# A CAPACITY APPROACH TO THE POINCARÉ INEQUALITY AND SOBOLEV IMBEDDINGS IN VARIABLE EXPONENT SOBOLEV SPACES

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A . We study the Poincaré inequality in Sobolev spaces with variable exponent. Under a rather mild and sharp condition on the exponent $p$ we show that the inequality holds. This condition is satisfied e.g. if the exponent $p$ is continuous in the closure of a convex domain. We also give a condition for the exponent $p$ when there exists an imbedding from the Sobolev space to the space of continuous functions. Finally we give an essential sharp condition for the exponent $p$ as to when there exists an imbedding from the Sobolev space to the space of bounded functions.

## 1. I

There has recently been a surge of interest in Sobolev spaces with variable exponent, cf. [4, 5, 7, 8, 9, 19, 20]. These spaces, introduced in [15], are the natural generalization of Sobolev spaces to the non-homogeneous situation, for instance they have been used for modeling electrorheological fluids, see the book of M. Růžička, [21]. The Lebesgue and Sobolev spaces with variable exponent share many properties with their classical equivalents. There is also some crucial differences. For instance the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$ if and only if the exponent $p$ is 0-Hölder continuous, see [4] and [19].

The Poincaré inequality, although of great importance in classical non-linear potential theory (especially in metric spaces) has not been previously studied in the case of variable exponent Sobolev spaces. Our first result, Theorem 2.2, is the following: If $D \subset \mathbb{R}^n$ is smooth domain, say a John domain, and the essential supremum of $p$ is less than the Sobolev conjugate of the essential infimum of $p$ then the Poincaré inequality

$$\|u - u_B\|_{L^{p(\cdot)}(D)} \leqslant C\|\nabla u\|_{L^{p(\cdot)}(D)} \tag{1.1}$$

holds for every $u \in W^{1,p(\cdot)}(D)$, where $u_B = \fint u(x)dx$. Here the constant $C$ depends on $n$, $p$, $\operatorname{diam}(D)$ and the John constant of $D$. We give an example which shows that the condition for $p$ is sharp even in a ball. We prove a corollary to this result, which shows that if $p$ is continuous in the closure of a convex domain then the Poincaré inequality holds (Corollary 2.16).

In the classical theory the constant of the Poincaré inequality is $C\operatorname{diam}(D)$. It is possible to achieve this also for variable exponent Sobolev spaces, as we prove in Corollary 2.22. The price we have to pay is that the exponent $p$ has to be 0-Hölder continuous.

Sobolev imbedding in variable exponent Sobolev spaces has been studied by many authors in the case when $p$ is less than the dimension, see [7, 20, 9, 8]. We give two results in the case when $p$ is greater than the dimension. We prove a result for continuity of the Sobolev functions, namely that every Sobolev function is continuous if the exponent is locally bounded away from the dimension. We show that if a domain satisfies a uniform interior cone condition and $p(x) \geqslant n + f(d(x, \partial G))$ for every $x$ and a certain increasing function $f$ then

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there exists an imbedding from the variable exponent Sobolev space to $L^\infty$. Our condition is essentially sharp.

**Notation.** We denote by $\mathbb{R}^n$ the Euclidean space of dimension $n \geqslant 2$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote an open ball with center $x$ and radius $r$ by $B(x, r)$.

Let $A \subset \mathbb{R}^n$ and $p\colon A \to [1, \infty)$ be a measurable function (called a *variable exponent* on $A$). We define $p_A^+ = \operatorname{ess\,sup}_{x \in A} p(x)$ and $p_A^- = \operatorname{ess\,inf}_{x\in A} p(x)$. If $A = \mathbb{R}^n$ we write $p^+ = p^+_{\mathbb{R}^n}$ and $p^- = p^-_{\mathbb{R}^n}$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We define the *generalized Lebesgue space* $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u\colon \Omega \to \mathbb{R}$ such that $\varrho_{p(\cdot)}(\lambda u) = \int_\Omega |\lambda u(x)|^{p(x)}\, dx < \infty$ for some $\lambda > 0$. The function $\varrho_{p(\cdot)}\colon L^{p(\cdot)}(\Omega) \to [0, \infty)$ is called the *modular* of the space $L^{p(\cdot)}(\Omega)$. One can define a norm, the so-called *Luxemburg norm*, on this space by the formula $\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\}$. Notice that if $p \equiv p_0$ then $L^{p(\cdot)}(\Omega)$ is the classical Lebesgue space, so there is no danger of confusion with the new notation.

The *generalized Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is the space of measurable functions $u\colon \Omega \to \mathbb{R}$ such that $u$ and the absolute value of the distributional gradient $\nabla u = (\partial_1 u, \dots, \partial_n u)$ are in $L^{p(\cdot)}(\Omega)$. The function $\varrho_{1,p(\cdot)}\colon W^{1,p(\cdot)}(\Omega) \to [0, \infty)$ is defined as $\varrho_{1,p(\cdot)}(u) = \varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(|\nabla u|)$. The norm $\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$ makes $W^{1,p(\cdot)}(\mathbb{R}^n)$ a Banach space.

See [15] for basic properties of variable exponent Lebesgue and Sobolev spaces.

## 2. T P ´

In this section we give a relatively mild condition on the exponent for the Poincaré inequality to hold. We also show that this condition is, in some sense, the best possible. For Sobolev functions with zero boundary values the Poincaré inequality was given in [8, Lemma 3.1] and considerably generalized in [12].

We recall the following well known Sobolev-Poincaré inequality. By $q^*$ we denote the Sobolev conjugate of $q < n$, $q^* = nq/(n - q)$.

2.1. **Lemma.** *Let $D \subset \mathbb{R}^n$ be a bounded John domain. Let $1 \leqslant p < n$ and $p \leqslant q \leqslant p^*$ be fixed exponents. Then*

$$\|u - u_D\|_q \leqslant C(n, p, \lambda)|D|^{1/n+1/q-1/p}\|\nabla u\|_p$$

*for all functions $u \in W^{1,p}(D)$, where $\lambda$ is the John constant.*

*If $p \geqslant n$ and $q < \infty$ then*

$$\|u - u_D\|_q \leqslant C(n, q, \lambda)|D|^{1/n+1/q-1/p}\|\nabla u\|_p$$

*for all functions $u \in W^{1,p}(D)$.*

*Proof.* The case $p < n$ and $q = p^*$ is by B. Bojarski [3, (6.6)]. The case $q < p^*$ follows from this by standard arguments: we choose $s \in [1, n)$ such that $s^* = q$ (or $s = 1$ if $q < 1^*$). By Hölder's inequality and Bojarski's result we obtain

$$\begin{aligned}
\left(\fint_D |u - u_D|^q dx\right)^{\frac{1}{q}} &\leqslant |D|^{-\frac{1}{s^*}}\left(\int_D |u - u_D|^{s^*} dx\right)^{\frac{1}{s^*}} \\
&\leqslant C|D|^{-\frac{1}{s^*}}\left(\int_D |\nabla u|^s dx\right)^{\frac{1}{s}} = C|D|^{\frac{1}{s}-\frac{1}{s^*}}\left(\fint_D |\nabla u|^s dx\right)^{\frac{1}{s}} \\
&\leqslant C|D|\left(\fint_D |\nabla u|^p dx\right)^{\frac{1}{p}},
\end{aligned}$$

which is clearly equivalent to the inequalities in the theorem. □

2.2. **Theorem.** *Let $D \subset \mathbb{R}^n$ be a bounded John domain, with constant $\lambda$. If $p_D^+ \leqslant (p_D^-)^*$ or $p_D^- \geqslant n$ and $p_D^+ < \infty$ then there exists a constant $C = C(n, p_D^-, p_D^+, \lambda)$ such that*

$$\|u - u_D\|_{p(\cdot)} \leqslant C(1 + |D|)^2 |D|^{\frac{1}{n} + \frac{1}{p_D^+} - \frac{1}{p_D^-}} \|\nabla u\|_{p(\cdot)} \tag{2.3}$$

*for every $u \in W^{1,p(\cdot)}(D)$.*

*Proof.* Assume first that $p_D^+ \leqslant (p_D^-)^*$. Since $p(x) \leqslant p_D^+ \leqslant (p_D^-)^*$ we obtain by [15, Theorem 2.8] and Lemma 2.1 that

$$\begin{aligned} \|u - u_D\|_{p(\cdot)} &\leqslant (1 + |D|)\, \|u - u_D\|_{p_D^+} \\ &\leqslant C(n, p_D^-, \lambda)\, (1 + |D|)\, |D|^{\frac{1}{n} + \frac{1}{p_D^+} - \frac{1}{p_D^-}} \|\nabla u\|_{p_D^-} \\ &\leqslant C(n, p_D^-, \lambda)\, (1 + |D|)^2\, |D|^{\frac{1}{n} + \frac{1}{p_D^+} - \frac{1}{p_D^-}} \|\nabla u\|_{p(\cdot)}. \end{aligned}$$

The case $p_D^- \geqslant n$ is similar, the only difference is that the constant in the second inequality in the above chain of inequalities is $C(n, p_D^+, \lambda)$. □

2.4. *Remark.* John domains are almost the right class of irregular domains for the classical Sobolev-Poincaré inequality, see [3], [1] and [2, Theorem 4.1].

Previous results on Sobolev imbeddings in the variable exponent setting have been derived in domains whose boundary is locally a graph of a Lipschitz continuous function, see [7], [9] and [8]. It is therefore of interest to note that every domain, whose boundary is locally a graph of a Lipschitz continuous function, is a John domain, see [17]. In particular every ball is a John domain.

If $D$ is a ball in Theorem 2.2, then the constant in inequality (2.3) is the classical Sobolev-Poincaré inequality in a ball, see for example [16, Corollary 1.64, p. 38].

The next example shows that if $p_D^- < n$ and $p_D^+ > (p_D^-)^*$ then there need not exist a constant $C > 0$ such that inequality (2.3) holds for every $u \in W^{1,p(\cdot)}(D)$.

Recall that the *variational capacity* for fixed $p$, $\mathrm{cap}_p(E, F; D)$, is defined for closed $E, F$ and open $D$ by

$$\mathrm{cap}_p(E, F; D) = \inf_{u \in L(E,F;D)} \int_D |\nabla u|^p \, dx,$$

where $L(E, F; D)$ is the set of $W^{1,p}(D) \cap C(D)$ functions $u$ that satisfy $u|_{E \cap D} = 1$ and $u|_{F \cap D} = 0$. We use the short-hand notation $\mathrm{cap}(E, F)$ for $\mathrm{cap}(E, F; \mathbb{R}^n)$, similarly for $L(E, F)$. For more information on capacities see [13, Chapter 2] or [18]. The following lemma will be used several times to estimate the gradient of variable exponent functions.

2.5. **Lemma.** [13, Example 2.12, p. 35] *For fixed $p \neq 1, n$, and every $x \in \mathbb{R}^n$ and $R > r > 0$ we have*

$$\mathrm{cap}_p(\mathbb{R}^n \setminus B(x, R), B(x, r)) = \omega_{n-1} \left| \frac{p - n}{p - 1} \right|^{p-1} \left| R^{(p-n)/(p-1)} - r^{(p-n)/(p-1)} \right|^{1-p}.$$

2.6. **Example.** Our aim is construct a sequence of functions in $B = B(0, 1) \subset \mathbb{R}^2$ for which the constant in the Poincaré inequality 2.3 goes to infinity. Let $B_i = B(2^{-i} e_1, \frac{1}{4} 2^{-i}) \subset \mathbb{R}^2$ and $B_i' = B(2^{-i} e_1, \frac{1}{8} 2^{-i^2}) \subset \mathbb{R}^2$ for every $i = 1, 2, \ldots$ and let $1 < p_1 < 2$. Choose a function $u_i \in C_0^\infty(B_i)$ with $u_i|_{B_i'} = 1$ be such that

$$\left(2\, \mathrm{cap}_{p_1}(B_i', \mathbb{R}^n \setminus B_i)\right)^{\frac{1}{p_1}} \geqslant \|\nabla u_i\|_{L^{p_1}(B_i)}. \tag{2.7}$$

Let $p_2 > p_1$ and define $p(x) = p_1\chi_{B_i\setminus B_i'}(x) + p_2\chi_{B_i'}(x)$ for $x \in B(0,1)$ with positive first coordinate. Since $\nabla u_i = 0$ in $B_i'$ we obtain

$$\|\nabla u_i\|_{L^{p(\cdot)}(B_i)} = \|\nabla u_i\|_{L^{p_1}(B_i)}. \tag{2.8}$$

Let $\tilde B_i = B(-2^{-i}e_1, \frac{1}{4}2^{-i})$. We extend $u_i$ to $B$ as an odd function of the first coordinate in $\tilde B_i$ and by zero elsewhere. We also extend $p$ to $B$ as an even function of the first coordinate. We denote the extensions by $\tilde u_i$ and $\tilde p$. By (2.7) and (2.8) we obtain

$$2^{1+\frac{1}{p_1}}\Big(\operatorname{cap}_{p_1}(B_i', \mathbb{R}^n \setminus B_i)\Big)^{\frac{1}{p_1}} \geqslant \|\nabla \tilde u_i\|_{L^{\tilde p(\cdot)}(B)}.$$

By Lemma 2.5 this yields

$$\|\nabla \tilde u_i\|_{L^{\tilde p(\cdot)}(B)} \leqslant C(p_1)\left|\tfrac{1}{4}2^{-i\frac{p_1-2}{p_1-1}} - \tfrac{1}{8}2^{-i^2\frac{p_1-2}{p_1-1}}\right|^{\frac{1-p_1}{p_1}}. \tag{2.9}$$

For large $i$ the right hand side is approximately equal to $C(p_1)2^{-i^2\frac{2-p_1}{p_1}}$.

Since $(\tilde u_i)_B = 0$, we obtain

$$\|\tilde u - (\tilde u_i)_B\|_{L^{\tilde p(\cdot)}(B)} = \|\tilde u\|_{L^{\tilde p(\cdot)}(B)} \geqslant |B_i'|^{\frac{1}{p_2}} \approx 2^{-i^2\frac{2}{p_2}}. \tag{2.10}$$

By the inequalities (2.9) and (2.10) we find that

$$\frac{\|\tilde u - (\tilde u_i)_B\|_{L^{\tilde p(\cdot)}(B)}}{\|\nabla \tilde u_i\|_{L^{\tilde p(\cdot)}(B)}} \geqslant C(p_1)2^{i^2(\frac{2}{p_1}-1-\frac{2}{p_2})} \to \infty$$

as $i \to \infty$ if $\frac{2}{p_1} - 1 - \frac{2}{p_2} > 0$, that is, if $p_2 > \frac{2p_1}{2-p_1} = p_1^*$.

We next show that the condition $p_D^+ \leqslant (p_D^-)^*$ in Theorem 2.2 can be replaced by a set of local conditions.

2.11. **Theorem.** *Let $D \subset \mathbb{R}^n$ be a bounded John domain. Assume that there exist John domains $G_i$, $i = 1, \ldots, j$, so that $G_i \subset D$ for every $i$, $D = \cup_{i=1}^j G_i$ and either $p_{G_i}^+ \leqslant (p_{G_i}^-)^*$ or $p_{G_i}^- \geqslant n$ for every $i$. Then there exists a constant $C > 0$ such that*

$$\|u - u_D\|_{p(\cdot)} \leqslant C\|\nabla u\|_{p(\cdot)} \tag{2.12}$$

*for every $u \in W^{1,p(\cdot)}(D)$. The constant $C$ depends on $n$,* diam$(D)$, *$|G_i|$, $p$ and the John constants of $D$ and $G_i$, $i = 1, \ldots, j$.*

*Proof.* Using the triangle inequality of the norm we obtain

$$\begin{aligned}\|u - u_D\|_{L^{p(\cdot)}(D)} &\leqslant \sum_{i=1}^j \|u - u_D\|_{L^{p(\cdot)}(G_i)} \\ &\leqslant \sum_{i=1}^j \|u - u_{G_i}\|_{L^{p(\cdot)}(G_i)} + \sum_{i=1}^j \|u_D - u_{G_i}\|_{L^{p(\cdot)}(G_i)}.\end{aligned} \tag{2.13}$$

We estimate the first part of the sum using Theorem 2.2. This yields

$$\begin{aligned}\|u - u_{G_i}\|_{L^{p(\cdot)}(G_i)} &\leqslant C(n, p_{G_i}, |G_i|, \lambda_i)\|\nabla u\|_{L^{p(\cdot)}(G_i)} \\ &\leqslant C(n, p_{G_i}, |G_i|, \lambda_i)\|\nabla u\|_{L^{p(\cdot)}(D)}\end{aligned} \tag{2.14}$$

for every $i = 1, \ldots, j$. Here $\lambda_i$ is the John constant of $G_i$. We next estimate the second part of the sum in (2.13) by [14, Lemma 2.3] (first inequality) and by the classical Poincaré inequality (second inequality). We obtain

$$
\begin{aligned}
\|u_D - u_{G_i}\|_{L^{p(\cdot)}(G_i)} &\leqslant \|1\|_{L^{p(\cdot)}(G_i)} \fint_D |u(x) - u_{G_i}| dx \\
&\leqslant \|1\|_{L^{p(\cdot)}(G_i)} |G_i|^{-1} \int_D |u(x) - u_D| dx \\
&\leqslant C(n, \operatorname{diam}(D), \lambda) |G_i|^{-1} \|1\|_{L^{p(\cdot)}(G_i)} \|\nabla u\|_{L^1(D)} \\
&\leqslant C(n, \operatorname{diam}(D), \lambda)(1 + |D|) |G_i|^{-1} \|1\|_{L^{p(\cdot)}(G_i)} \|\nabla u\|_{L^{p(\cdot)}(D)}
\end{aligned}
\tag{2.15}
$$

for every $i = 1, \ldots, j$. Here $\lambda$ is the John constant of $D$. Now the inequality (2.12) follows by the inequalities (2.13), (2.14) and (2.15). □

2.16. **Corollary.** *Let $D \subset \mathbb{R}^n$ be a bounded convex domain and let $p\colon \overline{D} \to [1, \infty)$ be a continuous exponent. Then there exists a constant $C > 0$ such that*

$$\|u - u_D\|_{p(\cdot)} \leqslant C \|\nabla u\|_{p(\cdot)}$$

*for every $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$.*

*Proof.* Since $p$ is continuous we find for every $x \in \overline{D}$ a constant $r(x) > 0$ such that either

$$p^+_{B(x,r(x))\cap D} \leqslant (p^-_{B(x,r(x))\cap D})^* \quad \text{or} \quad p^-_{B(x,r(x))\cap D} \geqslant n.$$

Since $\overline{D}$ is compact it is possible to find finite covering of $D$ with balls $B(x, r(x))$. It is easy to see that each $B(x, r(x)) \cap D$ is a John domain and hence the corollary follows by Theorem 2.11. □

Sometimes it is useful to have better control over the constant in the Poincaré inequality as the domain $D$ changes than in (2.3). In the fixed exponent case the constant of the Poincaré inequality is $C \operatorname{diam}(D)$. We show that this kind of constant is also possible for variable exponent Sobolev spaces. The price we have to pay for this is that the exponent $p$ has to satisfy a much stronger condition in Theorem 2.17 than in Theorem 2.2; in Theorem 2.2 the exponent $p$ could be discontinuous even in every point, but in Theorem 2.17 the exponent is 0-Hölder continuous.

2.17. **Theorem.** *Let $D \subset \mathbb{R}^n$ be a bounded uniform domain. Let $p\colon D \to \mathbb{R}$ be such that $1 < p^-_D \leqslant p^+_D < \infty$. Assume that there exists a constant $C > 0$ such that*

$$|p(x) - p(y)| \leqslant \frac{C}{-\ln|x - y|} \tag{2.18}$$

*for every $x, y \in D$ with $|x - y| \leqslant \frac{1}{2}$. Then the inequality*

$$\|u - u_D\|_{p(\cdot)} \leqslant C \operatorname{diam}(D) \left(1 + \max\left\{|D|^{1/p^+_D - 1/p^-_D}, |D|^{1/p^-_D - 1/p^+_D}\right\}\right) \|\nabla u\|_{p(\cdot)}, \tag{2.19}$$

*holds for every $u \in W^{1,p(\cdot)}(D)$. Here the constant $C$ depends on the dimension $n$, the uniform constant of $D$ and $p$.*

*Proof.* Since $W^{1,p(\cdot)}_0(D) \hookrightarrow W^{1,1}(D)$ we obtain as in the proof of [10, Theorem 11] for every $u \in W^{1,p(\cdot)}(D)$ that

$$|u(x) - u(y)| \leqslant C|x - y|(M\nabla u(x) + M\nabla u(y)) \tag{2.20}$$

for almost every $x, y \in D$. Here $M$ is the Hardy-Littlewood maximal operator:

$$M\nabla u(x) = \sup_{r>0} \fint_{B(x,r)} |\nabla u(y)| dy,$$

with the understanding that $\nabla u = 0$ outside $D$. The constant $C$ depends on the dimension $n$ and the uniform constants of $D$.

Integrating inequality (2.20) over $y$ we obtain

$$\begin{aligned}|u(x) - \fint_D u(y)dy| &\leqslant \fint_D |u(x) - u(y)|dy \\ &\leqslant C \operatorname{diam}(D)\Big(M\nabla u(x) + \fint_D M\nabla u(y)dy\Big).\end{aligned}$$

By Hölder's inequality [15, Theorem 2.1] this yields

$$|u(x) - u_D| \leqslant C \operatorname{diam}(D)\Big(M\nabla u(x) + \frac{C(p)\|1\|_{L^{p'(\cdot)}(D)}}{|D|}\|M\nabla u\|_{p(\cdot)}\Big).$$

Since the previous inequality holds point-wise, it is clear that we have an inequality also for the Lebesgue norms of both sides:

$$\begin{aligned}\|u - u_D\|_{p(\cdot)} &\leqslant C \operatorname{diam}(D)\left(\|M\nabla u\|_{p(\cdot)} + \frac{C}{|D|}\|1\|_{p'(\cdot)}\|1\|_{p(\cdot)}\|M\nabla u\|_{p(\cdot)}\right) \\ &\leqslant C \operatorname{diam}(D)\left(1 + \frac{\max\{|D|^{1+1/p_D^+-1/p_D^-}, |D|^{1+1/p_D^--1/p_D^+}\}}{|D|}\right)\|M\nabla u\|_{p(\cdot)} \\ &\leqslant C \operatorname{diam}(D)\left(1 + \max\left\{|D|^{1/p_D^+-1/p_D^-}, |D|^{1/p_D^--1/p_D^+}\right\}\right)\|M\nabla u\|_{p(\cdot)}.\end{aligned}$$

By [5, Remark 2.2] and [4, Theorem 3.5] the Hardy-Littlewood maximal operator is bounded, and so we obtain

$$\|u - u_D\|_{p(\cdot)} \leqslant C \operatorname{diam}(D)\left(1 + \max\left\{|D|^{1/p_D^+-1/p_D^-}, |D|^{1/p_D^--1/p_D^+}\right\}\right)\|\nabla u\|_{p(\cdot)},$$

where the constant $C$ depends on the dimension $n$, the uniform constant of $D$ and $p$. □

2.21. *Remark.* We refer to [17] for basic properties of uniform domains: Every uniform domain is a John domain. Every domain, whose boundary is locally a graph of a Lipschitz continuous function, is a uniform domain. In particular if $D$ is a ball then the constant in (2.19) depends on the dimension $n$ and $p$.

2.22. **Corollary.** *Let $p$ be as in the previous theorem. If $B$ is a ball with $|B| \leqslant 1$ then*

$$\|u - u_B\|_{p(\cdot)} \leqslant C \operatorname{diam}(B)\|\nabla u\|_{p(\cdot)},$$

*where the constant $C$ does not depend on $B$.*

*Proof.* Since $|B| \leqslant 1$ we have

$$\max\left\{|B|^{1/p_B^+-1/p_B^-}, |B|^{1/p_B^--1/p_B^+}\right\} = |B|^{1/p_B^+-1/p_B^-}.$$

Since $p$ is 0-Hölder continuous, (2.18), we obtain by [4, Lemma 3.2] that there exists a constant $C > 0$, depending only on the dimension $n$ and the constant in (2.18), such that $|B|^{1/p_B^+-1/p_B^-} \leqslant C$ for every ball $B$. Hence if $|B| \leqslant 1$ the constant in (2.19) is less than $C \operatorname{diam}(B)$. □

## 3. C

The functions in the classical Sobolev space $W^{1,p}$ are continuous if $p > n$. In this section we consider when functions in variable exponent Sobolev space are continuous.

3.1. **Theorem.** *Suppose that $p > n$ is locally bounded away from $n$ in $D$. Then $W^{1,p(\cdot)}(D) \subset C(D)$.*

*Proof.* Let $x \in D$ and consider the ball $B = B(x, \delta(x)/2)$. Define $q = \operatorname{essinf}_{y \in B} p(y)$. Then, by [15, Theorem 2.8],

$$W^{1,p(\cdot)}(B) \hookrightarrow W^{1,q}(B) \subset C(B).$$

Therefore every function in $W^{1,p(\cdot)}(D)$ is continuous at $x$, and since $x$ was arbitrary, the claim follows. □

The following corollary is immediate.

3.2. **Corollary.** *Suppose that $p$ is continuous in $D$. Then $W^{1,p(\cdot)}(D) \subset C(D)$ if $p(x) > n$ for every $x \in D$.*

We next use a classical example to show that the assumption that $p$ is locally bounded away from $n$ in $D$ is not superfluous when $p$ is not continuous.

3.3. **Example.** Let $B = B(0, 1/16)$, $\varepsilon > 0$ and suppose that

$$p(x) \leqslant n + (n - 1 - \varepsilon)\frac{\log_2 \log_2(1/|x|)}{\log_2(1/|x|)}$$

for $x \in B \setminus \{0\}$ and $p(0) > n$. We show that then $W^{1,p(\cdot)}(B) \not\subset C(B)$.

Define $u(x) = \cos(\log_2 |\log_2 |x||)$ for $x \in B \setminus \{0\}$ and $u(0) = 0$. Clearly $u$ is not continuous at the origin. So we have to show that $u \in W^{1,p(\cdot)}(B)$. It is clear that $u$ has partial derivatives, except in the origin.

Since $u$ is bounded it follows that $u \in L^{p(\cdot)}(B)$. We next estimate the gradient:

$$|\nabla u(x)| = \left| \sin(\log_2 |\log_2 |x||) \cdot \frac{1}{|x| \log_2 |x|} \right| \leqslant \left| \frac{1}{|x| \log_2 |x|} \right|.$$

We therefore find that

$$\begin{aligned}
\int_B |\nabla u(x)|^{p(x)} dx &\leqslant \int_B \frac{dx}{(|x|| \log_2 |x||)^{p(x)}} \\
&= \omega_{n-1} \int_0^{1/16} \frac{r^{n-1} dr}{(r| \log_2 r|)^{p(re_1)}} \\
&= \omega_{n-1} \sum_{i=5}^{\infty} \int_{2^{-i-1}}^{2^{i}} \frac{r^{n-1} dr}{(r| \log_2 r|)^{p(re_1)}}.
\end{aligned}$$

Since $1/(r| \log_2 r|) > 1$ we may increase the exponent $p$ for an upper bound. In the annulus $B(0, 2^{-i}) \setminus B(0, 2^{-i-1})$ we have $i \leqslant \log_2(1/|x|) \leqslant i + 1$. Since $y \to \log_2(y)/y$ is decreasing we find that

$$p(x) \leqslant n + (n - 1 - \varepsilon)\frac{\log_2 \log_2(1/|x|)}{\log_2(1/|x|)} \leqslant n + (n - 1 - \varepsilon)\frac{\log_2 i}{i}$$

in the same annulus. We can therefore continue our previous estimate by

$$\begin{aligned}
\int_B |\nabla u(x)|^{p(x)} dx &\leqslant \sum_{i=5}^{\infty} \int_{2^{-i-1}}^{2^{-i}} \frac{r^{n-1} dr}{(r| \log_2 r|)^{n+(n-1-\varepsilon)\log_2(i)/i}} \\
&\leqslant C \sum_{i=5}^{\infty} \int_{2^{-i-1}}^{2^{-i}} \frac{2^{-i(n-1)} dr}{(i 2^{-i})^{n+(n-1-\varepsilon)\log_2(i)/i}} \\
&= C \sum_{i=5}^{\infty} 2^{(n-1-\varepsilon)\log_2(i)} i^{-n-(n-1-\varepsilon)\log_2(i)/i} \\
&= C \sum_{i=5}^{\infty} i^{-1-\varepsilon} i^{-(n-1-\varepsilon)\log_2(i)/i} \leqslant C \sum_{i=5}^{\infty} i^{-1-\varepsilon} < \infty.
\end{aligned}$$

## 4. S

We start by introducing a relative variational $p(\cdot)$-pseudocapacity, and proving some basic properties for it. This capacity is quite similar to the Sobolev $p(\cdot)$-capacity studied by P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen in [11].

Let $F, E \subset \mathbb{R}^n$ be closed disjoint sets and $D$ be a domain in $\mathbb{R}^n$. The *variational $p(\cdot)$-pseudocapacity* is defined as

$$\psi_{p(\cdot)}(F, E; D) = \inf_{u \in L(F,E;D)} \|\nabla u\|_{L^{p(\cdot)}(D)},$$

where $L(F, E; D)$ is the set of functions $u \in W^{1,p(\cdot)}(D) \cap C(D)$ with $0 \leqslant u \leqslant 1$, $u|_{F\cap D} = 0$ and $u|_{E\cap D} = 1$. If $L(F, E; D) = \emptyset$ then we define $\psi_{p(\cdot)}(F, E; D) = \infty$. We write $L(E, x; D)$ for $L(F, \{x\}; D)$ etc.

4.1. *Remark.* Including $C(D)$ in the definition of the capacity is somewhat strange in this context, since we do not, in general, know whether continuous functions are dense in $W^{1,p(\cdot)}(D)$, but see [6]. However, since we are interested in the case when $p > n$, the assumption makes sense, by Theorem 3.1.

The reason for calling the function $\psi_{p(\cdot)}(F, E; D)$ a pseudocapacity is that it is defined as a capacity but using the norm instead of the modular. This corresponds to introducing an exponent $1/p$ to the capacity in the fixed exponent case. Because of this we cannot expect the pseudocapacity to have all the usual properties of a capacity. It nevertheless has many of them:

4.2. **Theorem.** *Let $F, E \subset \mathbb{R}^n$ be closed sets and $D$ be a domain in $\mathbb{R}^n$. Then the set function $(F, E) \mapsto \psi_{p(\cdot)}(F, E; D)$ has the following properties:*

(i) $\psi_{p(\cdot)}(\emptyset, E; D) = 0$.
(ii) $\psi_{p(\cdot)}(F, E; D) = \psi_{p(\cdot)}(E, F; D)$.
(iii) *If $E_1 \subset E_2$, then $\psi_{p(\cdot)}(F, E_1; D) \leqslant \psi_{p(\cdot)}(F, E_2; D)$.*
(iv) *If $E$ is a subset of $\mathbb{R}^n$, then*

$$\psi_{p(\cdot)}(F, E; D) = \inf_{\substack{E\subset U \\ U \text{ open}}} \psi_{p(\cdot)}(F, U; D).$$

(v) *If $K_1 \supset K_2 \supset \dots$ are compact, then*

$$\lim_{i\to\infty} \psi_{p(\cdot)}(F, K_i; D) = \psi_{p(\cdot)}\left(F, \bigcap_{i=1}^{\infty} K_i; D\right).$$

(vi) *Suppose that $p > n$ is locally bounded away from $n$. If $E_i \subset \mathbb{R}^n$ for every $i = 1, 2, \dots$, then*

$$\psi_{p(\cdot)}\left(F, \bigcup_{i=1}^{\infty} E_i; D\right) \leq \sum_{i=1}^{\infty} \psi_{p(\cdot)}(F, E_i; D).$$

*Proof.* Assertion (i) is clear since we may use a constant function.

Assertion (ii) is clear since if $u \in L(F, E; D)$ then $1 - u \in L(E, F; D)$.

Assertion (iii) follow by the facts $L(F, E_2; D) \subset L(F, E_1; D)$.

Next we prove (iv). It is clear that

$$\psi_{p(\cdot)}(F, E; D) \leqslant \inf_{\substack{E\subset U \\ U \text{ open}}} \psi_{p(\cdot)}(F, U; D).$$

Let $\varepsilon > 0$. Assume that $u \in L(F, E; D)$ is such that

$$\|\nabla u\|_{p(\cdot)} \leqslant \psi_{p(\cdot)}(F, E; D) + \varepsilon.$$

Since $u$ is continuous $\{u > 1-\varepsilon\}$ is an open set containing $E$. Hence we obtain

$$\begin{aligned}\inf_{\substack{E\subset U\\ U \text{ open}}} \psi_{p(\cdot)}(F,U;D) &\leqslant \psi_{p(\cdot)}(F,\{u>1-\varepsilon\};D)\\ &\leqslant \Big\|\nabla \min\{\frac{u}{1-\varepsilon},1\}\Big\|_{p(\cdot)} \le (1-\varepsilon)^{-1}\|\nabla u\|_{p(\cdot)}\\ &\leqslant (1-\varepsilon)^{-1}\psi_{p(\cdot)}(F,E;D)+\frac{\varepsilon}{1-\varepsilon}.\end{aligned}$$

Letting $\varepsilon \to 0$ yields assertion (iv).

We then prove (v). It is clear that

$$\psi_{p(\cdot)}(F,\cap_{i=1}^{\infty}K_i;D) \leqslant \lim_{i\to\infty}\psi_{p(\cdot)}(F,K_i;D)$$

Let $\varepsilon > 0$. Assume that $u \in L(F,\cap_{i=1}^{\infty}K_i;D)$ is such that

$$\|\nabla u\|_{p(\cdot)} \leqslant \psi_{p(\cdot)}(F,\cap_{i=1}^{\infty}K_i;D)+\varepsilon.$$

When $i$ is large the set $K_i$ lies in the closed set $\{u \geqslant 1-\varepsilon\}$; therefore

$$\begin{aligned}\lim_{i\to\infty}\psi_{p(\cdot)}(F,K_i;D) &\leqslant \psi_{p(\cdot)}(F,\{u\geqslant 1-\varepsilon\};D)\\ &\leqslant \Big\|\nabla \min\{\frac{u}{1-\varepsilon},1\}\Big\|_{p(\cdot)} \le (1-\varepsilon)^{-1}\|\nabla u\|_{p(\cdot)}\\ &\le (1-\varepsilon)^{-1}\psi_{p(\cdot)}(F,\cap_{i=1}^{\infty}K_i;D)+\frac{\varepsilon}{1-\varepsilon}.\end{aligned}$$

Letting $\varepsilon \to 0$ yields assertion (v).

To prove (vi) let $\varepsilon > 0$ and choose functions $u_i \in L(F,E_i;D)$ such that

$$\|\nabla u_i\|_{p(\cdot)} \leqslant \psi_{p(\cdot)}(F,E_i;D)+\varepsilon/2^i,$$

for $i = 1,\ldots$. Let $v_i = u_1 + \ldots + u_i$. Then $(v_i)$ is a Cauchy sequence, and so it converges to a function $v \in W^{1,p(\cdot)}(D)$. Define $\tilde v(x) = \max\{v(x),1\}$. It is clear that $\tilde v|_{F\cap D} = 0$ and $\tilde v|_{E\cap D} = 1$, where $E = \cup E_i$. Since $p > n$ is locally bounded away from $n$, it follows from Theorem 3.1 that every function in $W^{1,p(\cdot)}(D)$ is continuous, and so we have $\tilde v \in L(F,\bigcup E_i;D)$, from which the claim easily follows, since

$$\|\nabla \tilde v\|_{p(\cdot)} \leqslant \sum_{i=1}^{\infty}\|\nabla u_i\|_{p(\cdot)} \leqslant \sum_{i=1}^{\infty}\psi_{p(\cdot)}(F,E_i;D)+\varepsilon.$$

□

Using the pseudocapacity we can start our study of Sobolev-type imbeddings. The following result is the direct generalization of [18, 5.1.1, Theorem 1].

4.3. **Theorem.** *If $p^+ < \infty$ then the following two conditions are equivalent:*

(1) $W^{1,p(\cdot)}(D) \cap C(D) \hookrightarrow L^{\infty}(D)$.
(2) *There exist $r,k > 0$ such that $\psi_{p(\cdot)}(\overline{D}\setminus B(x,r),x;D) \geqslant k$ for every $x \in D$.*

*Proof.* Suppose that (2) holds, with constants $r,k > 0$. Let $u \in W^{1,p(\cdot)}(D)\cap C(D)$ and let $y \in D$ be a point with $u(y) \neq 0$. Fix a function $\eta \in C_0^{\infty}(B(0,1))$ with $0 \leqslant \eta \leqslant 1$ and $\eta(0) = 1$. Define $v(x) = \eta((x-y)/r)u(x)/u(y)$. It is clear that $v \in W^{1,p(\cdot)}(D)$ and since $v(y) = 1$ and $v(x) = 0$ for $x \notin B(y,r)$ we see that $v \in L(\overline{D}\setminus B(y,r),y;D)$. It follows that

$$k \leqslant \psi_{p(\cdot)}(\overline{D}\setminus B(y,r),y;D) \leqslant \|\nabla v\|_{p(\cdot)}.$$

Then we calculate that

$$\begin{aligned} k|u(y)| &\leqslant \|\nabla(u(x)\eta((x-y)/r))\|_{p(\cdot)} \\ &\leqslant \sup_{x\in D}\eta(x)\|\nabla u\|_{p(\cdot)} + \frac{1}{r}\sup_{x\in D}\nabla\eta(x)\|u\|_{p(\cdot)} \\ &\leqslant \max\Big\{\sup_{x\in D}\eta(x), \frac{1}{r}\sup_{x\in D}\nabla\eta(x)\Big\}\|u\|_{1,p(\cdot)}, \end{aligned}$$

so that $|u(y)|$ is bounded by a constant independent of $y$.

Suppose then conversely that (1) holds and let $C$ be a constant such that $\|u\|_\infty \leqslant C\|u\|_{1,p(\cdot)}$ for all $u \in W^{1,p(\cdot)}(D)$. For functions in $v \in L(\overline{D} \setminus B(x,r), x; D)$ this gives

$$1 = \|v\|_\infty \leqslant C\|v\|_{1,p(\cdot)} \leqslant C(\|\chi_{B(x,r)}\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}).$$

Since $p^+ < \infty$ we can choose $r$ small enough that $\|\chi_{B(x,r)}\|_{p(\cdot)} \leqslant 1/(2C)$. For such $r$ we have

$$\|\nabla u\|_{p(\cdot)} \geqslant 1/(2C).$$

It follows that

$$\psi_{p(\cdot)}(\overline{D} \setminus B(x,r), x; D) = \inf_{u\in L(\overline{D}\setminus B(x,r),x;D)} \|\nabla u\|_{p(\cdot)} \geqslant 1/(2C)$$

for the same $r$. □

4.4. *Remark.* Since we do not know whether $C^\infty(D)$ is dense in $W^{1,p(\cdot)}(D)$ we have only proved the theorem for continuous functions in $W^{1,p(\cdot)}(D)$. If $p$ is such that $C(D)$ is dense in $W^{1,p(\cdot)}(D)$, for instance if $p$ is locally bounded above $n$, then we may replace condition (1) by $W^{1,p(\cdot)}(D) \hookrightarrow L^\infty(D)$.

Define $D = B(1/16) \setminus \{0\}$ and let $p$ be as in Example 3.3. Then the standard example $u(x) = \log|\log(x)|$ shows that $W^{1,p(\cdot)}(D) \not\hookrightarrow L^\infty$, the calculations being as in the theorem. We next show that the exponent $p$ from the theorem is almost as good as possible. We need the following lemma.

4.5. **Lemma.** *Let $\{a_i\}$ be a partition of unity and $k > m-1$. Then*

$$\sum_{i=0}^{\infty} a_i^m i^k \geqslant \left(\sum_{i=0}^{\infty} i^{-k/(m-1)}\right)^{1-m}.$$

*Proof.* Fix an integer $i$ and consider the function

$$a \mapsto (a_i + a)^m i^k + (a_{i+1} - a)^m (i+1)^k,$$

for $-a_i < a < a_{i+1}$. We find that this function has a minimum at $a = 0$ if and only if

$$\left(\frac{a_i}{a_{i+1}}\right)^{m-1} = \left(\frac{i+1}{i}\right)^k. \tag{4.6}$$

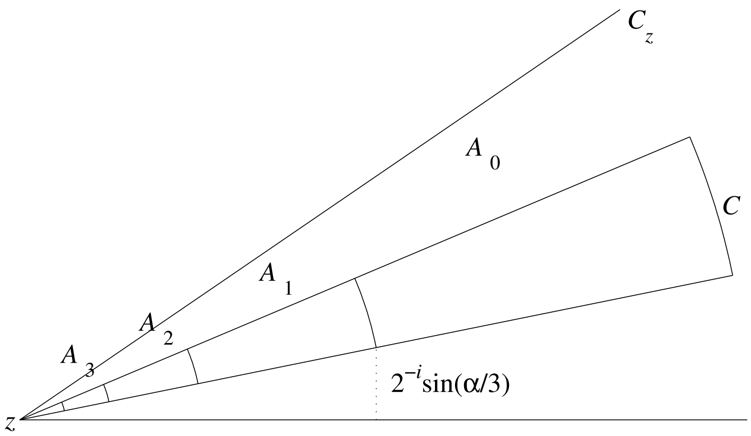
Let $\{a_i\}$ be a minimal sequence, so that (4.6) holds for every $i \geqslant 0$. This partition is given by $a_i = i^{-k/(m-1)} a_0$ for $i > 0$ and $a_0 = (\sum i^{-k/(m-1)})^{-1}$ and so we easily calculate the lower bound as given in the lemma. □

We next give a simple sufficient condition for the imbedding $W^{1,p(\cdot)}(D) \hookrightarrow L^\infty(D)$ to hold in a regular domain:

4.7. **Theorem.** *Suppose that $D$ satisfies a uniform interior cone condition. If $p^+ < \infty$ and*

$$p(x) \geqslant n + (n-1+\varepsilon)\frac{\log_2\log_2(c/\delta(x))}{\log_2(c/\delta(x))}$$

*for some fixed $0 < \varepsilon < n-1$ and constant $c > 0$ then $W^{1,p(\cdot)}(D) \hookrightarrow L^\infty(D)$. Here $\delta(x)$ denotes the distance of $x$ from the boundary of $D$*




F 1. The cone $C$ and the distance to the boundary

*Proof.* Note first that the claim trivially holds in compact subsets of $D$ which satisfy the cone condition, since $p$ is bounded away from $n$ in such sets. Therefore it suffices to prove the claim for $\delta(x)$ less than some constant.

By the uniform interior cone condition there exist real values $0 < \alpha < \pi/2$ and $r > 0$ and a unit vector field $v_x$ defined for $x \in D$ such that for every $x \in D$ the cone

$$C_x = \{y \in B(x, r) \colon \langle x - y, v_x\rangle > |x - y| \cos\alpha\}$$

lies completely in $D$, where $\langle\cdot,\cdot\rangle$ denotes the usual inner product.

Fix $z \in D$. Consider the cone

$$C = \{y \in B(z, r/2) \colon \langle z - y, v_z\rangle > |z - y| \cos(\alpha/3)\}$$

and, for $i = 2, 3, \ldots$, the annuli

$$A_i = (B(z, 2^{-i+1}r) \setminus B(z, 2^{-i}r)) \cap C.$$

To simplify notation let us assume that $z = 0$, $r = 1$ and $v_z = e_1$; the proof in the general case is essentially identical. Since $A_i \subset C \subset D$ we have $d(A_i, \partial D) \geqslant d(A_i, \partial C)$. We can estimate the latter distance as shown in Figure 1. This gives $d(A_i, \partial D) \geqslant 2^{-i}\sin(\alpha/3)$ so that

$$p(x) \geqslant n + (n - 1 + \varepsilon)\frac{\log_2(i + c)}{i + c}$$

for $x \in A_i$ and some $c$ depending on $\alpha$. Let us define $q_i = n + (n - 1 + \varepsilon)\frac{\log_2(i+c)}{i+c}$ and a new variable exponent by

$$q(x) = \begin{cases} q_i & \text{if } x \in A_i \\ p(x) & \text{otherwise} \end{cases}$$

By Theorem 4.3 we know that it suffices to find a lower bound for $\|\nabla u\|_{1,p(\cdot)}$ with $u \in L(\overline{D} \setminus B(0, r), 0; D)$ since by Theorem 3.1 $W^{1,p(\cdot)}(D) \subset C(D)$. Since $\|u\|_{1,p(\cdot)} \geqslant c\|u\|_{1,q(\cdot)}$, we see that it suffices to estimate $\psi_{q(\cdot)}(\overline{D} \setminus B(0, R), 0; B(0, R) \cap D)$ for small $R$ in order to prove the theorem. Moreover, by monotony, we need only consider $\psi_{q(\cdot)}(\overline{D} \setminus B(0, R), 0; B(0, R) \cap C)$. For every function $u \in W^{1,q(\cdot)}(C)$ we have

$$\|u\|_{1,q(\cdot)} \geqslant \min\{1, \varrho_{1,q(\cdot)}(u)\},$$

by [15, Theorem 2.8]. Therefore we see that it suffices to show that $\varrho_{1,q(\cdot)}(u) > c$ for every $u \in L(\overline{D} \setminus B(0, R), 0; B(0, R) \cap C)$ in order to get $\psi_{q(\cdot)}(\overline{D} \setminus B(0, R), 0; B(0, R) \cap C) \geqslant \min\{1, c\} > 0$, which will complete the proof.

It is clear that $|\nabla u| \geqslant |\partial u/\partial r|$, the radial component of the gradient, so that
$$\int_{A_i} |\nabla u|^{q_i} dx \geq \int_{A_i} \Big|\frac{\partial u}{\partial r}\Big|^{q_i} dx.$$
It is then easy to see that the function minimizing the sum over the integrals should depend only on the distance from the origin, not on the direction. For such a function let us denote the value at any point of distance $2^{-i}$ from the origin by $v_i$.

Consider then a function $v$ which equals $v_{i-1}$ on $S(0, 2^{-i+1})$ and $v_i$ on $S(0, 2^{-i})$. Using Lemma 2.5 we find that
$$\begin{aligned}\int_{A_i} |\nabla v|^{q_i} dx &\geqslant (v_{i-1} - v_i)^{q_i} \operatorname{cap}_{q_i}(\mathbb{R}^n \setminus B(0, 2^{-i+1}), B(0, 2^{-i}))\\ &= (v_{i-1} - v_i)^{q_i} \omega_{n-1} \left(\frac{q_i - n}{q_i - 1}\right)^{q_i - 1} (2^{(q_i-n)/(q_i-1)} - 1)^{1-q_i} 2^{i(q_i-n)}\\ &\geqslant c(v_{i-1} - v_i)^{q_i} 2^{i(q_i-n)},\end{aligned}$$
where the constant $c$ does not depend on $q_i$. It follows that
$$\varrho_{1,q(\cdot)}(v) = \sum_{i=0}^{\infty} \int_{A_i} |\nabla u|^{q_i} dx \geqslant c \sum_{i=0}^{\infty} (v_{i-1} - v_i)^{q_i} 2^{i(q_i-n)}.$$
Since the lower bound depends only on the $v_i$, we see that
$$\inf_{u \in L} \varrho_{1,q(\cdot)}(u) \geqslant c \inf_{\{v_i\}} \sum_{i=0}^{\infty} (v_{i-1} - v_i)^{q_i} 2^{i(q_i-n)},$$
where the second infimum is over sequences $\{v_i\}$ with $v_i \leqslant v_{i-1}$, $v_0 = 1$ and $\lim_{i\to\infty} v_i = 0$. Let us set $a_i = v_{i-1} - v_i$ so that $a_i \geqslant 0$ and $\sum a_i = 1$. Then we need to estimate
$$\inf_{\{a_i\}} \sum_{i=0}^{\infty} a_i^{q_i} 2^{i(q_i-n)},$$
with the infimum over partitions of unity $\{a_i\}$. By discounting a finite amount of terms, if necessary, we may assume that
$$\frac{\varepsilon}{3} \geqslant q_i - n = (n - 1 + \varepsilon)\frac{\log_2(i + c)}{i + c} \geqslant (n - 1 + \varepsilon/2)\frac{\log_2(i)}{i}.$$
Since $a_i \leqslant 1$ we have $a_i^{q_i} \geqslant a_i^{n+\varepsilon/3}$ for such terms. Then we find that
$$\inf_{\{a_i\}} \sum_{i=0}^{\infty} a_i^{q_i} 2^{i(q_i-n)} \geqslant \inf_{\{a_i\}} \sum_{i=0}^{\infty} a_i^{n+\varepsilon/3} i^{n-1+\varepsilon/2}$$
and so it follows from Lemma 4.5 that this sum is bounded from below. We have thus shown that the condition of Theorem 4.3 holds, which concludes the proof. □

## R

[1] Buckley, S. M. and P. Koskela: Sobolev-Poincaré implies John, *Mat. Res. Lett.* **2** (1995), 577–593.

[2] Buckley, S. M. and P. Koskela: Criteria for imbeddings of Sobolev-Poincaré type, *Int. Math. Research Notices* **18** (1996), 881–901.

[3] Bojarski, B. : Remarks on Sobolev imbedding inequalities in *Complex Analysis*, Lecture Notes In Math. 1351, Springer, Berlin, 1987, 52–68.

[4] Diening, L. : Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, *Mathematical Inequalities and Applications*, to appear.

[5] Diening, L. and M. Růžička: Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics, preprint.

[6] Edmunds, D. E. and J. Rákosník: Density of smooth functions in $W^{k,p(x)}(\Omega)$, *Proc. Roy. Soc. London Ser. A* **437** (1992), 229–236.

[7] Edmunds, D. E. and J. Rákosník: Sobolev embedding with variable exponent, *Studia Math.* **143** (2000), 267–293.

[8] Edmunds, D. E. and J. Rákosník: Sobolev embedding with variable exponent, II, *Math. Nachr.* **246–247** (2002), 53–67.

[9] Fan, X., J. Shen and D. Zhao: Sobolev embedding theorems for spaces $W^{k,p(x)}$, *J. Math. Anal. Appl.* **262** (2001), 749–760.

[10] Hajłasz, P. and O. Martio: Traces of Sobolev functions on fractal type sets and Characterization of extension domains, *J. Funct. Anal.* **143 (1)** (1997), 221–246.

[11] Harjulehto P., P. Hästö, M. Koskenoja and S. Varonen: Sobolev capacity on the space $W^{1,p(\cdot)}(\mathbb{R}^n)$, *J. Function Spaces and Applications*, to appear.

[12] Harjulehto P., P. Hästö, M. Koskenoja and S. Varonen: Dirichlet energy integral and Sobolev spaces with zero boundary values, preprint.

[13] Heinonen, J., T. Kilpeläinen, and O. Martio: *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 1993.

[14] Hurri, R.: Poincaré domain in $\mathbb{R}^n$, *Ann. Acad. Sci. Fenn. Math. Dissertationes* **71** (1988).

[15] Kováčik, O. and J. Rákosník: On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.* **41(116)** (1991), 592–618.

[16] Malý, J. and W. P. Ziemer: *Fine regularity of Solutions of Elliptic Partial Differential Equations*, Mathematical Surveys and Monographs, Volume 51, American Mathematical Society, 1997

[17] Martio, O. and J. Sarvas: Injectivity theorems in plane and space, *Ann. Acad. Sci. Fenn. Math.* **4** (1978/1979), 383–401.

[18] Maz'ya, V.: *Sobolev spaces*, Springer-Verlag, Berlin, 1985.

[19] Pick, L. and M. Růžička: An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded, *Expo. Math.* **19** (2001), 369–371.

[20] Rákosník, J.: Sobolev inequality with variable exponent, *Function spaces, differential operators and nonlinear analysis*, editors V. Mustonen and J. Rákosník, Mathematical institute of the Academy of the Czech Republic, Praha 2000, pp. 220–228.

[21] Růžička, M.: *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2000.

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