third author for the unit disc. We also obtain analogous assertions for the standard weighted Bergman projections, and, under the additional hypothesis that the domain be complete circular, also for the pluriharmonic Szegö projection.

1. Introduction

It is well known that the Szegö projection S is not bounded on $L^\infty(\partial\mathbf{D})$, and the Bergman projection P is not bounded on $L^\infty(\mathbf{D})$. Here $\mathbf{D}$ is the unit disc in the complex plane $\mathbf{C}$ and $\partial\mathbf{D}$ its boundary.

In his paper [15], J. Taskinen introduced certain locally convex spaces $L^\infty_V(\mathbf{D})$, $h^\infty_V(\mathbf{D})$ and $H^\infty_V(\mathbf{D})$ such that

(i) $L^\infty(\mathbf{D}) \subset L^\infty_V(\mathbf{D})$, and h^∞_V and H^∞_V are the (closed) subspaces of all harmonic and holomorphic functions in L^∞_V, respectively;

(ii) the Szegö projection is continuous from h^∞_V onto H^∞_V, and the Bergman projection is continuous from L^∞_V onto H^∞_V;

(iii) the topology in L^∞_V is given by a family of weighted sup-norms with radial weights (that is, by a collection of seminorms of the form

$$\|f\|_v := \sup_x v(x)|f(x)|$$

where v is a radial (i.e. $v(x) = v(|x|)$) positive continuous weight function on $\mathbf{D}$);

(iv) L^∞_V is the smallest space with the properties (i)–(iii), in the sense that it is contained continuously into any other such space.

Thus, the spaces L^∞_V, h^∞_V and H^∞_V are, as far as the boundedness of the Bergman or the Szegö projection is concerned, the best possible substitutes for L^∞, h^∞ and H^∞ on $\mathbf{D}$, respectively. The correct choice of the family of weights in (iii) turns out to be the collection of all radial $v : \mathbf{D} \to \mathbf{R}_+$ such that

$$\sup_{z\in\mathbf{D}} v(z)|\log(1-z)|^n < \infty \qquad \forall n = 0, 1, 2, \ldots.$$

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The aim of the present note is to extend the results of [15] in several ways. In more detail, our main results are:

- we extend the result on the boundedness of the Szegö projection to the pluriharmonic Szegö projections on complete circular domains in $\mathbf{C}^n$;
- we extend the result on the boundedness of the Bergman projection to the weighted Bergman projections P_α, $\alpha > -1$, on any smoothly bounded strictly pseudoconvex domain in $\mathbf{C}^n$;
- in the case of the unweighted Bergman projection, we extend the result concerning the minimality of the space L^∞_V to any smoothly bounded strictly pseudoconvex domain in $\mathbf{C}^n$;
- and similarly for the (pluriharmonic) Szegö case.

Though the proofs largely parallel the corresponding ones in [15], the main work consists in establishing the necessary generalizations of the various (often elementary) estimates on the disc to the domains in higher dimensions, where there is no explicit formula for the Bergman kernel etc. For instance, the proof of the boundedness of the Bergman projections includes a generalization of the classical Forelli-Rudin estimates [6].

The paper is organized as follows. In Section 2, we collect some preliminaries on strictly pseudoconvex domains, including the generalization of the Forelli-Rudin estimates. The spaces L^∞_V etc. are then defined in Section 3, together with a synopsis of a few auxiliary results from [15] that we will need. Section 4 contains the proofs of the second and the third items in the list above. Section 5 deals with the Szegö projections on complete circular domains (the first and the last items in the list). The final Section 6 briefly discusses some properties of the spaces introduced.

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Notation. For two functions f and g, we will say that f is dominated by g (denoted $f \lesssim g$) if there exists a constant $C > 0$, independent of the variables in question, such that $Cf \le g$; and that f and g are comparable (denoted $f \asymp g$) if $f \lesssim g$ and $g \lesssim f$.

Remark. In the final stages of writing this paper, the authors learnt of the paper [9], where the results of [15] are extended to the (unweighted) Bergman projection on the unit ball of $\mathbf{C}^n$. The method of proof of the boundedness is different from ours (actually seems much simpler), while the proof of the minimality of the space L^∞_V uses the same construction (going back to Bell [1]) as ours (applied to the trivial peaking function $a(z) = 1 - z_1$ for the ball; cf. the proof of Theorem 11 below). The weighted Bergman projections and the Szegö projection are not discussed. On a conference in Poznan in April 2003, the author also announced that he has extended his results to the pseudoconvex domains in $\mathbf{C}^n$, but as of this writing seems not to have any preprint available.

2. Pseudoconvex domains

Let Ω be a bounded domain in $\mathbf{C}^n$ with smooth ($= C^\infty$) boundary. A real-valued function $r \in C^\infty(\mathbf{C}^n)$ is called a *defining function* for Ω if

$$r > 0 \text{ on } \Omega, \quad r < 0 \text{ on } \mathbf{C}^n \setminus \Omega, \text{ and } r = 0 \text{ and } \|\operatorname{grad} r\| \neq 0 \text{ on } \partial\Omega.$$

The signed distance $\operatorname{dist}(\cdot,\partial\Omega)$ has always these properties, but it is often convenient to use other choices (such as $r(z)=1-|z|^2$ on the unit disc $\mathbf{D}$). The quadratic form on $\mathbf{C}^n$

$$L_\zeta(X) := -\sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(\zeta)\, X_j \overline{X}_k$$

is called the *Levi form* at ζ. The domain Ω is called *strictly pseudoconvex* if for each $\zeta\in\partial\Omega$, L_ζ is positive definite on the complex tangent space $\{X\in\mathbf{C}^n : \sum_j X_j \frac{\partial r}{\partial \overline{z}_j}(\zeta)=0\}$. It is known that this property is independent of the choice of the defining function r, and if Ω is strictly pseudoconvex, then one can even choose an r such that L_ζ is positive definite not only on the tangent space but on the whole $\mathbf{C}^n$, for all $\zeta\in\partial\Omega$; that is, one can choose r to be strictly plurisubharmonic on $\partial\Omega$. See e.g. Krantz [11] for more information on these matters.

For any $\alpha>-1$, the measure

$$d\mu_\alpha(z) := r(z)^\alpha\, dA(z) \qquad (dA = \text{the Lebesgue measure on } \mathbf{C}^n)$$

is finite, and one can consider the weighted Bergman spaces $L^2_{\mathrm{hol}}(\Omega, d\mu_\alpha)$ and the associated reproducing kernels $K_\alpha(x,y)$. The integral operator P_α on Ω given by

$$P_\alpha f(x) := \int_\Omega f(y) K_\alpha(x,y)\, d\mu_\alpha(y)$$

is called the *weighted Bergman projection.* (Note that for $\alpha\neq 0$, K_α and P_α depend also on the defining function r, though this is not reflected in the notation.)

The following theorem was proved for $\alpha=0$ (i.e. the ordinary Bergman kernel) by Fefferman [5], Boutet de Monvel-Sjöstrand [4] and Kerzman [10]; the general case is due to G. Komatsu (see [7]) and Peloso [12].

Theorem 1. *There exist functions a_α, b_α and r in $C^\infty(\mathbf{C}^n\times\mathbf{C}^n)$ such that*

(a) *$a_\alpha(x,y)=\overline{a_\alpha(y,x)}$, and $\partial a_\alpha(x,y)/\partial\overline{x}_j$ and $\partial a_\alpha(x,y)/\partial y_j$ as well as their partial derivatives of all orders vanish on the diagonal $x=y$; similarly for $b_\alpha(x,y)$ and $r(x,y)$;*

(b) *$r(x,x)=r(x)\ \forall x\in\mathbf{C}^n$ and $a_\alpha(x,x)>0\ \forall x\in\partial\Omega$;*

(c) *there exists a single-valued branch of $\log r(x,y)$ on $\Omega\times\Omega$, real-valued on the diagonal;*

(d) *there is $\delta>0$ such that on the set*

$$\mathcal{U}_\delta := \{(x,y)\in\Omega\times\Omega : r(x)+r(y)+|x-y|^2<\delta\}$$

the kernel $K_\alpha(x,y)$ satisfies

$$K_\alpha(x,y) = \frac{a_\alpha(x,y)}{r(x,y)^{n+\alpha+1}} + b_\alpha(x,y)\log r(x,y);$$

(e) *outside any $\mathcal{U}_\delta$, $K_\alpha(x,y)$ is C^∞ up to the boundary of $\overline{\Omega}\times\overline{\Omega}$.*

From now on, we will assume that Ω is strictly pseudoconvex and r is a strictly plurisubharmonic defining function for Ω.

Our main result in this section is the following generalization of the classical estimates of Forelli and Rudin [6].

Theorem 2. *Let $\alpha, \beta > -1$, $s > 0$. Then $\forall y \in \Omega$ and $0 < t < \frac{1}{2}\sup_\Omega r$,*

$$\int_{\{x:r(x)=t\}} |K_\alpha(x,y)|^s \, d\sigma(x) \asymp \begin{cases} [r(y)+t]^{n-q} & \text{if } n-q<0, \\ -\log[r(y)+t] & \text{if } n-q=0, \\ 1 & \text{if } n-q>0, \end{cases} \tag{1}$$

and

$$\int_\Omega |K_\alpha(x,y)|^s r(x)^\beta \, dA(x) \asymp \begin{cases} r(y)^{n-q+\beta} & \text{if } n-q+\beta+1<0, \\ -\log r(y) & \text{if } n-q+\beta+1=0, \\ 1 & \text{if } n-q+\beta+1>0, \end{cases} \tag{2}$$

where $q = (n+\alpha+1)s$.

Here $d\sigma$ denotes the $(2n-1)$-dimensional Hausdorff measure in $\mathbf{C}^n$.

Remark. Estimates similar to (2) can be found e.g. in [14], Lemma VII.7.3 (cf. also Lemma V.3.4 there) and go back to Phong and Stein (see [13], p. 698) and the references therein.

Proof. Looking at the Taylor expansion of r around x we see that

$$r(x,y) + r(y,x) - r(x) - r(y) = \frac{1}{2} L_x(x-y) + O(|x-y|^3).$$

Owing to the strict plurisubharmonicity of r, there exists therefore $\delta > 0$ such that

$$\operatorname{Re} r(x,y) \gtrsim r(x) + r(y) + |x-y|^2 \qquad \text{on } \mathcal{U}_\delta.$$

Similarly,

$$r(x,y) - r(y,x) = 2\operatorname{Im} \partial r(x)(x-y) + O(|x-y|^2),$$

so

$$|\operatorname{Im} r(x,y)| \gtrsim |\operatorname{Im} \partial r(x)(x-y)| \qquad \text{on } \mathcal{U}_\delta.$$

Consequently, by part (d) of the last theorem,

$$|K_\alpha(x,y)| \asymp [r(x) + r(y) + |x-y|^2 + |\operatorname{Im} \partial r(x)(x-y)|]^{-n-\alpha-1} \qquad \text{on } \mathcal{U}_\delta. \tag{3}$$

Also, by part (e) of the same, $|K_\alpha(x,y)|$ is bounded outside any $\mathcal{U}_\delta$. Therefore the integral in (1) satisfies

$$\int_{\mathcal{S}_t \setminus \mathcal{U}_\delta(y)} \lesssim 1$$

and

$$\int_{\mathcal{S}_t \cap \mathcal{U}_\delta(y)} \asymp \int_{\mathcal{S}_t \cap \mathcal{U}_\delta(y)} [r(x) + r(y) + |x-y|^2 + |\operatorname{Im} \partial r(x)(x-y)|]^{-q} \, d\sigma(x), \tag{4}$$

where $\mathcal{S}_t = \{x \in \Omega : r(x) = t\}$ and $\mathcal{U}_\delta(y) = \{x \in \Omega : r(x) + r(y) + |x-y|^2 < \delta\}$.

Let n_ξ denote the inward unit normal vector at $\xi \in \partial\Omega$. The smooth boundedness of Ω implies that for sufficiently small $\epsilon > 0$, the mapping

$$\Psi : (\xi, t) \mapsto \xi + t n_\xi \tag{5}$$

is a diffeomorphism of $\partial\Omega \times (-\epsilon, \epsilon)$ into $\mathbf{C}^n$, and $\mathcal{S}_t$ is the image of $\partial\Omega \times \{t\}$. Since it clearly suffices to prove (1) for $r(y) + t < \epsilon$ (the integral is a continuous function of $t > 0$ and $y \in \Omega$), we will assume this from now on. Writing $y = \eta + r(y) n_\eta$ and noting that $d\sigma|_{\mathcal{S}_t} \to d\sigma|_{\partial\Omega}$ as $t \searrow 0$, it follows that the integral on the right-hand side of (4) is

$$\asymp \int_{\{\xi\in\partial\Omega:\xi+tn_\xi\in\mathcal{U}_\delta(y)\}} [t + r(y) + |\xi + tn_\xi - \eta - r(y)n_\eta|^2 \\ + |\operatorname{Im}\partial r(\xi + tn_\xi)(\xi + tn_\xi - \eta - r(y)n_\eta)|]^{-q}\, d\sigma(\xi).$$

Denote for brevity $t + r(y) =: d$. Clearly, as $d \searrow 0$ the integration over $\{\xi \in \partial\Omega : |\xi - \eta| > \epsilon_0\}$ gives a contribution of order $O([d + (\epsilon_0 - d)^2]^{-q}) = O(1)$, for any $\epsilon_0 > 0$. Hence it suffices to deal with integration over $|\xi - \eta| < \epsilon_0$. Since (5) is a diffeomorphism, we then have $|\xi + tn_\xi - \eta - r(y)n_\eta|^2 \asymp |\xi - \eta|^2 + |t - r(y)|^2$, so $\delta + r(y) + |\xi + tn_\xi - \eta - r(y)n_\eta|^2 \asymp d + |\xi - \eta|^2$. Finally, for any $u \in \mathbf{C}^n \simeq \mathbf{R}^{2n}$, $\operatorname{Im}\partial r(x)(u) = \langle v(x), u\rangle_{\mathbf{R}^{2n}}$ for a certain vector field $v(x)$ which is tangential to $\partial\Omega$ (annihilates the defining function r). Employing another diffeomorphism to identify the neighbourhood $|\xi - \eta| < \epsilon_0$ of η in $\partial\Omega$ with a small neighbourhood of the origin in $\mathbf{R}^{2n-1}$ and $v(x)$ with $(0, 0, \dots, 0, 1)$, we are thus finally reduced to evaluating the integral

$$\int_{\{z\in\mathbf{R}^{2n-1}:|z|<\epsilon_0\}} [d + |z|^2 + |z_{2n-1}|]^{-q}\, dA(z).$$

Passing to polar coordinates in the first $2n - 2$ variables $z' := (z_1, \dots, z_{2n-2})$, this gives

$$\int_{-\epsilon_0}^{\epsilon_0} \int_0^{\sqrt{\epsilon_0^2-|z_{2n-1}|^2}} (d + \rho^2 + |z_{2n-1}|^2 + |z_{2n-1}|)^{-q} \rho^{2n-3}\, d\rho\, dz_{2n-1},$$

that is (setting $x = \rho^2$)

$$\frac{1}{2} \int_{-\epsilon_0}^{\epsilon_0} \int_0^{\epsilon_0^2-|z_{2n-1}|^2} (d + x + |z_{2n-1}|^2 + |z_{2n-1}|)^{-q} x^{n-2}\, dx\, dz_{2n-1}. \tag{6}$$

Consider now, quite generally, the integral

$$\int_0^{\epsilon_1} (x + c)^a x^b\, dx \qquad (b > -1, a \in \mathbf{R}) \tag{7}$$

where $c \searrow 0$. Making the substitution $x = sc$ leads to

$$c^{a+b+1} \int_0^{\epsilon_1/c} (1 + s)^a s^b ds.$$

For $s > 1$, $(1 + s)^a \asymp s^a$, so the last integral is

$$\int_0^1 + \int_1^{\epsilon_1/c} \asymp 1 + \int_1^{\epsilon_1/c} s^{a+b} \asymp \begin{cases} 1 + (1/c)^{a+b+1} & \text{if } a + b + 1 > 0 \\ 1 + \log(1/c) & \text{if } a + b + 1 = 0 \\ 1 & \text{if } a + b + 1 < 0. \end{cases}$$

Consequently,

$$(7) \asymp \begin{cases} 1 + c^{a+b+1} & \text{if } a + b + 1 < 0 \\ 1 + \log(1/c) & \text{if } a + b + 1 = 0 \\ 1 & \text{if } a + b + 1 > 0. \end{cases} \tag{8}$$

Applying this to the inner integral in (6), and noting that $|z_{2n-1}|^2 + |z_{2n-1}| \asymp |z_{2n-1}|$, we see that this inner integral is

$$\asymp \begin{cases} 1 + [d + |z_{2n-1}|]^{n-q-1} & \text{if } n-q-1<0 \\ 1 - \log[d + |z_{2n-1}|] & \text{if } n-q-1=0 \\ 1 & \text{if } n-q-1>0. \end{cases}$$

Upon integration with respect to z_{2n-1}, using again (8) as well as the elementary fact that

$$\int_0^{\epsilon_0} |\log(d+x)|\,dx \asymp 1 \qquad \text{as } d \searrow 0,$$

we get

$$(6) \asymp \begin{cases} d^{n-q} & \text{if } n-q<0 \\ -\log d & \text{if } n-q=0 \\ 1 & \text{if } n-q>0, \end{cases}$$

as $d \searrow 0$, which proves (1).

Finally, as

$$\int_\Omega |K_\alpha(x,y)|^s r(x)^\beta \, dA(x) = \int_0^{t_{max}} \Big(\int_{\mathcal{S}_t} |K_\alpha(x,y)|^s \, d\sigma(x) \Big) t^\beta \, dt,$$

the result just proved, together with (8) and the elementary fact that

$$\int_0^{\epsilon_0} |\log(s+t)|\, t^\beta \, dt \asymp 1 \qquad \text{as } s \searrow 0,$$

imply the second estimate (2). This completes the proof of the theorem. □

Note that the proof used, in effect, only the estimate (3) for K_α. In view of the parts (d) and (e) of Theorem 1, we therefore have also the following slightly more general version:

Theorem 3. *For any multiindices ν, κ, the assertion of the last theorem remains in force even with $K_\alpha(x,y)$ is replaced by $(\partial/\partial x)^\nu (\partial/\partial \overline{y})^\kappa K_\alpha(x,y)$, if q is replaced by $(n+\alpha+|\nu|+|\kappa|+1)s$.*

Finally, for the same reason and since $r \asymp r_1$ for any other defining function r_1 for Ω, we also have

Theorem 4. *The last theorem remains in force if r is replaced by any other defining function for Ω (e.g. not necessarily strictly plurisubharmonic on the boundary).*

3. The spaces $L^\infty_V, H^\infty_V, h^\infty_V$ and ph^∞_V

Following [15], we now define a family V of weights on $(-\infty, 1)$, and an associated family $V_{\Omega,r}$ of weights on Ω. Denote

$$l(t) = \max\{1, \log \frac{1}{1-t}\}.$$

Definition 5. A continuous strictly positive function v on $(-\infty,1)$ belongs to V if and only if $\sup_{t<1} v(t) l(t)^n < \infty$ for all $n = 0, 1, 2, \dots$.

A function w on a domain Ω with a defining function r belongs to $V_{\Omega,r}$ if $w(z) = v(1 - r(z))$ for some $v \in V$.

Clearly, $\lim_{t\nearrow 1} v(t) = 0\ \forall v \in V$. The following properties of V and $V_{\Omega,r}$ were established in [15], Lemma 2.1 and Proposition 2.2.

Proposition 6. *Let $v \in V$ be arbitrary.*

1° *If $w \in V$, then $v + w \in V$ and $\max\{v, w\} \in V$.*
2° *The decreasing function $w(t) := \sup_{s\geq t} v(s)$ belongs to V.*
3° *There exists a decreasing weight $w \geq v$ and a $C > 0$ such that for every $0 < \delta < 1$,*

$$w(1 - \frac{\delta}{2}) > C\, w(1 - \delta).$$

4° *For every $n = 0, 1, 2, \ldots$ one can find a $w \geq v$, $w \in V$ such that the function $l^k w$ is decreasing for every $0 \leq k \leq n$. Moreover, if v has the property 3°, then so has w.*
5° *There exists $w \geq v$, $w \in V$, and $C > 0$ such that for all $0 < \delta < \frac{1}{2}$,*

$$\int_{1-\delta}^{1} \frac{dt}{w(t)} \leq \frac{C\delta}{w(1-\delta)}.$$

Further, if v is decreasing then we may take w also decreasing.
6° *For every $\alpha > 0$, the weight v^α (pointwise power) belongs to V.*

Recall that a function f is called *pluriharmonic* if it is locally of the form $f = g + \overline{h}$, with g, h holomorphic; equivalently, if it is smooth and $\partial^2 f/\partial z_j \partial \overline{z}_k = 0$ for all $j, k = 1, \ldots, n$. A pluriharmonic function is always harmonic, but the converse is not true unless $n = 1$.

Now we are ready to define our main spaces of functions, as follows.

Definition 7. For $v \in V$ and $w = v \circ (1 - r) \in V_{\Omega,r}$, set

$$\|f\|_w := \|f\|_v := \sup_{z\in\Omega} w(z)\,|f(z)|. \tag{9}$$

Let

$$H^\infty_{V,r}(\Omega) = \{f : \Omega \to \mathbf{C} \text{ holomorphic}, \ \|f\|_w < \infty\ \forall w \in V_{\Omega,r}\}$$

and let $h^\infty_{V,r}(\Omega), ph^\infty_{V,r}(\Omega)$ and $L^\infty_{V,r}(\Omega)$ be defined in the same way replacing the word "holomorphic" by "harmonic", "pluriharmonic" and "measurable", respectively (take "ess-sup" in the last case).

Since Ω and r are fixed most of the time, we will usually omit them from the notation. (The omission of r will actually be justified in a moment, when we show that these spaces are, in fact, independent of the choice of the defining function.)

With the topology induced by the seminorms (9), these spaces become Hausdorff locally convex topological vector spaces which are not metrizable. Further, L^∞_V and H^∞_V are topological algebras under the pointwise multiplication. (The proof is identical to the one for $\mathbf{D}$ in [15], Theorem 5.1.)

It is clear from their definition that the spaces L^∞_V, ph^∞_V and H^∞_V possess the properties (i) and (iii) from the introduction. [1] The remainder of this paper is devoted to the investigation of (ii) and (iv).

We finish this section by showing that the above spaces are in fact independent of the choice of the defining function.

[1] We will say nothing about h^∞_V in this paper.

Corollary 8. *The spaces $L^\infty_{V,r}, H^\infty_{V,r}, h^\infty_{V,r}$ and $ph^\infty_{V,r}$ are independent of the choice of the defining function r, in the sense that for any two defining functions the corresponding spaces are equal as sets and have the same topology.*

Proof. If r, r_1 are two defining functions for Ω, then there exists an integer $k \geq 0$ such that

$$2^{-k} r_1 \leq r \leq 2^k r_1 \qquad \text{on } \Omega.$$

Let $v \in V$, and use part 3° of Proposition 6 repeatedly (k times) to produce $w \in V$ decreasing and such that, for some $C > 0$,

$$w(1 - 2^{-k}\delta) > C\, w(1 - \delta) \qquad \forall \delta > 0.$$

Then

$$\begin{aligned} \|f\|_{v,r_1} \leq \|f\|_{w,r_1} &= \sup_{\delta>0} \Big(\sup_{r_1(z)=\delta} |f(z)| \Big) w(1-\delta) \\ &\leq \sup_{\delta>0} \Big(\sup_{2^{-k}\delta \leq r(z) \leq 2^k \delta} |f(z)| \Big) w(1-\delta) \\ &\leq \sup_{\delta>0} \left(\frac{\|f\|_{w,r}}{\inf_{2^{-k}\delta \leq s \leq 2^k\delta} w(1-s)} \right) w(1-\delta) \\ &= \sup_{\delta>0} \left(\frac{\|f\|_{w,r}}{w(1-2^{-k}\delta)} \right) w(1-\delta) \\ &\leq C\, \|f\|_{w,r}. \end{aligned}$$

Thus $L^\infty_{V,r} \subset L^\infty_{V,r_1}$ continuously, and similarly for H^∞_V, h^∞_V and ph^∞_V. Interchanging the roles of r and r_1, the assertion follows. ☐

4. The Bergman projections

Our estimates from Section 2 can now be used to prove the continuity of the Bergman projections on L^∞_V on any smoothly bounded strictly pseudoconvex domain Ω. Fix a defining function r and let the spaces L^∞_V, H^∞_V, as well as the weighted Bergman projections P_α, be defined using this r.

Theorem 9. *For any $\alpha \geq 0$, the weighted Bergman projection P_α is a continuous operator on L^∞_V (onto H^∞_V).*

Proof. Let $v \in V$. Recall that we have denoted $l(t) = \max\{1, -\log(1-t)\}$. Use Proposition 6 to choose $u \in V$, $u \geq lv$ which is decreasing, and then further to choose $w \in V$, $w \geq u$ decreasing so that 5° is satisfied (with u in the place of v). Let $f \in L^\infty_V$ be such that $\|f\|_w \leq 1$. We show that $\|P_\alpha f\|_v \leq C$, i.e.

$$|f(z)| v(1 - r(z)) \leq C \qquad \forall z \in \Omega, \tag{10}$$

for some $C < \infty$ (independent of f).

Clearly it is enough to prove (10) for z close to the boundary, i.e. for $r(z) < \epsilon$ for some small $\epsilon > 0$. Let us choose ϵ so small that, as in the proof of Theorem 2, the mapping (5) is a diffeomorphism of $\partial\Omega \times [-\epsilon, \epsilon]$ onto a neighbourhood of $\partial\Omega$, and let $z \in \Omega$ satisfy $r(z) < \epsilon$. Thus $z = \Psi(x, r(z))$, with some $x \in \partial\Omega$ and $0 < r(z) < \epsilon$. Denote $z_0 = \Psi(x, \epsilon) \in \Omega$. We first prove the following lemma.

Lemma 10. *For any holomorphic function g on Ω,*

$$v(1-r(z))|g(z)-g(z_0)| \le C\sum_{j=1}^{n} \sup_{0<r(x)<\epsilon} |\partial_j g(x)| r(x) w(1-r(x)),$$

for some constant $C<\infty$ (independent of g and z).

Here (and in the sequel) $\partial_j \equiv \dfrac{\partial}{\partial z_j}$.

Proof of the lemma. Note that for any $s \in (0,\epsilon)$,

$$\int_0^1 \frac{dt}{\epsilon-(\epsilon-s)t} = \frac{\log\epsilon - \log s}{\epsilon - s} \le C\, l(1-s),$$

for some C depending only on ϵ. Since $r(z_0+t(z-z_0)) = \epsilon-(\epsilon-r(z))t$ and $|z_j - z_{0j}| \le \|z-z_0\| < \epsilon$, we thus have

$$\begin{aligned}
|g(z)-g(z_0)| &= \Big|\int_0^1 \frac{d}{dt} g(z_0+t(z-z_0))\,dt\Big| \\
&= \Big|\sum_j \int_0^1 (z_j-z_{0j})(\partial_j g)(z_0+t(z-z_0))\,dt\Big| \\
&\le \sum_j |z_j-z_{0j}| \Big(\sup_{t\in[0,1]} (r\,\partial_j g)(z_0+t(z-z_0))\Big) \int_0^1 \frac{dt}{r(z_0+t(z-z_0))} \\
&\le C\, l(1-r(z)) \sum_j \sup_{t\in[0,1]} (r\,\partial_j g)(z_0+t(z-z_0)).
\end{aligned}$$

Hence, since u decreases,

$$\begin{aligned}
&u(1-r(z))|g(z)-g(z_0)| \\
&\qquad \le C\, l(1-r(z)) \sum_j \sup_{t\in[0,1]} u(1-r(z)) \cdot (r\,\partial_j g)(z_0+t(z-z_0)) \\
&\qquad \le C\, l(1-r(z)) \sum_j \sup_{t\in[0,1]} (u(1-r)\, r\,\partial_j g)(z_0+t(z-z_0)),
\end{aligned}$$

so, as $u \le w$ and $r(z_0+t(z-z_0)) \in (0,\epsilon)$,

$$\frac{u(1-r(z))}{l(1-r(z))} |g(z)-g(z_0)| \le C\sum_j \sup_{0<r(x)<\epsilon} |\partial_j g(x)| r(x) w(1-r(x)).$$

Since $u/l \ge v$, the claim follows. □

Continuing the proof of Theorem 9, we apply the lemma to the function $g = P_\alpha f$. This gives, using the reproducing property of the Bergman kernels,

$$\begin{aligned} v(1-r(z))|P_\alpha f(z)| &\le v(1-r(z))|P_\alpha f(z_0)| \\ &\qquad + C\sum_j \sup_{0<r(x)<\epsilon} |\partial_j P_\alpha f(x)| w(1-r(x)) r(x) \\ &\le v(1-r(z))|P_\alpha f(z_0)| + C\sum_j \sup_{0<r(x)<\epsilon} w(1-r(x)) r(x) \\ &\qquad \cdot \Big| \int_\Omega f(\zeta) \frac{\partial}{\partial x_j} K_\alpha(x,\zeta)\, d\mu_\alpha(\zeta) \Big|. \end{aligned} \tag{11}$$

Since $\|f\|_w \le 1$ by hypothesis, we can bound the last integral by

$$\begin{aligned} &\int_\Omega \frac{1}{w(1-r(\zeta))} \left| \frac{\partial K_\alpha}{\partial x_j}(x,\zeta) \right| d\mu_\alpha(\zeta) \\ &\qquad = \int_0^{\max r} \frac{1}{w(1-\rho)} \int_{r(\zeta)=\rho} \left| \frac{\partial K_\alpha}{\partial x_j}(x,\zeta)\right| d\sigma(\zeta)\, \rho^\alpha\, d\rho. \end{aligned} \tag{12}$$

By Theorem 4,

$$\int_{\{\zeta : r(\zeta)=\rho\}} \left| \frac{\partial}{\partial z_j} K_\alpha(z,\zeta) \right| d\sigma(\zeta) \asymp [r(z)+\rho]^{n-q},$$

with $q = n+\alpha+2$, whence $n-q = -\alpha-2 < -1$. Consequently, (12) is bounded by

$$\begin{aligned} &\int_0^{\max r} \frac{1}{w(1-\rho)} [r(x)+\rho]^{-\alpha-2}\, \rho^\alpha\, d\rho \\ &\qquad \le \int_0^{\max r} \frac{1}{w(1-\rho)} [r(x)+\rho]^{-2}\, d\rho = \int_0^{r(x)} + \int_{r(x)}^{\max r} \\ &\qquad \le r(x)^{-2} \int_0^{r(x)} \frac{d\rho}{w(1-\rho)} + \frac{1}{w(1-r(x))} \int_{r(x)}^{\max r} \frac{d\rho}{[r(x)+\rho]^2} \\ &\qquad \le r(x)^{-2} \frac{C\, r(x)}{w(1-r(x))} + \frac{1}{w(1-r(x))} \frac{1}{2r(x)} \\ &\qquad \le \frac{C}{r(x)\, w(1-r(x))}, \end{aligned}$$

where the penultimate estimate follows from the property 5° of w (for the first summand) and the computation

$$\int_a^b \frac{d\rho}{(a+\rho)^2} = \frac{-1}{a+\rho}\Big|_{\rho=a}^b = \frac{1}{2a} - \frac{1}{a+b} = \frac{b-a}{2a(a+b)} \le \frac{1}{2a}$$

for the second summand (with $a = r(x)$ and $b = \max r$); cf. the arguments in the proof of Theorem 4.6 in [15].

Substituting this into (11), we thus get

$$v(1-r(z))|P_\alpha f(z)| \le v(1-r(z))|P_\alpha f(z_0)| + C.$$

Finally, since $K(x,y)$ is uniformly bounded if one of the arguments stays in the compact subset $\{z : r(z) \ge \epsilon\}$ of Ω, we have

$$\begin{aligned} |P_\alpha f(z_0)| &\le \int_\Omega |f(\zeta)|\,|K_\alpha(\zeta,z_0)|\,d\mu_\alpha(\zeta) \\ &\le C\int_\Omega \frac{1}{w(1-r(\zeta))}\,d\mu_\alpha(\zeta) \\ &\le C\int_0^{\max r} \frac{1}{w(\rho)}\rho^\alpha\,d\rho \le C < \infty, \qquad \text{by the property } 5^\circ, \end{aligned}$$

for some C depending only on ϵ. Putting everything together, we thus obtain

$$v(1-r(z))|P_\alpha f(z)| \le C\,v(1-r(z)) + C \lesssim C.$$

Hence $\|P_\alpha f\|_v \le C$ for some $C<\infty$ depending only on ϵ, and the proof is complete. $\square$

Remark. It is possible to extend the last theorem also to the remaining range $-1<\alpha<0$, as follows. Let $w'(t) = (1-t)^{-\alpha}w(t)$. Assume that $\|f\|_{w'} \le 1$. Note that w' is decreasing (since w is and $\alpha<0$) and $(1-t)w = (1-t)^{1+\alpha}w'$. Therefore the above argument used to estimate $\|P_\alpha f\|_v$ still works, granted $w(1-r(z))$ and $r(z)$ are replaced on the appropriate places by $w'(1-r(z))$ and $r(z)^{1+\alpha}$, respectively. This leads to

$$\begin{aligned} \|P_\alpha f\|_v &\lesssim 1 + \sum_j \sup_{0<r(x)<\epsilon} w(1-r(x))r(x)\int_0^{\max r} \frac{1}{w'(1-\rho)}\frac{\rho^\alpha}{(r(x)+\rho)^{\alpha+2}}\,d\rho \\ &= 1 + \sum_j \sup_{0<r(x)<\epsilon} w(1-r(x))r(x)\int_0^{\max r} \frac{1}{w(1-\rho)\,(r(x)+\rho)^{\alpha+2}}\,d\rho. \end{aligned}$$

Splitting the last integral again into $\int_0^{r(x)}$ and $\int_{r(x)}^{\max r}$ and estimating these as before yields

$$\int_0^{\max r} \frac{1}{w(\rho)\,(r(x)+\rho)^{\alpha+2}}\,d\rho \lesssim \frac{1}{w(1-r(x))r(x)^{\alpha+1}}.$$

Since $\alpha<0$, this is $\le 1/[w(1-r(x))r(x)]$, and the desired conclusion follows. $\square$

In order to have a complete analogue of Theorem 4.6 in [15], it remains to prove the "minimality" of the space L^∞_V. We show this only for the unweighted Bergman projection (i.e. $\alpha=0$). Arguing as in the proof of the last part of Theorem 4.6 in [15] reveals that it is enough to produce a sequence of functions $f_n \in L^\infty_V$ such that, for each n,

(13) $f_n(x) \lesssim 1 + |\log r(x)|^n$, but there exists a sequence $x_j \in \Omega$, $\lim_{j\to\infty} r(x_j) = 0$, such that $|Pf_n(x_j)| \asymp |\log r(x_j)|^{n+1}$.

To that end, fix an arbitrary point $z_0 \in \partial\Omega$. Recall that a function $a(z)$ is called a *peaking function* for z_0 if a is continuous on $\overline{\Omega}$ and holomorphic on Ω, $a(z_0)=1$, and $|a(z)|<1$ $\forall z \in \overline{\Omega}\setminus\{z_0\}$. It is known that for strictly pseudoconvex domains, every boundary point has a peaking function, and in fact a extends holomorphically to a neighbourhood of $\overline{\Omega}$.

Since $r(x)$ is a defining function for Ω, there exist functions $\phi_j \in C^\infty(\mathbf{C}^n)$, $j=1,2,\dots,n$, such that $\phi_j \ge 0$ on $\mathbf{C}^n$, $\sum_j \phi_j = 1$ near $\partial\Omega$, and $\partial_j r \ne 0$ on the support of ϕ_j. Set

$$h_n(x) = \Big(\log \frac{1}{1-a(x)}\Big)^{n+1}$$

and

$$f_n = h_n - \sum_j \partial_j \Big(\phi_j \frac{r}{\partial_j r} h_n\Big).$$

We claim that these functions f_n have the required properties.

Theorem 11. *The functions f_n satisfy* (13).

Remark. The construction of the functions f_n was inspired by Lemma 2 in [1].

Proof. We can assume that $z_0 = 0$. The function $\operatorname{Re} a(x)$ has a global maximum on $\partial\Omega$ at $x = 0$; therefore its first order derivatives in all tangential directions to $\partial\Omega$ vanish, while the matrix of the second order derivatives is negative definite. On the other hand, the derivative in the direction of the interior normal at 0 is positive, by Hopf's Lemma (see e.g. [11], p. 71). Hence, near the origin,

$$\operatorname{Re} a(x) \le 1 - cr(x) - c_1|x|^2 + O(r(x)^2), \tag{14}$$

with some $c, c_1 > 0$. Further, as a takes values in the unit disc, $\operatorname{Im}\log(1-a)$ is bounded on Ω. Consequently,

$$\begin{aligned} |\log(1-a(x))| &\lesssim 1 + |\operatorname{Re}\log(1-a(x))| \\ &\asymp 1 - \log|1-a(x)| \\ &\le 1 - \log \operatorname{Re}(1-a(x)) \\ &\le 1 - \log[cr(x) + O(r(x)^2)] \\ &\asymp 1 - \log r(x) \end{aligned}$$

as $x \to 0$. It follows that

$$h_n = O(1 + |\log r|^{n+1}). \tag{15}$$

This estimate is sharp, since for $x_t = tn_0$, where n_0 is the unit interior normal vector to $\partial\Omega$ at $x = 0$, we have $\operatorname{Re} a(x) = 1 - ct + O(t^2)$, by (14), whence $|\log(1-a(x_t))| \asymp 1 - \log t$, or

$$|h_n(x_t)| \asymp 1 + |\log t|^{n+1}. \tag{16}$$

Let now v be any function in $C^\infty(\overline{\Omega})$ which vanishes on $\partial\Omega$. We claim that

$$P[\partial_j(vh_n)] = 0. \tag{17}$$

To see this, let f be any function continuous on $\overline{\Omega}$ and holomorphic in Ω, and let $\epsilon > 0$. Then, by integrating by parts,

$$\int_{\{x:r(x)>\epsilon\}} \overline{f}\partial_j(vh_n)\,dA = \oint_{\{x:r(x)=\epsilon\}} \overline{f}vh_n n_j\,d\sigma - \int_{\{x:r(x)>\epsilon\}} vh_n\partial_j\overline{f}\,dA.$$

The last integral vanishes, since $\partial\overline{f} = 0$ for holomorphic f, and the surface integral tends to zero as $\epsilon \to 0$ owing to (15) and the hypotheses on f and v. Since the possible functions f form a dense subset in $L^2_{\mathrm{hol}}(\Omega)$, the claim follows.

Taking $v = \phi_j r/\partial_j r$ in (17), we see that

$$Pf_n = h_n.$$

In view of (16), the functions f_n therefore satisfy the second condition in (13). To verify also the first condition, use the Leibniz rule to write

$$f_n = \Big(h_n - \sum_j \phi_j h\Big) - r\sum_j h_n \partial_j\Big(\frac{\phi_j}{\partial_j r}\Big) - r\sum_j \frac{\phi_j h}{\partial_j r}\partial_j h_n.$$

The first summand vanishes identically in a neighbourhood of $\partial\Omega$. The second summand is $O(rh_n) = O(r + r|\log r|^{n+1}) = O(1)$, by (15). Finally,

$$\partial_j h_n = \frac{(n+1)\partial_j a}{1-a} h_{n-1},$$

so the third summand is $O(rh_{n-1}/(1-a)) = O(h_{n-1}) = O(1 + |\log r|^n)$, by (15) again, since

$$|1 - a(x)| \geq \operatorname{Re}(1 - a(x)) \geq cr(x) + O(r(x)^2) \qquad \text{as } x \to 0$$

by (14) and so $r/(1-a)$ is bounded. $\square$

We can thus summarize all our findings so far in the following corollary, which is a complete analogue of Theorem 4.6 in [15].

Corollary 12. *Let Ω be a smoothly bounded strictly pseudoconvex domain. Then the space $L^\infty_{V,r}(\Omega)$ defined in the beginning of this section is the smallest locally convex space $\mathcal{X}$ of functions on Ω (in the sense of [15]) with the following properties:*

1° $L^\infty(\Omega) \subset \mathcal{X}$

2° *the Bergman projection P is bounded on $\mathcal{X}$*

3° *the topology in $\mathcal{X}$ is given by a family of radially weighted sup-norms, where "radial" means "depending only on r".*

It would be interesting to extend this corollary also to the weighted projections P_α.

5. Complete circular domains: the Szegö projection

Recall that a domain $\Omega \subset \mathbf{C}^n$ is called *complete circular* if $z \in \Omega$ and $t \in \mathbf{C}$, $|t| \leq 1$ implies $tz \in \Omega$. Every such domain is star-like, and the function

$$\rho(z) := \inf\{t > 0 : z/t \in \Omega\}$$

satisfies $0 \leq \rho < 1$ on Ω, $\rho = 1$ on $\partial\Omega$, and $\rho > 1$ on the complement of $\overline{\Omega}$. If Ω is bounded and has smooth boundary, then $\rho \in C^\infty(\mathbf{C}^n)$ and $r = 1 - \rho$ is a defining function for $\partial\Omega$.

Note that since Ω is star-like (hence with trivial homology), any pluriharmonic function on it can be written as $h + \overline{g}$ with h, g holomorphic on Ω (in general this is true only locally). The functions h and g are uniquely determined modulo constants.

Theorem 13. *Let Ω be a complete circular domain, and let the spaces H^∞_V and ph^∞_V be defined using the "defining" function $r = 1 - \rho$. Then the pluriharmonic Szegö projection*

$$S : h + \overline{g} \mapsto h + \overline{g(0)} \qquad (h, g \text{ holomorphic on } \Omega)$$

and the pluriharmonic conjugation operator

$$T : h + \overline{g} \mapsto h - \overline{g} + 2\overline{g(0)}$$

are continuous from ph^∞_V onto H^∞_V and from ph^∞_V onto itself, respectively.

Proof. Let $v \in V$ be arbitrary, and choose $w' \geq v$ as in Proposition 6, 3°, and then $w \geq w'$ as in Proposition 6, 4° (with $n = 1$ and w' instead of v). It suffices to prove the assertion concerning T. This, in turn, will follow if we show that

$$\|Tf\|_w \leq C\|f\|_{lw} \qquad \forall f \in ph_V^\infty.$$

It is no restriction to assume that f is real-valued, $f(0) = 0$, and $\|f\|_{lw} \leq 1$. Define $F = f + iTf$, so F is holomorphic and $F(0) = 0$. Fix $z \in \Omega$ and let $R := \rho(z)$. By the complete circularity, the function $G(t) := F(tz/R)$ is holomorphic on $\mathbf{D}$, and $g(t) := \operatorname{Re} G(t) = f(tz/R)$. Applying the argument as in [15], (15), it follows that

$$\begin{aligned}
|F(z)| = |G(R)| &\leq 4 \int_{1-R}^{1} \frac{1}{s} (\sup\nolimits_{|t|=1-(s/2)} |g(t)|)\, ds \\
&\leq 4 \int_{1-R}^{1} \frac{1}{s} (\sup\nolimits_{\rho(\zeta)=1-(s/2)} |f(\zeta)|)\, ds \\
&\leq 4 \int_{1-R}^{1} \frac{1}{s} \frac{1}{w(1-\frac{s}{2}) l(\frac{s}{2})}\, ds \\
&\leq \frac{C}{w(R)}
\end{aligned}$$

with C depending only on v, w', w (hence, independent of z and f). Thus $\|F\|_w \leq C$, completing the proof. □ □

We conclude by proving also the corresponding minimality result.

Theorem 14. *Under the same hypothesis as in Theorem 11, the functions*

$$R_n = \operatorname{Re}(\log(1-a))^{n+1}, \quad I_n = -\operatorname{Im}(\log(1-a))^{n+1}, \qquad n = 0, 1, 2, \ldots,$$

satisfy

$$I_n = O(1 + |\log r|^n), \tag{18}$$

$$\exists x_j \in \Omega,\ \lim_{j\to\infty} r(x_j) = 0,\ |R_n(x_j)| \asymp 1 + |\log r(x_j)|^{n+1}, \tag{19}$$

and

$$TI_n = R_n.$$

Proof. The function $R_n - iI_n$ is holomorphic, so the last property is obvious. Also, as we have already noted before, $\operatorname{Im}\log(1-a)$ is bounded by $\pi/2$, since a maps Ω into the unit disc. On the other hand, we have also seen, in course of the proof of Theorem 11, that $|\operatorname{Re}\log(1-a)| = O(1 + |\log r|)$ and this estimate is sharp (being attained for the sequence $x_j = n_0/j$, with n_0 the interior unit normal vector at 0). Thus (18) and (19) hold for $n = 0$, and as $R_n - iI_n = (R_0 - iI_0)^{n+1}$, the assertion for the general n follows by an easy application of the binomial formula. □ □

Remark. For $\Omega = \mathbf{D}$, the last paragraph provides a much simpler proof of the corresponding part of Theorem 4.1 in [15].

Corollary 15. *Let Ω be a complete circular smoothly bounded strictly pseudoconvex domain. Then the space $ph_V^\infty(\Omega)$ defined in the beginning of Section 5 (using the "defining" function $1 - \rho(x)$) is the smallest locally convex space $\mathcal{X}$ of functions on Ω (in the sense of [15]) with the following properties:*

1° *$\mathcal{X} \supset ph^\infty(\Omega)$, the subspace of all pluriharmonic functions in $L^\infty(\Omega)$;*

2° *the pluriharmonic Szegö projection S (or, equivalently, the pluriharmonic conjugation operator T) is bounded on $\mathcal{X}$;*
3° *the topology in $\mathcal{X}$ is given by a family of radially weighted sup-norms, where "radial" means "depending only on ρ".*

6. The basic locally convex structure

We follow the lines of [15] to show that the spaces H_V^∞ and ph_V^∞ are Schwartz spaces and describe their duals, as well as the predual of L_V^∞. We assume throughout that Ω is smoothly bounded and strictly pseudoconvex, and when dealing with the Szegö projection, also that Ω is complete circular.

To that end, let us define the decreasing weight sequence $\mathcal{V} := (v_n)_{n=1}^\infty$ on Ω by $v_n(z) := l(1-r(z))^{-n}$. Let us also define the Banach spaces $H_{v_n}^\infty := \{f : \Omega \to \mathbb{C} \text{ analytic} \mid \|f\|_{v_n} < \infty\}$.

According to [2], $\mathcal{V}$ satisfies the condition (s), if for all $n \in \mathbb{N}$ one can find $m \in \mathbb{N}$ such that

$$\lim_{r(z)\to 0} \frac{v_m(z)}{v_n(z)} = 0.$$

This is clearly satisfied in our case. Moreover, the weight family V is the associated family of $\mathcal{V}$ in the sense of [3]. Hence, the following is a consequence of [3]:

Theorem 16. *The space H_V^∞ is the locally convex inductive limit of the Banach spaces $H_{v_n}^\infty$, and it is a Schwartz space.*

Next define the increasing weight sequence $W := (w_n)_{n=1}^\infty$ on Ω by $w_n(z) := l(1-r(z))^n$, and the Fréchet space

$$A_W^1 := A_W^1(\Omega) := \Big\{ f : \Omega \to \mathbb{C} \text{ analytic} \Big| \|f\|^{(n)} := \int_\Omega |f| w_n dA < \infty \text{ for all } n \Big\},$$

and the same for ph_W^1 and L_W^1 with "pluriharmonic" and "measurable" replacing "analytic".

Following the reasoning of [15], Proposition 6.4 and Theorem 6.5, one can now prove the topological isomorphisms

$$L_W^1 \cong \lambda_1((L^1(D_k, dA)_{k=1}^\infty), \qquad L_V^\infty \cong K_\infty((L^\infty(D_k, dA)_{k=1}^\infty),$$

and, as a consequence, the fact that L_V^∞ is the strong dual of L_W^1. Here the vector valued Köthe sequence space $\lambda_1((L^1(D_k, dA)_{k=1}^\infty)$ is defined as in the proof of Proposition 6.4 of [15], except that the Köthe matrix $\mathcal{A} := (a_{nk})_{n,k=1}^\infty$ is defined by

$$a_{nk} := \sup_{D_k} |w_n(z)|,$$

where $D_k := \{z \mid 1/k \geq r(z) \geq 1/(k+1)\} \subset \Omega$. The definition of the space $K_\infty((L^\infty(D_k, dA)_{k=1}^\infty)$ is analogous.

Theorem 17. *The Bergman projection P is continuous from L_W^1 onto A_W^1. As a corollary, the dualities $(A_W^1)'_b \cong H_V^\infty$ and $(H_V^\infty)' \cong A_W^1$ hold, with dual pairing $\langle f, g\rangle := \int_\Omega f\bar{g}\, dA$. The space A_W^1 is thus a Fréchet-Schwartz space.*

To follow the reasoning of [15], Proposition 6.6 and Corollary 6.7, the only thing is to remark that P is self dual with respect to $\langle\cdot,\cdot\rangle$. This is so since P is the orthogonal projection with respect to the same inner product.

The statement about the Schwartz property is a standard duality property, see for example [8].

Corollary 18. *The pluriharmonic Szegö projection and the pluriharmonic conjugation operator are continuous on ph^1_W. As a consequence, $(ph^1_W)'_b \cong ph^\infty_V$ and $(ph^\infty_V)' \cong ph^1_W$ with pairing $\int_\Omega f\bar{g}\,dA$.*

The proof is again the same as in [15].

Remark. In the case of the unit disc it was proved in [16] that H^∞_V is not nuclear. The same is probably true also for the pseudoconvex case, but unfortunately no easy proof of this seems to exist, nor any easy general result about the relationship between these spaces on different domains. Hence we postpone the discussion of nuclearity to a future occasion, and restrict ourselves here only to the following remark.

Assume that Ω has the following property: there exist complex numbers $a_2, \dots a_n$ such that the domain $D := \{z \in \mathbb{C}|\ (z, a_2, \dots, a_n) \in \Omega\}$ is simply connected and $D \times \mathbb{C}^{n-1}$ contains Ω. Form the space $H^\infty_V(D)$. Since D has a smooth boundary and is simply connected, the Riemann conformal map $\psi : \mathbb{D} \to D$ induces a topological isomorphism from $H^\infty_V(D)$ onto $H^\infty_V(\mathbb{D})$: since D has smooth boundary, $1 - |z|$ is equivalent to $\mathrm{dist}(\psi(z), \partial D)$. Hence, $H^\infty_V(D)$ is not nuclear, by [16].

On the other hand $H^\infty_V(D)$ is isomorphic to a subspace Y of $H^\infty_V(\Omega)$; hence, the latter space is also not nuclear.

For Y one chooses the subspace of functions constant in the 2nd–nth coordinates. The map $\Psi : Y \to H^\infty_V(D)$, $(\Psi f)(z_1) := f(z_1, a_2, \dots, a_n)$, $z_1 \in D$, is an isomorphism. Namely, Ψ is continuous: given $v \in V_D$ and $z \in \Omega$, we have $z_1 \in D$, and the weight $w(z) := v(z_1)$ belongs to V_Ω, since $r(z) \lesssim \mathrm{dist}(z, \partial\Omega) \leq \mathrm{dist}(z_1, \partial D)$. Hence,

$$\|\Psi f\|_v = \sup_{z_1 \in D} |f(z_1, a_2, \dots, a_n)| v(z_1) = \sup_{z \in \Omega} |f(z_1, a_2, \dots, a_n)| w(z) = \|f\|_w.$$

Also Ψ^{-1} (which is nothing else than the identity mapping) is continuous, because given $w \in V_\Omega$, then also

$$v(z_1) := \sup_{z_2, \dots, z_n} w(z_1, z_2, \dots, z_n) \in V_D,$$

and we obtain $\|\Psi^{-1} g\|_w \leq \|g\|_v$ for functions $g \in H^\infty_V(D)$. (Notice that we cannot directly apply an open mapping theorem here, since, although $H^\infty_V(\Omega)$ is bornological, the same is not, in general, true for its closed subspaces.)

Of course, the above is true as well if the construction can be done with respect to some other coordinate instead of the first one.

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