Teichmüller's Extremal Ring Problem

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Abstract. An algorithmic solution to Teichmüller's extremal ring problem is given.

1 Introduction

A ring is a domain in the euclidean space $\mathbb{R}^n, n \geq 2$, characterized by the property that its complement has two components in $\overline{\mathbb{R}}^n$. We let $R(E,F)$ stand for a ring domain with complementary components E, F and denote its conformal capacity and modulus by $\operatorname{cap} R(E,F)$ and $\operatorname{mod} R(E,F)$, respectively. For these notions see F.W. Gehring [G]. The conformal capacity of a ring is a real number, which reflects the shape and relative size of the components with respect to each other. We shall mainly consider the case $n = 2$ and identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. For $n = 2$ a ring can be conformally mapped onto an annulus $\{z \in \mathbb{C} : 1 < |z| < t\}$; in this case the real numbers $2\pi/\log t$ and $\log t$ are its capacity and modulus, respectively. Both capacity and modulus are examples of conformal invariants, which are widely used in geometric function theory, see for instance L.V. Ahlfors [A1], [A2], J. Jenkins [J], G.V. Kuz'mina [K1], [K2], O. Lehto and K.I. Virtanen [LV].

The purpose of this paper is to study the following problem posed by O. Teichmüller, one of the pioneers of geometric function theory and quasiconformal mappings, in 1938 in [T1, p. 638], [T2]. For $z \in \mathbb{C} \setminus \{0,1\}$, find the minimal capacity $p(z)$ of all ring domains with complementary components E, F such that $0, 1 \in E$, $z, \infty \in F$. In other words, Teichmüller considers the problem of evaluating the values of the function $p : \mathbb{C} \setminus \{0,1\} \to (0,\infty)$ defined for $z \in \mathbb{C} \setminus \{0,1\}$ by

$$p(z) \equiv \inf \operatorname{cap} R(E,F) \tag{1.1}$$

where the infimum is taken over all rings $R(E,F)$ with $0, 1 \in E$ and $z, \infty \in F$. We call p the Teichmüller function. We will give in this article an algorithmic solution to this problem. Before we proceed to describe our main results, we give a brief review of some earlier work.

The starting point of these developments was Teichmüller's work, where he used a symmetrization method to prove a lower bound for $p(z)$. Note that Teichmüller considered the modulus of a ring whereas we prefer the capacity. This leads to some minor notational differences. For $0 < r < 1$, set $r' = \sqrt{1 - r^2}$ and

$$\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}, \tag{1.2}$$

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(1.3) $$\tau_2(t) = \pi/\mu(1/\sqrt{1+t}), t > 0.$$

1.4. **Theorem.** [T1, p. 638] *(1) For $z \in \mathbb{R}$, $z > 1$,*

(1.5) $$p(z) = \tau_2(|z-1|)\,.$$

(2) For $z \in \mathbb{C} \setminus \{0,1\}$,

(1.6) $$p(z) \geq \max\{\tau_2(|z-1|), \tau_2(|z|)\} = \max\{p(1+|z-1|), p(-|z|)\}\,.$$

The second part has a geometric interpretation: $-|z|$ and $1+|z-1|$ are the points where circles centered at 0 and 1 through z intersect the negative and positive x-axis, resp. This geometric interpretation is one of the reasons for the several applications of this result. It is also important to note that both lower bounds can be easily computed if we use the arithmetic-geometric mean iteration of Gauss for the complete elliptic integral $\mathcal{K}(r)$, see Section 5 below.

Teichmüller's extremal problem for ring domains was the subject of many studies. M. Schiffer [S] proved that a ring domain with minimal capacity exists and described its shape in terms of a conformal mapping. Further results were obtained by H. Wittich [W] and J. Krzyż [Kr]. A detailed description of the extremal ring domain with an explicit formula for Teichmüller's function in terms of complete elliptic integrals $\mathcal{K}$ of complex arguments can be found in Chapter 5 of a book by G.V. Kuz'mina, in particular in [K1, Theorem 5.2, p. 192].

1.7. **Theorem.** [K1, p. 192] *For $z \in \mathbb{C} \setminus \{0,1\}$,*

(1.8) $$p(z) = \frac{2\pi}{\log M(2z-1)},$$

where

$$\log M(a) = \pi \mathrm{Im}\left\{ i\frac{\mathcal{K}'(r)}{\mathcal{K}(r)} \right\},\ \ r^2 = \frac{2}{1+a}.$$

Here the integrals $\mathcal{K}(r)$ and $\mathcal{K}'(r) \equiv \mathcal{K}(\sqrt{1-r^2})$ are understood to be positive for $r^2 \in (0,1)$ with the explicit formula (1.2), defined for $\mathrm{Im}\,\{r^2\} \neq 0$ by analytic continuation along any path not intersecting the real axis of the r^2-plane, and defined for $\mathrm{Im}\,\{r^2\} = 0$ and $r^2 \notin [0,1]$ by analytic continuation along any path in the lower half-plane $\mathrm{Im}\,\{r^2\} \leq 0$.

Unlike Teichmüller's theorem 1.4 this result does not seem to have a geometric interpretation. As far as we know, there is no simple way of proving Theorem 1.4 as a corollary of Theorem 1.7. And it is not even clear how to use this result for the numerical evaluation of the value $p(z)$ for a given point z.

A. Yu. Solynin and M. Vuorinen proved in [SV] the following duplication formula for the Teichmüller function. It will play a central role in this paper.

1.9. **Theorem.** [SV, Theorem 1.7] *For* $z \in I, I = \{x+iy \in \mathbb{C} \,|\, x \geq 1/2, y \geq 0\} \setminus \{(1,0)\}$, *the function* $p(z)$ *satisfies*

$$p(z) = 2p(w^4)$$

where $w = \sqrt{z} + \sqrt{z-1}$, *and the branches of the square roots are chosen so that* $0 \leq \arg\sqrt{z} \leq \pi/2$ *and* $0 \leq \arg\sqrt{z-1} \leq \pi$ *when* $z \in I$.

Kuz'mina's recent survey [K2] of extremal problems of geometric function theory contains a section, where she reviews what currently is known about Teichmüller's problem and provides relevant references.

The following theorem is crucial for this paper.

1.10. **Theorem.** *For* $|z| > 1$, *we have the inequalities*

$$p(-|z|) \leq p(z) \leq p(|z|)\,. \tag{1.11}$$

Theorem 1.10 is part of the more general result Theorem 3.20. The lower bound follows from Teichmüller's Theorem 1.4 and is based on symmetrization. The upper bound is due to Gehring, see the remarks following Theorem 3.20 below.

The bounds in (1.11) have explicit formulas, as we will see in Section 3, and it is also known that $p(z)$ is monotone when z moves along some arcs of algebraic plane curves, see [K2], [SV], and references therein.

Theorem 1.10 is sharp in the sense that both bounds are attained, but however, at different values of z. Furthermore, both bounds in Theorem 1.10 are asymptotically sharp in the sense that for large $|z|$ the relative error in (1.11) tends to zero.

1.12. **Theorem.** *The relative error in (1.11) tends to* 0, *i.e.*

$$(p(|z|) - p(-|z|))/p(|z|) \to 0\,, |z| \to \infty. \tag{1.13}$$

In this paper we construct two sequences of continuous functions $L_k : \mathbb{C} \setminus \{0,1\} \to (0,\infty)$, $U_k : \mathbb{C} \setminus \{0,1\} \to (0,\infty)$ such that

$$L_k \leq p(z) \leq U_k,\ \ k = 1,2,3,\dots\,. \tag{1.14}$$

For the sake of brevity we write in (1.14) and in what follows L_k, U_k instead of $L_k(z)$, $U_k(z)$. The definition of the sequences $\{L_k\}, \{U_k\}$, is given in Theorem 4.10. We write

$$L = \lim_{k\to\infty} L_k\,, \quad U = \lim_{k\to\infty} U_k. \tag{1.15}$$

One of the consequences of our main result, Theorem 4.10, states that these limits are the same.

1.16. **Theorem.** $L = U = p(z)$.

Furthermore, we obtain an explicit estimate for the speed of convergence $U_k - L_k \to 0$, as $k \to \infty$, which shows that the convergence is uniform in compact subsets of $\mathbb{C} \setminus \{0, 1\}$. Theorem 1.16 provides an algorithmic solution to Teichmüller's problem. This new solution complements the old solution given in Theorem 1.7 and has the following interesting properties. The first property is that both sequences L_k, U_k depend only on the sides $|z|, |z-1|$ of the triangle $0, 1, z$. The second one is that the limiting process in Theorem 1.16 gives a computationally effective method for the computation of the numerical values of $p(z)$ which only involves real numbers. A third property is that some old estimates of capacities of ring domains now appear in a new light, because even the first few majorant and minorant functions U_k and L_k provide new bounds for Teichmüller's function in terms of the usual comparison functions, elliptic integrals.

We have made an effort to present our solution to Teichmüller's problem in a way as selfcontained and as easily accessible as possible. Some notation is introduced in Section 2 and the necessary background information about Teichmüller's function is given in Section 3. In Section 5 we compare our approximation to another method of computing Teichmüller's function from [AVV].

The key ideas of this paper, which lead to the proofs of Theorems 1.16 and 4.10, may be structured into three logical parts. The first idea is to repeatedly apply the duplication formula for the function p in Theorem 1.9, due to A. Solynin and M. Vuorinen [SV, Theorem 1.7].

The second idea is to use a lower and an upper bound for $p(z)$ which are "accurate enough" for large $|z|$. The simplest examples of such bounds are the bounds (1.11). The third idea is to express the majorant and minorant function in (1.11) in terms of complete elliptic integrals and other special functions. These formulas, in combination with the properties of special functions from [AVV], enable us to prove the convergence of our algorithm. For a more detailed description of these three ideas, see 4.2.

2 Notation

Several special functions will play a crucial role in this paper. Perhaps the most basic of these is the hypergeometric function. Given complex numbers a, b, and c with $c \neq 0, -1, -2, \ldots$, the *Gaussian hypergeometric function* is the analytic continuation to the slit plane $\mathbb{C} \setminus [1, \infty)$ of

$$(2.1) \qquad F(a,b;c;z) = {}_2F_1(a,b;c;z) \equiv \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \quad |z| < 1.$$

Here $(a, 0) = 1$ for $a \neq 0$, and (a, n) is the *shifted factorial function*

$$(2.2) \qquad (a,n) \equiv a(a+1)(a+2)\cdots(a+n-1)$$

for $n = 1, 2, 3, \ldots$. An important special case is

$$(2.3) \qquad \mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}},$$

which is called *a complete elliptic integral of the first kind.* In what follows the argument is usually a real number with $0 \le r < 1$. For $r \in [0,1]$ we sometimes write $r' = \sqrt{1-r^2}$. We will make use of the growth estimate

$$\log \frac{4}{r'} < \mathcal{K}(r) < \log \frac{e^{\pi/2}}{r'} \tag{2.4}$$

for $r \in (0,1)$, from [AVV, Theorem 3.21 (3)].

We also write, for $0 < r < 1$,

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(r')}{\mathcal{K}(r)}. \tag{2.5}$$

The function $\mu(r)$ satisfies the following functional identities

$$\begin{cases} \mu(r)\mu(r') = \dfrac{\pi^2}{4} \ , \\ \mu(r)\mu\Big(\dfrac{1-r}{1+r}\Big) = \dfrac{\pi^2}{2} \ , \\ \mu(r) = 2\mu\Big(\dfrac{2\sqrt{r}}{1+r}\Big) \ . \end{cases} \tag{2.6}$$

These functions have been studied systematically in [AVV]. We will require below for instance the following formulas for $0 < r < 1$ [AVV, p. 82 (5.9)]

$$\mu'(r) = -\frac{\pi^2}{4rr'^2\mathcal{K}(r)^2} \ , \tag{2.7}$$

and [AVV, Theorem 5.13 (4) p. 84 and (5.30)]

$$\log \frac{1+3r'}{r} < \operatorname{arth} \sqrt[4]{r'} < \mu(r) < \log \frac{2(1+r')}{r} \ . \tag{2.8}$$

3 The Teichmüller function

3.1. **The Teichmüller and Mori rings.** The complementary components of *the Teichmüller ring* are $[-1,0]$ and $[s,\infty]$, $s > 0$. The complementary components of *the Mori ring* are $[0,\infty]$ and $\{e^{it} : t \in [\pi - 2\varphi, \pi + 2\varphi]\}$, where $2\varphi \in [0,\pi/2]$. These two ring domains are conformally equivalent to extremal rings for Teichmüller's extremal problem in particular cases, see [K1, p. 192], which we shall now describe. The first particular extremal ring for $p(z)$ is the Teichmüller ring, when $z \in \mathbb{R} \setminus [0,1]$, while the second one is the Mori ring when $\operatorname{Re}\{z\} = \frac{1}{2}$.

For $t > 0$ the capacity $\tau_2(t)$ of the Teichmüller ring can be expressed by [K1, p. 192], [Vu2, 5.60 (1)] as

$$\tau_2(t) = \frac{\pi}{\mu(1/\sqrt{1+t})} = p(1+t) = p(-t), \tag{3.2}$$

whereas the capacity of Mori's ring has the form [AVV, Theorem 8.54], [LV, p. 59]

$$\nu_2(\varphi) = \frac{2\pi}{\mu(\sin(\varphi))} = p(\frac{1}{2} + i\frac{1}{2\tan(2\varphi)})\,. \tag{3.3}$$

These canonical ring domains have several applications to quasiconformal mappings, see e.g. [A1] and [LV].

3.4. **Lemma.** *For $t > 0$,*

$$\tau_2'(t) = -\frac{\pi}{2t\,\mathcal{K}\left(\sqrt{\frac{t}{1+t}}\right)^2}\,. \tag{3.5}$$

In particular, both $\tau_2(t)$ and $|\tau_2'(t)|$ are strictly decreasing on $(0,\infty)$ and tend to zero as $t \to \infty$, and for $0 < t < s$,

$$\frac{\pi(s-t)}{2s\,\mathcal{K}\left(\sqrt{\frac{s}{1+s}}\right)^2} < \tau_2(t) - \tau_2(s) < \frac{\pi(s-t)}{2t\,\mathcal{K}\left(\sqrt{\frac{t}{1+t}}\right)^2}\,. \tag{3.6}$$

Proof. The formula (3.5) follows from (2.7) and (3.2). The second part of the lemma follows from [AVV, Theorem 3.21 (2)] and (3.6) follows from the mean value theorem and (3.5). $\square$

3.7. **The duplication formula for $\tau_2(t)$.** On the basis of the duplication formula for Teichmüller's function in Theorem 1.9 it is clear that there exists a duplication formula for $\tau_2(t)$ too. Following [AVV, 5.19] we will give below in (3.8)-(3.10) this formula in a form which is more convenient for our purposes than using Theorem 1.9 directly. The definition (3.2) combined with the identities (2.6) for the function $\mu(r)$ yield the following expedient formulas

$$\frac{2\pi}{\mu(1/s)} = 2\tau_2(s^2-1) = \tau_2\left(\frac{(s-1)^2}{4s}\right), \quad s > 1\,, \tag{3.8}$$

$$\tau_2\left(\frac{s-1}{2}\right) = 2\tau_2((s+\sqrt{s^2-1})^2-1), \quad s > 1, \tag{3.9}$$

$$\tau_2(s) = 2\tau_2((\sqrt{s}+\sqrt{s+1})^4-1), \quad s > 0. \tag{3.10}$$

3.11. **Lemma.** *For $s > 1$, let $t = s + \sqrt{s^2-1}$. Then*

$$\tau_2(\frac{s-1/2}{2}) < 2\,\tau_2(t^2) < \tau_2(\frac{s-1}{2})\,. \tag{3.12}$$

Proof. Fix $s > 1$. The upper bound follows from (3.9) because τ_2 is strictly decreasing by Lemma 3.4. Observing that $s = (1+t^2)/(2t)$ we see by (3.8) that the lower bound is equivalent to

$$\tau_2(\frac{s-1/2}{2}) = \tau_2(\frac{1+t^2-t}{4t}) < 2\tau_2(t^2) = \tau_2(\frac{(\sqrt{1+t^2}-1)^2}{4\sqrt{1+t^2}})$$

and further to

$$u \equiv \frac{t^4}{(\sqrt{1+t^2}+1)^2\sqrt{1+t^2}} < \frac{1+t^2-t}{t}.$$

Because $(1+t)\sqrt{1+t^2} > 1+t^2$ for all $t > 0$, we see that

$$\frac{t^4\,(1+t)}{(\sqrt{1+t^2}+1)^2\,(1+t^2)} < \frac{t^4\,(1+t)}{(1+t)^2\,(1+t^2)} < \frac{1+t^2-t}{t},$$

where the third inequality follows after some elementary manipulation. □

3.13. **The tilde operation.** From the definition it is clear that Teichmüller's function has the following symmetries: the values at $z \in \mathbb{C}\setminus\{0,1\}$ and z_1 are the same if z_1 is obtained from z by a reflection in the real axis or in the line $\{z : \mathrm{Re}\{z\} = 1/2\}$. Thus all the values of $p(z)$ are determined by the values of $p(z)$ in the set I as defined in Theorem 1.9. We now define an argument reduction operation, which for each $z \in \mathbb{C}\setminus\{0,1\}$ defines its representative in I . For $z \in \mathbb{C}\setminus\{0,1\}$, denote

$$\tilde{z} = |\mathrm{Re}\,\{z\} - 1/2| + 1/2 + i|\mathrm{Im}\,\{z\}|. \tag{3.14}$$

Then $\tilde{z} \in I$ and

$$\{|z|, |z-1|\} = \{|\tilde{z}|, |\tilde{z}-1|\} \tag{3.15}$$

since the mapping $z \mapsto \tilde{z}$ consists of possibly one reflection in the real axis and possibly one reflection in the line $\{x+iy \in \mathbb{C} \mid x = 1/2\}$. Note that $p(\tilde{z}) = p(z)$ for all $z \in \mathbb{C}\setminus\{0,1\}$.

The duplication transformation $z \mapsto w(z)^4 \equiv (\sqrt{z}+\sqrt{z-1})^4$ from Theorem 1.9 will have a crucial role in the sequel. In particular, we will need the following formulas.

3.16. **Lemma.** [BV, (2.8), (2.9)] *Let $z \in \mathbb{C}\setminus\{0,1\}$, $r = |z|$ and $s = |z-1|$. Then*

$$|w(z)|^4 = (r+s+\sqrt{(r+s)^2-1})^2, \tag{3.17}$$

and

$$|w(z)^4 - 1| = 4\sqrt{rs}\,(r+s+\sqrt{(r+s)^2-1}). \tag{3.18}$$

If we set $z_0 = \tilde{z}, z_{k+1} = \widetilde{w(z_k)^4}$, then $p(z) = 2^k p(z_k)$ and

$$\begin{aligned}\{|z_{k+1}|, |z_{k+1}-1|\} \;=\; &\{(|z_k| + |z_k-1| + \sqrt{(|z_k|+|z_k-1|)^2-1})^2,\\ &4\sqrt{|z_k||z_k-1|}\,(|z_k|+|z_k-1|+\sqrt{(|z_k|+|z_k-1|)^2-1})\}\end{aligned} \tag{3.19}$$

with $|z_{k+1}| \geq |z_{k+1}-1|$.

Proof. For the proof of (3.17) we proceed as follows:

$$\begin{aligned}
|w(z)|^2 &= |\sqrt{z}+\sqrt{z-1}|^2\\
&= (\sqrt{z}+\sqrt{z-1})(\sqrt{\bar z}+\sqrt{\bar z-1})\\
&= \sqrt{z}\sqrt{\bar z}+\sqrt{z-1}\sqrt{\bar z-1}+(\sqrt{z}\sqrt{\bar z-1}+\sqrt{\bar z}\sqrt{z-1})\\
&= r+s+(\sqrt{z}\sqrt{\bar z-1}+\sqrt{\bar z}\sqrt{z-1})\\
&= r+s+\sqrt{z(\bar z-1)+\bar z(z-1)+2\sqrt{z(\bar z-1)\bar z(z-1)}}\\
&= r+s+\sqrt{z\bar z+(z\bar z-z-\bar z)+2\sqrt{r^2s^2}}\\
&= r+s+\sqrt{r^2+(z\bar z-z-\bar z)+2rs}\\
&= r+s+\sqrt{r^2+s^2-1+2rs}\\
&= r+s+\sqrt{(r+s)^2-1}\,.
\end{aligned}$$

On the other hand, $w(z)^2 = z+z-1+2\sqrt{z}\sqrt{z-1}$. Therefore

$$\begin{aligned}
w(z)^4-1 &= (w(z)^2-1)(w(z)^2+1)\\
&= 2(z-1+\sqrt{z}\sqrt{z-1})\,2(z+\sqrt{z}\sqrt{z-1})\\
&= 4(z-1+\sqrt{z}\sqrt{z-1})(z+\sqrt{z}\sqrt{z-1})\\
&= 4(z(z-1)+z\sqrt{z}\sqrt{z-1}+(z-1)\sqrt{z}\sqrt{z-1})\\
&= 4\sqrt{z}\sqrt{z-1}(2\sqrt{z}\sqrt{z-1}+z+(z-1))\\
&= 4\sqrt{z}\sqrt{z-1}w(z)^2\,.
\end{aligned}$$

Hence $|w(z)^4-1| = 4\sqrt{rs}\,(r+s+\sqrt{(r+s)^2-1})$.

Now (3.19) follows from (3.17) and (3.18) with (3.15). The last claim follows from the fact that $z_{k+1}\in I$. $\square$

Note that it follows from (3.17) that w maps the segment $[0,1]$ into the unit circumference, because $r+s=1$ at the points of this segment and that $|z_k|>1$ for all $k=1,2,3,...$

Teichmüller's problem also makes sense in $\mathbb{R}^n, n>2$, and in this case we denote Teichmüller's function by $p_n(z), z\in\mathbb{R}^n\setminus\{0,e_1\}$, where e_i is the i^{th} coordinate unit vector in $\mathbb{R}^n$. In dimensions $n>2$ much less is known than for $n=2$ about the conformal capacity in general and Teichmüller's function in particular. For instance, there is no formula for $p_n(z), n>2$, like the formula in Theorem 1.7. The methods applied to the case $n>2$ are often geometric in character and use e.g. the symmetrization and polarization methods, see F. W. Gehring [G], M. Vuorinen [Vu2, Chapter 8], D. Betsakos [Be], and D. Betsakos and M. Vuorinen [BV]. Many upper and lower bounds have been found for $p_n(z)$ in terms of the sides of the triangle with vertices at $0, e_1, z$. Symmetrization is a transformation that to each ring domain in $\mathbb{R}^n$ associates another ring domain in $\mathbb{R}^n$ with specific rules. Teichmüller proved that the capacity of a ring domain in $\mathbb{C}$ decreases under symmetrization and this method was extended by Gehring [G] to the case of $\mathbb{R}^3$. The reader interested in the applications of symmetrization methods to complex analysis is referred to a survey of A. Baernstein [Ba].

3.20. **Theorem.** *For $z \in \mathbb{R}^n$, $|z| > 1$, the following inequalities hold:*

$$\tau_n(|z|) = p_n(-|z|\,e_1) \le p_n(z) \le p_n(|z|\,e_1) = \tau_n(|z|-1)\,, \tag{3.21}$$

where $\tau_n(t)$, $t > 0$, is the capacity of the Teichmüller ring in $\mathbb{R}^n$ with complementary components $[-e_1, 0]$ and $\{se_1 : s \ge t\}$. The lower and upper bounds in (3.21) hold with equality if $z = -se_1, s > 0$, or $z = se_1, s > 1$, respectively. Furthermore, for $z \in \mathbb{R}^n \setminus \{0, e_1\}$, the upper bound may be refined to

$$p_n(z) \le \tau_n\left(\frac{|z| + |z - e_1| - 1}{2}\right) \le \tau_n(|z|-1)\,, \tag{3.22}$$

with equality in the first inequality both for $z = -se_1, s > 0$, and $z = se_1, s > 1$.

Proof. The lower bound in (3.21) is due to Teichmüller [T1] for $n = 2$, Gehring for $n = 3$ [G], and G. D. Mostow [M] for $n \ge 4$. The upper bound in (3.21) was conjectured by Vuorinen and its proof, first published in [Vu1, Lemma 2.58], is due to Gehring (the same proof is also given in [Vu2, Lemma 5.27]). The inequality (3.22) was proved by Vuorinen in [Vu3]. □

Refined versions of (1.11) for the dimensions $n \ge 2$ were proved in [BV], and [Be].

Proof of Theorem 1.12 . The proof follows from (3.21) and Lemma 3.4 . □

3.23. **Open problem.** Find a duplication formula like the one in Theorem 1.9 for p_n.

The next result, which deals with the case $n = 2$, shows that the maximal and minimal values of p on an ellipse with foci at 0 and 1 occur at the end points of the semiaxes. The minimal value, attained at the end point of a smaller semiaxis, corresponds to the case when the extremal ring is Mori's ring. The maximal value is attained at the end point of a greater semiaxis and corresponds to Teichmüller's ring. This well-known result was also used in [SV].

3.24. **Lemma.** *Let $z \in \mathbb{C} \setminus \{0, 1\}$ and let w_1 be the point of intersection of an ellipse through z with foci at $0, 1$ with the positive real axis and w_2 the point of intersection of this ellipse with the line $x = \frac{1}{2}$ in the upper half plane. If $s = |z| + |z - 1|$, then*

$$p(w_2) \le p(z) \le p(w_1) \tag{3.25}$$

with

$$p(w_1) = \tau_2((s-1)/2) = 2\tau_2((s + \sqrt{s^2-1})^2 - 1), \tag{3.26}$$

and

$$p(w_2) = \frac{2\pi}{\mu\left(\sqrt{\frac{1}{2s(s+\sqrt{s^2-1})}}\right)} = 2\tau_2((s + \sqrt{s^2-1})^2) > \tau_2(\frac{s - 1/2}{2}). \tag{3.27}$$

Proof. Let $z \in \mathbb{C} \setminus \{0,1\}$. By symmetry we may assume that $\mathrm{Re}\{z\} \geq 1/2$ and $\mathrm{Im}\{z\} \geq 0$. Assume first that $z \notin [1/2, 1)$. Then, by [SV, 1.5.], $p(w_1) \geq p(z) \geq p(w_2)$ where z and w_1, w_2 are on the same ellipse with foci at 0 and 1, $\mathrm{Im}\{w_2\}$ is maximal, $\mathrm{Im}\{w_1\} = 0$, $\mathrm{Re}\{w_1\} = (s+1)/2$. Because $p(w_1) = \tau_2((s-1)/2)$ by (3.2), the upper bound follows from (3.9). Note that the upper bound also follows from (3.22). It remains to prove the lower bound. Clearly $\mathrm{Re}\{w_2\} = 1/2$. If $z \in [1/2, 1)$, then we use the formula (1.8) together with the fact [SV, (3.21)] that $\log M(a)$ is decreasing for $a \in (0,1)$ to obtain

$$p(z) = \frac{2\pi}{\log M(2z-1)} \geq \frac{2\pi}{\log M(0)} = p(1/2) = 2\,\tau_2(1) = 4.$$

Hence also in this case we may write $p(z) \geq p(w_2)$ with $w_2 = 1/2$.

Now $|w_2| = |w_2 - 1| = s/2$. Denote $v = \mathrm{Im}\,\{w_2\}$. Then $v^2 + (1/2)^2 = (s/2)^2$ and it follows that $v = (\sqrt{s^2-1})/2$. Hence $w_2 = (1 + i\sqrt{s^2-1})/2$.

Let α be the angle between the segments $[w_2, 0]$ and $[w_2, 1]$. By (3.3) we get

$$\nu_2(\alpha) = 2\pi/\mu(\sin\alpha) = 2\tau_2(1/\sin^2\alpha - 1).$$

We have

$$\cos 2\alpha = \frac{v}{u} = \frac{\sqrt{s^2-1}}{s}$$

and by elementary trigonometry

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2} = \left(1 - \frac{\sqrt{s^2-1}}{s}\right)/2$$

and

$$\begin{aligned}
\frac{1}{\sin^2\alpha - 1} &= \frac{1-\sin^2\alpha}{\sin^2\alpha} \\
&= \left(2 - \left(1 - \frac{\sqrt{s^2-1}}{s}\right)\right) / \left(1 - \frac{\sqrt{s^2-1}}{s}\right) \\
&= \frac{s+\sqrt{s^2-1}}{s-\sqrt{s^2-1}} = (s+\sqrt{s^2-1})^2.
\end{aligned}$$

Hence by (3.12)

$$p(z) \geq p(w_2) = 2\tau_2((s+\sqrt{s^2-1})^2) > \tau_2(\frac{s-1/2}{2})\,. \qquad \square$$

3.28. **Particular cases of $p(z)$.** We know by (3.2) and (3.3) expressions for $p(z)$ in terms of well-known functions in the two particular cases $z > 1$ and $\mathrm{Re}\,\{z\} = 1/2$. For later use, we record one more such case from [SV, p. 4107, (3.21)] : if $0 < x < 1$, then

$$\begin{aligned}
p(x) &= 2\,\frac{\mathcal{K}(\sqrt{x})^2 + \mathcal{K}(\sqrt{1-x})^2}{\mathcal{K}(\sqrt{x})\mathcal{K}(\sqrt{1-x})} = \frac{4}{\pi}\left(\mu(\sqrt{1-x}) + \mu(\sqrt{x})\right) \\
&= \tau_2(\frac{1-x}{x}) + \tau_2(\frac{x}{1-x})\,.
\end{aligned} \tag{3.29}$$

In particular, $p((1/2, 0)) = 4$. Observe that some additional particular cases will follow, if we apply the duplication transformation in Theorem 1.9.

4 Convergence

4.1. **The sequence** (z_k)**.** Throughout this section we shall assume that $z \in \mathbb{C} \setminus \{0,1\}$ is fixed. We set $z_0 = \tilde{z}, z_{k+1} = \widetilde{w(z_k)^4}, k = 0,1,2,...$ where $\tilde{}$ is the mapping defined in (3.14) and w is the duplication transformation from Theorem 1.9. In this section we will give an iterative algorithm, defined in terms of the sequence (z_k), for the computation of $p(z)$ and prove the convergence of the algorithm.

4.2. **Preliminary considerations.** As already noted in Theorem 3.20, the lower and upper bounds in (3.21) hold with equality for certain specific choices of the argument z, i.e. the inequality is sharp. We will show next that this inequality also is asymptotically sharp for large $|z|$. To clarify this statement observe first that both bounds of inequality (3.21) have the same limit 0 when $|z| \to \infty$. Next introduce the notation

$$\varepsilon_n(t) \equiv (\tau_n(t) - \tau_n(t+1))/\tau_n(t)\,, \quad t > 0\,. \tag{4.3}$$

With this notation we see that $\varepsilon_n(|z|-1)$ is an upper bound for the relative error in (3.21). From (3.2), (2.5), (3.5), and the mean value theorem we get

$$\varepsilon_2(t) \le \frac{\pi}{2t\,\mathcal{K}(\sqrt{t/(t+1)})^2}\frac{\mu(1/\sqrt{t+1})}{\pi} = \frac{\pi}{4}\frac{1}{t\,\mathcal{K}(\sqrt{t/(t+1)})\mathcal{K}(1/\sqrt{t+1})}\,. \tag{4.4}$$

It follows from [AVV, Lemma 3.32 (1)] that this last function is monotone decreasing on $[1,\infty)$ with limit 0 as $t \to \infty$. Thus the relative error in (3.21) tends to 0 when $|z| \to \infty$ and inequality (3.21) is asymptotically sharp for large $|z|$.

The bounds for the function $p(z)$ that we will give in this paper are based on this observation and on the duplication formula $p(z) = 2^k p(z_k), k = 0,1,2,...$ Note that the relative error in the estimate (3.21) for $p(z_k)$ is much smaller than $p(z)$, provided that $|z_k|$ is much larger than $|z|$. Instead of applying (3.21) directly to the estimation of $p(z)$ we may apply it to $p(z_k)$. A possible drawback in this idea is that the growth of the coefficient 2^k may spoil the increased accuracy given by the duplication formula, unless there is a suitable balance. It is our goal in this section to show that such a balance holds and hence the idea works.

We first obtain a growth estimate for the sequence (z_k) in 4.1.

4.5. **Lemma.** *If we write* $M_k = \frac{1}{16}\exp\left(2^{k+1}\pi/p(z)\right)$, *then for all* $k = 0,1,2,...$ *we have*

$$M_k \le |z_k| \le 1 + M_k. \tag{4.6}$$

Proof. Suppose first that $|z_k| > 1$. It follows from (3.21) that

$$\tau_2(|z_k|) \le p(z_k) \le \tau_2(|z_k|-1)\,. \tag{4.7}$$

Then, from (2.8) and (3.2) we get

$$\frac{\pi}{\log(2(\sqrt{1+t}+\sqrt{t}))} \le \tau_2(t) \le \frac{\pi}{\log(\sqrt{1+t}+3\sqrt{t})}\,. \tag{4.8}$$

The inequalities (4.7) and (4.8) together give

$$\frac{\pi}{\log(4\sqrt{|z_k|})} \le p(z_k) \le \frac{\pi}{\log(4\sqrt{|z_k|-1})}\,. \tag{4.9}$$

Next, the duplication formula gives $p(z) = 2^k p(z_k)$ which together with (4.9) gives the desired inequality. It remains to consider the case $|z_k| < 1$. The above proof of the lower bound in (4.6) is valid in this case, too. Hence it remains to consider the upper bound in (4.6), but it holds trivially in this case. □

It follows from Lemma 4.5 that $z_k \to \infty$ as $k \to \infty$.

4.10. **Theorem.** *Let $z \in \mathbb{C} \setminus \{0,1\}$ and let (z_k) be the sequence from 4.1. Let $U_k \equiv 2^k\,\tau_2(|z_k|-1)$, $L_k \equiv 2^k\,\tau_2(|z_k|)$, $k = 1,2,\ldots$ (recall that $|z_k| > 1$ for $k = 1,2,3,\ldots$). Then for all $k = 1,2,\ldots$,*

$$L_k \le p(z) \le U_k, \tag{4.11}$$

and

$$U_k - L_k \le \frac{2^{k-1}\,\pi}{(|z_k|-1)\mathcal{K}\left(\sqrt{\frac{|z_k|-1}{|z_k|}}\right)^2} < \frac{2^{-k-2}\,p(z)^2}{|z_k|-1}. \tag{4.12}$$

If k is so large that $2^k \ge p(z)$, then

$$U_k - L_k \le 2^{1-k}\,p(z)^2 \exp\left(-\frac{2^{k+1}\pi}{p(z)}\right). \tag{4.13}$$

Proof. The inequality (4.11) follows from Theorem 1.9 and (3.21). The formula (3.5) gives

$$2^k\,\tau_2(|z_k|-1) - 2^k\,\tau_2(|z_k|) < \frac{2^k\,\pi}{2(|z_k|-1)\,\mathcal{K}(\sqrt{(|z_k|-1)/|z_k|})^2}.$$

Next, by (2.4) and Lemma 4.5 we get $\mathcal{K}(\sqrt{(|z_k|-1)/|z_k|}) > 2^{k+1}\pi/p(z)$ and (4.12) follows. These inequalities also give

$$(|z_k|-1)\mathcal{K}\left(\sqrt{\frac{|z_k|-1}{|z_k|}}\right)^2 \ge \left(\frac{1}{16}\exp\left(\frac{2^{k+1}\pi}{p(z)}\right) - 1\right)\left(\frac{2^{k+1}\pi}{p(z)}\right)^2.$$

Next, because $2^k \ge p(z)$ we have

$$\frac{1}{16}\exp\left(\frac{2^{k+1}\pi}{p(z)}\right) - 1 > \frac{1}{32}\exp\left(\frac{2^{k+1}\pi}{p(z)}\right)$$

which together with the earlier results completes the proof of (4.13). □

The following lemma will be applied in the monotonicity proof below.

4.14. **Lemma.** *If $z \in \mathbb{C} \setminus \{0,1\}$ with $\operatorname{Re}\{z\} \geq 1/2$, then for $r = |z|$ and $s = |z-1|$ we have that*

$$(r+s+\sqrt{(r+s)^2-1})^2 < (\sqrt{r}+\sqrt{r+1})^4 - 1 \tag{4.15}$$

and that

$$16rs < (\sqrt{r}+\sqrt{r+1})^4 - 1. \tag{4.16}$$

Proof. Since $\operatorname{Re}\{z\} \geq 1/2$, we have $r \geq s$ and it follows that

$$\begin{aligned}(r+s+\sqrt{(r+s)^2-1})^2 &\leq (2r+\sqrt{(2r)^2-1})^2 \\ &= 8r^2+4r\sqrt{(2r)^2-1}-1 \\ &< 8r^2+4r\sqrt{(2r)^2}-1 \\ &= 16r^2-1 = (2\sqrt{r})^4-1 \\ &< (\sqrt{r}+\sqrt{r+1})^4-1\end{aligned}$$

and that

$$\begin{aligned}(\sqrt{r}+\sqrt{r+1})^4-1 &= (2r+2\sqrt{r(r+1)}+1)^2-1 \\ &> (4r+1)^2-1 \\ &= 16r^2+8r \\ &\geq 16rs. \qquad \square\end{aligned}$$

We summarize the results of this section in the following theorem.

4.17. **Theorem.** *Let $z \in \mathbb{C} \setminus \{0,1\}$ and let (z_k) be the sequence from 4.1. Then for all $k = 1, 2, \ldots,$*

$$L_k \equiv 2^k\,\tau_2(|z_k|) \leq p(z) \leq 2^k\,\tau_2(|z_k|-1) \equiv U_k \tag{4.18}$$

and the sequences (L_k), (U_k) both converge to $p(z)$. Moreover, $L_k < L_{k+1}$ and $U_k \geq U_{k+1}$.

Proof. The formula (4.18) and the convergence were proved in Theorem 4.10.

Denote $r_k = |z_k|$, $s_k = |z_k - 1|$, $v_k = r_k + s_k$, $w_k = r_k s_k$, $t_k = \sqrt{w_k}$, and $A_k = v_k + \sqrt{v_k^2-1}$. We prove that for $k \in \mathbb{N} \setminus \{0\}$, $L_k < L_{k+1}$. By Theorem 3.16 this is to say that

$$2\tau_2(\max\{A_k^2, 4t_kA_k\}) > \tau_2(r_k).$$

By (3.10) we see that

$$\tau_2(r_k) = 2\tau_2((\sqrt{r_k}+\sqrt{r_k+1})^4-1).$$

Hence we need to prove that

$$\max\{A_k^2, 4t_kA_k\} < (\sqrt{r_k}+\sqrt{r_k+1})^4-1.$$

In the case when $A_k^2 \geq 4t_kA_k$ this holds by (4.15) since $z_k \in I$. Assume then that $4t_kA_k > A_k^2$. Then, by (4.15), $A_k < \sqrt{(\sqrt{r_k}+\sqrt{r_k+1})^4-1}$, so after squaring it is enough to prove that

$$16r_ks_k < (\sqrt{r_k}+\sqrt{r_k+1})^4-1.$$

But this is true by (4.16).

It remains to prove that $U_k \geq U_{k+1}$ for $k = 1, 2, \ldots$. This is equivalent to

$$\tau_2(r_k - 1) \geq 2\tau_2(\max\{A_k^2, 4t_k A_k\} - 1).$$

Using (3.9) we see that

$$\begin{aligned} 2\tau_2(\max\{A_k^2, 4t_k A_k\} - 1) &\leq 2\tau_2(A_k^2 - 1) \\ &= \tau_2\left(\frac{v_k - 1}{2}\right), \end{aligned}$$

so it suffices to prove that

$$r_k - 1 \leq \frac{v_k - 1}{2},$$

or equivalently,

$$r_k \leq s_k + 1,$$

which is true by the triangle inequality. $\square$

4.19. **Theorem.** *Let $z \in \mathbb{C} \setminus \{0, 1\}$ and let (z_k) be the sequence from 4.1. Then for all $k = 1, 2, \ldots,$*

$$\begin{aligned} l_k &\equiv 2^{k+1}\,\tau_2((|z_k| + |z_k - 1| + \sqrt{(|z_k| + |z_k - 1|)^2 - 1})^2) \\ &\leq p(z) \leq 2^k\,\tau_2(\frac{|z_k| + |z_k - 1| - 1}{2}) \equiv u_k \end{aligned} \tag{4.20}$$

and the sequences (l_k), (u_k) both converge to $p(z)$. Moreover, $l_k \geq L_k$, $u_k \leq U_k$, and $u_k \geq u_{k+1}$.

Proof. We use the notation from the proof of Theorem 4.17.

The inequalities in (4.20) follow from Theorem 1.9 with Lemma 3.24 and (3.22).

By the triangle inequality we have that $r_k \leq s_k + 1$, which implies that

$$\tau_2\left(\frac{r_k + s_k - 1}{2}\right) \leq \tau_2(r_k - 1),$$

which is to say that $u_k \leq U_k$ for all $k = 1, 2, \ldots$. Again, by the triangle inequality, $s_k \leq r_k + 1$. We apply (3.9) to obtain

$$2\tau_2\left(\left(v_k + \sqrt{v_k^2 - 1}\right)^2\right) = \tau_2\left(\frac{v_k - 1}{2}\right) \geq \tau_2(r_k),$$

which implies that $l_k \geq L_k$ for all $k = 1, 2, \ldots$. Hence the convergence follows from Theorem 4.17.

The claim $u_k \geq u_{k+1}$ is equivalent to

$$\tau_2\left(\frac{v_k - 1}{2}\right) \geq 2\,\tau_2\left(\frac{v_{k+1} - 1}{2}\right).$$

We have that $r_{k+1} = \max\{A_k^2, 4t_kA_k\} \geq A_k^2$. Hence, by (3.9) we get

$$\begin{aligned}\tau_2\left(\frac{v_k-1}{2}\right) &= 2\,\tau_2\left(\left(v_k+\sqrt{v_k^2-1}\right)^2-1\right)\\ &= 2\,\tau_2(A_k^2-1)\\ &\geq 2\,\tau_2(r_{k+1}-1).\end{aligned}$$

Thus it is enough to show that

$$r_{k+1}-1 \leq \frac{v_{k+1}-1}{2} = \frac{r_{k+1}+s_{k+1}-1}{2},$$

which is equivalent to

$$r_{k+1} \leq s_{k+1}+1.$$

This holds by the triangle inequality. □

5 The algorithm

The purpose of this section is to express the iterative procedure in Section 4 as an algorithm that could be easily implemented in a programming language. Because this algorithm will involve also the function $\tau_2(t)$ and hence complete elliptic integrals, we also briefly recall classical facts about the numerical computation of these functions; see the software supplement to [AVV] for more details.

5.1. **The arithmetic-geometric mean.** The *arithmetic-geometric mean* of positive numbers a, b is the limit

(5.2) $$AG(a,b) = \lim a_n = \lim b_n,$$

where $a_0 = a$, $b_0 = b$, and for $n = 0, 1, 2, 3, ...,$

$$a_{n+1} = A(a_n, b_n) \equiv (a_n+b_n)/2, \quad b_{n+1} = G(a_n, b_n) \equiv \sqrt{a_n b_n},$$

are the arithmetic and geometric means of a_n and b_n, resp. It is a basic fact that if $0 < b < a$, then $b_n < b_{n+1} < a_{n+1} < a_n$ for all $n = 0, 1, 2, ...$

More than two centuries ago, the identity

(5.3) $$AG(1, r') = \frac{\pi}{2\mathcal{K}(r)},\ 0 < r < 1\,,$$

was used by Lagrange and Gauss for the numerical computation of complete elliptic integrals, see [AVV, 4.16, p. 79].

If we combine (2.5), (5.3), and (3.2), then we get the following corollary.

5.4. **Corollary.** *For $z \in \{x \in \mathbb{R} : x > 1\}$ we have the following formula*

$$p(z) = \tau_2(|z| - 1) = \frac{2\,AG(1,r)}{AG(1,r')} = \frac{2F(\frac{1}{2},\frac{1}{2};1;\frac{1}{z})}{F(\frac{1}{2},\frac{1}{2};1;\frac{z-1}{z})}, \quad r = 1/\sqrt{z}. \tag{5.5}$$

5.6. **Remark.** An algorithm for the computation of Teichmüller's function was given in a software supplement to [AVV]. This algorithm was based on an extension of Corollary 5.4 to the case of complex argument and seemed to be adequate for numerical purposes. However, no formal justification or convergence proof like the above theorem was given for that algorithm. As far as we know, Theorems 4.10 and 5.10 below give the first rigorously proved algorithm for the computation of the values of Teichmüller's function. Note also that complex numbers are not used in the iteration step of the algorithm, but only their absolute values.

5.7. **The algorithm.** We consider the sequence 4.1, use the notation from the proof of Theorem 4.17, and set $y_k = \sqrt{|z_k|} + \sqrt{|z_k - 1|}$. An elementary calculation using (3.19) shows that

$$\frac{|z_{k+1}| + |z_{k+1} - 1| - 1}{2} = y_k^2 A_k - 1.$$

Furthermore, we have that

$$|z_k| = \max\{A_{k-1}^2, 4t_{k-1}A_{k-1}\}$$

for $k = 2, 3, \ldots$. Hence we may write the following algorithm for the computation of L_k, l_k, U_k, and u_k.

```
r = |z|;
s = |z - 1|;
for j = 1 to k do
   t = sqrt{rs};
   y = sqrt{r} + sqrt{s};
   A = r + s + sqrt{(r + s)^2 - 1};
   r = A^2;
   s = 4tA;
end for;
v = y^2 A - 1;
w = (r + s + sqrt{(r + s)^2 - 1})^2;
rr = max{r,s};
L_k = 2^k tau_2(rr);
l_k = 2^(k+1) tau_2(w);
U_k = 2^k tau_2(rr-1);
u_k = 2^k tau_2(v);
```

Note that the iterative step of our algorithm only uses basic arithmetic operations of real numbers. At the end of the algorithm we compute the function $\tau_2(t)$ at four different arguments. For that purpose Corollary 5.4 is used.

Instead of using a fixed number `k` of iterations, we can control the number of iterations by using a stopping criterion based on an accuracy requirement as follows. Recall that by (4.12),

$$U_k - L_k \le \frac{2^{k-1}\,\pi}{(|z_k|-1)\mathcal{K}\left(\sqrt{\frac{|z_k|-1}{|z_k|}}\right)^2}. \tag{5.8}$$

Using (2.4), we see that for $r > 1$, $t = \sqrt{(r-1)/r}$,

$$\mathcal{K}(t)^2 > \left(\log \frac{4}{\sqrt{1-t^2}}\right)^2 = (\log(4\sqrt{r}))^2.$$

This together with (5.8) implies that

$$U_k - L_k < \frac{2^{k-1}\,\pi}{(|z_k|-1)\log(4\sqrt{|z_k|})^2}.$$

Since $z_k \in I$ for all $k \in \mathbb{N}$, we see that after each iteration of the algorithm, we have $|z_k| = \max\{\mathtt{r},\mathtt{s}\}$. If we use, say, the limit $\lim_{k\to\infty} L_k$ to compute $p(z)$, and require an accuracy of `acc`, the algorithm takes the following form.

```
r = |z|;
s = |z - 1|;
kp = 0.5;
do
   kp = 2kp;
   t = sqrt{rs};
   A = r + s + sqrt{(r + s)^2 - 1};
   r = A^2;
   s = 4tA;
while kp*pi/((max(r,s)-1)log(4sqrt(max(r,s)))^2) >= acc;
rr = max{r,s};
L_k = 2^k tau_2(rr);
```

5.9. **Explicit estimates for convergence.** It follows from [K1, Corollary 5.4, p. 206] that for $t \in (0,1/2)$, $p(1+te^{i\theta})$ increases with θ on $(0,\pi)$. Therefore it is clear that for $\varepsilon \in (0,1/2)$ the largest values of $p(z)$ in $\mathbb{C}_\varepsilon \equiv \mathbb{C} \setminus (\mathbb{B}^2(0,\varepsilon) \cup \mathbb{B}^2(1,\varepsilon))$ are attained at $(\varepsilon,0)$ and $(1-\varepsilon,0)$, where we write $\mathbb{B}^2(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$, for $a \in \mathbb{C}, r > 0$. By (3.29) we see that

$$\sup\{p(z) : z \in \mathbb{C}_\varepsilon\} = p(1-\varepsilon) = \tau_2(\varepsilon/(1-\varepsilon)) + \tau_2((1-\varepsilon)/\varepsilon).$$

A computation based on (5.5) shows that $p((0.97,0)) < 5$ and hence

$$p(z) < 5 \text{ for all } z \in \mathbb{C}_{\varepsilon_1},\ \varepsilon_1 = 0.03.$$

From (4.12) and (4.6) we conclude that

$$U_k - L_k < \frac{13 \cdot 2^{-k}}{|z_k| - 1} < 52 \cdot 2^{-k} e^{-2^k} \equiv \delta_k$$

all $z \in \mathbb{C}_{\varepsilon_1}$. Because $\delta_k < 10^{-13}$ for all $z \in \mathbb{C}_{\varepsilon_1}$ and $k \geq 5$, the algorithm in Theorem 4.10 converges very fast in $\mathbb{C}_{\varepsilon_1}$. This argument proves the following theorem.

5.10. **Theorem.** *For all initial values* $z \in \mathbb{C}_{\varepsilon_1}$, $\varepsilon_1 = 0.03$, *the iteration in Theorem* 4.10 *satisfies*

$$U_k - L_k < 10^{-13} \, for \, all \, k \geq 5 \, .$$

5.11. **Practical implementations.** We have implemented the new algorithm on two platforms, namely C++ and Mathematica®. The source code for test programs on both platforms is available from

`http://mat-173.math.helsinki.fi/teich.html.`

This page also contains WWW-interfaces to some executable programs related to the function p.

Running the above mentioned test programs with several different parameter configurations shows that, to compute $p(z)$, the new algorithm takes only $50-60\%$ of the time taken by the algorithm implemented in the software supplement to [AVV].

6 Epilogue

The approximation procedure, which was on the basis of Theorem 4.10, is largely independent of the particular form of the function $p(z)$ to which it was applied. It is our aim here to outline a general form of this approximation procedure.

Our goal is to study an unknown continuous function $\rho : \mathbb{C} \to (0, \infty)$ and we want to express its values in terms of a known homeomorphism $\tau : (0, \infty) \to (0, \infty)$. We assume that τ is decreasing with $\tau(t) \to 0, t \to \infty$, and

$$\tau(|z|) \leq \rho(z) \leq \tau(|z| - 1) \tag{6.1}$$

for all $z \in \mathbb{C}, |z| > 1$. The main property of the function ρ is that it satisfies a functional equation: there exists a number $c \in (1, \infty)$ and for each $z \in \mathbb{C}$ a sequence (z_k) such that

$$\rho(z) = c^k \rho(z_k) \, , \quad k = 1, 2, 3, \ldots \tag{6.2}$$

and

$$c^k(\tau(|z_k| - 1) - \tau(|z_k|)) \to 0 \text{ as } k \to \infty \, . \tag{6.3}$$

Then it is clear by the proof of Theorem 4.10 that for a fixed $z \in \mathbb{C}$ the limit of $c^k \tau(|z_k|)$ exists and equals $\rho(z)$.

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