**RANDOM LOGARITHM AND HOMOGENEITY**

Tapani Hyttinen*

**Abstract**

We give an example of a non-elementary class of structures which is homogeneous for non-trivial reasons and which arises from classical mathematics. In addition, this work serves as an example of the use of homogeneous model theory in the study of a concrete class of structures.

In [Zi2], B. Zilber has studied model theoretic properties of (very) weak structures that arise from the structure $(\mathbb{C}, +, \cdot, exp)$. He showed that the class of these structures is not homogeneous although apart from this, the class behaves nicely. In this paper we study the possibilities of seeing the full structure $(\mathbb{C}, +, \cdot, exp)$ as a member of a homogeneous class of structures i.e. as a member of a class of structures that has a homogeneous universal domain (see Section 1 for the precise definition).

Let $\mathbf{K}^e$ be the class of algebraically closed fields $F$ of characteristic 0 which in addition contain an epimorphism $exp : (F, +) \to (F - \{0\}, \cdot)$ such that the kernel of $exp$ is $\pi\mathbf{Z}$ for some transcendental element $\pi$ and such that Schanuel's conjecture holds. In [Zi1], Zilber pointed out that Schanuel's conjecture gives us a Hrushovski style predimension that seems to be an effective tool to study the class $\mathbf{K}^e$. It also suggests that using the argument from [Hy1] Example 3.3, one can show that $\mathbf{K}^e$ is homogeneous allowing us to use the machinery from homogeneous model theory to study the class. However soon one notices that problems arise. It seems that some kind of logarithm (i.e. inverse for the function $exp$) is needed to push the argument through. Also the proof of the result from [Zi2] mentioned above suggests this.

(*) Partially supported by the Academy of Finland, grant 40734. The author wishes to express his gratitude to T. Huuskonen and K. Suominen for helpful discussions on the topic of this paper.
2000 Mathematics subject classification: primary 03C52 secondary 03C60, 03C68
Key words and phrases: homogeneous model theory, complex exponentiation

But if one adds any particular logarithm to $(\mathbb{C},+,\cdot,exp)$, like the principal one, one makes the reals definable and so also the metric is definable etc. This changes the structure far too much. So one should add a logarithm that increases the expressive power minimally. This kind of problems arise regularly in descriptive finite model theory. There is also a standard technique that often solves the problem. Instead of adding some particular function, one adds a random one. In our case this idea seems a good one, intuitively. The problem is that it is nonsense: what does it mean to make randomly arbitrarily many choices so that each choice is made from a countable set with equal probability between the elements of the set.

Below we will extract some properties of the 'intuitive' random logarithm, show that the properties can be realized and show that if we add such function to the models from $\mathbf{K}^e$, we get a homogeneous class of structures. In the last three sections we study the structure of our class of models. In particular, in the last section a non-structure theorem is proved for $\mathbf{K}^e$.

We want to point out that our random logarithm can also be used to make various modifications of $\mathbf{K}^e$ homogeneous. E.g. if we drop the requirement that $exp$ satisfies Schanuel's conjecture from the definition of $\mathbf{K}^e$ we get a class that becomes homogeneous by adding the random logarithm to the models (as far as we know, the class may be homogeneous also without the logarithm).

We want to point out also that our proofs do not assume (that $(\mathbb{C},+,\cdot,exp)$ satisfies) Schanuel's conjecture. For this paper, the only consequence of the failure of Schanuel's conjecture is that then there is no function $log$ such that

$$(\mathbb{C},+,\cdot,exp,log,2\pi i,\mathbf{Z},\mathbf{Q},q)_{q\in\mathbf{Q}}$$

belongs to our class of structures.

This work can be seen as an attempt to explore the boundaries of homogeneous framework. In particular we will give a non-trivial example of a homogeneous class of structures which, in addition, arises from 'core mathematics'. Also this work can be seen to suggest one possibility of what the structure $(\mathbb{C},+,\cdot,exp)$ looks like. Since very little is known about the structure, it is not even known whether it has a non-trivial automorphism (there are two trivial ones), one can not hope of proving a theorem that describes the structure; at least not, one assumes, by using only methods from model theory.

We assume that the reader is familiar with the basic techniques from stability theory as well as with the basic ideas behind homogeneous model theory, see e.g. [HS2].

## 1. Homogeneous class of structures

In this section we explain what we mean by a homogeneous class of structures. Our notion of a homogeneous class of structures is from [Hy1] and it is rather more general than the usual one (the class of all models of a good finite diagram, see

[Sh1]). However, modulo the current state of homogeneous model theory, there is no difference between the two notions.

Let $\mathbf{K}$ be a collection of $\tau$-models for a fixed similarity type $\tau$. We say that a collection $\mathcal{L}$ of first-order $\tau$-formulas is *regular* for $\mathbf{K}$ if the following holds:

(i) $\mathcal{L}$ contains all atomic $\tau$-formulas,

(ii) $\mathcal{L}$ is closed under freely substituting free variables by terms,

(iii) $\mathcal{L}$ is closed under negation,

(iv) if for $i < j < \alpha$, $\mathcal{A}_i \in \mathbf{K}$ is an $\mathcal{L}$-elementary submodel of $\mathcal{A}_j \in \mathbf{K}$ (see below), then for all $i < \alpha$, $\mathcal{A}_i$ is $\mathcal{L}$-elementary submodel of $\cup_{i<\alpha}\mathcal{A}_i$.

We will not distinguish notationally finite sequences from elements, e.g. unless explicitly stated $a \in A$ means that $a$ is a finite sequence of elements from $A$. Let $\mathcal{A}$ and $\mathcal{B}$ be models and $f$ a partial map from $\mathcal{A}$ to $\mathcal{B}$. We say that $f$ is $\mathcal{L}$-*elementary* iff for all formulas $\phi(x) \in \mathcal{L}$ and all sequences $a \in dom(f)$ of the same length as $x$, $\mathcal{A} \models \phi(a)$ iff $\mathcal{B} \models \phi(f(a))$. We say that $\mathcal{A}$ is an $\mathcal{L}$-*elementary submodel* of $\mathcal{B}$ and write $\mathcal{A} \prec_{\mathcal{L}} \mathcal{B}$ if $\mathcal{A}$ is a substructure of $\mathcal{B}$ and $id_{\mathcal{A}}$ is $\mathcal{L}$-elementary. Also the notion of an $\mathcal{L}$-*type* $t_{\mathcal{L}}(a/A)$ of a finite sequence $a \in \mathcal{A}$ over a set $A \subseteq \mathcal{A}$ is defined as usual.

Let $\mathbf{K}$ be a class of $\tau$-models for fixed similarity type $\tau$. Suppose $\mathbf{K}$ is closed under isomorphisms and contains arbitrary large (infinite) structures. We say that $\mathbf{K}$ is *homogeneous* if for some regular collection $\mathcal{L}$ of formulas for $\mathbf{K}$ the following holds:

- AP ($\mathcal{L}$-*amalgamation property*): Assume $\mathcal{A} \in \mathbf{K}$ and $\mathcal{B} \in \mathbf{K}$ and for all $\mathcal{L}$-formulas $\phi(x)$ and sequences $a \in \mathcal{A} \cap \mathcal{B}$ (possibly empty) we have $\mathcal{A} \models \phi(a)$ iff $\mathcal{B} \models \phi(a)$. Then there are a model $\mathcal{C} \in \mathbf{K}$ and a function $f : \mathcal{A} \cup \mathcal{B} \to \mathcal{C}$ such that $f \restriction \mathcal{A}$ and $f \restriction \mathcal{B}$ are $\mathcal{L}$-elementary.
- sI (*Strong* $\mathcal{L}$-*inductivity*): Assume that for all $\beta < \alpha$, $\mathcal{A}_\beta \in \mathbf{K}$ and that for all $\gamma < \beta < \alpha$, $\mathcal{A}_\gamma \prec_{\mathcal{L}} \mathcal{A}_\beta$. Then $\mathcal{A} = \cup_{\beta<\alpha}\mathcal{A}_\alpha \in \mathbf{K}$ (and for all $\beta < \alpha$, $\mathcal{A}_\beta \prec_{\mathcal{L}} \mathcal{A}$).
- Co ($\mathcal{L}$-*Completeness*): For all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ and sentences $\phi \in \mathcal{L}$, $\mathcal{A} \models \phi$ iff $\mathcal{B} \models \phi$.

Such a class of structures is called homogeneous because the Fraïssé-construction immediately gives the following:

**1.1 Proposition.** *Suppose $\mathbf{K}$ is a homogeneous class of models witnessed by $\mathcal{L}$. Then for all cardinals $\kappa$, there is a model $\mathcal{A} \in \mathbf{K}$ of power $\geq \kappa$ such that*

*(i) $\mathcal{A}$ is strongly $\mathcal{L}$-$\kappa$-homogeneous i.e. for all $\mathcal{L}$-elementary partial maps $f : \mathcal{A} \to \mathcal{A}$ the following holds: if $dom(f)$, $rng(f)$ are of power $< \kappa$ then there is an automorphism $g$ of $\mathcal{A}$ such that $f \subseteq g$.*

*(ii) $\mathcal{A}$ is $\mathcal{L}$-$\kappa$-universal for $\mathbf{K}$ i.e. for all models $\mathcal{C} \in \mathbf{K}$ of power $\leq \kappa$ there is an $\mathcal{L}$-elementary function $f : \mathcal{C} \to \mathcal{A}$.* □

The following lemma is the main ingredient behind the nice behaviour of the homogeneous classes of models.

**1.2 Lemma.** *Suppose $\mathcal{L}$ is a collection of first-order formulas closed under negation and $\mathbf{M}$ is a (strongly) $\mathcal{L}$-$\kappa$-homogeneous model of power $\geq \kappa$. Let $A \subseteq \mathbf{M}$ be of power $< \kappa$ and $p$ a complete $\mathcal{L}$-type over $A$. If for all finite $B \subseteq A$, $p \restriction B$ is realized in $\mathbf{M}$, then $p$ is realized in $\mathbf{M}$.*

**Proof.** Easy. □

## 2. Basic definitions with some observations

We start by defining a class of structures that is not our 'target' class of structures. The class is needed to provide a platform to do some constructions we will use in the proofs later. In the constructions we need to work with partial interpretations for *exp*. However, later we want *exp* to be a function symbols. This e.g. simplifies the definition of our collection $\mathcal{L}$ of formulas as well as the proof of $\mathcal{L}$-amalgamation.

We let $\tau$ be the similarity type which consists of the following symbols: a constant $\pi$ and for $q \in \mathbf{Q}$, a constant $c_q$, unary predicates $Z$ and $Q$, a binary predicate *exp* and binary functions $+$ and $\cdot$. We let $\mathbf{K}^v$ be the class of all $\tau$-models $\mathcal{A}$ with the following properties:

(1) $F_{\mathcal{A}} = \mathcal{A} \restriction \{+, \cdot\}$ is an algebraically closed field of characteristic 0 and

(a) $\mathcal{A} \models Q(a)$ iff $a$ is a rational element of $F_{\mathcal{A}}$ and $\mathcal{A} \models Z(a)$ iff $a$ is an integer in $F_{\mathcal{A}}$,

(d) the interpretation of $\pi$ is a transcendental element of $F_{\mathcal{A}}$ and for $q \in \mathbf{Q}$, the interpretation of $c_q$ is the rational $q$ as calculated in $F_{\mathcal{A}}$ (we will not distinguish $c_q$ and $q$).

(2) The interpretation of the predicate *exp* satisfies the following:

(a) if for all $i < 3$, $\mathcal{A} \models exp(a_i, b_i)$ and $\mathcal{A} \models a_0 + a_1 = a_2$, then $\mathcal{A} \models b_0 \cdot b_1 = b_2$,

(b) if $\mathcal{A} \models exp(a, b)$ then for all $c \in \mathcal{A}$, $\mathcal{A} \models exp(c, b)$ iff $\mathcal{A} \models \exists x(Z(x) \wedge c = a + x \cdot \pi)$,

(c) $\mathcal{A} \models exp(\pi, 1) \wedge \forall x_0 \forall x_1 \forall x_2((exp(x_0, x_1) \wedge exp(x_0, x_2)) \rightarrow x_1 = x_2)$,

(d) if for $i < 2$, $\mathcal{A} \models exp(a_i, b_i) \wedge Q(q_i)$, then there is $b_2 \in \mathcal{A}$ such that $\mathcal{A} \models exp(q_0 \cdot a_0 + q_1 \cdot a_1, b_2)$.

Suppose that $\mathcal{A} \in \mathbf{K}^v$ and $X \subseteq \mathcal{A}$. By $dim(X)$ we mean the dimension of $X$ as a subset of a vector space over $\mathbf{Q}$ ($\mathcal{A}$ can be seen as a vector space over $\mathbf{Q}$ the natural way). Notice that it is first-order expressible in $\mathcal{A}$ (with a negation of an existential formula) that $dim(X) = |X|$ for sets $X$ of fixed finite size. By $deg(X)$ we mean the transcendence degree of $X$ as a subset of $F_{\mathcal{A}}$. For $X \subseteq \mathcal{A}$, by $exp(X)$ we mean the set of all $b \in \mathcal{A}$ such that for some $a \in X$, $\mathcal{A} \models exp(a, b)$. Similarly, for an element $a \in \mathcal{A}$, by $exp(a)$ we mean the unique $b \in \mathcal{A}$ such that $\mathcal{A} \models exp(a, b)$ (if exists). By $dom(exp) = dom_{\mathcal{A}}(exp)$ we denote the set of all $a \in \mathcal{A}$ such that for some $b \in \mathcal{A}$, $\mathcal{A} \models exp(a, b)$.

We let $\mathbf{K}^s$ be the class of all $\mathcal{A} \in \mathbf{K}^v$ such that

(3) for all finite $X \subseteq dom_{\mathcal{A}}(exp)$,

$$\delta(X) = deg(X \cup \{1,\pi\} \cup exp(X \cup \{1,\pi\})) - dim(X \cup \{1,\pi\}) \geq 0.$$

The properties of this kind Hrushovski style predimension $\delta$ are well-known, see e.g. [BS] (and notice that in the definition of $\delta$, $dim(X \cup \{1,\pi\})$ is a dimension that is based on a modular pregeometry). We assume that the reader is familiar with these properties i.e. we will use the properties regularly and usually without proving them.

Suppose $\mathcal{A} \in \mathbf{K}^s$. If $B$ is a subset of $dom_{\mathcal{A}}(exp)$ and $dim(B)$ is finite, then $\delta(B)$ is defined as in (3) above. If $C \subseteq B$, then by $\delta(B/C)$ we mean $\delta(B) - \delta(C)$. We say that $B$ is a *relevant* subset of $\mathcal{A}$ if $B \subseteq dom_{\mathcal{A}}(exp)$, $B$ contains the rationals and $\pi$, $B$ is closed under $+$ and for all $b \in B$ and rationals $q$, $q \cdot b \in B$. We say that a relevant set $B \subseteq \mathcal{A}$ is *closed* (in $\mathcal{A}$) if for all relevant $D \subseteq \mathcal{A}$ which have finite dimension, $\delta(D/B \cap D) \geq 0$.

Notice that if $B \subseteq dom_{\mathcal{A}}(exp)$ is a subset of the relevant set $A$ generated by finite $X$ (i.e. $A$ is the least relevant set which contains $X$) and also $B$ generates $A$, then $\delta(B) = \delta(X)$.

**2.1 Lemma.** *Suppose $\mathcal{A} \in \mathbf{K}^s$. Then there is $\mathcal{B} \in \mathbf{K}^s$ such that*
*(i) $\mathcal{A}$ is a substructure of $\mathcal{B}$,*
*(ii) $dom_{\mathcal{A}}(exp)$ is closed in $\mathcal{B}$,*
*(iii) for all $a \in \mathcal{B}$ there is $b \in \mathcal{B}$ such that $\mathcal{B} \models exp(a,b)$,*
*(iv) for all $b \in \mathcal{B} - \{0\}$ there is $a \in \mathcal{B}$ such that $\mathcal{B} \models exp(a,b)$.*

**Proof.** Clearly it is enough to prove the Claims 2.1.1 and 2.1.2 below.

**2.1.1 Claim.** *If $a \in \mathcal{A} - dom_{\mathcal{A}}(exp)$, then there is $\mathcal{B} \in \mathbf{K}^s$ such that (i) and (ii) from Lemma 2.1 hold and $a \in dom_{\mathcal{B}}(exp)$.*

**Proof.** Choose $\mathcal{B}'$ so that (i) from Lemma 2.1 holds,

$$dom_{\mathcal{B}'}(exp) = dom_{\mathcal{A}}(exp)$$

and there is $b \in \mathcal{B}'$ such that $b$ is transcendental over $\mathcal{A}$. For $q \in \{x \in \mathbf{Q} |\ x > 0\}$, pick elements $b_q \in \mathcal{B}'$ so that $b_1 = b$ and for all $q \in \{x \in \mathbf{Q} |\ x > 0\}$ and $n \in \mathbb{N} - \{0\}$, $(b_q)^n = b_{q \cdot n}$. Then we get $\mathcal{B}$ by extending $exp$ so that $dom_{\mathcal{B}}(exp)$ is the relevant set generated by $dom_{\mathcal{A}}(exp) \cup \{a\}$ and for all $c \in dom_{\mathcal{A}}(exp)$ and non-zero rational $q$, $\mathcal{B} \models exp(c + q \cdot a, d)$ iff either

$q > 0$ and $\mathcal{B} \models e \cdot b_q = d$

or

$q < 0$ and $\mathcal{B} \models e \cdot b_{-q}^{-1} = d$,

where $e$ is such that $\mathcal{B} \models exp(c,e)$. It is easy to see that this $\mathcal{B}$ satisfies (i) and (ii) and that $\mathcal{B} \in \mathbf{K}^s$. (E.g. Suppose that $e \cdot b_q = e' \cdot b_{q'}$, $e, e' \in \mathcal{A}$. Then $e/e' = b_{q'}/b_q$ and since if $q \neq q'$, then $b_{q'}/b_q$ is transcendental over $\mathcal{A}$, we get that $q = q'$ and $e = e'$. Thus $\mathcal{B}$ satisfies (2)(b).) $\square$ Claim 2.1.1.

**2.1.2 Claim.** *If $a \in \mathcal{A} - exp(dom_{\mathcal{A}}(exp))$, then there is $\mathcal{B} \in \mathbf{K}^s$ such that (ii) and (iii) from Lemma 2.1 hold and $a \in exp(dom_{\mathcal{B}}(exp))$.*

**Proof.** This can be proved exactly as Claim 2.1.1 as soon as we have shown that for all positive integers $z$, $a^z \notin exp(dom_{\mathcal{A}}(exp))$. Suppose not, say $a^z = b$ and $\mathcal{A} \models exp(c,b)$. For all integers $0 \leq n < z$, let $c_n = c/z + n \cdot \pi/z$. Let $b_n$ be such that $\mathcal{A} \models exp(c_n, b_n)$. Clearly $(b_n)^z = b$. Also for $n \neq m$, $b_n \neq b_m$. Since there are only $z$ many $d$ such that $d^z = b$, for some $n$, $a = b_n$, a contradiction. (For (2)(b) notice that if $exp(e + q \cdot b) = 1$, then $a_q = exp(-e)$ and thus for some integer $z$, $a^z \in exp(dom_{\mathcal{A}}(exp))$ (if $q \neq 0$), a contradiction.) □ Claim 2.1.2.

□

A structure $\mathcal{B} \in \mathbf{K}^s$ is called *full* if it satisfies (iii) and (iv) from Lemma 2.1.

Let $\mathcal{A} \in \mathbf{K}^v$, $a \in \mathcal{A}$ and $A, B \subseteq \mathcal{A}$. We say that $a$ is independent from $B$ over $A$ and write $a \downarrow_A B$ if $a$ is independent from $B$ over $A$ in $F_{\mathcal{A}}$ in the sense of non-forking. If we consider $F_{\mathcal{A}}$ as a vector space over $\mathbf{Q}$, we talk about linear independence.

**2.2 Lemma.** *Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}^s$ are such that*
*(i) $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ is a substructure of both $\mathcal{A}$ and $\mathcal{B}$,*
*(ii) $\mathcal{C}$ is full,*
*(iii) $\mathcal{C} = dom_{\mathcal{C}}(exp)$ is closed both in $\mathcal{A}$ and in $\mathcal{B}$.*
*Then there is full $\mathcal{D} \in \mathbf{K}^s$ such that $\mathcal{A}$ and $\mathcal{B}$ are substructures of $\mathcal{D}$ and both $dom_{\mathcal{A}}(exp)$ and $dom_{\mathcal{B}}(exp)$ are closed in $\mathcal{D}$.*

**Proof.** By Lemma 2.1, we may assume that also both $\mathcal{A}$ and $\mathcal{B}$ are full. Also by Lemma 2.1, it is enough to find $\mathcal{D} \in \mathbf{K}^v$ such that $\mathcal{A}$ and $\mathcal{B}$ are substructures of $\mathcal{D}$ and

(a) the property (3) from the definition of $\mathbf{K}^s$ holds,
(b) $\mathcal{A} = dom_{\mathcal{A}}(exp)$ is closed in $\mathcal{D}$,
(c) $\mathcal{B} = dom_{\mathcal{B}}(exp)$ is closed in $\mathcal{D}$.

We let $F_{\mathcal{D}}$ be the free amalgamation of $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ over $F_{\mathcal{C}}$ (i.e. the algebraic closure of the tensor product over $\mathcal{C}$). This determines $\mathcal{D}$ excluding the interpretation of $exp$. The relation $exp$ is defined to be the extension of $exp^{\mathcal{A}} \cup exp^{\mathcal{B}}$ such that $dom_{\mathcal{D}}(exp)$ is the relevant subset of $\mathcal{D}$ generated by $dom_{\mathcal{A}}(exp) \cup dom_{\mathcal{B}}(exp)$ and for all $a \in dom_{\mathcal{A}}(exp)$ and $b \in dom_{\mathcal{B}}(exp)$,

$\mathcal{D} \models exp(a+b, c)$ iff $\mathcal{D} \models exp(a) \cdot exp(b) = c$.

Then it is easy to check that $\mathcal{D} \in \mathbf{K}^v$ and clearly $\mathcal{A}$ and $\mathcal{B}$ are substructures of $\mathcal{D}$. We are left to prove that (a)-(c) hold.

A simple calculation shows that (b) implies (a) and by symmetry, (b) implies (c). Thus it suffices to prove (b). For a contradiction, assume that $D$ is a relevant subset of $dom_{\mathcal{D}}(exp)$ such that $\delta(D/A) < 0$, where $A = D \cap \mathcal{A}$. Then for all relevant subsets $A'$ of $\mathcal{A}$ with finite dimension, if $A \subseteq A'$, then $\delta(D \cup A'/A') < 0$. Thus we may assume the following (keeping in mind that $\mathcal{A}$ is full):

(d) $A \cup exp(A)$ is independent from $\mathcal{B}$ over $(A \cup exp(A)) \cap \mathcal{C}$,
(e) for all $d \in D$ there are $a \in A$ and $b \in \mathcal{B}$ such that $d = a + b$.

Let $d_i$, $i < n$, be a linear base for $D$. Let $a_i$ and $b_i$ be as given by (e) for $d_i$. In addition choose these so that if $d_i \in \mathcal{C}$, then $a_i = 0$.

Then

$$(*) \quad \delta(\{b_0, ..., b_{n-1}\} \cup A/A) < 0.$$

Clearly we may assume that in addition $B \cap \mathcal{C} = A \cap \mathcal{C}$, where $B \subseteq \mathcal{B}$ is the relevant set generated by $\{b_0, ..., b_{n-1}\}$. Then by (d), $(*)$ contradicts the assumption that $dom_{\mathcal{C}}(exp)$ is closed in $\mathcal{B}$. $\square$

By a canonical $s$-structure $\mathcal{A}^s_C$ be mean the obvious structure we get from the complex number ($\pi$ is interpreted to $2\pi i$, of course). Clearly, $\mathcal{A}^s_C \in \mathbf{K}^v$ but it is not known whether $\mathcal{A}^s_C$ satisfies (3) or not (Schanuel's conjecture is an open question).

Now we turn to define our target class $\mathbf{K}$. We let $\rho$ be the similarity type be which consists of the following symbols: constant $\pi$ and for $q \in \mathbf{Q}$, constant $c_q$, unary predicates $Z$ and $Q$, unary functions $exp$ and $log$ and binary functions $+$ and $\cdot$. We let $\mathbf{K}^{-3}$ be the class of all $\rho$-models $\mathcal{A}$ such that the following holds:

(1) $F_{\mathcal{A}} = \mathcal{A} \restriction \{+, \cdot\}$ is an algebraically closed field of characteristic $0$.

(2) $\mathcal{A} \models Q(a)$ iff $a$ is a rational element of $F_{\mathcal{A}}$ and $\mathcal{A} \models Z(a)$ iff $a$ is an integer in $F_{\mathcal{A}}$.

(3) The interpretation of $\pi$ is a transcendental element of $F_{\mathcal{A}}$ and for $q \in \mathbf{Q}$, the interpretation of $c_q$ is the rational $q$ as calculated in $F_{\mathcal{A}}$ (again we will not distinguish $c_q$ and $q$).

(4) $exp$ is a homomorphism from $(F_{\mathcal{A}}, +)$ onto $(F_{\mathcal{A}} - \{0\}, \cdot)$ with kernel $\pi Z$.

(5) $\mathcal{A} \models \forall x(exp(log(x)) = x) \wedge log(0) = 0$.

Clearly every $\mathcal{A} \in \mathbf{K}^{-3}$ determines a unique structure $\mathcal{A}^s \in \mathbf{K}^v$ (in fact, it suffices that $\mathcal{A}$ is a $\rho - \{log\}$-model satisfying (1)-(4)). We let $\mathbf{K}^{-2}$ be the class of all $\mathcal{A} \in \mathbf{K}^{-3}$ such that

(6) $\mathcal{A}^s \in \mathbf{K}^s$ (i.e. those which satisfy Schanuel's conjecture).

The structures $\mathcal{A}^s$ allow us to transfer the concepts defined for models from $\mathbf{K}^s$ to models from $\mathbf{K}^{-2}$. In particular, we have the notion of predimension $\delta$ and the notion of a relevant set for models $\mathcal{A} \in \mathbf{K}^{-2}$.

We let $\mathbf{K}^-$ be the set of all $\mathcal{A} \in \mathbf{K}^{-2}$ such that the following holds:

(7) for all relevant sets $A \subseteq \mathcal{A}$ with finite dimension and rationals $q \neq 1$ and $r$, there are at most finitely many $a, b \in A - \pi Q$ such that

$$\mathcal{A} \models (b = q \cdot a + r \cdot \pi) \wedge (log(exp(a)) = a) \wedge (log(exp(b)) = b).$$

This property is the key technical 'trick' behind the proof of homogeneity of our class of structures. The point here is the following: The difficult step in the homogeneity proof is Claim 2.8.3. There a certain type $p = t_{\mathcal{L}}(a/\mathcal{A} \cap \mathcal{B})$ is considered. Using the property (7), we can choose $a$ so that there is a formula $\phi$ such that

(i) there are only finitely many sequences that satisfy $\phi$,

(ii) the sequence $a$ satisfies $\phi$.

With this we can see that $a$ can be chosen so that $p$ is 'principle'. This is what is needed in the proof of the claim.

**2.3 Lemma.** *Suppose $\mathcal{A}$ is a $(\rho-\{log\})$-model such that it satisfies (1)-(4) above, $B$ and $C$ are subsets of $\mathcal{A}$ such that either $B=C=\emptyset$ or both are relevant and $\pi Q \subseteq B \cap C$ and we have a function $log^*$ defined on $exp(B) \cup exp(C)$ such that for all $a \in exp(B) \cup exp(C)$, $exp(log^*(a)) = a$ and (7) holds for $log^*$ for all relevant $A$ such that $A$ has finite dimension and $A \subseteq B$ or $A \subseteq C$. Then there is a function $log : \mathcal{A} \to \mathcal{A}$ such that*

*(i) $log \restriction (exp(B) \cup exp(C)) = log^*$,*

*(ii) if $\mathcal{A}^+$ is the model one gets by adding $log$ to $\mathcal{A}$, then $\mathcal{A}^+$ satisfies condition (7) and so if in addition $\mathcal{A}$ satisfies (6), then $\mathcal{A}^+ \in \mathbf{K}^-$.*

**Proof.** Clearly we may assume that $\pi Q \subseteq B \cap C$ ($log \restriction exp(\pi Q)$ is easy to define, since (5) is the only requirement). Choose a linear base $D = \{d_i |\ i < \alpha\}$ for $\mathcal{A}$ so that $d_0 = \pi$, $D \cap B$ is a base for $B$ and $D \cap C$ is a base for $C$. If $X = \{x_0, ..., x_n\} \subseteq D - \{d_0\}$, then by $\mathcal{A}_X$ we denote the set of all $a \in \mathcal{A}$ such that there are non-zero $q_0, ..., q_n \in \mathbf{Q}$ and $r \in \mathbf{Q}$ (possibly zero) such that $a = q_0 \cdot x_0 + ... + q_n \cdot x_n + r \cdot \pi$.

**2.3.1 Claim.** *There is a function $log : \mathcal{A} \to \mathcal{A}$ such that*

*(a) $log \restriction (exp(B) \cup exp(C)) = log^*$,*

*(b) for all $a \in \mathcal{A}$, $log(exp(a)) = a$,*

*(c) for all finite $X \subseteq B$ and rationals $q \neq 1$ and $r$, there are at most finitely many $a, b \in \mathcal{A}_X - \pi Q$ such that $b = q \cdot a + r \cdot \pi$, $log(exp(a)) = a$ and $log(exp(b)) = b$.*

**Proof.** Easy diagonalization argument. □ Claim 2.3.1.

It suffices to prove that $log$ from Claim 2.3.1 satisfies the condition (7): Suppose not. Let $A$, $q$ and $r$ witness this. Then $q \neq 0$. Since every relevant $A' \supseteq A$ with finite dimension still witnesses the failure of (7), we may assume that $A \cap D$ is a base for $A$. Clearly,

(*) for all $X \subseteq (A \cap D) - \{\pi\}$, if $a \in \mathcal{A}_X$, then $q \cdot a + r \cdot \pi \in \mathcal{A}_X$.

By this and the pigeon hole principle, there is $X \subseteq (A \cap D) - \{\pi\}$ such that for infinitely many $a, b \in \mathcal{A}_X - \pi Q$, $b = q \cdot a + r \cdot \pi$, $log(exp(a)) = a$ and $log(exp(b)) = b$. This contradicts Claim 2.3.1 (c). □

By Lemma 2.3 there is a function $log$ such that

$$\mathcal{A}_C = (\mathbb{C}, +, \cdot, exp, log, 2\pi i, \mathbf{Z}, \mathbf{Q}, q)_{q \in \mathbf{Q}}$$

satisfies (1)-(5) and (7). We fix such a function and call $\mathcal{A}_C$ the canonical structure. Notice that $(\mathcal{A}_C)^s = \mathcal{A}_C^s$.

For all $\rho$-models $\mathcal{A}$ which satisfy (1)-(5), we let $P(\mathcal{A})$ be the least substructure of $\mathcal{A}$ such that $F_{P(\mathcal{A})}$ is algebraically closed. We fix some countable $\rho$-model $\mathcal{A}_{pr}$ so that for some $\mathcal{A} \in \mathbf{K}^-$, $\mathcal{A}_{pr} = P(\mathcal{A})$ (by Lemmas 2.1 and 2.3, $\mathbf{K}^- \neq \emptyset$).

If Schanuel's conjecture holds, the natural choice for $\mathcal{A}_{pr}$ is $P(\mathcal{A}_C)$. Then we let $\mathbf{K}$ be the class of all $\mathcal{A} \in \mathbf{K}^-$ such that

(8) $P(\mathcal{A})$ is isomorphic to $\mathcal{A}_{pr}$.

We want to point out that one can define $log \restriction \mathcal{A}_{pr}$ anyway one wants (i.e. it does not need to satisfy (7)) and still the class is homogeneous if in the requirement (7), $\pi Q$ is replaced by $P(\mathcal{A})$.

## 3. Homogeneity of K

In this section we show that $\mathbf{K}$ is a homogeneous class of models. The proof uses ideas from [Hy1] Example 3.3.

We start by defining a collection $\mathcal{L}$ of first-order formulas. These formulas express roughly the following: Let $\mathcal{A} \in \mathbf{K}$ and $a_0, ..., a_n \in \mathcal{A}$. Let $\mathcal{B} \supseteq \{a_0, ..., a_n\}$ be the least substructure of $\mathcal{A}$ such that it is algebraically closed as a subset of $F_{\mathcal{A}}$ and closed as a relevant subset of $\mathcal{A}$ (such $\mathcal{B}$ exists). Then each (positive) formula (in $n+1$ free variables) from $\mathcal{L}$ state about $(a_0, ..., a_n)$ that there is some finite sequence of elements in $\mathcal{B}$ and describe this sequence up to (finite) quantifier free information together with complete information on linear dependences (and independences). The linear dependences can be described since we can quantify over the rationals.

Suppose $\mathcal{A} \in \mathbf{K}$. We say that relevant $C \subseteq \mathcal{A}$ is *unclosed* in relevant $B \subseteq \mathcal{A}$ with finite dimension if for all relevant $D$ the following holds: if $C \subseteq D \subseteq B$ and $D \neq B$, then $\delta(B/D) < 0$. We say that relevant $B$ is an *intrinsic closure* of relevant $A$ with finite dimension if $A$ is unclosed in $B$ and $B$ is closed. If $A$ and $B$ are just subsets of $\mathcal{A}$ of finite dimension, then these notions are defined for them by looking the relevant subsets generated by them.

**3.1 Lemma.**

*(i) If $A \subseteq \mathcal{A} \in \mathbf{K}$ is relevant and of finite dimension, then the intrinsic closure of $A$ exists and is of finite dimension.*

*(ii) Suppose $B$, $C$, $B'$ and $C'$ are relevant subsets of $\mathcal{A} \in \mathbf{K}$ with finite dimension, $B \subseteq C$, $B'$ is an intrinsic closure of $B$ and $C'$ is an intrinsic closure of $C$. Then $B' \subseteq C'$. In particular, if $B = C$, then $B' = C'$ i.e. the intrinsic closure is unique.*

**Proof.** (i): Let $B \subseteq \mathcal{A}$ be such that

(a) $A \subseteq B$ and $B$ is relevant and of finite dimension,

(b) $\delta(B)$ is minimal among those that satisfy (a).

In addition choose $B$ so that

(c) $dim(B)$ is minimal among those which satisfy (a) and (b).

Clearly $B$ is an intrinsic closure of $A$.

(ii): Immediate by the definition of intrinsic closure. □

We say that a first-order formula $\phi(x_0, ..., x_n, ..., x_{n+k})$ is $n$-*unclosed* if the following holds:

(i) $\phi = \psi \wedge \theta$, where $\psi$ is quantifier free and $\theta$ is a universal formula which describes the linear dependences within $\pi, 1, x_0, ..., x_{n+k}$. In particular, $\theta$ fixes the dimension of $\{\pi, 1, x_0, ..., x_{n+k}\}$.

(ii) There are $\mathcal{A} \in \mathbf{K}$ and $a_0, ..., a_{n+k} \in \mathcal{A}$ such that $\mathcal{A} \models \phi(a_0, ..., a_{n+k})$ and among $\pi, 1, a_0, ..., a_{n+k}, exp(1), exp(a_0), ..., exp(a_{n+k})$ there are no other algebraic dependences than those explicitly stated in $\psi$ (i.e. the transcendence degree is maximal). Such a sequence is called *generic* for $\phi$.

(iii) If $a_0, ..., a_{n+k} \in \mathcal{A} \in \mathbf{K}$ and $b_0, ..., b_{n+k} \in \mathcal{B} \in \mathbf{K}$ are generic for $\phi$ then the subfields generated by $\{\pi, 1, a_0, ..., a_{n+k}, exp(1), exp(a_0), ..., exp(a_{n+k})\}$ and by $\{\pi, 1, b_0, ..., b_{n+k}, exp(1), exp(b_0), ..., exp(b_{n+k})\}$ are isomorphic.

(iv) If $a_0, ..., a_{n+k} \in \mathcal{A} \in \mathbf{K}$ is generic for $\phi$ then $\{\pi, 1, a_0, ..., a_n\}$ is unclosed in $\{\pi, 1, a_0, ..., a_{n+k}\}$.

Notice that if $a_0, ..., a_{n+k} \in \mathcal{A} \in \mathbf{K}$ are distinct elements, and $\{\pi, 1, a_0, ..., a_n\}$ is unclosed in $\{\pi, 1, a_0, ..., a_{n+k}\}$, then there is an $n$-unclosed formula $\phi$ such that $(a_0, ..., a_{n+k})$ is generic for $\phi$.

**3.2 Lemma.**

*(i) If $\phi(x_0, ..., x_n, ..., x_{n+k})$ is an $n$-unclosed formula, then for all $0 \leq p \leq k$, $\phi(x_0, ..., x_n, ..., x_{n+k})$ is an $(n+p)$-unclosed formula.*

*(ii) Suppose $\mathcal{A}, \mathcal{B} \in \mathbf{K}$, $\mathcal{A} \cap \mathcal{B}$ is a relevant subset of $\mathcal{A}$ and of $\mathcal{B}$ and for all $n$-unclosed $\phi(x_0, ..., x_{n+k})$, $n < \omega$, and $a \in \mathcal{A} \cap \mathcal{B}$, $\mathcal{A} \models \exists x_{n+1} ... \exists x_{n+k} \phi(a)$ iff $\mathcal{B} \models \exists x_{n+1} ... \exists x_{n+k} \phi(a)$. Let $A \subseteq \mathcal{A}$ be the intrinsic closure of $B = A \cap \mathcal{B}$ and let $a_i \in A$, $i < n+k$, be such that $\{a_0, ..., a_n\}$ generate $B$, $\{a_0, ..., a_{n+k}\}$ generate $A$ and for all $m \leq k$, $a_{n+m} \notin B$. If $\phi(x_0, ..., x_{n+k})$ is such an $n$-unclosed formula that $(a_0, ..., a_{n+k})$ is generic for $\phi$, then*

*(a) for all $b_{n+1}, ..., b_{n+k} \in \mathcal{A}$, if $\mathcal{A} \models \phi(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$, then $(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$ is generic for $\phi$,*

*(b) for all $b_{n+1}, ..., b_{n+k} \in \mathcal{B}$, if $\mathcal{B} \models \phi(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$, then $(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$ is generic for $\phi$.*

**Proof.** (i): Immediate by the definitions.

(ii): We note first that by basic properties of $\delta$,

(*) if $C \subseteq \mathcal{A}$ is relevant, has finite dimension and $B \subseteq C$, then $\delta(C) \geq \delta(A)$ (if $D$ is the relevant set generated in $\mathcal{A}$ by $C \cup \{a_{n+1}, ..., a_{n+k}\}$, then $\delta(D/C) \leq 0$). Since $\phi$ decides the linear dependences within $a_0, ..., a_n, b_{n+1}, ..., b_{n+k}$, the existence of a sequence that witnesses the failure of (a) would contradict (*). Similarly, if there is a sequence that witnesses the failure of (b), then there is $D \subseteq \mathcal{B}$ such that $D$ is the intrinsic closure of $B$ and $\delta(D) < \delta(A)$. But then by the assumptions, there is $C \subseteq \mathcal{A}$ such that $\delta(C) < \delta(A)$ and $B \subseteq C$ which as before contradicts (*). □

We say that $\theta(x_0, ..., x_{n+k})$ is *$n$-algebraic* if $\theta$ is quantifier free and for all $0 < i \leq k$, $\theta$ implies $x_{n+i}$ is algebraic over $\{x_0, ..., x_{n+i-1}\}$ or $x_{n+i} = exp(x)$ for some $x \in \{x_0, ..., x_{n+i-1}\}$ or $x_{n+i} = log(x)$ for some $x \in \{x_0, ..., x_{n+i-1}\}$.

We define the class of *extended $n$-unclosed* formulas, $n \in \mathbb{N}$, as follows:

(a) every $n$-unclosed formula is extended $n$-unclosed formula,

(b) $\phi = \exists x_{n+1} ... \exists z_{n+k} \theta(x_0, ..., x_{n+k})$ in an extended $n$-unclosed formula if $\theta$ is $n$-algebraic,

(c) if $\phi(x_0, ..., x_{n+k})$ is an extended $n$-unclosed formula, $\psi(x'_0, ..., x'_{m+p})$ is an extended $m$-unclosed formula and no $x_i$, $i \leq n+k$, appears in $\psi$ then

$$\phi(x_0, ..., x_{n+k}) \wedge \exists x'_{m+1} ... \exists x'_{m+p} \psi(t_0, ..., t_m, x'_{m+1}, ..., x'_{m+p})$$

in an extended $n$-unclosed formula, where each $t_i = t_i(x_0, ..., x_{n+k})$ is a term. We let $\mathcal{L}^e$ be the class of all formulas $\phi(x_0, ..., x_m)$ of the form

$$\phi(x_0, ..., x_m) = \exists x'_{n+1} ... \exists x'_{n+k} \psi(t_0, ..., t_n, x'_{n+1}, ..., x'_{n+k}),$$

where $\psi = \psi(x'_0, ..., x'_{n+k})$ is an extended $n$-unclosed formula, $n \in \mathbb{N}$, and each $t_i = t_i(x_0, ..., x_m)$ is a term and no $x_i$, $i \leq m$, appears in $\psi$.

Finally, we let

$$\mathcal{L} = \mathcal{L}^e \cup \{\neg\phi \mid \phi \in \mathcal{L}^e\}.$$

**3.3 Lemma.** *$\mathcal{L}$ is regular for* $\mathbf{K}$.

**Proof.** It is routine to check that $\mathcal{L}$ satisfies the requirement (iv). The other requirements are explicit in the definition of $\mathcal{L}$. □

**3.4 Lemma.** *Suppose $\mathcal{A}, \mathcal{B} \in \mathbf{K}$. Then $\mathcal{A} \prec_{\mathcal{L}} \mathcal{B}$ iff $\mathcal{A}$ is a substructure of $\mathcal{B}$ and $\mathcal{A}$ is closed in $\mathcal{B}$.*

**Proof.** Follows easily from the basic properties of $\delta$, especially Lemma 3.1, and the definition of $\mathcal{L}$. □

The following result is the main theorem of this paper.

**3.5 Theorem.** $\mathbf{K}$ *is $\mathcal{L}$-complete and strongly $\mathcal{L}$-inductive and it has the $\mathcal{L}$-amalgamation property.*

We will prove the theorem in a series of lemmas.

**3.6 Lemma.** $\mathbf{K}$ *is $\mathcal{L}$-complete.*

**Proof.** Suppose $\mathcal{A}, \mathcal{B} \in \mathbf{K}$. By the requirement (8) in the definition of $\mathbf{K}$, there is an isomorphism $f : P(\mathcal{A}) \to P(\mathcal{B})$. Clearly it suffices to prove that as a partial function from $\mathcal{A}$ to $\mathcal{B}$, $f$ is $\mathcal{L}$-elementary. By Lemma 3.4, it suffices to prove that for every $\mathcal{C} \in \mathbf{K}$, $P(\mathcal{C})$ is a closed subset of $\mathcal{C}$. But this is clear since for every finite $X \subseteq P(\mathcal{C})$, there is finite $Y \subseteq P(\mathcal{C})$ such that $X \subseteq Y$ and $\delta(Y) = 0$. □

**3.7 Lemma.** *Suppose $\mathcal{A} \in \mathbf{K}$, $a_0, ..., a_{n+k} \in \mathcal{A}$, $A$ the relevant subset of $\mathcal{A}$ generated by $\{a_0, ..., a_n\}$ and $B$ the relevant subset of $\mathcal{A}$ generated by $\{a_0, ..., a_{n+k}\}$. If $B$ is the intrinsic closure of $A$, then there is $n$-unclosed formula $\phi$ such that* *(i)* $(a_0, ...., a_{n+k})$ *is generic for* $\phi$ *(in particular,* $\mathcal{A} \models \phi(a_0, ..., a_{n+k})$*),*

*(ii) for all* $\mathcal{B} \in \mathbf{K}$ *and* $b_0, ..., b_n \in \mathcal{B}$*, there are at most finitely many* $b_{n+1}, ..., b_{n+k} \in \mathcal{B}$ *such that* $(b_0, ..., b_{n+k})$ *is generic for* $\phi$*,*

*(iii) for all* $\mathcal{B} \in \mathbf{K}$ *and* $b_0, ..., b_{n+k} \in \mathcal{B}$*, if* $(b_0, ..., b_{n+k})$ *is generic for* $\phi$*, then* $\{b_0, ..., b_n\}$ *is unclosed in* $\{b_0, ..., b_{n+k}\}$*.*

**Proof.** If needed, we can replace $a_i$ by any $a_i + z \cdot \pi$, $z \in Z$, since these are interdefinable and generate the same relevant set (even $exp(a_i) = exp(a_i + z \cdot \pi)$). So we may assume that for all $i \leq n + k$, $log(exp(a_i)) = a_i$. For all $i \leq n + k$ such that $a_i \notin \pi Q$, choose some $q_i, r_i \in Q$ so that $q_i \neq 0, 1$ and

(*) $log(exp(q_i \cdot a_i + r_i \cdot \pi)) = q_i \cdot a_i + r_i \cdot \pi$.

Let $\phi$ be an $n$-unclosed formula such that (i) holds and for all $i \leq n + k$, if $a_i \in \pi Q$, then $\phi$ expresses that $x_i = q \cdot \pi$ for some $q \in Q$ and otherwise $\phi$ expresses (*) and that $log(exp(x_i)) = x_i$. It is clear that (iii) holds. We prove that (ii) holds.

Let $\mathcal{B} \in \mathbf{K}$ and $b_0, ..., b_n \in \mathbf{K}$. Let $A'$ be the relevant set generated by $\{b_0, ..., b_n\}$. Let $B'$ be the intrinsic closure of $A'$. By the property (7) and the finite dimension of $B'$, it suffices to show that if $b_{n+1}, ..., b_{n+k} \in \mathcal{B}$ are such that $(b_0, ..., b_{n+k})$ is generic for $\phi$, then for all $n < i \leq n + k$, $b_i \in B'$. This is clear by (iii) and the basic properties of $\delta$. □

**3.8 Lemma.** $\mathbf{K}$ *has the* $\mathcal{L}$*-amalgamation property.*

**Proof.** Let $\mathcal{A}$ and $\mathcal{B}$ be as in the definition of AP. Clearly (by 'moving' the universe of $\mathcal{B}$) we may assume that

(*) there is no partial $\mathcal{L}$-elementary map $f : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{A} \cap \mathcal{B} \subseteq dom(f) \neq \mathcal{A} \cap \mathcal{B}$ and $f \restriction (\mathcal{A} \cap \mathcal{B}) = id$.

**3.8.1 Claim.** $\mathcal{A} \cap \mathcal{B}$ *contains the interpretations of the constants from* $\rho$*.*

**Proof.** Immediate by Lemma 3.3 and the assumptions on $\mathcal{A}$ and $\mathcal{B}$. □ Claim 3.8.1.

**3.8.2 Claim.** $\mathcal{A} \cap \mathcal{B}$ *is a substructure of* $\mathcal{A}$ *(and of* $\mathcal{B}$*).*

**Proof.** Immediate by Lemma 3.3. □ Claim 3.8.2.

**3.8.3 Claim.** $\mathcal{A} \cap \mathcal{B}$ *is closed in* $\mathcal{A}$ *(and in* $\mathcal{B}$*).*

**Proof.** Suppose not. By Lemmas 3.7 and 3.3 (ii), choose $a_0, ..., a_{n+k} \in \mathcal{A}$, $0 < k, n < \omega$ and $n$-unclosed $\theta$ so that if $A$ is the relevant subset of $\mathcal{A}$ generated by $\{a_0, ..., a_n\}$ and $B$ is the relevant subset of $\mathcal{A}$ generated by $\{a_0, ..., a_{n+k}\}$, then

(i) $B \neq A$ is the intrinsic closure of $A$, $A = B \cap \mathcal{B}$ (and so $A$ is closed in $\mathcal{A} \cap \mathcal{B}$) and the elements $a_0, ..., a_{n+k}$ are linearly independent,

(ii) $\mathcal{A} \models \theta(a_0, ..., a_{n+k})$,

(iii) there are at most finitely many $b_{n+1}, ..., b_{n+k} \in \mathcal{A}$ ($b_{n+1}, ..., b_{n+k} \in \mathcal{B}$) such that $\mathcal{A} \models \theta(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$ ($\mathcal{B} \models \theta(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$),

(iv) for all $b_{n+1},...,b_{n+k} \in \mathcal{A}$ ($b_{n+1},...,b_{n+k} \in \mathcal{B}$), if

$$\mathcal{A} \models \theta(a_0,...,a_n,b_{n+1},...,b_{n+k}) \quad (\mathcal{B} \models \theta(a_0,...,a_n,b_{n+1},...,b_{n+k})),$$

then $\{a_0,...,a_n\}$ is unclosed in $\{a_0,...,a_n,b_{n+1},...,b_{n+k}\}$.
By Lemma 3.1, it is easy to see that

(v) if $b_{n+1},...,b_{n+k} \in \mathcal{A}$ ($\in \mathcal{B}$) and

$$\mathcal{A} \models \theta(a_0,...,a_n,b_{n+1},...,b_{n+k}) \quad (\mathcal{B} \models \theta(a_0,...,a_n,b_{n+1},...,b_{n+k})),$$

then $b_i \notin \mathcal{A} \cap \mathcal{B}$ for any $k < i \leq n+k$.
For $b_{n+1},...,b_{n+k} \in \mathcal{A}$ such that $\mathcal{A} \models \theta(a_0,..,a_n,b_{n+1},...,b_{n+k})$, we let

$$\Phi(b_{n+1},...,b_{n+k})$$

be the set

$$\{\exists y_0...\exists y_r \phi(a_0,...,a_n,c_0,...,c_p,x_{n+1},...,x_{n+k},y_0,...,y_r)|$$

$$\exists x_{n+1}...\exists x_{n+k}\exists y_0...\exists y_r \phi \in \mathcal{L}^e,$$

$$c_0,...,c_p \in \mathcal{A} \cap \mathcal{B},$$

$$\mathcal{A} \models \exists y_0...\exists y_r \phi(a_0,...,a_n,c_0,...,c_p,b_{n+1},...,b_{n+k})\}.$$

By (iii), we can choose the elements $a_0,...,a_{n+k}$ so that in addition

(vi) $\Phi(a_{n+1},...,a_{n+k})$ is maximal.

By Lemma 3.1 and the definition of $\mathcal{L}^e$,

(vii) for all $\psi_i \in \Phi(a_{n+1},...,a_{n+k})$, $i < s < \omega$, there is $\psi \in \Phi(a_{n+1},...,a_{n+k})$ such that $\models \forall x_{n+1}...\forall x_{n+k}(\psi \to \wedge_{i<s}\psi_i)$.

Then by (iii) and (vii) we can find $b_{n+1},...,b_{n+k} \in \mathcal{B}$ so that for all formulas $\phi(x_0,...,x_{m+k}) \in \mathcal{L}^e$ and $c_0,...,c_m \in \mathcal{A} \cap \mathcal{B}$,

(a) if $\mathcal{A} \models \phi(c_0,...,c_m,a_{n+1},...,a_{n+k})$, then $\mathcal{B} \models \phi(,c_0,...,c_p,b_{n+1},...,b_{n+k})$.
By repeating this argument, it is easy to see that for all formulas $\phi(x_0,...,x_{m+k}) \in \mathcal{L}^e$ and $c_0,...,c_m \in \mathcal{A} \cap \mathcal{B}$,

(b) if $\mathcal{B} \models \phi(c_0,...,c_m,b_{n+1},...,b_{n+k})$, then $\mathcal{A} \models \phi(,c_0,...,c_p,a_{n+1},...,a_{n+k})$.
Items (a) and (b) together mean that there is a partial $\mathcal{L}$-elementary map $h : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{A} \cap \mathcal{B} \subseteq dom(f)$ and by (v), $A \neq \mathcal{A} \cap \mathcal{B}$. This contradicts (*).
□ Claim 3.8.3.

**3.8.4 Claim.** *$\mathcal{A} \cap \mathcal{B}$ is an algebraically closed subset of $F_{\mathcal{A}}$.*

**Proof.** Let $\mathcal{A}^* \subseteq \mathcal{A}$ be the least submodel of $\mathcal{A}$ such that it is algebraically closed as a subset of $F_{\mathcal{A}}$. and $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}^*$. $\mathcal{B}^* \subseteq \mathcal{B}$ is defined similarly. Let $\mathcal{L}^a$ be the class of all formulas $\phi$ such that

(i) $\phi = \exists x_{n+1} ... \exists x_{n+k} \theta(x_0, ..., x_{n+k})$, where $\theta$ is $n$-algebraic (i.e. $\theta$ is quantifier free and implies that for all $0 < i \leq k$, $x_{n+i}$ is algebraic over $\{x_0, ..., x_{n+i-1}\}$ or $x_{n+i} = exp(x)$ for some $x \in \{x_0, ..., x_{n+i-1}\}$ or $x_{n+i} = log(x)$ for some $x \in \{x_0, ..., x_{n+i-1}\}$).

Let $f : \mathcal{A} \to \mathcal{B}$ be a partial function such that

(a) $\mathcal{A} \cap \mathcal{B} \subseteq dom(f)$ and $f \restriction (\mathcal{A} \cap \mathcal{B}) = id$,

(b) $f$ is $\mathcal{L}^a$-elementary,

(c) $dom(f) \subseteq \mathcal{A}^*$,

(d) $f$ is maximal among those that satisfy (a)-(c).

**3.8.4.1 Subclaim.** *$dom(f) = \mathcal{A}^*$ and $rng(f) = \mathcal{B}^*$.*

**Proof.** By the definition of $\mathcal{L}^e$, $id : \mathcal{A} \cap \mathcal{B} \to \mathcal{A} \cap \mathcal{B}$ satisfies (a)-(c). But then one can prove that $dom(f) = \mathcal{A}^*$ essentially as the related claim was proved in the proof of Claim 3.8.3. By the definition of $\mathcal{B}^*$, this implies that $rng(f) = \mathcal{B}^*$. □ Subclaim 3.8.4.1.

So by Subclaim 3.8.4.1, $f$ is an isomorphism between $\mathcal{A}^*$ and $\mathcal{B}^*$. By (*) and Lemma 3.4 (and symmetry) it suffices to prove the following subclaim.

**3.8.4.2 Subclaim.** *$\mathcal{A}^*$ is closed in $\mathcal{A}$.*

**Proof.** Let $A$ be a relevant subset of $\mathcal{A}$ with finite dimension. Let $B = A \cap \mathcal{B}$ and $C = A \cap \mathcal{A}^*$. We need to prove that $\delta(A/C) \geq 0$. By the basic properties of $\delta$, it suffices to prove this for any relevant $A' \supseteq A$ such that $A'$ has finite dimension and is the relevant subset of $\mathcal{A}$ generated by $A \cup C'$ for some relevant $C' \subseteq \mathcal{A}^*$ such that $C \subseteq C'$. By the definition of $\mathcal{A}^*$, we can find $C'$ so that it is contained in the least submodel $\mathcal{C}$ of $\mathcal{A}$ such that $C' \cap \mathcal{B} \subseteq \mathcal{C}$ and $\mathcal{C}$ is algebraically closed in $F_{\mathcal{A}}$. So we may also assume that $\delta(C/B) = 0$ ($\delta(C/B) \geq 0$ follows from Claim 3.8.3).

Now

$$\delta(A/C) = \delta(A/B) - \delta(C/B).$$

Since by Claim 3.8.3, $\delta(A/B) \geq 0$, $\delta(A/C) \geq 0$. □ Subclaim 3.8.4.2.

□ Claim 3.8.4.

By Claims 3.8.3 and 3.8.4, $\mathcal{A} \cap \mathcal{B} \in \mathbf{K}$ is a closed substructure of both $\mathcal{A}$ and $\mathcal{B}$. So by Lemmas 2.2 and 2.3, there is $\mathcal{C} \in \mathbf{K}$ such that both $\mathcal{A}$ and $\mathcal{B}$ are closed substructures of $\mathcal{C}$ i.e. $\mathcal{C}$ and $id : \mathcal{A} \cup \mathcal{B} \to \mathcal{C}$ are as required in the $\mathcal{L}$-amalgamation property. □

**3.9 Lemma.** *$\mathbf{K}$ is strongly $\mathcal{L}$-inductive.*

**Proof.** Immediate by the definition of $\mathbf{K}$ (all the requirements are local). □

This finishes the proof of Theorem 3.5.

**3.10 Conclusion.** *For all cardinals $\kappa$, there is a model $\mathcal{A} \in \mathbf{K}$ of power $\geq \kappa$ such that $\mathcal{A}$ is strongly $\mathcal{L}$-$\kappa$-homogeneous and $\mathcal{L}$-$\kappa$-universal for $\mathbf{K}$.*

## 4. On independence in K

In this section we will study mainly the meaning and behaviour of dividing in the class $\mathbf{K}$, see e.g. [BL].

Let $\kappa$ be a relatively large cardinal and $\mathbf{M} \in \mathbf{K}$ be a structure as given by Conclusion 3.10, i.e. $\mathbf{M}$ is strongly $\mathcal{L}$-$\kappa$-homogeneous and $\mathcal{L}$-$\kappa$-universal for $\mathbf{K}$. We will use concepts and notation from [HS2], [HS3] and [BL] freely (in particular, $A \subseteq \mathbf{M}$ means that $A$ is a subset of $\mathbf{M}$ and $|A| < \kappa$ or even $|A| \ll \kappa$ if needed). Notice that for all $a \in \mathbf{M}$ and $A \subseteq \mathbf{M}$ $t_{\mathcal{L}}(a/A)$ determines the first-order type $t(a/A)$ of $a$ over $A$.

**4.1 Remark.** $\mathbf{M}$ *is unstable (see e.g. [HS2]).*

**Proof.** Immediate by the proof of Lemma 2.3. □

Now Remark 4.1 implies that $\mathbf{K}$ has all the non-structure properties one can think of. However, since all these properties arise from the added 'random' function, this is not very interesting. Also the class may be simple (it is as we will see) and as we know from the elementary case, from such classes one may obtain a lot of structural information. So we try to do better. In a sense, we look at how these structures behave if we remove the random function from the structures. (However, we do use the fact that the function can be added making the behaviour of the class nice.)

Below we define some concepts for models from $\mathbf{K}^s$. As before, this defines the same concepts for $\mathbf{M}$.

**4.2 Definition.** *Suppose $\mathcal{C} \in \mathbf{K}^s$, $a \in \mathcal{C}$ and $A, B \subseteq \mathcal{C}$.*

*(i) By $scl(A) = scl_{\mathcal{C}}(A)$ (small closure operator, see [BL]; in [HS2] $scl$ is denoted by $acl$) we denote the least submodel $\mathcal{A} \supseteq A$ of $\mathcal{C}$ such that $\mathcal{A}$ is algebraically closed as a subset of $F_{\mathcal{C}}$, is closed as a relevant subset of $\mathcal{C}$ and for all elements $b, c \in \mathcal{C}$, if $\mathcal{C} \models exp(b, c)$, then $b \in \mathcal{A}$ iff $c \in \mathcal{A}$ (i.e. in $\mathbf{M}$, the least submodel of $\mathbf{M}$ which contains all the intrinsic closures of finite subsets of $A$ and which is algebraically closed as a subset of $F_{\mathbf{M}}$, see the proof of Subclaim 3.8.4.2).*

*(ii) We define $R(a/A) \in \mathbb{N}$ as follows: If $a \in scl(A)$, then $R(a/A) = 0$ and otherwise $R(a/A)$ is the least $n$ such that there is $C \subseteq scl(A \cup a)$ for which the following holds:*

*(a) $C \subseteq scl(A \cup a)$ is a relevant set and has finite dimension,*

*(b) $a \in C$,*

*(c) $C \cap scl(A)$ is closed,*

*(d) $n = \delta(C/C \cap scl(A)) + 1$.*

*If $C$ satisfies (a)-(c) and $\delta(C/C \cap scl(A)) + 1 = R(a/A)$, then we say that $C$ is an $R$-witness for $(a, A)$.*

*(iii) We write $a \downarrow^d_A B$ if $scl(a \cup A) \downarrow_{scl(A)} scl(A \cup B)$ and the relevant set generated by $scl(a \cup A) \cup scl(A \cup B)$ is closed. For (possibly infinite) sets $C \subseteq \mathcal{C}$, $C \downarrow^d_A B$ is defined similarly.*

Notice that for all elements $a \in \mathbf{M}$ and $A \subseteq \mathbf{M}$, $0 \leq R(a/A) \leq 2$. Notice also that $scl(A) = \cup\{scl(B) | \ B \subseteq A \text{ finite}\}$.

Our goal is to prove that $t(a/B)$ does not divide over $A \subseteq B$ iff $a \downarrow^d_A B$.

**4.3 Lemma.**

*(i) For all $a \in \mathbf{M}$ and $A \subseteq B \subseteq \mathbf{M}$, $a \downarrow^d_A B$ iff $scl(a \cup A) \cap scl(B) = scl(A)$ and $R(a/B) = R(a/A)$.*

*(ii) For all $a \in \mathbf{M}$ and $B \subseteq \mathbf{M}$ there is countable (or finite) $A \subseteq B$ such that $a \downarrow^d_A B$.*

*(iii) For all $a \in \mathbf{M}$ and $A \subseteq B \subseteq \mathbf{M}$ there is $b \in \mathbf{M}$ such that $t(b/A) = t(a/A)$ and $b \downarrow^d_A B$.*

*(iv) Suppose $A = scl(A) \subseteq B = scl(B) \subseteq \mathbf{M}$, $t(a/A) = t(b/A)$, $a \downarrow^d_A B$ and $b \downarrow^d_A B$. Then there is an isomorphism*

$$f : scl(B \cup a) \restriction (\rho - \{log\}) \to scl(B \cup b) \restriction (\rho - \{log\})$$

*such that $f \restriction B = id$ and $f(a) = b$.*

*(v) Suppose $A \subseteq B \subseteq C \subseteq \mathbf{M}$. Then $a \downarrow^d_A C$ iff $a \downarrow^d_A B$ and $a \downarrow^d_B C$.*

*(vi) For all $A \subseteq \mathbf{M}$ and $a, b \in \mathbf{M}$, $a \downarrow^d_A b$ iff $b \downarrow^d_A a$.*

*(vii) For all $A \subseteq B \subseteq \mathbf{M}$ and $C \subseteq \mathbf{M}$, $C \downarrow^d_A B$ iff for all (finite sequences) $c \in C$, $c \downarrow^d_A B$.*

**Proof.** (i): "$\Rightarrow$": Clearly it suffices to prove that $R(a/B) = R(a/A)$. For this, let $C$ be the relevant set generated by $scl(A \cup a) \cup scl(B)$, $D$ an $R$-witness for $(a, A)$ and $E$ an $R$-witness for $(a, B)$. Since $C$ is closed, $\delta(E \cap C/E \cap scl(B)) \leq \delta(E/E \cap scl(B))$. Thus we may assume that $E \subseteq C$. As in the proof of Lemma 2.2, we can find $E' \subseteq C$ such that $E \subseteq E'$, $\delta(E'/E' \cap scl(B)) \leq \delta(E/E \cap scl(B))$ and $\delta(E' \cap scl(A \cup a)/E' \cap scl(A)) \leq \delta(E'/E' \cap scl(B))$. But then $\delta(D/D \cap scl(A)) \leq \delta(E/E \cap scl(B))$.

"$\Leftarrow$": Let $C$ be the relevant set generated by $scl(A \cup a) \cup scl(B)$ and $D$ an $R$-witness for $(a, A)$. Since $R(a/A) = R(a/B)$, $D$ is an $R$-witness for $(a, B)$. Thus if $E$ is the relevant set generated by $scl(B) \cup D$, $E$ is closed. As in the proof of Lemma 2.1, one can see that every $C' \subseteq C$ with finite dimension is contained in $C'' \subseteq C$ such that $C''$ has finite dimension, $C' \subseteq C''$ and $\delta(C''/C'' \cap E) = 0$. As before, this implies that $C$ is closed.

Let $D$ be as above and $E$ a relevant set generated by $D \cup scl(A)$. Again, for every $C \subseteq scl(A \cup a)$ with finite dimension, there is $C' \subseteq scl(A \cup a)$ such that $C'$ has finite dimension, $C \subseteq C'$ and $\delta(C'/C' \cap E) = 0$. With this, since $scl(B)$ and $scl(A \cup a)$ are linearly independent over $scl(A)$, a simple calculation shows that if $scl(A \cup a) \not\downarrow_{scl(A)} scl(B)$, then $R(a/B) < R(a/A)$.

(ii): By (i), just choose countable $A \subseteq B$ such that for some $R$-witness $C$ for $(a, B)$, $C \subseteq scl(A \cup a)$ and $scl(C \cup A) \cap scl(B) = scl(A)$.

(iii): Let $\mathcal{A} = (scl(B))^s \in \mathbf{K}^s$ and let $\mathcal{B} = scl(A \cup a)^s \in \mathbf{K}^s$. Let $\mathcal{C}$ be the 'free' amalgamation of $\mathcal{A}$ and $\mathcal{B}$ over $scl(A)$ from the proof of Lemma 2.2. By Lemma 2.3, there is $\mathcal{D} \in \mathbf{K}$ such that $\mathcal{D}^s = \mathcal{C}$ and both $id : scl(B) \to \mathcal{D}$ and $id : scl(A \cup a) \to \mathcal{D}$ are $\mathcal{L}$-elementary. By the universality and the homogeneity of $\mathbf{M}$, there is $\mathcal{L}$-elementary $g : \mathcal{D} \to \mathbf{M}$ such that $g \circ (f \restriction scl(B)) = id$. Then a simple predimension calculation (as in the proof Lemma 2.2) shows that $b = g(f(a))$ is as required.

(iv): For $X \subseteq \mathbf{M}$, by $acl(X)$ we denote the algebraic closure of $X$ in $F_{\mathbf{M}}$. Clearly it is enough to find, by induction on $\alpha < \alpha^*$, sets $A_\alpha \subseteq scl(B \cup a)$, elements $a_\alpha \in scl(B \cup a)$ and functions $f_\alpha : acl(A_\alpha \cup exp(A_\alpha)) \to scl(B \cup b)$ such that the following holds:

(a) $A_0$ is the relevant set generated by $B \cup scl(A \cup a)$, $A_{\alpha+1}$ is the relevant set generated by $A_\alpha \cup \{a_\alpha\}$ and for limit $\alpha$, $A_\alpha = \cup_{\beta<\alpha} A_\beta$,

(b) for all $\alpha$, $a_\alpha \notin A_\alpha$ and either $a_\alpha \in acl(A_\alpha \cup exp(A_\alpha))$ or $exp(a_\alpha) \in acl(A_\alpha)$,

(c) $f_\alpha$ is an isomorphism between $dom(f_\alpha) \restriction \{+, \cdot\}$ and $rng(f_\alpha) \restriction \{+, \cdot\}$ and for all $x \in A_\alpha$, $f_\alpha(exp(x)) = exp(f_\alpha(x))$,

(d) $f_0 \restriction B = id$ and $f_0(a) = b$,

(e) for $\beta < \alpha < \alpha^*$, $f_\beta \subseteq f_\alpha$,

(f) $\cup_{\alpha<\alpha^*} A_\alpha = scl(B \cup a)$.

The existence of these objects is easy to see: For $\alpha = 0$, $A_\alpha$ is determined by (a) and one can let $f_0$ be any isomorphism between $acl(B \cup a) \restriction \{+, \cdot\}$ and $acl(B \cup b) \restriction \{+, \cdot\}$ such that (d) holds (such a function exists since $a \downarrow_A B$ and $b \downarrow_A B$).

For $\alpha = \beta+1$ we do the following: If $\cup_{\alpha<\alpha^*} A_\alpha \neq scl(B \cup a)$, there is $a_\beta \notin A_\beta$ such that $a_\beta \in acl(A_\beta \cup exp(A_\beta))$ or $exp(a_\beta) \in acl(A_\beta)$ (since the relevant set generated by $B \cup scl(A \cup a)$ is closed). We assume that $a_\beta \in acl(A_\beta \cup exp(A_\beta))$, the other case is similar. Now $a_\beta$ determines $A_\alpha$ via (a).

**4.3.1 Claim.** $exp(a_\beta) \notin acl(A_\beta \cup exp(A_\beta))$.

**Proof**. For a contradiction, suppose $exp(a_\beta) \in acl(A_\beta \cup exp(A_\beta))$. Using (a) and (b), one can find a relevant $C \subseteq A_\beta$ such that $C$ has finite dimension, $\delta(C/C \cap A_0) = 0$ and $a_\beta, exp(a_\beta) \in acl(C \cup exp(C))$. Since $a_\beta \notin A_\beta$, $\delta(C \cup \{a_\beta\}/C) < 0$. But then $\delta(C \cup \{a_\beta\}/C \cap A_0) < 0$. This contradicts the assumption that $A_0$ is closed (notice that $a_\beta$ is linearly independent from $A_0 \cup C$). □ Claim 4.3.1.

Similarly, we can see that $exp(f_\beta(a_\beta)) \notin dom(f_\beta)$. So there is $f_\alpha$ such that $f_\beta \subseteq f_\alpha$, $f_\alpha(exp(a_\beta)) = exp(f_\beta(a_\beta))$ and $f_\beta$ is an isomorphism between $acl(dom(f_\beta) \cup \{exp(a_\beta)\}) \restriction \{+, \cdot\}$ and $acl(rng(f_\beta) \cup \{exp(f_\beta(a_\beta))\}) \restriction \{+, \cdot\}$. Clearly $f_\alpha$ is as required.

(v): It is easy to see that $c \downarrow^d_D E$ iff $c \downarrow^d_{scl(D)} scl(E)$. Thus we may assume that $A = scl(A)$, $B = scl(B)$ and $C = scl(C)$. Then from right to left, the claim is clear by (i). The other direction follows from this and (the proof of) (iv).

(vi): Immediate by the definition of $\downarrow^d$.

(vii): Follows easily from the definition of $\downarrow^d$. □

We can improve Lemma 4.3 (ii):

**4.4 Proposition.** *For all $a$ and $B$ there is finite $A \subseteq B$ such that $a \downarrow^d_A B$.*

**Proof.** Let $\mathcal{B} = scl(B)$, $C$ an $R$-witness for $(a, B)$ and $\mathcal{A} = scl(C \cap B)$. Since $\delta(C/C \cap scl(a \cup \mathcal{A})) \geq 0$, we may assume that $C \subseteq scl(a \cup \mathcal{A})$. Clearly, it suffices to prove that $a \downarrow^d_{\mathcal{A}} \mathcal{B}$. For a contradiction, suppose that this is not the case. By Lemma 4.3 (i), this means that $scl(a \cup \mathcal{A}) \cap \mathcal{B} \neq \mathcal{A}$.

Then we can find $d_i \in scl(\mathcal{A} \cup a)$, $i \leq n$, and $p < k \leq l < n$ such that

(a) $(d_i)_{i \leq n}$ is linearly independent,

(b) if $D$ is the relevant set generated by $\{d_0, ..., d_n\}$, then $D \cap \mathcal{A}$ is generated by $\{d_0, ..., d_{k-1}\}$,

(c) if $E$ is the relevant set generated by $\{d_0, ..., d_{n-1}\}$, then $E \cap \mathcal{B} = E \cap \mathcal{A}$,

(d) $D \cap \mathcal{A}$ is closed,

(e) $\{d_0, ..., d_p, d_k, ..., d_l\}$ generate the relevant set $C$ and for all $l < i \leq n$, either $d_i$ or $exp(d_i)$ is algebraic over $\{d_0, ..., d_{i-1}, exp(d_0), ..., exp(d_{i-1})\}$,

(f) $d_n, exp(d_n) \in \mathcal{B}$.

Then

$$deg((d_k, ..., d_{n-1}, exp(d_k), ..., exp(d_{n-1}))/\mathcal{B}) <$$

$$deg((d_k, ..., d_{n-1}, exp(d_k), ..., exp(d_{n-1}))/\mathcal{A}) = n - k$$

and $dim((d_k, ..., d_{n-1})/\mathcal{B}) = n - k$. This contradicts the choice of $C$. □

For the definition of an arbitrary model being $\lambda$-stable, see e.g. [Sh1].

**4.5 Corollary.** $\mathbf{M} \restriction (\rho - \{log\})$ *is $\lambda$-stable for all $\lambda \geq 2^\omega$.*

**Proof.** This follows immediately from Lemma 4.3 (iv) and Proposition 4.4 together with the following claim:

**4.4.1 Claim.** *Suppose $a, b \in \mathbf{M}$ and that there is an isomorphism $f : scl(a) \restriction (\rho - \{log\}) \to scl(b) \restriction (\rho - \{log\})$ such that $f(a) = b$. Then $(\mathbf{M} \restriction (\rho - \{log\}), a)$ is back and forth equivalent with $(\mathbf{M} \restriction (\rho - \{log\}), b)$.*

**Proof.** Immediate by $\lambda$-$\kappa$-homogeneity and $\lambda$-$\kappa$-universality of $\mathbf{M}$ together with Lemmas 2.3 and 3.4. □ Claim 4.4.1.

□

We say that $t(a/B)$ divides weakly over $A \subseteq B$, if there is $C \supseteq B$ such that for all $b$, $t(b/B) = t(a/B)$ implies that $t(b/C)$ divides over $A$.

**4.6 Lemma.** *Suppose $A \subseteq B \subseteq \mathbf{M}$. If $a \not\downarrow^d_A B$, then $t(a/B)$ divides weakly over $A$.*

**Proof.** If $scl(A \cup a) \not\downarrow_{scl(A)} scl(B)$, it is easy to see that $t(a/B)$ divides weakly over $A$. So we assume that $scl(A \cup a) \downarrow_{scl(A)} scl(B)$. Then the relevant set generated by $scl(B) \cup scl(A \cup a)$ is not closed. Adapting notation from [BL], let $b$ be a (possibly infinite) sequence such that $B = rng(b)$. Let $\lambda$ be a large enough cardinal, $\mathcal{A} = scl(A)$ and let $b_i$, $i < \lambda$, be such that

(a) $(b_i)_{i<\lambda}$ is $\mathcal{A}$-indiscernible,

(b) $b_0 = b$,

(c) for all $i < \lambda$, $b_i \downarrow^d_{\mathcal{A}} \cup_{j<i} b_j$,

(d) for all $i < \lambda$, the relevant set $B^r_i$ generated by $\cup_{j<i} scl(b_j)$ is closed.

By (the proof of) Lemma 4.3 (iii), homogeneity and Erdös-Rado theorem, such sequences $b_i$ exist ((d) follows from (c)).

For a contradiction, suppose $a^*$ is such that for all $i < \lambda$, $t(a^* \frown b_i/\emptyset) = t(a \frown b/\emptyset)$. Let $\mathcal{A}^* = scl(\mathcal{A} \cup a^*)$, $\mathcal{B}_i = scl(b_i)$ and $\mathcal{B}^*_i = scl(b_i \cup a^*)$. Clearly we may assume that for all $i < \lambda$,

(e) $\mathcal{B}_i \downarrow_{\mathcal{A}} scl(\cup_{j<i} \mathcal{B}^*)$ (not $\downarrow^d$!).

Also there is $n < \omega$ such that for all $i < \omega$, $0 \leq R(a^*/B^r_i) < n$. Thus the following claim implies the required contradiction.

**4.6.1 Claim.** *Let $i < \omega$, and let $C$ be the relevant set generated by $\mathcal{B}_i \cup \mathcal{A}^*$ and $D$ be the relevant set generated by $scl(\cup_{j<i} \mathcal{B}^*_j) \cup C$. Then (i) and (ii) below hold.*

*(i) If $D$ is closed, then so is $C$ (i.e. $D$ is not closed).*

*(ii) If $D$ is not closed, then $R(a^*/B^r_{i+1}) < R(a^*/B^r_i)$.*

**Proof.** (i): This can be proved essentially as the related claim was proved in Lemma 2.2. However, since there is a small difference, we repeat the proof for the reader's convenience.

Suppose $E \subseteq D$ is a relevant set and has finite dimension. Since $D$ is closed, it suffices to prove that $\delta(E/E \cap C) \geq 0$. Let $X = \{e_0, ..., e_n\}$ be a linear base for $E$ such that $X \cap C = \{e_k, ..., e_n\}$ is a base for $E \cap C$. Since we may increase $E \cap C$ without increasing $\delta(E/E \cap C)$ we may assume that for all $e \in E$, there is $c \in C \cap E$ and $b \in scl(\cup_{j<i} \mathcal{B}^*_j)$ such that $e = c + b$. Thus we may assume that $X$ was chosen so that for all $m < k$, $e_m \in scl(\cup_{j<i} \mathcal{B}^*_j)$. By (e) we may also assume that writing $E'$ for $E \cap scl(\cup_{j<i} \mathcal{B}^*_j)$,

$$E' \cup exp(E') \downarrow_{(E \cap \mathcal{A}^*) \cup exp(E \cap \mathcal{A}^*)} C \cup exp(C).$$

But then $\delta(E'/E' \cap \mathcal{A}^*) = \delta(E/E \cap C)$. Since $\mathcal{A}^*$ is closed, $\delta(E'/E' \cap \mathcal{A}^*) \geq 0$.

(ii): If $D$ is not closed, then there is a relevant set $E \subseteq scl(D) = scl(B^r_{i+1})$ such that it has finite dimension and

(*) $\delta(E/E \cap D) < 0$.

Notice that as before, we may increase $E$ by increasing the set $E \cap D$ and (*) remains true. Also as before, for every $N \subseteq scl(\cup_{j<i} \mathcal{B}^*_j)$ with finite dimension there is $N^* \subseteq scl(\cup_{j<i} \mathcal{B}^*_j)$ such that $N \cup (E \cap scl(\cup_{j<i} \mathcal{B}^*_j) \subseteq N^*$ and $N^*$ is an $R$-witness for $R(a^*/B^r_i)$. So since $scl(\cup_{j<i} \mathcal{B}^*_j) \subseteq D$, we may assume that $E' = E \cap scl(\cup_{j<i} \mathcal{B}^*_j)$ is an $R$-witness for $R(a^*/B^r_i)$. Furthermore, since every elements in $D$ depends linearly from $H = scl(\cup_{j<i} \mathcal{B}^*_j) \cup \mathcal{B}_i$, we may assume that every element of $E \cap D$ depends linearly from $E \cap H$. Thus, letting $D'$ be the relevant set generated by $scl(B^r_i) \cup \mathcal{B}_i$, by (e), we may assume that

(1) $\delta(E \cap D/E \cap D') = R(a^*/B^r_i)$.

Since both $B^r_i$ and $B^r_{i+1}$ are closed, essentially repeating the argument above, we can see that

(2) $D'$ is closed.

Then by (1) and (*), $\delta(E/E\cap D') < R(a^*/B^r_i)$. By (2), $\delta(E\cap scl(B^r_{i+1})/E\cap D') \geq 0$. Thus, $\delta(E/E\cap scl(B^r_{i+1})) \leq \delta(E/E\cap D') < R(a^*/B^r_i)$. $\square$ Claim 4.6.1.

$\square$

**4.7 Lemma.** *Suppose $A \subseteq B \subseteq \mathbf{M}$. If $a \downarrow^d_A B$, then $t(a/B)$ does not divide over $A$.*

**Proof.** Again, let $b$ be a (possibly infinite) sequence such that $B = rng(b)$. We need to prove that $t(a/b)$ does not divide over $A$. For this let $(b_i)_{i<\alpha}$ be $A$-indiscernible sequence such that $t(b_0/A) = t(b/A)$ and $\alpha$ is an infinite ordinal. Notice that it is enough to find $a^*$ and $b'_i$, $i<\alpha$, such that

(i) there is a ($\mathcal{L}$-)elementary $F : \cup_{i<\alpha} b'_i \to b_i$ such that $F \restriction A = id$ and for all $i<\alpha$, $F(b'_i) = b_i$,

(ii) for all $i<\alpha$, $t(a^* \frown b'_i/\emptyset) = t(a \frown b_0/\emptyset)$.

By homogeneity we may assume that $b_0 = b$. Let $\mathcal{A} = scl(A)$. Again by homogeneity and the usual Erdös-Rado-argument, we may assume that there are sets $\mathcal{B}_i$, $i<\alpha$, such that

(a) $scl(\mathcal{B}_i) = \mathcal{B}_i$,

(b) $scl(b) = \mathcal{B}_0$,

(c) for all $i<\alpha$, there is an isomorphism $f_i : \mathcal{B}_0 \to \mathcal{B}_i$ such that $f_i \restriction \mathcal{A} = id$ and $f_i(b_0) = b_i$,

(d) $(\mathcal{B}_i)_{i<\alpha}$ is $\mathcal{A}$-indiscernible.

It is enough to find $a^*$ such that for all $i<\alpha$, $t(a^*/\mathcal{B}_i) = f_i(t(a/\mathcal{B}_0))$.

Let $(I,<)$ be an ordering such that it extends $(\alpha,<)$, the order type of $I-\alpha$ is $\omega_1$ and for all $x \in I-\alpha$ and $y\in\alpha$, $x<y$. Choose (using homogeneity) $\mathcal{B}_i$, $i \in I-\alpha$, so that $(\mathcal{B}_i)_{i\in I}$ is $\mathcal{A}$-indiscernible. Let $\mathcal{C} = scl(\cup_{i\in I-\alpha}\mathcal{B}_i)$ and for $i<\alpha$, $\mathcal{C}_i = scl(\mathcal{B}_i \cup \mathcal{C})$. By Lemma 4.3 (ii), (v) and (vi), for all $i<\alpha$ and finite subsequences $b'$ of $b_i$,

$$b' \downarrow^d_{\mathcal{C}} \cup_{j<i}\mathcal{C}_j.$$

Thus by Lemma 4.3 (vi) and the definition of $\downarrow^d$, for all $i<\alpha$,

$$\mathcal{C}_i \downarrow^d_{\mathcal{C}} \cup_{j<i}\mathcal{C}_j.$$

By Lemma 4.3 (iii) and (v), we may assume that $a \downarrow^d_{\mathcal{C}} \mathcal{C}_0$. So, and this was the purpose the considerations above, we may assume that for all $i<\alpha$,

$$(*) \quad \mathcal{B}_i \downarrow^d_{\mathcal{A}} \cup_{j<i}\mathcal{B}_j.$$

For all $i<\alpha$, find $\mathcal{D}_i \in \mathbf{K}$ such that

(e) $\mathcal{D}_0 = scl(\mathcal{B}_0 \cup a)$,

(f) for all $i<\alpha$, there is an isomorphism $g_i : \mathcal{D}_0 \to \mathcal{D}_i$ such that $f_i \subseteq g_i$ and $g_i \restriction scl(\mathcal{A}\cup a) = id$ (and $g_0 = id$).

By $\mathcal{A}^*$ we denote $scl(A \cup a)$. For all $i \leq \alpha$, we choose models $\mathcal{E}_i \in \mathbf{K}^s$ so that the following holds:

(g) $\mathcal{E}_0 = (\mathcal{A}^*)^s$, $\mathcal{E}_1 = (\mathcal{D}_0)^s$ and for limit $i$, $\mathcal{E}_i = \cup_{j<i}\mathcal{E}_j$,

(h) $\mathcal{E}_{i+1}$, $i > 0$, is the free amalgamation (i.e. the one from the proof of Lemma 2.2) of $\mathcal{E}_i$ and $(\mathcal{D}_i)^s$ over $\mathcal{A}^*$.

We denote $\mathcal{E} = \mathcal{E}_\alpha$ and let $\mathcal{B}^* = \cup_{i<\alpha} scl(\cup_{j\leq i}\mathcal{B}_j)$ and $\mathcal{B}^+ = \cup_{i<\alpha} scl_{\mathcal{E}}(\cup_{j\leq i}\mathcal{B}_j)$. By Lemma 4.3 (iv) and (*) above, there is an isomorphism $h : \mathcal{B}^+ \to (\mathcal{B}^*)^s$ such that for all $i < \omega$, $h \restriction \mathcal{B}_i = id$. By Lemmas 3.4 and 3.8, the existence of $a^*$ follows if there is $\mathcal{E}^* \in \mathbf{K}$ (i.e. there is an interpretation $log^*$ for $log$ on $\mathcal{E}$) such that $(\mathcal{E}^*)^s = \mathcal{E}$,

(i) for all $x, y \in \mathcal{B}^+$, $\mathcal{E}^* \models log(x) = y$ iff $\mathbf{M} \models log(h(x)) = h(y)$

and

(j) for all $x, y \in \mathcal{D}_0$ and $i < \alpha$, $\mathcal{D}_0 \models log(x) = y$ iff $\mathcal{E}^* \models log(g_i(x)) = g_i(y)$.

We prove this.

By the construction, it is easy to see that for all $i < \alpha$, $\mathcal{D}_i \cap \mathcal{B}^+ = \mathcal{B}_i$ and that $\mathcal{D}_i \cap (\cup_{j\in(\alpha-\{i\})}\mathcal{D}_j) = \mathcal{A}^*$. Thus for all $i < \alpha$, $\mathcal{D}_i \cap (\mathcal{B}^+ \cup \bigcup_{j\in(\alpha-\{i\})}\mathcal{D}_j) \subseteq \mathcal{A}^* \cup \mathcal{B}^+$. So one can define $log^* \restriction C$ so that (i) and (j) hold, where $C = (\mathcal{B}^+ \cup \bigcup_{j<\alpha}\mathcal{D}_j)$.

We are left to define $log^* \restriction (\mathcal{E} - C)$ so that $log^*$ satisfies the requirements (5) and (7) from the definition of $\mathbf{K}$ (notice that $exp(C) = C$).

Choose a linear base $X = \{x_i |\ i < \alpha^*\}$ for $\mathcal{E}$ so that $x_0 = \pi$. For finite $Y \subseteq X - \{x_0\}$, $\mathcal{E}_Y$ is defined as in the proof of Lemma 2.3. By the proof of Lemma 2.3, it is enough to define $log^* \restriction (\mathcal{E} - C)$ so that ((5) holds and) (7) holds for $\mathcal{E}_Y$ for all finite $Y \subseteq X - \{x_0\}$. Since $C$ is a union of relevant sets,

(k) for all $a \in \mathcal{E}_Y$, $a \in C$ iff for some $q \in \mathbf{Q} - \{0\}$ and $r \in \mathbf{Q}$, $q \cdot a + r \cdot \pi \in C$ iff for all $q \in \mathbf{Q}$ and $r \in \mathbf{Q}$, $q \cdot a + r \cdot \pi \in C$.

With this, a simple diagonalization argument, as in the proof of Lemma 2.3, gives $log^* \restriction (\mathcal{E} - C)$ such that (5) holds and (7) holds for $\mathcal{E}_Y - C$ for all finite $Y \subseteq X - \{x_0\}$. But then, by (k) and the pigeon hole principle, (7) holds for $\mathcal{E}_Y$ if there is finite $J \subseteq \alpha$ such that $\mathcal{E}_Y \cap C \subseteq \mathcal{B}^+ \cup \bigcup_{j\in J}\mathcal{D}_j$. This is clear by (*) above (i.e. $(\mathcal{D}_i)_{i<\alpha}$ is linearly independent over $\mathcal{A}^*$) together with finite dimension of $E_Y$. □

So we have proved:

**4.8 Proposition.** *$t(a/B)$ divides weakly over $A \subseteq B$ iff $a \downarrow^d_A B$.*

In particular, we have proved that $\mathbf{M}$ (or more precisely, the finite diagram of $\mathbf{M}$) is supersimple in the sense of [BL].

## 5. On a-prime models

This section is an adaptation of S. Shelah's theory of $a$-prime models from [Sh2] to our context.

**5.1 Definition.**

*(i) We write $t^e(a/A) = t^e(b/A)$ if there is an isomorphism $f : scl(a \cup A) \restriction (\rho - \{log\}) \to scl(b \cup A) \restriction (\rho - \{log\})$ such that $f \restriction scl(A) = id$ and $f(a) = b$.*

*(ii) For $B \subseteq A$, we write $t^e(a/B) \vdash t^e(a/A)$ if for all $b$, $t^e(b/B) = t^e(a/B)$ implies $t^e(b/A) = t^e(a/A)$.*

*(iii) We write $t(a/A) \in F^{ae}_\kappa(B)$ and say that $t(a/A)$ is $ae$-$\kappa$-isolated if $B \subseteq A$ has power $< \kappa$ and for all $b$, if $t^e(b/B) = t^e(a/B)$, then $b \downarrow^d_B A$.*

*(iv) By $\kappa(\mathbf{M})$ we denote the least $\kappa$ such that for all $a$ and $A$ there is $B \subseteq A$ of power $< \kappa$ such that $a \downarrow^d_B A$.*

*(v) We say that $t(a/A)$ is $ae$-isolated if it is $ae$-$\kappa(\mathbf{M})$-isolated and write $F^{ae}$ in place of $F^{ae}_{\kappa(\mathbf{M})}$.*

Notice that by Lemma 4.4, $\kappa(\mathbf{M}) = \omega$.

**5.2 Lemma.** *For all $B \subseteq A$, $a$ and $\kappa > |B|$, the following are equivalent:*

*(i) If there is an isomorphism $f : scl(a \cup B) \restriction (\rho - \{log\}) \to scl(b \cup B) \restriction (\rho - \{log\})$ such that $f \restriction scl(B) = id$ and $f(a) = b$, then there is an isomorphism $g : scl(a \cup A) \restriction (\rho - \{log\}) \to scl(b \cup A) \restriction (\rho - \{log\})$ such that $f \subseteq g$ and $g \restriction scl(A) = id$.*

*(ii) $t^e(a/B) \vdash t^e(a/A)$.*

*(iii) $t(a/A) \in F^{ae}_\kappa(B)$.*

**Proof.** (i) implies (ii) trivially and the implication (ii)$\Rightarrow$(iii) follows immediately from Lemma 4.3. For (iii)$\Rightarrow$(i), suppose (iii) holds but (i) is not true. Let $b$ and $f$ witness the failure of (i). Since (iii) holds, $scl(b \cup B) \downarrow^d_{scl(B)} scl(A)$. Thus by Lemma 2.3, there is $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{C} \restriction (\rho - \{log\}) = scl(b \cup A) \restriction (\rho - \{log\})$, for all $c, d \in scl(A)$, $log(c) = d$ iff $\mathcal{C} \models log(c) = d$ and for all $c, d \in scl(b \cup B)$, $log(f^{-1}(c)) = f^{-1}(d)$ iff $\mathcal{C} \models log(c) = d$. By universality and homogeneity of $\mathbf{M}$, there are $b' \in \mathbf{M}$ and an isomorphism $g : scl(b' \cup A) \to \mathcal{C}$ such that $g(b') = b$ and $g \restriction scl(A) = id$. The existence of this $b'$ contradicts Lemma 4.3 (iv) (or more precisely, the proof of the lemma). □

**5.3 Lemma.**

*(i) Suppose $f : scl(A \cup a) \restriction (\rho - \{log\}) \to scl(B \cup b) \restriction (\rho - \{log\})$ is an isomorphism such that $f[A] = B$ and $f(a) = b$. Then for all $C \subseteq A$, $t(a/A) \in F^{ae}_\kappa(C)$ iff $t(b/B) \in F^{ae}_\kappa(f[C])$.*

*(ii) If $A \subseteq B$, $t^e(a/A) = t^e(b/A)$, $a \downarrow^d_A B$ and $b \downarrow^d_A B$, then $t^e(a/B) = t^e(b/B)$.*

**Proof.** (i): By symmetry, it suffices to prove the implication from left to right. Suppose $t(b/B) \notin F^{ae}_\kappa(f[C])$. Let $c$ witness this. By Lemma 2.3, there is $\mathcal{C} \in \mathbf{K}$ such that there is an isomorphism $g : \mathcal{C} \restriction (\rho - \{log\}) \to scl(B \cup c) \restriction (\rho - \{log\})$ and for all $d, d' \in scl(B)$, $\mathcal{C} \models log(g^{-1}(d)) = g^{-1}(d')$ iff $log(f^{-1}(d)) = f^{-1}(d')$. By the properties of $\mathbf{M}$, there are $c' \in \mathbf{M}$ and an isomorphism $h : \mathcal{C} \to scl(A \cup c')$ such that for all $d \in scl(B)$, $h(g^{-1}(d)) = f^{-1}(d)$ and $h(g^{-1}(c)) = c'$. Then $c'$ witnesses that $t(a/A) \notin F^{ae}_\kappa(C)$.

(ii): Easy consequence of Lemmas 2.3 and 4.3 (iv) together with the properties of $\mathbf{M}$ (see the the proof of (iii)$\Rightarrow$(i) in Lemma 5.2). □

**5.4 Definition.**

*(i) We say that* $\mathcal{A} \subseteq \mathbf{M}$ *is* $ae$-$\kappa$*-saturated if for* $A \subseteq \mathcal{A}$ *and* $a$ *the following holds: if* $t(a/A)$ *is* $ae$-$\kappa$*-isolated, then there is* $b \in \mathcal{A}$ *such that* $t^e(b/A) = t^e(a/A)$.

*(ii) We say that* $A$ *is* $ae$-$\kappa$*-constructible over* $B \subseteq A$, *if there are* $a_i$ *and* $A_i$, $i < \alpha$, *such that* $A = B \cup \{a_i | \ i < \alpha\}$ *and for all* $i < \alpha$, $t(a_i/B \cup \{a_j | \ j < i\}) \in F^{ae}_\kappa(A_i)$. *Then we also say that* $(B, (a_i, A_i)_{i<\alpha})$ *is an* $F^{ae}_\kappa$*-construction.*

*(iii) We say that* $\mathcal{A}$ *is* $ae$-$\kappa$*-primary if it is* $ae$-$\kappa$*-constructible and* $ae$-$\kappa$*-saturated.*

*(iv) We write* $ae$*-isolated for* $ae$-$\kappa(\mathbf{M})$*-isolated. The notions* $ae$*-saturated,* $ae$*-constructible and* $ae$*-primary are defined similarly.*

**5.5 Lemma.** $\mathcal{A}$ *is* $ae$-$\kappa$*-saturated if for all* $A \subseteq \mathcal{A}$ *of power* $< \kappa$ *and* $a$ *there is* $b \in \mathcal{A}$ *such that* $t^e(b/A) = t^e(a/A)$.

**Proof.** Immediate by Lemma 5.2. □

**5.6 Lemma.** *Suppose* $\mathcal{A}$ *is* $ae$-$\kappa$*-saturated and* $A$ *is* $ae$-$\kappa$*-constructible over* $B \subseteq \mathcal{A}$. *Then there is a one to one function* $f : scl(A) \to \mathcal{A}$ *such that* $f \restriction B = id$, $f$ *is an isomorphism between* $scl(A) \restriction (\rho - \{log\})$ *and* $rng(f) \restriction (\rho - \{log\})$ *and* $scl(rng(f)) = rng(f)$.

**Proof.** Let sequences $a_i$ and sets $A_i$, $i < \alpha$, witnesses the assumption that $A$ is $ae$-$\kappa$-constructible. By induction on $i < \alpha$, we define a function $f_i$ such that $dom(f_i) = scl(B \cup \{a_j | \ j < i\})$, $scl(rng(f_i)) = rng(f_i)$, $f_i \restriction B = id$ and $f_i$ is an isomorphism between $dom(f_i) \restriction (\rho - \{log\})$ and $rng(f_i) \restriction (\rho - \{log\})$.

For $i = 0$, we let $f_0 = id_{scl(B)}$ and for limit $i$, $f_i = \cup_{j<i} f_j$. Clearly these are as required. So suppose $f_j$ is defined and we define $f_i$ for $i = j+1$. By Lemma 2.3, there is $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{C} \restriction (\rho - \{log\}) = scl(dom(f_j) \cup a_j) \restriction (\rho - \{log\})$ and for all $c, d \in dom(f_j)$, $\mathcal{C} \models log(c) = d$ iff $log(f_j(c)) = f_j(d)$. Thus there is $b \in \mathbf{M}$ and an isomorphism $g_i : scl(dom(f_j) \cup a_j) \restriction (\rho - \{log\}) \to scl(rng(f_j) \cup b) \restriction (\rho - \{log\})$ such that $f_j \subseteq g_i$ and $g_i(a_j) = b$. By Lemma 5.3, $t(b/rng(f_j))$ is $ae$-$\kappa$-isolated. Since $\mathcal{A}$ is $ae$-$\kappa$-saturated, there is $c \in \mathcal{A}$ such that $t^e(c/rng(f_j)) = t^e(b/rng(f_j))$. Let $g'_i$ be the isomorphism that witnesses this. Then $f_i = g'_i \circ g_i$ is as wanted. □

**5.7 Lemma.** *For all regular* $\kappa$ *and* $A$ *there exists a model* $\mathcal{A}$ *such that it is* $ae$-$\kappa$*-primary over* $A$.

**Proof.** Immediate by Lemmas 4.4 and 5.5. □

**5.8 Lemma.** *Suppose* $\mathcal{A}$ *is* $ae$*-primary over* $A$.

*(i) For all* $a \in \mathcal{A}$, *there is* $B \subseteq A$ *such that* $t(a/A) \in F^{ae}(B)$.

*(ii)* $\mathcal{A}$ *is* $ae$*-primary over any* $B$ *such that* $A \subseteq B \subseteq scl(A)$.

**Proof.** (i): Suppose not. Let sequences $a_i$ and sets $A_i$, $i < \alpha$, witness the fact that $\mathcal{A}$ is $ae$-constructible over $A$. Then we can find (essentially as in the proofs above, see e.g. the proof of Lemma 5.3) $I \subseteq \alpha, B \subseteq A$ (both of power $< \kappa(\mathbf{M})$) and $b_i$, $i \in I$, such that

(a) there is an isomorphism $f : scl(B \cup \{a_i|\ i \in I\}) \restriction (\rho - \{log\}) \to scl(B \cup \{b_i|\ i \in I\}) \restriction (\rho - \{log\})$ such that $f \restriction scl(B) = id$ and for all $i \in I$, $f(a_i) = b_i$,

(b) for all $i \in I$, $t(a_i/A \cup \{a_j|\ j < i, j \in I\}) \in F^{ae}(B \cup \{a_j|\ j < i, j \in I\})$,

(c) $\{b_i|\ i \in I\} \not\downarrow^d_B A$.

By choosing these so that $I$ is minimal, we may assume that there is $i^* \in I$ such that $I \subseteq i^* + 1$ and letting $I^* = I - \{i^*\}$,

(d) $\{b_i|\ i \in I^*\} \downarrow^d_B A$.

By (d) and (the proof of) Lemma 4.3 (iv), there is an isomorphism $g : scl(A \cup \{a_i|\ i \in I^*\}) \restriction (\rho - \{log\}) \to scl(A \cup \{b_i|\ i \in I^*\}) \restriction (\rho - \{log\})$ such that $f \restriction scl(B \cup \{a_i|\ i \in I^*\}) \subseteq g$ and $g \restriction scl(A) = id$. Also it is routine to find $c$ such that there is an isomorphism $h : scl(A \cup \{a_i|\ i \in I\}) \restriction (\rho - \{log\}) \to scl(A \cup \{b_i|\ i \in I^*\} \cup \{c\}) \restriction (\rho - \{log\})$ such that $g \subseteq h$ and $h(a_{i^*}) = c$. Then $b_{i^*}$ witnesses that $t(c/A \cup \{b_i|\ i \in I^*\}) \notin F^{ae}(B \cup \{b_i|\ i \in I^*\})$. This contradicts Lemma 5.3.

(ii): Let sequences $a_i$ and sets $A_i$, $i < \alpha$, witness the fact that $\mathcal{A}$ is $ae$-constructible over $A$. It suffices to prove that for all $i < \alpha$, $t(a_i/B \cup \{a_j|\ j < i\})$ is $ae$-isolated. This is clear since for all $c$ and $C \subseteq D$, $c \downarrow^d_C D$ iff $c \downarrow^d_C scl(D)$. □

By $\mathfrak{c}$ we denote the cardinal $2^\omega$.

**5.9 Lemma.** *Suppose $\mathcal{A}$ is $ae$-primary over $A$ and $(a_i)_{i<\alpha}$ is such that for $j < i < \alpha$, $a_j \neq a_i$, for all $i < \alpha$, $a_i \downarrow^d_A \cup\{a_j|\ j < i\}$ and for some $B \subseteq A$, $t(a_i/A) \in F^{ae}(B)$ for all $i < \alpha$. Then $\alpha < \mathfrak{c}^+$.*

**Proof.** For a contradiction, suppose $\alpha = \mathfrak{c}^+$. Then we may also assume that for all $j < i < \alpha$, $t(a_j/scl(B)) = t(a_i/scl(B))$. Let $\mathcal{B}$ be a model such that it is $ae$-primary over $A \cup \{a_i|\ i < \omega_1\}$. By Lemmas 5.6 and 5.8 (ii), there is an embedding $f : \mathcal{A} \restriction (\rho - \{log\}) \to \mathcal{B} \restriction (\rho - \{log\})$ such that $scl(rng(f)) = rng(f)$ and $f \restriction scl(A) = id$.

**5.9.1 Claim.** *For all $i < \mathfrak{c}^+$, $f(a_i) \not\downarrow^d_A \{a_j|\ j < \omega_1\}$.*

**Proof.** By Lemma 5.8 (i), there is $C \subseteq A \cup \{a_j|\ j < \omega_1\}$ such that

(*) $t(f(a_i)/A \cup \{a_j|\ j < \omega_1\}) \in F^{ae}(C)$.

Let $j < \omega_1$ be such that for all $k < \omega_1$, if $a_k \in C$, then $k < j$. If $f(a_i) \downarrow^d_A \{a_k|\ k < \omega_1\}$, then for all $j < k < \omega_1$, $t^e(a_k/C) = t^e(f(a_i)/C)$. Clearly this contradicts (*). □ Claim 5.9.1.

By Claim 5.9.1, Lemma 4.3 and the pigeon hole principle, we can find finite $I \subseteq \omega_1$ and $J \subseteq \mathfrak{c}^+$ of power $\mathfrak{c}^+$ such that for all $j \in J$, $f(a_j) \not\downarrow^d_A \cup_{i \in I} a_i$. Since for all $j \in J$, $f(a_j) \downarrow^d_A \{f(a_k)|\ k < j,\ k \in J\}$, this contradicts the basic properties of $\downarrow^d$ and the fact that $\kappa(\mathbf{M}) \leq \omega_1$, see Lemma 4.3. □

**5.10 Lemma.** *Suppose $B, C \subseteq A$, $t(a/A \cup \{b\}) \in F^{ae}(B)$ and $t(b/A) \in F^{ae}(C)$. Then $t(b/A \cup \{a\}) \in F^{ae}(C)$.*

**Proof.** Suppose $t^e(c/C) = t^e(b/C)$. For a contradiction, suppose $t^e(c/A \cup \{a\}) \neq t^e(b/A \cup \{a\})$. As before, we can find $a' \in \mathbf{M}$ such that $t^e((a', b)/A) = t^e((a, c)/A)$. Then $t^e(a'/B) = t^e(a/B)$ but $t^e(a'/A \cup \{b\}) \neq t^e(a/A \cup \{b\})$, a contradiction. □

**5.11 Lemma.** *Let $(A,(a_i,B_i)_{i<\alpha})$ be an ae-construction and $s:\beta\to\alpha$ be one-one and onto. If for all $i<\beta$, $B_{s(i)}\subseteq A\cup\bigcup_{j<i}a_{s(j)}$, then $(A,(a_{s(i)},B_{s(i)})_{i<\beta})$ is an ae-construction.*

**Proof.** Let $i<\beta$. For all $j\le\alpha$, we write $D_j=A\cup B_{s(i)}\cup\bigcup\{a_{s(k)}|\ k<i,\ s(k)<j\}$. By induction on $j\le\alpha$, we show that $t(a_{s(i)},D_j)\in F^{ae}(B_{s(i)})$. This is enough, since $D_\alpha=A\cup\bigcup_{k<i}a_{s(k)}$.

If $j\le s(i)+1$, then $D_j\subseteq A\cup\bigcup_{k<s(i)}a_k$ and so the claim clear. If $j$ is limit, then the claim follows from the induction assumption (using the definition of $ae$-isolation, not the equivalent forms from Lemma 5.2). So assume $j=k+1$ and $k>s(i)$. We may also assume that $D_j=D_k\cup\{a_k\}$, since otherwise there is nothing to prove. Then there is $m<i$ such that $s(m)=k$. By the assumption on $s$, $B_k\subseteq D_k$. Then $t(a_k,D_k\cup a_{s(i)})\in F^{ae}(B_k)$. By the induction assumption and Lemma 5.10, the claim follows. □

Let $(A,(a_i,B_i)_{i<\alpha})$ be an $ae$-construction. We say that $X\subseteq\alpha$ is closed if for all $i\in X$, $B_i\subseteq A\cup\{a_j|\ j\in X\}$.

**5.12 Lemma.** *For $l\in\{1,2\}$, let $(A,(a^l_i,B^l_i)_{i<\alpha^l})$ be an ae-construction of an ae-primary model $C^l$ over $A$. Assume that $f$ is a function and $X^l\subseteq\alpha^l$, $i\in\{1,2\}$, are closed sets such that $f:scl(A\cup\bigcup_{i\in X^1}a^1_i)\restriction(\rho-\{log\})\to scl(A\cup\bigcup_{i\in X^2}a^2_i)\restriction(\rho-\{log\})$ is an isomorphism and $f\restriction scl(A)=id$. Then for all $i^*<\alpha^1$, there are a function $g\supseteq f$ and closed $Y^l\subseteq\alpha^l$ such that $X^1\cup\{i^*\}\subseteq Y^1$, $X^2\subseteq Y^2$ and $g:scl(A\cup\bigcup_{i\in Y^1}a^1_i)\restriction(\rho-\{log\})\to scl(A\cup\bigcup_{i\in Y^2}a^2_i)\restriction(\rho-\{log\})$ is an isomorphism.*

**Proof.** The claim follows easily from Lemmas 5.11 and 5.6. □

We finish this section with two results that are not needed in this paper, but are needed if one wants to improve the non-structure results of this paper, see Remark 6.8.

**5.13 Proposition.** *If $B$ and $C$ are ae-primary models over $A$, then there is an isomorphism $f:B\restriction(\rho-\{log\})\to C\restriction(\rho-\{log\})$ such that $f\restriction scl(A)=id$.*

**Proof.** Let $(A,(b_i,B_i)_{i<\beta})$ and $(A,(c_i,C_i)_{i<\gamma})$ be $ae$-constructions of $B$ and $C$, respectively. By induction on $i\le\alpha=max\{\beta,\gamma\}$, we choose functions $f_i$ and closed sets $X_i\subseteq\beta$ and $Y_i\subseteq\gamma$ such that

(i) $f_0=id_{scl(A)}$ and $X_0=Y_0=\emptyset$,

(ii) for all $i<j$, $f_i\subseteq f_j$, $X_i\subseteq X_j$ and $Y_i\subseteq Y_j$,

(iii) $f_i:scl(A\cup\bigcup_{k\in X_i}b_k)\restriction(\rho-\{log\})\to scl(A\cup\bigcup_{k\in Y_i}c_k)\restriction(\rho-\{log\})$ is an isomorphism,

(iv) if $i<\beta$, then $i\in X_{i+1}$ and if $i<\gamma$, then $i\in Y_{i+1}$.

If $i$ is limit, we let $f_i=\cup_{j<i}f_j$, $X_i=\cup_{j<i}X_j$ and $Y_i=\cup_{j<i}Y_j$. Clearly these are as wanted. If $i=j+1$, then the existence of $f_i$, $X_i$ and $Y_i$ follows from Lemma 5.12. Clearly $f_\alpha$ is as required. □

**5.14 Lemma.** *Suppose $A$ is ae-saturated and $B$ is ae-primary over $A\cup\{a\}$. If $b\downarrow^d_A a$, then $b\downarrow^d_A B$.*

**Proof**. By Lemma 5.8, this can be proved essentially as Lemma 5.3. $\square$

## 6. Non-structure of $\mathbf{K}^e$

In this section we prove non-structure theorems for the class $\mathbf{K}^e = \{\mathcal{A} \restriction \rho - \{log\} |\ \mathcal{A} \in \mathbf{K}\}$.

**6.1 Definition.** *We say that $t(a/B)$ is orthogonal to $A \subseteq B$ if for all $C \supseteq B$ and $b$ the following holds: If $a \downarrow^d_B C$ and $b \downarrow^d_A C$, then $a \downarrow^d_C b$.*

**6.2 Lemma.** *$\mathbf{K}^e$ has 'dop' (see e.g. [Sh2]) i.e. there are $\mathcal{A}_i \subseteq \mathbf{M}$, $i < 4$, and $a \in \mathbf{M}$ such that*

*(i) for all $i < 4$, $\mathcal{A}_i = scl(\mathcal{A}_i)$,*

*(ii) $\mathcal{A}_0 = \mathcal{A}_1 \cap \mathcal{A}_2$, $\mathcal{A}_1 \downarrow^d_{\mathcal{A}_0} \mathcal{A}_2$ and $\mathcal{A}_3 = scl(\mathcal{A}_1 \cup \mathcal{A}_2)$,*

*(iii) $a \notin \mathcal{A}_3$ and $t(a/\mathcal{A}_3)$ is orthogonal to both $\mathcal{A}_1$ and $\mathcal{A}_2$.*

**Proof**. Choose $\mathcal{A}_i$, $i < 4$, so that (i) and (ii) hold and that there are elements $a_i \in \mathcal{A}_i - \mathcal{A}_0$ for $i \in \{1, 2\}$ such that $R(a_i/\mathcal{A}_0) = 2$.

**6.2.1 Claim.** *There is $a \notin \mathcal{A}_3$ such that $exp(a) = a \cdot exp(a_1) \cdot exp(a_2)$.*

**Proof**. Choose $a$ so that $a \notin \mathcal{A}_3$ and choose for rationals $q > 0$, $b_q$ so that $b_1 = a \cdot exp(a_1) \cdot exp(a_2)$ and for all integers $z > 0$, $(b_q)^z = b_{zq}$. For negative rationals $q$, let $b_q = (b_{-q})^{-1}$. Let $A$ be the relevant set generated by $\mathcal{A}_3 \cup \{a\}$ and define $f : A \to \mathbf{M}$ so that for all $b \in \mathcal{A}_3$ and rationals $q \neq 0$, $f(b + q \cdot a) = exp(b) \cdot b_q$. It suffices to prove that $f$ is a homomorphism from $(A, +)$ to $(\mathbf{M} - \{0\}, \cdot)$ and that the kernel of $f$ is $\pi Z$.

It is routine to check that $f$ is a homomorphism. So we prove only that $Ker(f) = \pi Z$. For this, let $b \in \mathcal{A}_3$ and $q \in \mathbf{Q}$ be such that $f(b + q \cdot a) = 1$. It suffices to show that $q = 0$. Suppose not. Then $exp(b) \cdot b_q \in \mathcal{A}_3$. Thus $b_q \in \mathcal{A}_3$. Since $\mathcal{A}_3$ is algebraically closed, $a \in \mathcal{A}_3$, a contradiction. $\square$ Claim 6.2.1.

We are left to prove the following claim:

**6.2.2 Claim.** *Let $a$ be as in Claim 6.2.1. Then $t(a/\mathcal{A}_3)$ is orthogonal to both $\mathcal{A}_1$ and $\mathcal{A}_2$.*

**Proof**. Let $\mathcal{A} \supseteq \mathcal{A}_3$ be such that $scl(\mathcal{A}) = \mathcal{A}$ and $a \downarrow^d_{\mathcal{A}_3} \mathcal{A}$. Then $R(a/\mathcal{A}) = 1$. Let $b$ be such that $b \downarrow^d_{\mathcal{A}_1} \mathcal{A}$. We need to prove that $a \downarrow^d_{\mathcal{A}} b$.

Let $\mathcal{B} = scl(\mathcal{A} \cup b)$. We show first that $a, exp(a) \notin \mathcal{B}$. For this it suffices to prove the following subclaim.

**6.2.2.1 Subclaim.**

*(i) Let $B$ be the (closed) relevant set generated by $\mathcal{A} \cup scl(\mathcal{A}_1 \cup b)$. Then $a$ and $exp(a)$ are transcendental over $B \cup exp[B]$.*

*(ii) Suppose $C \supseteq B$ is a closed relevant set and $a$ and $exp(a)$ are transcendental over $C \cup exp[C]$. Let $c$ be such that $exp(c)$ is algebraic over $C \cup exp[C]$ and $D$ the relevant set generated by $C \cup \{c\}$. Then $a$ and $exp(a)$ are transcendental over $D \cup exp[D]$.*

*(iii) Suppose $C \supseteq B$ is a closed relevant set and $a$ and $exp(a)$ are transcendental over $C \cup exp[C]$. Let $c$ be such that it is algebraic over $C \cup exp[C]$ and $D$ the relevant set generated by $C \cup \{c\}$. Then $a$ and $exp(a)$ are transcendental over $D \cup exp[D]$.*

**Proof.** (i): We show first that $a \notin B$. For a contradiction, suppose $a = c + d$ for some $c \in scl(\mathcal{A}_1 \cup \{b\})$ and $d \in \mathcal{A}$. Then

(1) $exp(c) \cdot exp(d) = a \cdot exp(a_1) \cdot exp(a_2) = c \cdot exp(a_1) \cdot exp(a_2) + d \cdot exp(a_1) \cdot exp(a_2)$.

So $c$ and $exp(c)$ realize a linear equation over $\mathcal{A}$. Since $scl(\mathcal{A}_1 \cup \{b\}) \downarrow_{\mathcal{A}_1} \mathcal{A}$, $c$ and $exp(c)$ realize a linear equation over $\mathcal{A}_1$ i.e. there are $p, r \in \mathcal{A}_1$ such that $exp(c) = p \cdot c + r$. Then (1) gives

(2) $c \cdot (exp(a_1) \cdot exp(a_2 - d) - p) + d \cdot exp(a_1) \cdot exp(a_2 - d) - r = 0$.

Since $c \notin \mathcal{A}$, (2) implies that

(3) $exp(a_1) \cdot exp(a_2 - d) - p = 0$.

This means that $exp(a_2 - d) \in \mathcal{A}_1$ i.e. $a_2 - d \in \mathcal{A}_1$. So we may assume that $d = a_2$. Then

(4) $c = a - a_2$ and (by (1),) $exp(c) = a \cdot exp(a_1)$.

Clearly these do not realize any linear equation over $\mathcal{A}_1$, a contradiction.

But then $a$ is transcendental over $B \cup exp[B]$ since otherwise

$deg((a, exp(a))/B \cup exp[B]) = 0$

and $dim(a/B) = 1$ contradicting the fact that $B$ is closed. Similarly, $exp(a)$ is transcendental over $B \cup exp[B]$.

(ii): Again we show first that $a \notin D$. For a contradiction, suppose there are $d \in C$ and $q \in \mathbf{Q}$ such that $a = d + qc$. Then '$exp(a) = exp(d) \cdot exp(c)^q$'. This is impossible since by our assumptions, '$exp(d) \cdot exp(c)^q$' is algebraic over $C \cup exp[C]$. Since $D$ is closed, as one can easily see, the claim follows as in (i).

(iii): The proof goes essentially as in (ii) (but is easier). □ Subclaim 6.2.2.1.

Now we continue as in the proof of Lemma 4.4: By Lemma 4.3, if $a \not\downarrow^d_{\mathcal{A}} b$, then $scl(\mathcal{A} \cup \{a\}) \cap \mathcal{B} \not\subseteq \mathcal{A}$. For a contradiction, suppose that this is the case. Then we can find $d_i \in scl(\mathcal{A} \cup \{a\})$, $i \leq n$, and $k < n$ such that

(a) $(d_i)_{i \leq n}$ is linearly independent,

(b) if $D$ is the relevant set generated by $\{d_0, ..., d_n\}$, then $D \cap \mathcal{A}$ is generated by $\{d_0, ..., d_{k-1}\}$,

(c) if $C$ is the relevant set generated by $\{d_0, ..., d_{n-1}\}$, then $C \cap \mathcal{B} = C \cap \mathcal{A}$,

(d) $D \cap \mathcal{A}$ is closed,

(e) $d_k = a$ and for all $k < i \leq n$, either $d_i$ or $exp(d_i)$ is algebraic over $\{d_0, ..., d_{i-1}, exp(d_0), ..., exp(d_{i-1})\}$,

(f) $d_n, exp(d_n) \in \mathcal{B}$.

Then

$$deg((d_k, ..., d_{n-1}, exp(d_k), ..., exp(d_{n-1}))/\mathcal{B}) <$$

$$deg((d_k, ..., d_{n-1}, exp(d_k), ..., exp(d_{n-1}))/\mathcal{A}) = n - k$$

and $dim((d_k,...,d_{n-1})/\mathcal{B}) = n-k$. This contradicts the fact that $\mathcal{B}$ is closed. □ Claim 6.2.2.

□

We can now prove an easy non-structure theorem.

**6.3 Proposition.** *For all regular uncountable cardinals $\kappa$, there are $2^\kappa$ pairwise non-isomorphic models of power $\kappa$ in $\mathbf{K}^e$.*

**Proof.** Let $I = (I,<)$ be a linear ordering and choose $\mathcal{A}_i$, $i<4$, $a_i \in \mathcal{A}_i - \mathcal{A}_0$, $i \in \{1,2\}$ and $a$ as in (the proof of) Lemma 6.2. Clearly we may assume in addition that $\mathcal{A}_0 = \mathcal{A}_{pr}$ and for $i \in \{1,2\}$, $\mathcal{A}_i = scl(\mathcal{A}_0 \cup \{a_i\})$ $(= scl(\{a_i\}))$. For all $i \in I$, choose $b_i$ and $c_i$ so that for all $i \in I$,

(i) $(b_i,c_i) \downarrow^d_{\mathcal{A}_0} \{b_j,c_j |\ j<i\}$,

(ii) there is an automorphism $F$ of $\mathbf{M}$ such that $F \restriction \mathcal{A}_0 = id$, $F(b_i) = a_1$ and $F(c_i) = a_2$.

By homogeneity, we may choose these so that

(iii) $((b_i,c_i)_{i\in I}$ is order indiscernible over $\mathcal{A}_0$.

By Lemma 4.3 (iv) and (i) above, for all $i,j \in I$, $i \neq j$,

(iv) there is an isomorphism

$$f : scl(\mathcal{A}_0 \cup \{b_i,c_j\}) \restriction (\rho - \{log\}) \to scl(\{a_1,a_2\}) \restriction (\rho - \{log\})$$

such that $f \restriction \mathcal{A}_0 = id$, $f(b_i) = a_1$ and $f(c_j) = a_2$.

So for all $i,j \in I$, $i<j$, we can find $d_{ij}$ so that $d_{ij} \notin scl(\{b_i,c_j\})$ and $exp(d_{ij}) = d_{ij} \cdot exp(b_i) \cdot exp(c_j)$. Clearly these can be chosen so that for all $i<j$,

$$d_{ij} \downarrow^d_{scl(\{b_i,c_j\})} \{b_k,c_k |\ k \in I\} \cup \{d_{kl} |\ k<l,\ (i,j) \neq (k,l) \in I^2\}$$

(in fact we can not choose the elements in any other way). Again by homogeneity, we can find these so that in addition,

(v) for all partial isomorphism $g : I \to I$ there is an automorphism $f$ such that $f \restriction \mathcal{A}_0 = id$, for all $i \in dom(g)$, $f(b_i) = b_{g(i)}$ and $f(c_i) = c_{g(i)}$ and for all $i,j \in I$, $i<j$, $f(a_{ij}) = a_{g(i)g(j)}$.

We let $\mathcal{A}_I$ be $scl(\{b_i,c_i |\ i \in I\} \cup \{a_{ij} |\ i,j \in I,\ i<j\})$.

**6.3.1 Claim.** *For all $i,j \in I$, $i<j$ iff there is $d \in \mathcal{A}_I$ such that $d \notin scl(\{b_i,c_j\})$ and $exp(d) = d \cdot exp(b_i) \cdot exp(c_j)$.*

**Proof.** If $i<j$, then clearly $d$ exists. So assume that $j \leq i$. Let $d$ $(\in \mathbf{M})$ be such that $d \notin scl(\{b_i,c_j\})$ and $exp(d) = d \cdot exp(b_i) \cdot exp(c_j)$. We show that $d \notin \mathcal{A}_I$.

Let $A = \{b_k,c_l |\ k,l \in \mathrm{I}, k \neq i,\ l \neq j\} \cup \{d_{k,l} |\ k,l \in I,\ k<l,\ k \neq i,\ l \neq j\}$. Then $A \downarrow^d_{\mathcal{A}_0} \{b_i,c_j\}$. Since $t(d/scl(\{b_i,c_j\}))$ is orthogonal to $\mathcal{A}_0$, $d \notin scl(\{b_i,c_j\} \cup A)$. Denote $A_b = scl(\{b_i\} \cup A)$ and $A_c = scl(\{c_j\} \cup A)$. By the basic properties of $\downarrow^d$, $A_b \downarrow^d_{scl(A)} A_c$. Thus, by the proof of Lemma 6.2, $t(d/scl(\{b_i,c_j\} \cup A))$ is orthogonal to both $A_b$ and $A_c$.

Let $B = \{d_{ik} | \ i < k\}$ and $C = \{d_{kj} | \ k < j\}$. A standard application of the basic properties of our independence notion gives that

(a) $B \downarrow^{d}_{A_b} scl(\{b_i, c_j\} \cup A)$,

(b) $C \downarrow^{d}_{A_c} scl(\{b_i, c_j\} \cup A \cup B)$.

By (a) and Lemma 6.2, $d \notin scl(B \cup A \cup \{b_i, c_j\})$ and $t(d/scl(B \cup A \cup \{b_i, c_j\}))$ is orthogonal to $A_c$. So by (b), $d \notin scl(A \cup B \cup C \cup \{b_i, c_j\}) = \mathcal{A}_I$. □ Claim 6.3.1.

Let $\phi(x, y)$ be the $L_{\omega_1\omega}$-formula in the similarity type $\rho - \{log\}$, which says that there is $z$ such that $z \notin scl(\{x, y\})$ and $exp(z) = z \cdot exp(x) \cdot exp(y)$. Let $\psi(x_0, x_1, y_0, y_1) = \phi(x_0, y_1) \wedge \neg\phi(x_1, y_0)$. By Claim 6.3.1 and (v) above, it is easy to see that if $I$ is a dense linear ordering, then $((b_i, c_i))_{i \in I}$ is weakly $(\omega, \psi)$-skeleton like in $\mathcal{A}_I$ (see [Sh3]). Thus the claim in our proposition follows immediately from [Sh3] (see e.g [Hy2] where this kind of argument is done in details). □

Since the models in Proposition 6.3 are not necessarily existentially closed in the class $\mathbf{K}^e$, the proposition is not completely satisfactory. So we try to do better.

**6.4 Lemma.** *For all linear orderings $\eta = (\eta, <)$, there is an $ae$-saturated model $\mathcal{A} = \mathcal{A}_\eta$ for which the following holds: There are elements $b_i, c_i \in \mathcal{A} - scl(\emptyset)$, $i \in \eta$, such that*

*(i) $b_i \downarrow^{d}_{\emptyset} \{b_j, c_j | \ j < i\}$, $c_i \downarrow^{d}_{\emptyset} \{b_j, c_j | \ j < i\} \cup \{b_i\}$ and both $\{b_i\}$ and $\{c_i\}$ are closed,*

*(ii) there are sets $I_{ji} = \{a^{ji}_k | \ k < \mathfrak{c}^+\}$, $i, j \in \eta$ and $j < i$, such that*

*(a) $\mathcal{A}$ is $ae$-primary over $\{b_i, c_i | \ i \in \eta\} \cup \bigcup\{I_{ji} | \ j, i \in \eta, \ j < i\}$,*

*(b) if $g$ is a partial order-preserving function from $\eta$ to $\eta$, then there is an automorphism $f$ of $\mathbf{M}$ such that for all $j < i$ from $\eta$ and $k < \mathfrak{c}^+$, $f(b_j) = b_{g(j)}$, $f(c_j) = c_{g(j)}$ and $f(a^{ji}_k) = a^{g(j)g(i)}_k$,*

*(c) for all $j < i$, $I_{ji} \downarrow^{d}_{scl(\{c_j, b_i\})} \{b_k, c_k | \ k \in \eta\} \cup \bigcup\{I_{kl} | \ k < l, (k, l) \neq (j, i)\}$,*

*(iii) $j < i$ iff $(*)_{ji}$ below holds:*

*$(*)_{ji} = (*)^{\mathcal{A}}_{c_j, b_i}$ : there are $a_k \in \mathcal{A}$, $k < \mathfrak{c}^+$, such that*

*(a) $a_k \notin scl(\{c_j, b_i\})$,*

*(b) $a_k \downarrow^{d}_{scl(\{c_j, b_i\})} \{a_p | \ p < k\}$,*

*(c) $exp(a_k) = a_k \cdot exp(c_j) \cdot exp(b_i)$.*

**Proof.** First, we want to choose elements $b_i, c_i \in \mathbf{M} - scl(\emptyset\}$, $i \in \eta$, so that (i) holds and for all $j < i$, we want to choose choose $I_{ji} = (a^{ji}_k)_{k < \mathfrak{c}^+}$ so that (a)-(c) from (iii) hold for these elements. In addition, we want to choose these so that (ii) (c) holds. If $\eta$ is an ordinal, it is easy to see that such elements can be found. But then by Erdös-Rado-theorem (and homogeneity of $\mathbf{M}$), we can find these for any linear ordering $\eta$ and in addition so that (ii) (b) holds. Letting $\mathcal{A}$ be an $ae$-primary model over $\{b_i, c_i | \ i < \kappa\} \cup \bigcup\{I_{ji} | \ j < i < k\}$, (i) and (ii) hold. We show that also (iii) holds.

If $j < i$, then clearly $(*)_{ji}$ holds. So suppose $i \leq j$ and for a contradiction, assume that $(*)_{ji}$ holds. Let $a_k$, $k < \mathfrak{c}^+$, witness the assumption that $(*)_{ji}$ holds.

Let $A = \{c_k|\ k \in \eta,\ k \neq j\} \cup \{b_k|\ k \in \eta,\ k \neq i\} \cup \bigcup\{I_{kl}|\ k < l,\ k \neq j,\ l \neq i\}$, $A_c = scl(A \cup \{c_j\})$ and $A_b = scl(A \cup \{b_i\})$. Then it is easy to see that $A_c \downarrow^d_{scl(A)} A_b$. Also by (the proof of) Lemma 6.2, for all $k < \mathfrak{c}^+$, $t(a_k/\{c_j, b_i\})$ is orthogonal to $\emptyset$ and since $A \downarrow^d_\emptyset \{c_j, b_i\}$, it is easy to see that

(iv) $\{a_k|\ k < \mathfrak{c}^+\} \downarrow^d_{scl(\{c_j,b_i\})} scl(A \cup \{c_j, b_i\})$.

Thus by applying Lemma 6.2 again, we see that

(v) for all $k < \mathfrak{c}^+$, $t(a_k/scl(A \cup \{c_j, b_i\}))$ is orthogonal to both $A_c$ and $A_b$.

Let $C = \cup\{I_{jl}|\ j < l\}$ and $B = \cup\{I_{ki}|\ k < i\}$. Then $C \downarrow^d_{A_c} scl(A_c \cup A_b)$ and $B \downarrow^d_{A_b} scl(C \cup A_c \cup A_b)$. From this and (v) a routine argument gives

(vi) $\{a_k|\ k < \mathfrak{c}^+\} \downarrow^d_{scl(A_c \cup A_b)} scl(A_c \cup A_b \cup C \cup B)$.

So by (iv) and (vi), for all $k < \mathfrak{c}^+$, $a_k \downarrow^d_{scl(A_c \cup A_b \cup C \cup B)} \{a_p|\ p < k\}$. Also the proofs of (iv) and (vi) imply that for all $k < \mathfrak{c}^+$, $t(a_k/scl(A_c \cup A_b \cup C \cup B)) \in F^{ae}(\{c_j, b_i\})$. This contradicts Lemma 5.9. □

**6.5 Lemma.** *There is a(n $L_{\mathfrak{c}^{2+}\mathfrak{c}^{2+}}$-)formula $\Phi(x, y)$ in the similarity type $\rho - \{log\}$ such that for all submodels $\mathcal{A} = scl(\mathcal{A})$ of $\mathbf{M}$ and $c, b \in \mathcal{A}$, $\mathcal{A} \models \Phi(c, b)$ iff $\{c, b\}$ is closed and $(*)^{\mathcal{A}}_{c,b}$ holds. Furthermore for all submodels $\mathcal{B} \subseteq \mathcal{C}$ of $\mathbf{M}$, if $scl(\mathcal{B}) = \mathcal{B}$ and $\mathcal{B} \models \Phi(c, b)$, then $\mathcal{C} \models \Phi(c, b)$.*

**Proof.** Clearly 'is $\{x, y\}$ a closed set?' can be decided in $\mathbf{M} \restriction (\rho - \{log\})$ (and it is expressible in $L_{\omega_1\omega}$). Also it is easy to see that if $\{c, b\}$ is closed and $scl(\mathcal{A}) = \mathcal{A}$ then $(*)^{\mathcal{A}}_{c,b}$ is expressible with an $L_{\mathfrak{c}^{2+}\mathfrak{c}^{2+}}$-formula and in the similarity type $\rho - \{log\}$ (notice that by Lemma 4.3, if $\{c, b\}$ is closed, then for $a_k \downarrow^d_{scl(\{c,b\})} \{a_p|\ p < k\}$ it suffices to say that if $z \in scl(\{c, b\} \cup \{a_p|\ p < k\}) \cap scl(\{c, b, a_k\})$ then $z \in scl(\{c, b\})$ and that for $X = \{c, b\}$ or $X = \{c, b, a_k\}$ or $X = \{c, b\} \cup \{a_p|\ p < k\}$, $scl(X)$ is the least $Y \subseteq \mathcal{A}$ such that $X \subseteq Y$, $Y$ is closed under $exp$ and $exp^{-1}$ and $Y$ is algebraically closed subset of $F_{\mathcal{A}}$). The furthermore part is immediate by the definition of $(*)^{\mathcal{A}}_{c,b}$. □

**6.6 Theorem.** *Suppose $\kappa > (2^{\mathfrak{c}})^+$ is a regular cardinal. Then there are ae-saturated models $\mathcal{A}_i \in \mathbf{K}$, $i < 2^\kappa$, of power $\kappa$ such that for all $i < j < 2^\kappa$, $\mathcal{A}_i \restriction (\rho - \{log\}) \not\cong \mathcal{A}_j \restriction (\rho - \{log\})$.*

**Proof.** Let $\eta$ be a linear ordering and $\mathcal{A}_\eta$ as in Lemma 6.4. By Lemma 5.11, Lemma 5.6 and Lemma 6.5, it is easy to see that if $\eta$ is $\mathfrak{c}^+$-dense, then $((b_i, c_i))_{i \in \eta}$ is weakly $(\kappa(\mathbf{M}), \Phi(x_1, y_0) \wedge \neg\Phi(x_0, y_1))$-skeleton like in $\mathcal{A}_\eta$. Thus the claim follows from [Sh3] (again, see e.g. [Hy2]). □

**6.7 Remark.** *By working harder, the assumptions on $\kappa$ in Theorem 6.6 can be weakened a lot (e.g. under suitable cardinal arithmetic assumptions $\mathfrak{c}$ is a possible value for $\kappa$).*

**6.8 Remark.** *The ideas behind the construction in the proof of Theorem 6.6 can be used to get non-structure theorems for $\mathbf{K}^e$ that are much stronger than Theorem 6.6, see [HT] or [HS4] and e.g. [FHR] and [HHR].*

## References


[BS] J. Baldwin and N. Shi, Stable generic structures, Annals of Pure and Applied Logic, vol. 79, 1996, 1-35.

[BL] S. Buechler and O. Lessmann, Simple homogeneous models, Journal of the American Mathematical Society, to appear.

[FHR] Sy D. Friedman, T. Hyttinen and M. Rautila, Classification theory and $0^{\#}$, The Journal of Symbolic Logic, to appear.

[HHR] T. Huuskonen, T. Hyttinen and M. Rautila, Potential isomorphism and nonstructure, Archive for Mathematical Logic, to appear.

[Hy1] T. Hyttinen, Canonical finite diagrams and quantifier elimination, Mathematical Logic Quarterly, to appear.

[Hy2] T. Hyttinen, On nonstructure of elementary submodels of a stable homogeneous structure, Fundamenta Matematicae, vol. 156, 1998, 167-182.

[Hy3] T. Hyttinen, On canonical finite diagrams and quantifier elimination, Mathematical Logic Quarterly, vol. 48, 2002, 533-554.

[HS1] T. Hyttinen and S. Shelah, On the number of elementary submodels of an unsuperstable homogeneous structure, Mathematical Logic Quarterly, vol. 44, 1998, 354-358.

[HS2] T. Hyttinen and S. Shelah, Strong splitting in stable homogeneous models, Annals of Pure and Applied Logic, vol. 103, 2000, 201-228.

[HS3] T. Hyttinen and S. Shelah, Main gap for locally saturated elementary submodels of a homogeneous structure, The Journal of Symbolic Logic, vol. 66, 2001, 1286-1302.

[HS4] T. Hyttinen and S. Shelah, Constructing strongly equivalent nonisomorphic models for unsuperstable theories, part C, The Journal of Symbolic Logic, vol. 64, 1999, 634-642.

[HT] T. Hyttinen and H. Tuuri: Constructing strongly equivalent nonisomorphic models for unstable theories, Annals of Pure and Applied Logic, vol. 52, 1991, 203-248.

[Sh1] S. Shelah, Finite diagrams stable in power, Annals of Mathematical Logic, vol. 2, 1970, 69-118.

[Sh2] S. Shelah, Classification Theory, Studies in Logic and Foundations of Mathematics, no. 92, North-Holland, Amsterdam, 1990, 2nd revised edition.

[Sh3] S. Shelah, Non-structure Theory, to appear.

[Zi1] B. Zilber, Fields with pseudo-exponentiation, Proceedings of Maltsev Conference, Novosibirsk, to appear.

[Zi2] B. Zilber, Covers of the multiplicative group of an algebraically closed field of characteristic zero, manuscript.



Department of Mathematics
P.O. Box 4
00014 University of Helsinki
Finland